THE CONVERGENCE OF MEASURES ON PARAMETRIC SURFACES

BY

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1. **Introduction.** In [2], H. Federer has proved several theorems, one of which is the following. Let X be a compact differentiable k-manifold of class ∞ and f be a continuous map of X into R^n. To each smooth map f_i of X into R^n corresponds a measure μ_i over the middle space M_f of f, whose values are k-dimensional currents in R^n. This measure associates with any continuous, real-valued function χ on M_f the current $f_{i\#}\{X \wedge (\chi \circ m_f)\}$ given by the formula

$$f_{i\#}\{X \wedge (\chi \circ m_f)\}(\phi) = \int_X (\chi \circ m_f) \wedge f_i^{\#}(\phi),$$

whenever ϕ is a differential k-form of class ∞ on R^n. m_f denotes the monotone component in the monotone light factorization of f.

If f has finite Lebesgue area and either $k = 2$ or the range of f has $k+1$ dimensional Hausdorff measure zero, then there exists a unique current valued measure μ over M_f such that for every sequence of smooth maps f_i, which converge uniformly to f and whose areas are bounded, the measures μ_i converge weakly to μ.

Federer also treats the case where X is not compact but f is proper; i.e., $f^{-1}(Y)$ is compact for every compact $Y \subset R^n$.

It is the purpose of this paper to generalise this theorem (except for the special case $k = 2$) to cover the case of a manifold with boundary. (Actually, an oriented pseudo-manifold is used.) The following result is obtained.

Let M be a k-dimensional pseudo-manifold ($k \geqq 1$) and C an integral k-chain on M such that the support of $\partial(c)$ is the boundary of M. Let f be a continuous mapping of $|M|$ into R^n with middle space $\mathscr{M}$ and monotone light factorization $f = l_f \circ m_f$. As with Federer's smooth maps, there corresponds to each quasi-linear mapping f_i of $|M|$ into R^n, a measure μ_i over M whose values are k-dimensional currents in R^n. For each continuous function χ on M,

$$\{\mu_i(\chi)\}(\phi) = \sum_\sigma \int_{A_\sigma} c(T_\sigma) \cdot \chi[m\{T_\sigma(\chi)\}] \wedge [(f_i \circ T_\sigma)^{\#}(\phi)]$$

where ϕ is a differential k-form of class ∞ on R^n, the summation is taken over all k-simplexes σ of M, each T_σ is a linear nonsingular mapping of a subset A_σ of

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R^k onto σ and $c(T_\sigma)$ is the coefficient of the chain c on the orientation of σ corresponding to T_σ.

If f has finite Lebesgue area and the range of f has $k+1$ dimensional Hausdorff measure zero, then there exists a unique current valued measure μ over $\mathscr{M}$ such that for every sequence of quasi-linear mappings f_i which converge uniformly to f and whose elementary areas converge to the Lebesgue area of f, the measures μ_i converge weakly to μ.

No attempt is made to generalise the other theorems of Federer's paper.

2. **Preliminaries.** Throughout the paper we are concerned with (finite) geometric complexes and their integral chains. The definition adopted for a geometric complex will be that of Pontrjagin [8], in which the simplexes are closed sets.

We follow Lefschetz [5] in defining subdivisions; i.e., we define K_1 to be a subdivision of K if $|K_1| = |K|$ and for every simplex ζ of K_1, there exists a simplex σ of K containing ζ.

As in Whitney [9], a pseudo-manifold of dimension k (where $k \geqq 1$) is a geometric complex K with the following two properties:

(i) every simplex of K is a face of at least one k-simplex;

(ii) every $(k-1)$-simplex is a face of at most two k-simplexes.

No connectivity properties are assumed. The boundary $\mathscr{B}(K)$ of K is the subcomplex of K formed by the closure of the set of all those $(k-1)$-simplexes, each of which is a face of only one k-simplex.

2.1. Quasi-linear mappings. For the purposes of this paper a quasi-linear mapping of a geometric complex L into a euclidean space R^n is a continuous mapping g of $|L|$ into R^n for which there exists a subdivision L_1 of L with g linear on each simplex of L_1. Clearly

2.1.1. If L_* is a geometric complex with $|L_*| \subseteq |L|$, then the restriction of g to $|L_*|$ is quasi-linear with respect to L_*.

