COERCIVENESS IN L^p

BY

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1. **Introduction.** The concept of coerciveness was introduced by Aronszajn [5] who considered necessary and sufficient conditions for the inequality

$$\| u \|_{m,2} \leqq \text{const.}\left(\sum \| A_k u \|_{0,2} + \| u \|_{0,2} \right) \tag{1.1}$$

to hold for all functions(2) u in a bounded domain G, where the A_k are linear partial differential operators of order m and $\| u \|_{s,p}$ is the sum of the L^p norms of u and its derivatives up to order s. This result was later generalized by Agmon [1] and Schechter [17] to inequalities of the form

$$\| u \|_{m,2} \leqq \text{const.}\left(\sum \| A_k u \|_{0,2} + \sum \langle B_j u \rangle_{m-\nu_j-1/2,2} + \| u \|_{0,2} \right), \tag{1.2}$$

where each B_j is a boundary differential operator of order $\nu_j < m$ and the $\langle \cdot \rangle_{t,2}$ are appropriate boundary norms. Agmon [1] also considered inequalities of the form

$$\| u \|_{m,2}^2 \leqq \text{const.}\left(\operatorname{Re}[u,u] + \sum \langle B_j u \rangle_{m-\nu_j-1/2,2}^2 + \| u \|_{0,2}^2 \right), \tag{1.3}$$

where $[u,v]$ is an integro-differential bilinear form of order m (cf. §5). In general, if

$$\| u \|_{m,2}^2 \leqq \text{const.}\left(\operatorname{Re}[u,u] + \| u \|_{0,2}^2 \right) \tag{1.4}$$

for all functions u in a set U, it is said that the form $[u,v]$ is coercive over U.

In 1959 Agmon proved the inequality

$$\| u \|_{m,p} \leqq \text{const.}\left(\sum \| A_k u \|_{0,p} + \sum \langle B_j u \rangle_{m-\nu_j-1/p,p} + \| u \|_{0,p} \right) \tag{1.5}$$

for $1 < p < \infty$ under the same hypotheses employed for (1.2). Agmon-Douglis-Nirenberg [4] and Browder [8] had previously proved special cases. Independently and by a different method, Smith [22] established the L^p analogue of (1.1) requiring stronger hypotheses on the A_k and weaker assumptions on the boundary than did Aronszajn. For smooth domains the result of Smith is contained in that of Agmon.

In this paper we are concerned with generalizing (1.2) and (1.3) in several directions. First we remove the restriction that the A_k be of the same order and

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(2) In this, and other quoted results we do not specify the regularity assumptions made.

define the norm $\|\cdot\|_{s,p}$ for negative integers s as well. We prove, under virtually the same hypotheses as employed for (1.2), that

$$(1.6)\qquad \| u \|_{s,p} \leqq \text{const.}\big(\sum \| A_k u \|_{s-m_k,p} + \sum \langle B_j u \rangle_{s-\nu_j-1/p,p} + \| u \|_{s-m,p}\big),$$

for any integer s and real p, $1 < p < \infty$, where m_k is the order of A_k and $m = \max m_k$. In particular, if $s = m_0$, the minimum order of the A_k, we have

$$(1.7)\qquad \| u \|_{m_0,p} \leqq \text{const.}\big(\sum \| A_k u \|_{0,p} + \sum \langle B_j u \rangle_{m_0-\nu_j-1/p,p} + \| u \|_{0,p}\big).$$

These inequalities are new even for $p = 2$. We also prove a slightly stronger result when there is only one operator A_1 (cf. §6). Special cases are considered in Lions-Magenes [14].

In order to obtain a counterpart of (1.3), we set

$$[u]_{t,p} = \operatorname{lub} \frac{|[u,v]|}{\| v \|_{m-t,p'}},$$

where the lub is taken over all functions v and $p' = p/(p-1)$. Then under the same hypotheses that were used in proving (1.3) we have(3)

$$(1.8)\qquad \| u \|_{s,p} \leqq \text{const.}\big([u]_{s-m,p} + \sum \langle B_j u \rangle_{s-\nu_j-1/p,p} + \| u \|_{s-m,p}\big)$$

for any integer $s \leqq m$.

We go one step further and define the norm $\|\cdot\|_{s,p}$ for all real values of s. This is done by means of complex interpolation methods introduced by Calderon [9] and Lions [13] (cf. §2). We prove that for each real $s \leqq m$

$$(1.9)\qquad \| u \|_{s,p} \leqq \text{const.}\big(\sum \| A_k u \|_{s-m_k,p} + \| u \|_{s-m,p}\big)$$

and

$$(1.10)\qquad \| u \|_{s,p} \leqq \text{const.}\big([u]_{s-m,p} + \| u \|_{s-m,p}\big)$$

for all u satisfying

$$B_j u = 0 \text{ for each } j$$

on the boundary. The hypotheses are the same as those for (1.6) and (1.8), respectively.

Our interest in the problem began when A. Zygmund brought our attention to the inequality due to Friedrichs [24]

$$\| u \|_{1,2} \leqq \text{const.} \sum_{k=1}^{n-1} \left\| \frac{\partial u}{\partial x_k} \right\|_{0,2}$$

holding for harmonic functions satisfying

$$\int_G u\,dx = 0.$$

(3) Actually we prove a slightly stronger inequality (cf. §8).

This is equivalent to

$$\| u \|_{1,2} \leqq \text{const.} \left(\| \Delta u \|_{-1,2} + \sum_{k=1}^{n-1} \left\| \frac{\partial u}{\partial x_k} \right\|_{0,2} + \| u \|_{0,2} \right)$$

holding for all functions u. Our present theorems give the corresponding result for $p \neq 2$ (cf. §3).

We are indebted to Professors A. Zygmund and A. P. Calderon for several interesting discussions.

2. **Complex interpolation spaces.** Let X_0 and X_1 be Banach spaces and denote by $\mathscr{H}(X_0,X_1)$ the set of functions $f(x + iy)$ having values in $X_0 + X_1$ which are analytic in $0 < x < 1$, continuous and bounded in $0 \leqq x \leqq 1$, and such that $f(iy) \in X_0$, $f(1 + iy) \in X_1$. Set

$$\| f \|_{\mathscr{H}(X_0,X_1)} = \max \left[\operatorname{lub}_y \| f(iy) \|_{X_0}, \ \operatorname{lub}_y \| f(1 + iy) \|_{X_1} \right].$$

For $0 \leqq \theta \leqq 1$, the set $[X_0, X_1; \delta(\theta)]$ consists of those elements of $X_0 + X_1$ which are equal to $f(\theta)$ for some $f \in \mathscr{H}(X_0, X_1)$. Under the norm

$$\| u \|_{[X_0,X_1;\delta(\theta)]} = \operatorname{glb}_{f(\theta)=u} \| f \|_{\mathscr{H}(X_0,X_1)},$$

the set $[X_0, X_1; \delta(\theta)]$ becomes a Banach space. This method of interpolation was introduced by Calderon [9] and Lions[13].

3. **Inequalities for formally positive forms.** Let G be a bounded domain in Euclidean n-space E^n with boundary ∂G of class C^∞. Let $C^2(\bar{G})$ denote the set of complex valued functions infinitely differentiable in the closure $\bar{G}$ of G. For i a non-negative integer and $p > 1$ we employ the norm

$$\| u \|_{i,p} = \left(\sum_{|\mu| \leqq i} \int_G |D^\mu u|^p dx \right)^{1/p}, \tag{3.1}$$