2.2. Induced measures. Let K be a geometric complex and c a nonzero integral k-chain on K ($k \geqq 1$). Denote by M and N the sub-complexes of K that support c and $\partial(c)$ respectively. Let g be a quasi-linear mapping of M into R^n. We associate with g, in the following way, a measure $\nu(g)$ on the space C of real-valued continuous functions on $|M|$ whose values are k-dimensional currents in R^n. For each $\psi \in C$ and each differential k-form ϕ of R^n of class ∞,

$$[\{\nu(g)\}(\psi)](\phi) = \sum_\sigma \int_{A_\sigma} c(T_\sigma)\cdot\psi[T_\sigma(x)] \wedge [(g\circ T_\sigma)^*(\phi)]\,dx,$$

where the summation is taken over all k-simplexes σ of M, each T_σ is a linear nonsingular mapping of a subset A_σ of R^k onto σ and $c(T_\sigma)$ is the coefficient of the chain c on the orientation of σ given by the mapping T_σ. {The notation in the above formula is that of [3].}

If B is a Borel subset of $|M|$, then the elementary area of g on B is

$$a(g, B) = \sum_{\sigma^k \in M_1} \left[\sum_{P \in P_k^n} \{k\text{-dimensional measure of } (P \circ g)(B \cap \sigma)\}^2 \right]^{1/2}$$

where M_1 is a subdivision of M such that g is linear on each simplex of M_1 and P_k^n denotes the collection of all those projections from R^n to R^k formed by deleting $n-k$ of the coordinates.

2.3. Lebesgue area. Let K, c, M and N be as in 2.2. Let f be a continuous mapping of $|M|$ into R^n. For the purposes of this paper, the Lebesgue area of f will be

$$L(f) = \inf \lim_{r \to \infty} \inf a(f^{(r)}, |M|),$$

where the infimum is taken over all sequences $\{f^{(r)}\}$ of quasi-linear mappings of M into R^n such that $f^{(r)} \to f$ uniformly as $r \to \infty$.

3. **Retraction of the boundary.** In this section we prove a theorem to the effect that the boundary of a surface with finite Lebesgue area makes only a small contribution to the area.

We will be dealing with geometric complexes and the following notation is adopted. $\Pi(K)$ denotes the barycentric subdivision of a geometric complex K. We denote the closure of a collection L of simplexes of K by $\mathrm{Cl}(L)$. Of the several definitions in use for the star $\mathrm{St}(\sigma)$ of a simplex σ of K we select the one that defines $\mathrm{St}(\sigma)$ to be the collection of all these simplexes with σ for a face (σ being regarded as a face of itself). K^r will denote the r-dimensional skeleton of K.

3.1. Pseudo-manifolds. We obsesve that a pseudo-manifold K has the following properties:

3.1.1. If $\dim k \geqq 1$, then either $\mathscr{B}(K) = \emptyset$ or $\dim \mathscr{B}(K) = \dim K - 1$.

3.1.2. $\Pi(K)$ is a pseudo-manifold and $\mathscr{B}\{\Pi(K)\} = \Pi\{\mathscr{B}(K)\}$.

3.2. Definition. We define an F-complex to be a nonempty pseudo-manifold K of dimension $k \geqq 1$ and such that: for every pair σ_1^k, σ_2^k of distinct k-simplexes of K, there exists a sequence

$$\sigma_1^k = \xi_0^k, \xi_1^k, \cdots, \xi_r^k = \sigma_2^k$$

of distinct k-simplexes of K such that for each $i = 1, \cdots, r$, ξ_{i-1} and ξ_i have a common $(k-1)$-face.

Evidently

3.2.1. If K is an F-complex, then $\Pi(K)$ is also an F-complex.

3.3. Definition. Let K be a pseudo-manifold with dimension $k \geqq 1$. There exists a unique set $K_1, K_2, \cdots, K_p$ of subcomplexes of K with the following properties:

(i) each K_i is an F-complex,

(ii) no two of $K_1, K_2, \cdots, K_p$ have a k-simplex or a $(k-1)$-simplex in common;

(iii) $K = K_1 \cup K_2 \cdots \cup K_p$.

$K_1, K_2, \cdots, K_p$ will be called the F-components of K.

Evidently

3.3.1. $\mathscr{B}(K) = \mathscr{B}(K_1) \cup \mathscr{B}(K_2) \cup \cdots \cup \mathscr{B}(K_p)$.

3.4. Lemma. *If K is an F-complex of dimension $k \geqq 1$ and σ^{k-1} is a $(k-1)$-simplex of $\mathscr{B}(K)$, then there exists a continuous mapping ϕ of $|K|$ into $|K^{k-1}|$ such that*

$$\phi(x) = x$$

for all $x \in |\mathscr{B}(K) \sim \sigma^{k-1}| \cup |K^{k-2}|$.

Proof. Since K is an F-complex we can arrange its k-simplexes into a sequence

$$\sigma_1^k, \sigma_2^k, \cdots, \sigma_p^k$$

in such a way that σ^{k-1} is a face of σ_1^k and for $i = 2, 3, \cdots, p$, σ_i^k has a $(k-1)$-face σ_i^{k-1} in common with one of $\sigma_1^k, \sigma_2^k, \cdots, \sigma_{i-1}^k$. Put $\sigma_1^{k-1} = \sigma^{k-1}$. For each $i = 1, 2, \cdots, p$, let K_i be the subcomplex obtained by removing $\sigma_1^k, \sigma_2^k, \cdots, \sigma_i^k$, $\sigma_1^{k-1}, \cdots, \sigma_i^{k-1}$ from K. Put $K_0 = K$.

For each $i = 1, \cdots, p$ there exists a continuous mapping ψ_i of σ_i^k into $|\mathscr{B}(\sigma_i^k) \sim \sigma_i^{k-1}|$ such that $\psi_i(x) = x$ for all $x \in |\mathscr{B}(\sigma_i^k) \sim \sigma_i^{k-1}|$. Define

$$\begin{aligned} \phi_i(x) &= \psi_i(x) \text{ for } x \in \sigma_i^k \\ &= x \text{ for } x \in |K_{i-1}| \sim \sigma_i^k. \end{aligned}$$

Then ϕ_i is a continuous mapping of $|K_{i-1}|$ into $|K_i|$ such that $\phi_i(x) = x$ for all $x \in |K_i|$. Define

$$\phi = \phi_p \phi_{p-1} \cdots \phi_1.$$

Then ϕ is a continuous mapping of $|K_0| = |K|$ into $|K_p| \subseteq |K^{k-1}|$ such that

$$\phi(x) = x$$

for all $x \in |K_p| \supseteq |\mathscr{B}(K) \sim \sigma^{k-1}| \cup |K^{k-2}|$.

3.5. Lemma. *Let K be a pseudo-manifold of dimension $k \geqq 1$ and let L be a sub-pseudo-manifold of K such that each F-component of L has a $(k-1)$-simplex in common with $\mathscr{B}(K)$.*

There exists a continuous mapping ψ of $|K|$ into $|K|$ and a subdivision $\Pi^r(K)$ of K such that

(i) $\psi(|L|) \subseteq |L^{k-1}|$;

(ii) $\psi(x) = x$ *for all* $x \in |\mathrm{Cl}(K \sim L)|$ *and*

(iii) *for each k-simplex ξ^k of $\Pi^r(L)$, there exists a $(k-1)$-simplex σ^{k-1} of L^{k-1} with $\psi(\xi^k) \subseteq \sigma^{k-1}$ and with ψ linear on ξ^k.*

Proof. Let $L_1, L_2, \cdots, L_p$ be the F-components of L. Let σ_i^{k-1} be a $(k-1)$-simplex that L_i has in common with $\mathscr{B}(K)$. Then $\sigma_i^{k-1} \in \mathscr{B}(L_i)$, hence by 4.4, there is a continuous mapping ϕ_i of $|L_i|$ into $|L_i^{k-1}|$ such that

$$\phi_i(x) = x$$

for all $x \in |\mathscr{B}(L_i) \sim \sigma_i^{k-1}|$. Define

$$(1) \qquad \begin{aligned} \phi(x) &= x \qquad \text{if } x \in |\mathrm{Cl}(K \sim L)| \\ &= \phi_i(x) \quad \text{if } x \in L_i. \end{aligned}$$

Then ϕ is a continuous mapping of $|K|$ into $|K|$ and

$$(2) \qquad \phi(|L|) \subseteq |L^{k-1}|.$$

Choose a positive integer r such that for every vertex x of $\Pi^r(L)$, there exists a vertex y of L^{k-1} with

$$(3) \qquad \phi\{|\mathrm{St}(x) \text{ in } \Pi^r(L)|\} \subseteq S(y)$$

where $S(y)$ denotes the union of the interiors of the simplexes of $\mathrm{St}(y)$ in L^{k-1}. Define a function ψ from $\{\Pi^r(L)\}^0$ to $|L|$ such that

$$(4) \qquad \psi(x) = x$$

if $x \in \{\Pi^r(L)\}^0 \cap |\mathrm{Cl}(K \sim L)|$ and such that when $x \in \{\Pi^r(L)\}^0 \sim |\mathrm{Cl}(K \sim L)|$ $\psi(x)$ is a vertex of L^{k-1} with

$$(5) \qquad \phi\{|\mathrm{St}(x) \text{ in } \Pi^r(L)|\} \subseteq S\{\psi(x)\}.$$