where summation is taken over all derivatives $D^\mu u$ of order $|\mu| \leqq i$. We let $H^{i,p}(G)$ denote the completion of $C^\infty(\bar{G})$ with respect to the norm (3.1). For any real number s such that $i < s < i + 1$ we define $H^{s,p}(G)$ to be the space $[H^{i,p}(G), H^{i+1,p}(G); \delta(\theta)]$, where $\theta = s - i$. For s real and negative $H^{s,p}(G)$ is defined as the completion of $C^\infty(\bar{G})$ with respect to the norm

$$\| u \|_{s,p} = \operatorname{lub}_{v \in C^\infty(\bar{G})} \frac{|(u,v)|}{\| v \|_{-s,p'}},$$

where $(u,v) = \int_G u\bar{v}dx$ and $p' = p/(p-1)$. We consider the following boundary norms. For $\phi \in C^\infty(\partial G)$ and s real and positive we define

$$\langle \phi \rangle_{s,p} = \operatorname{glb} \| u \|_{s+1/p,p},$$

where the glb is taken over all $u \in C^\infty(\bar{G})$ which equal ϕ on ∂G. For s negative, we write

$$\langle \phi \rangle_{s,p} = \operatorname{lub} \left| \int_{\partial G} \phi \bar{\psi} \, d\sigma \right| \langle \psi \rangle_{-s,p'}^{-1}, \tag{3.2}$$

where the lub is taken over all $\psi \in C^\infty(\partial G)$.

Let $\{A_k\}$ and $\{B_j\}$ be two finite systems of linear partial differential operators with coefficients in $C^\infty(\bar{G})$. The set $\{B_j\}$ may be void. Let m_k denote the order of A_k and ν_j the order of B_j. Set $m = \max m_k$ and $\nu = \max \nu_j$. We make the following assumptions.

(a) The orders of the B_j are distinct, and $\nu < m$.

(b) The boundary ∂G is noncharacteristic to each B_j at every point.

(c) At each point $x \in \bar{G}$ the characteristic polynomials $P_k(x,\xi)$ of the A_k do not vanish simultaneously for any real vector $\xi \neq 0$.

(d) The B_j cover the A_k. This means the following. At each point x^0 of ∂G let $N \neq 0$ be a vector orthogonal to ∂G at x^0 and $T \neq 0$ a tangential vector. Let $z_1, \cdots, z_h$ denote the complex roots with positive imaginary parts common to the polynomials $P_k(z) \equiv P_k(x^0, T + zN)$. If $Q_j(x,\xi)$ denotes the characteristic polynomial of B_j, then it is assumed that there are h polynomials among the $Q_j(z) \equiv Q_j(x^0, T + zN)$ which are linearly independent modulo the polynomial $(z - z_1)(z - z_2) \cdots (z - z_h)$. If the set $\{B_j\}$ is empty, it is assumed that there are no such roots z_i.

(e) At each boundary point x^0, ∂G is noncharacteristic for some operator A_k of order m.

Theorem 3.1. *Assume that the systems $\{A_k\}$, $\{B_j\}$ satisfy hypotheses* (a)-(e). *Then for each integer s and each set of real numbers $s_k \geqq s - m_k$, $t_j \geqq s - \nu_j - 1/p$ there is a constant C such that*

$$\| u \|_{s,p} \leqq C \left(\sum_k \| A_k u \|_{s_k,p} + \sum_j \langle B_j u \rangle_{t_j,p} + \| u \|_{s-m,p} \right) \tag{3.3}$$

for all $u \in C^\infty(\bar{G})$.

Corollary 3.1. *If $s \geqq m$, then hypothesis* (e) *is unnecessary in Theorem* 3.1.