We will now prove that if $v_0, v_1, \cdots, v_k$ are the vertices of a k-simplex of $\Pi^r(L)$, then there exists a simplex σ of L^{k-1} such that

$$(6) \qquad \psi(v_0), \cdots, \psi(v_k) \in \sigma.$$

Since $r \geqq 1$, there is at least one v_i not in $|\mathrm{Cl}(K \sim L)|$. If no v_i is in $|\mathrm{Cl}(K \sim L)|$, then (6) follows immediately from (5). If $v_0, \cdots, v_s$ are in $|\mathrm{Cl}(K \sim L)|$ but $v_{s+1}, \cdots, v_k$ are not, then by (4), there exists a simplex σ_1 of L^{k-1} containing $\psi(v_0), \cdots, \psi(v_s)$. But

$$v_0 \in |\mathrm{St}(v_j) \text{ in } \Pi^r(L)|$$

so that by (5)

$$\phi(v_0) \in S\{\psi(v_j)\}$$

for $j = s+1, \cdots, k$; hence since $\phi(v_0) = v_0$ it follows from (4) that

$$\psi(v_0) \in S\{\psi(v_j)\}, \qquad j = s+1, \cdots, k.$$

Then

$$\sigma_1 \in \mathrm{St}\{\psi(v_j)\} \text{ in } L^{k-1}$$

for $j = s+1, \cdots, k$; i.e., each $\psi(v_j)$ is a vertex of σ. Thus (6) is true.

We can now extend ψ to a continuous mapping of $|K|$ into $|K|$ by putting

$$\psi(x) = x \tag{7}$$

for all $x \in |\mathrm{Cl}(K \sim L)|$ and extending ψ linearly over each k-simplex of $\Pi^r(L)$. It follows immediately from (6) and (7), that ψ has the required properties.

3.6. THEOREM. *Let K be a pseudo-manifold of dimension $k \geqq 1$ and c be a nonzero integral k-chain on K such that $\partial(c)$ is supported by $\mathscr{B}(K)$. Let f be a continuous mapping of $|K|$ into R^n $(n \geqq k)$ with finite Lebesgue area and $\{f^{(r)}\}$ be a sequence of quasi-linear mappings of K into R^n such that*

$$\|f^{(r)} - f\| \to 0$$

and

$$a(f^{(r)}, |K|) \to L(f)$$

as $r \to \infty$. Let $\varepsilon > 0$ and B be a dense subset of R^1.

There exists a finite number $J_1, \cdots, J_p$ of closed intervals of R^n such that:

(i) *$J_1, \cdots, J_p$ have mutually disjoint interiors and the coordinates of their vertices are all in B;*

$$f(|\mathscr{B}(K)|) \subseteq \mathrm{Int}\left\{\bigcup_{j=1}^{p} J_j\right\}; \tag{ii}$$

(iii) *for every $\delta > 0$, there exists a subdivision $\Pi^l(K)$ of K and a sub-pseudo-manifold L of $\Pi^l(K)$ such that every point of $f(|\mathscr{B}(L)|)$ has a distance less than δ from $\bigcup_{j=1}^{p} Fr(J_j)$ and*

$$\limsup_{r \to \infty} a(f^{(r)}, |K| \sim |L|) < \varepsilon.$$

Proof. We can construct an infinite sequence $\mathscr{J}_1, \mathscr{J}_2, \cdots$ with the following property:

(A) each $\mathscr{J}_t$ is a finite collection of closed intervals of R^n with mutually disjoint interiors, with the coordinates of all their vertices in B, with diameters less than $1/t$, with

$$f\{|\mathscr{B}(K)|\} \subseteq \mathrm{Int}\left\{\bigcup_{J \in \mathscr{J}_t} J\right\}$$

and all intersecting $f\{|\mathscr{B}(K)|\}$.

For each $J \in \mathscr{J}_t$ let V_J be the set consisting of all those points x of $f^{-1}(\mathrm{Int}\, J)$ with the property:

(B) there exists a subdivision $\Pi^v(K)$ of K and a subcomplex M of $\Pi^v(K)$ such that M is an F-complex with a $(k-1)$-simplex in common with $\mathscr{B}\{\Pi^v(K)\} = \Pi^v\{\mathscr{B}(K)\}$ and $x \in |M| \subseteq f^{-1}(\mathrm{Int}\, J)$.

For each r, let K_r be a subdivision of K such that $f^{(r)}$ is linear on each simplex of K_r. A point x belongs to V_J if and only if

(C) for each r, there exists a subdivision $\Pi^{\nu_r}(K_r)$ and a subcomplex M of $\Pi^{\nu_r}(K_r)$ such that M is an F-complex with a $(k-1)$-simplex in common with $\mathscr{B}\{\Pi^{\nu_r}(K_r)\} = \Pi^{\nu_r}\{\mathscr{B}(K_r)\}$ and $x \in |M| \subseteq f^{-1}(\operatorname{Int} J)$.

Put

$$V_t = \bigcup_{J \in \mathscr{J}_t} V_J. \tag{1}$$

We will prove that

$$\lim_{t\to\infty}\left[\limsup_{r\to\infty} a\{f^{(r)}, V_t\}\right] = 0. \tag{2}$$

To prove (2), we suppose it is not true. Then there exists an $\varepsilon' > 0$ and two subsequences $\{r_s\}$, $\{t_s\}$ of the positive integers such that

$$a\{f^{(r_s)}, V_{t_s}\} > \varepsilon' \tag{3}$$

for all $s = 1, 2, \cdots$. For all positive integers q, s and all $J \in \mathscr{J}_{t_s}$, let $L_{qs}(J)$ denote the sub-pseudo-manifold of $\Pi^q(K_{r_s})$ that is the closure of the set of all those k-simplexes that are contained in V_J. Let $M_{qs}(J)$ denote the subcomplex of $L_{qs}(J)$ consisting of the union of all those F-components of $L_{qs}(J)$ that have a $(k-1)$-simplex in common with $\mathscr{B}\{(\Pi^q(K_{r_s})\} = \Pi^q\{\mathscr{B}(K_{r_s})\}$. Then

$$\lim_{q\to\infty} \bigcup_{J \in \mathscr{J}_{t_s}} |M_{qs}(J)| = V_{t_s},$$

hence by (3) we can choose, for each s, a positive integer q_s such that

$$a\left\{f^{(r_s)}, \bigcup_{J \in \mathscr{J}_{t_s}} |M_{q_s s}(J)|\right\} > \frac{1}{2}\varepsilon'. \tag{4}$$

By 3.5, there exists for each $J \in \mathscr{J}_{t_s}$ a continuous mapping ψ_J of $|K|$ into $|K|$ and a subdivision $\Pi^{\nu_s}(K_{r_s})$ of $\Pi^{q_s}(K_{r_s})$ such that:

(i) $\psi_J\{|M_{q_s s}(J)|\} \subseteq |\{M_{q_s s}(J)\}^{k-1}|$;

(ii) $\psi_J(x) = x$ for all $x \in |\operatorname{Cl}\{\Pi^{q_s}(K_{r_s}) \sim M_{q_s s}(J)\}|$;

(iii) for each k simplex ξ^k of $\Pi^{\nu_s - q_s}\{M_{q_s s}(J)\}$ there exists a $(k-1)$-simplex σ^{k-1} of $\{M_{q_s s}(J)\}^{k-1}$ such that $\psi_J(\xi^k) \subseteq \sigma^{k-1}$ and ψ_J is linear on ξ^k.

By forming a composition of the mappings ψ_J $(J \in \mathscr{J}_{t_s})$ and $f^{(r_s)}$, we arrive at a quasi-linear mapping $g^{(r_s)}$ of K into R^n with the following properties:

$$\| g^{(r_s)} - f^{(r_s)} \| < \frac{1}{t_s}; \tag{α}$$

$$a(g^{(r_s)}, |K|) = a\left(f^{(r_s)}, |K| \sim \bigcup_{J \in \mathscr{J}_{t_s}} |M_{q_s s}(J)|\right). \tag{β}$$

By (α) and hypothesis

$$\|g^{(r_s)} - f\| \to 0$$

as $s \to \infty$ and by (β) and (4)

$$\limsup_{s \to \infty} a(g^{(r_s)}, |K|) \leqq L(f) - \varepsilon'$$

contrary to hypothesis. Thus (2) is true.

By (2) we can choose a positive integer t' such that

$$\limsup_{r \to \infty} a\{f^{(r)}, V_{t'}\} < \varepsilon \tag{5}$$

and we let $J_1, \cdots, J_p$ be the members of $\mathscr{J}_{t'}$.