Theorem 3.2. *Under hypotheses* (a)-(e), *for every set of real $s \leqq m$ and $s_k \geqq s - m_k$ there is a constant C such that*

$$\| u \|_{s,p} \leqq C \left(\sum_k \| A_k u \|_{s_k,p} + \| u \|_{s-m,p} \right) \tag{3.4}$$

for all $u \in C^\infty(\bar{G})$ satisfying

$$B_j u = 0 \ \text{on} \ \partial G \ \text{for each} \ j. \tag{3.5}$$

Let m_0 be the minimum order of the A_k. Then an important special case of Theorem 3.2 is

COROLLARY 3.2. *Under hypotheses* (a)-(e),

$$\| u \|_{m_0,p} \leqq C\Big(\sum_k \| A_k u \|_{0,p} + \| u \|_{0,p}\Big)$$

for all $u \in C^\infty(\bar{G})$ *satisfying* (3.5).

REMARK 3.1. For s an integer and $\geqq m$, inequality (3.4) was first proved by Agmon (to appear). Subsequently and independently, Smith [22] proved a slightly weaker result for more general domains.

REMARK 3.2. When there is only one operator A_1, stronger theorems can be proved. Some are stated and proved in §6.

A simple example of Theorem 3.2 which is of interest is the inequality

$$\| u \|_{1,p} \leqq c\Big(\| Au \|_{-1,p} + \sum_{i=1}^{n-1} \Big\| \frac{\partial u}{\partial x_i} \Big\|_{0,p} + \| u \|_{0,p}\Big)$$

holding for all $u \in C^\infty(\bar{G})$, where A is any second order elliptic operator. This generalizes an L^2 inequality of Friedrichs [24] (cf. §1).

4. **Some known results.** An immediate consequence of the definition of a complex interpolation space is

LEMMA 4.1 [9; 13]. *If X_i and Y_i are Banach spaces, $i = 0, 1$, and T is a bounded linear map of X_0 into Y_0 and X_1 into Y_1, then it can be extended to be a bounded linear map of $[X_0, X_1; \delta(\theta)]$ into $[Y_0, Y_1; \delta(\theta)]$.*

The following result is due to Lions-Magenes [14].

LEMMA 4.2. *If ∂G is of class C^∞ and s_1 and s_2 are non-negative real numbers,*

$$[H^{s_1,p}(G),\ H^{s_2,p}(G);\ \delta(\theta)] = H^{s_3}(G),$$

where $s_3 = (1-\theta)s_1 + \theta s_2$.

LEMMA 4.3. *Let E be a linear differential operator of order t with coefficients in $C^\infty(\bar{G})$. Then for s real and non-negative*

$$\| Eu \|_{s,p} \leqq \text{const.} \| u \|_{s+t,p}$$

for all $u \in C^\infty(\bar{G})$.

Proof. When s is an integer, this follows from the definition of the norm. Otherwise, let i be the integer such that $i < s < i+1$. Then E is a bounded linear mapping of $H^{i+t,p}(G)$ into $H^{i,p}(G)$ and of $H^{i+1+t,p}(G)$ into $H^{i+1,p}(G)$. Hence it is a bounded linear map of $[H^{i+t,p}(G), H^{i+1+t,p}(G); \delta(\theta)]$ into

$$[H^{i,p}(G),\ H^{i+1,p}(G);\ \delta(\theta)],$$

with $\theta = s - i$. We apply Lemma 4.2 to complete the proof.

LEMMA 4.4. *Under the same hypotheses if E contains only derivatives in tangential directions on ∂G, then*

$$\langle E\phi\rangle_{s,p} \leqq \text{const.}\ \langle\phi\rangle_{s+t,p}$$

for all $\phi \in C^\infty(\partial G)$.

Proof. We have by Lemma 4.3

$$\begin{aligned}\langle E\phi\rangle_{s.p} &\leqq \ \text{glb}\,\| Eu \|_{s+1/p.p}\\ &\leqq \ \text{const. glb}\,\| u \|_{s+t+1/p,p} = \text{const.}\ \langle\phi\rangle_{s+t,p},\end{aligned}$$

where the glb is taken over all $u \in C^\infty(\bar{G})$ which equal ϕ on the boundary.

We denote the normal derivative of u on ∂G of order t by $\gamma_t u\,(\gamma_0 u = u)$.