To prove (iii), take an arbitrary $\delta > 0$ and let C be the subset of $|K|$ consisting of all those points x for which $x \in V_{t'}$ and $f(x)$ has a distance $\geqq \frac{1}{2}\delta$ from $\bigcup_{j=1}^{p} Fr(J_j)$. Then C is compact. For each $j = 1, \cdots, p$, let L_{qj} be the subcomplex of $\Pi^q(K)$ that is the closure of the set of all those k-simplexes that are contained in V_{J_j}. Let M_{qj} denote the subcomplex of L_{qj} consisting of the union of all those F-components of L_{qj} that have a $(k-1)$-simplex in common with $\mathscr{B}\{\Pi^q(K)\} = \Pi^q\{\mathscr{B}(K)\}$. Let

$$M_q = \bigcup_{j=1}^{p} M_{qj}.$$

Then

$$\lim_{q \to \infty} [\text{Int of } |M_q| \text{ in } V_{t'}] = V_{t'},$$

hence there exists an l such that $C \subseteq |M_l|$. Put

$$L = \mathrm{Cl}\{\Pi^l(K) \sim M_l\},$$

and L has the required properties.

4. **The main convergence theorem.** Throughout §4, K will be a geometric complex, c a nonzero integral k-chain ($k \geqq 1$) on K and M,N the subcomplexes of K that support c and $\partial(c)$ respectively. f is a continuous mapping of $|M|$ into R^n.

f has a unique monotone light factorization (see [2] or [10]),

$$f = l \circ m \quad m : |M| \to \mathscr{M} \quad l : \mathscr{M} \to R^n,$$

whose middle space $\mathscr{M}$ consists of the maximal continua of constancy of f. M can be metrized by defining

$$d(\xi,\eta) = \inf\{\operatorname{diam} f(C); C \text{ is a continuum of } |M| \text{ containing } \xi \cup \eta\},$$

when ξ,η both belong to the same component of $|M|$ and

$$d(\xi,\eta) = 1 + \operatorname{diam} f(|M|)$$

otherwise. $\mathscr{M}$ is evidently compact.

Let $\mathscr{C}$ be the Banach space of all real-valued continuous functions on $\mathscr{M}$ with the norm of uniform convergence.

For each quasi-linear mapping of g of M into R^n, there was defined in 2.2 a current-valued measure $\nu(g)$ over $|M|$. This induces a measure $\mu(g)$ over $\mathscr{M}$ as follows:

$$\{\mu(g)\}(\chi) = \{\nu(g)\}(\chi \circ m), \qquad \chi \in \mathscr{C}.$$

4.1. THEOREM. *If f has finite Lebesgue area, $f(|M|)$ has $k+1$ dimensional Hausdorff measure zero and $\partial(c) = 0$, then there exists a unique current-valued measure $\mu(f)$ over $\mathscr{M}$ such that for every sequence $\{f^{(r)}\}$ of quasi-linear mappings of M into R^n which converge uniformly to f and have bounded areas, the measures $\mu(f^{(r)})$ converge weakly to $\mu(f)$.*

This theorem is just 3.9 of [2] without the special case $k = 2$. Although Federer's theorem is stated for differentiable k-manifolds of class ∞, the proof is valid for k-dimensional geometric complexes.

4.2. LEMMA. *Let M be a pseudo-manifold, $N = \mathscr{B}(M)$, f have finite Lebesgue area and $f(|M|)$ have $k+1$ dimensional Haudsorff measure zero. Let $\mathscr{C}_0$ be the subspace of $\mathscr{C}$ consisting of all $\chi \in \mathscr{C}$ for which $\chi \circ m$ vanishes on $|N|$. Then there exists a unique current-valued measure $\mu_0(f)$ on $\mathscr{C}_0$ such that for every sequence $\{f^{(r)}\}$ of quasi-linear mappings of M into R^n which converge uniformly to f and have bounded areas, the restrictions of the measures $\mu(f^{(r)})$ to $\mathscr{C}_0$ converge weakly to $\mu_0(f)$ on $\mathscr{C}_0$.*

Proof. Let M_* be a k-dimensional pseudo-manifold containing M as a subcomplex and such that:

(A) $\mathscr{B}(M_*) = \emptyset$;

(B) if $L = \mathrm{Cl}(M_* \sim M)$, then $M \cap L = N = \mathscr{B}(M) = \mathscr{B}(L)$;

(C) there exists an isomorphism θ of $|M|$ onto $|L|$ such that $\theta(\sigma) = \sigma$ for all $\sigma \in \mathscr{B}(M)$.

Let η denote the corresponding homeomorphism. The chain c extends to an integral k-cycle c_* on M_*.

For each continuous mapping h of $|M|$ into R^n, let h_* be the continuous mapping of $|M_*|$ into R^n given by

$$\begin{aligned} h_*(x) &= h(x), & x \in |M|, \\ &= h\{\eta^{-1}(x)\}, & x \in |L|. \end{aligned}$$

For a quasi-linear mapping $g : |M| \to R^n$ we evidently have

$$a(g_*, |M_*|) = 2a(g, |M|),$$

hence

$$L(f_*, |M_*|) \leqq 2L(f, |M|).$$

Denote the middle space of f_* by $\mathscr{M}_*$. By 4.1, there exists a unique measure μ_* over $\mathscr{M}_*$ such that for every sequence $\{q_*^{(r)}\}$ of quasi-linear mappings of $|M_*|$ into R^n, which converge uniformly to f_* and have bounded areas, the measures $\mu(q_*^{(r)})$ converge weakly to μ_*. For each $\chi \in \mathscr{C}_0$, let χ_* be the continuous function on M_* given by

$$\begin{aligned} \chi_*(\zeta) &= \chi(|M| \cap \zeta) \text{ if } |M| \cap \zeta \neq \emptyset, \\ &= 0 \text{ if } |M| \cap \zeta = \emptyset. \end{aligned}$$

Define

$$\{\mu_0(f)\}(\chi) = \mu_*(\chi_*), \qquad \chi \in \mathscr{C}_0.$$

If $\{f^{(r)}\}$ is a sequence of quasi-linear mappings of M into R^n, which converge uniformly to f and have bounded areas, then $f^{(r)}$ converges uniformly to f_* and $a(f_*^{(r)}, |M_*|)$ is bounded, hence, for each $\chi \in \mathscr{C}_0$

$$\lim_{r\to\infty} \{\mu(f_*^{(r)})\}(\chi_*) = \mu_*(\chi_*),$$

so that

$$\lim_{r\to\infty} \{\mu(f^{(r)})\}(\chi) = \{\mu_0(f)\}(\chi).$$

4.3. THEOREM. *If M is a pseudo-manifold, $N = \mathscr{B}(M)$, f has finite Lebesgue area and $f(|M|)$ has $k+1$ dimensional Hausdorff measure zero, then there exists a unique current-valued measure $\mu(f)$ over M such that, for every sequence $\{f^{(r)}\}$ of quasi-linear mappings of M into R^n, which converge uniformly to f and whose areas converge to the Lebesgue area of f, the measures $\mu(f^{(r)})$ converge weakly to $\mu(f)$.*

Proof. Since every uniformly bounded sequence of measures over $\mathscr{M}$ has a convergent subsequence, it will be sufficient to prove that if $\{g^{(r)}\}$, $\{h^{(r)}\}$ are two sequences of quasi-linear mappings of M into R^n, which converge uniformly to f, whose areas converge to the Lebesgue area of f and which are such that the corresponding sequences of measures $\{\mu(g^{(r)})\}$, $\{\mu(h^{(r)})\}$ converge weakly to measures μ and μ', then $\mu = \mu'$.

Suppose this statement is not true, i.e., there exists a $\chi \in \mathscr{C}$ such that $\mu(\chi) \neq \mu'(\chi)$. Then

$$\left| \lim_{r\to\infty} \{\nu(g^{(r)})\}(\chi \circ m) - \lim_{r\to\infty} \{\nu(h^{(r)})\}(\chi \circ m) \right| = 5\varepsilon > 0. \tag{1}$$

Put

$$K = 1 + \sup_{\zeta \in \mathscr{M}} \chi(\zeta).$$

We can assume that there exists a countable dense subset Y of R^1 such that if we put for $i = 1, \cdots, n$ and $t \in Y$

$$A_i = \{x; x \in R^n \text{ and } x_i < t\},$$

then each of the limits

$$\lim_{r\to\infty} a\{g^{(r)}, f^{-1}(A_i)\} = \rho_i(t)$$

$$\lim a\{h^{(r)}, f^{-1}(A_i)\} = \rho_i^*(t)$$

exists; because, if not, we could choose subsequences for which these limits existed. Since ρ_i and ρ_i^* are monotone increasing there exists a subset Z of R^1 such that $R^1 \sim Z$ has zero measure and $\rho_i'(t), \rho_i^{*\prime}(t)$ exist for all $t \in Z$ amd all i. Then, for every interval J of R^n with the coordinates of its vertices all in Z, we have

$$(2) \qquad \lim_{\lambda\to 0+} \limsup_{r\to\infty} \left[a\{g^{(r)}, f^{-1}(B_\lambda)\} + a\{h^{(r)}, f^{-1}(B_\lambda)\}\right] = 0,$$

where

$$B_\lambda = \{x;\ x \in R^n \text{ and } d[x, Fr(J)] < \lambda\}.$$

By 3.6, there exists a finite number $J_1, \cdots, J_p$ of closed intervals of R^n such that