LEMMA 4.5 [11; 12; 21]. *For every integer* $i > 0$ *and every set of functions* $\phi_0, \phi_1, \cdots, \phi_{i-1}$ *in* $C^\infty(\partial G)$ *there is a function* $u \in C^\infty(\bar{G})$ *such that*

$$\gamma_t u = \phi_t, \qquad 0 \leqq t < i,$$

and

$$\| u \|_{i,p} \leqq \text{const.} \sum_{t=0}^{i-1} \langle\phi_t\rangle_{i-t-1/p,p}$$

where the constant does not depend on u *or the* ϕ_t.

Let $W^{s,p}(\partial G)$ denote the completion of $C^\infty(\partial G)$ with respect to the norm $\langle\cdot\rangle_{s,p}$.

LEMMA 4.6. *If* s_1 *and* s_2 *are positive real numbers, then*

$$[W^{s_1,p}(\partial G),\ W^{s_2,p}(\partial G);\ \delta(\theta)] \supseteq W^{s_3,p}(\partial G)$$

where $s_3 = (1-\theta)s_1 + \theta s_2$. *Moreover, equality holds when* $s_i + 1/p \geqq 1$, $i = 1,2$.

Proof. By the definition of the norms, we know that γ_0 is a bounded linear mapping of $H^{s_1+1/p,p}(G)$ into $W^{s_1,p}(\partial G)$ and of $H^{s_2+1/p,p}(G)$ into $W^{s_2,p}(\partial G)$. Hence by Lemmas 4.1 and 4.2 it is a bounded linear map of $H^{s_3+1/p,p}(G)$ into $[W^{s_1,p}(\partial G), W^{s_2,p}(\partial G); \delta(\theta)]$. Denote the latter space by X. Thus if $\phi \in C^\infty(\partial G)$, $u \in C^\infty(\bar{G})$ and $u = \phi$ on ∂G, then

$$\| \phi \|_X \leqq \text{const.}\ \| u \|_{s_3+1/p,p}\,.$$

Hence

$$\| \phi \|_X \leqq \text{const.}\ \langle\phi\rangle_{s_3,p}$$

and the first part of the lemma is proved. In order to prove the second part, we first assume that the $s_i + 1/p$ are integers $i = 1,2$. Then by Lemma 4.5 there is a bounded linear mapping T of $W^{s_i,p}(\partial G)$ into $H^{s_i+1/p,p}(G)$, $i = 1,2$(4). Thus T is a bounded linear map of X into $H^{s_3+1/p,p}(G)$. This means that if $\phi \in C^\infty(\partial G)$, then

(4) One must observe that T does not depend on s_1 or s_2. This follows from the proofs of Lemma 4.5.

$$\| T\phi \|_{s_3+1/p,p} \leqq \text{const.} \| \phi \|_X$$

and hence

$$\langle \phi \rangle_{s_3,p} \leqq \text{const.} \| \phi \|_X,$$

showing that $X = W^{s_3,p}(\partial G)$. Thus T is a bounded linear map of $W^{s,p}(\partial G)$ into $H^{s+1/p,p}(G)$ for all real $s \geqq 1 - 1/p$. Repeating the reasoning above we now obtain $X = W^{s_3,p}(G)$ whenever s_1 and s_2 are real and $\geqq 1 - 1/p$.

We shall call a set $\{B_j\}_{j=1}^r$ of boundary operators a *normal set of order* $< m$ if it satisfies hypotheses (a) and (b) of §3. Clearly $r \leqq m$.

LEMMA 4.7. *Let $\{B_j\}_{j=1}^r$ be a normal set of order $< m$. Then for every real $s \geqq m$ and every set $\{\phi_j\}_{j=1}^r$ of functions in $C^\infty(\partial G)$ there is a $u \in C^\infty(\bar{G})$ such that*

$$B_j u = \phi_j \text{ on } \partial G, \qquad 1 \leqq j \leqq r,$$

and

$$\| u \|_{s,p} \leqq \text{const.} \sum_{j=1}^{r} \langle \phi_j \rangle_{s-\nu_j-1/p,p},$$

where the constant does not depend on u or the ϕ_j.