(A) $J_1, \cdots, J_p$ have mutually disjoint interiors and the coordinates of their vertices are all in Z;

$$(\mathrm{B}) \qquad f(|N|) \subseteq \mathrm{Int}\left\{\sum_{j=1}^{p} J_j\right\}; \text{ and}$$

(C) for every $\delta > 0$, there exists a subdivision $\Pi^1(M)$ of M and a sub-pseudo-manifold L of $\Pi^1(M)$ such that every point of $f(|\mathscr{B}(L)|)$ has a distance less than δ from $\bigcup_{j=1}^{p} Fr(J_j)$ and

$$\limsup_{r\to\infty} a(g^{(r)}, |M| \sim |L|) < \varepsilon \cdot K^{-1},$$

$$\limsup_{r\to\infty} a(h^{(r)}, |M| \sim |L|) < \varepsilon \cdot K^{-1}.$$

By (2) one can choose a $\delta_1 > 0$ and a positive integer r_1 such that the sets

$$U_j = \{x;\ x \in R^n \text{ and } d(x, Fr(J_j)) < \delta_1\}$$

are sufficiently small that

$$(3) \qquad a\left[g^{(r)}, f^{-1}\left(\bigcup_{j=1}^{p} U_j\right)\right] < \varepsilon \cdot K^{-1}$$

and

$$(4) \qquad a\left[h^{(r)}, f^{-1}\left(\bigcup_{j=1}^{p} U_j\right)\right] < \varepsilon \cdot K^{-1}$$

for all $r \geqq r_1$. Let $\Pi^l(M)$ be a subdivision of M and L a sub-pseudo-manifold of $\Pi^l(M)$ such that

$$|\mathscr{B}(L)| \subseteq f^{-1}\left(\bigcup_{j=1}^{p} U_j\right) \tag{5}$$

and

$$\limsup_{r\to\infty} a(g^{(r)}, |M| \sim |L|) < \varepsilon \cdot K^{-1}, \tag{6}$$

$$\limsup_{r\to\infty} a(h^{(r)}, |M| \sim |L|) < \varepsilon \cdot K^{-1}. \tag{7}$$

Let $\mathscr{M}_1$ be the middle space of the restriction of f to $|L|$, η be the inclusion mapping of $\mathscr{M}_1$ into $\mathscr{M}$ and $f = l_1 \circ m_1$ be the monotone light factorization of f with respect to $|L|$. Since

$$\mathscr{E}_1 = l_1^{-1}\left\{R^n \sim \bigcup_{j=1}^{p} U_j\right\}$$

is a closed subset of $\mathscr{M}_1$ which does not intersect $m_1\{|\mathscr{B}(L)|\}$, there exists a continuous function χ_1 on $\mathscr{M}_1$ such that

$$\begin{aligned} \chi_1(\zeta) &= \chi\{\eta(\zeta)\}, \qquad && \zeta \in \mathscr{E}_1 \\ &= 0 && \zeta \in m\{|\mathscr{B}(L)|\}, \end{aligned}$$

and

$$|\chi_1(\zeta)| \leqq K$$

for all $\zeta \in \mathscr{M}_1$. By 4.2

$$\lim_{r\to\infty} \{v(g^{(r)})\}(\chi_1 \circ m_1) = \lim_{r\to\infty} \{v(h^{(r)})\}(\chi_1 \circ m_1). \tag{8}$$

But

$$|\{v(g^{(r)})\}(\chi \circ m - \chi_1 \circ m_1)| \leqq 2Ka\left\{g^{(r)}, f^{-1}\left\{\bigcup_{j=1}^{p} U_j\right\} \cup (|M| \sim |L|)\right\},$$

hence by (3) and (6),

$$\limsup_{r\to\infty} |\{v(g^{(r)})\}(\chi \circ m - \chi_1 \circ m_1)| < 2\varepsilon,$$

and similarly

$$\limsup_{r\to\infty} |\{v(h^{(r)})\}(\chi \circ m - \chi_1 \circ m)| < 2\varepsilon$$

so that by (8)

$$\left|\lim_{r\to\infty} \{v(g^{(r)}\}(\chi \circ m) - \lim_{r\to\infty} \{v(h^{(r)})\}(\chi \circ m)\right| < 4\varepsilon$$

contradicting (1).

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