Proof. We first assume that s is an integer $\geqq m$. By adding $m-r$ appropriate operators and considering the corresponding ϕ_j to be zero, we see that we need only consider the case $r = m$. By rearrangement if necessary, we may assume that $\nu_j = j-1$. Now we have (cf., e.g., [6; 18])

$$B_j = \sum_{t=1}^{j} \Gamma_{jt}\gamma_{t-1}, \qquad 1 \leqq j \leqq m, \tag{4.1}$$

$$\gamma_{t-1} = \sum_{i=1}^{t} \Lambda_{ti} B_i, \qquad 1 \leqq t \leqq m, \tag{4.2}$$

where Γ_{jj} and Λ_{tt} are nonvanishing functions and Γ_{it} and Λ_{it} are operators of order $\leqq i-t$ involving only derivatives in tangential directions. Thus

$$\sum_{t=1}^{j} \Gamma_{jt}\Lambda_{ti} = \delta_{ji}, \qquad 1 \leqq i \leqq j \leqq m,$$

where δ_{ji} is the Kronecker delta. Now by Lemma 4.5, there is a $u \in C^\infty(\bar{G})$ such that

$$\gamma_{t-1}u = \sum_{i=1}^{t} \Lambda_{ti}\phi_i, \qquad 1 \leqq t \leqq s,$$

and

$$\| u \|_{s,p} \leqq \text{const.} \sum_{t=1}^{m} \langle \gamma_{t-1}u \rangle_{s-t+1-1/p,p}.$$

But

$$B_j u = \sum_{t=1}^{j} \Gamma_{jt} \sum_{i=1}^{t} \Lambda_{ti}\phi_i = \sum_{i=1}^{j} \phi_i \sum_{t=i}^{j} \Gamma_{jt}\Lambda_{ti} = \phi_j$$

and by Lemma 4.4

$$\langle \Lambda_{ti}\phi_i \rangle_{s-t+1-1/p,p} \leqq \text{const.}\ \langle \phi_i \rangle_{s-i+1-1/p,p},$$

proving the lemma when s is an integer. When s is not an integer, let i be the integer such that $m \leqq i < s < i+1$. Now the mapping just constructed is a bounded one from

$$\prod_j W^{i-\nu_j-1/p,p}(\partial G) \quad \text{into} \quad H^{i,p}(G)$$

and from

$$\prod_j W^{i+1-\nu_j-1/p,p}(\partial G) \quad \text{into} \quad H^{i+1,p}(G).$$

Hence, by Lemmas 4.2 and 4.6, it is a bounded linear mapping of

$$\prod_j W^{s-\nu_j-1/p,p}(\partial G) \quad \text{into} \quad H^{s,p}(G)$$

(cf. [13, Theorem 3]). This completes the proof.

Let $C_0^\infty(G)$ denote the set of those $v \in C^\infty(\bar{G})$ having compact support in G. Denote its closure in $H^{s,p}(G)$ by $H_0^{s,p}(G)$.

Lemma 4.8. *If $v \in C^\infty(\bar{G})$ and $\gamma_j v = 0$ for $0 \leqq j < s$, then $v \in H_0^{s,p}(G)$.*

Proof. When s is an integer, the lemma is well known (cf., e.g., [7, p. 48]). When s is not an integer, let i be the integer such that $i < s < i+1$. Then $\gamma_j v = 0$ for $0 \leqq j \leqq i$ and hence

$$v \in H_0^{i+1,p}(G) \subset H_0^{s,p}(G).$$

Lemma 4.9. *Let $\{B_j\}_{j=1}^r$ be a normal set, and let V be the set of those $v \in C^\infty(\bar{G})$ which satisfy*

$$B_j v = 0 \quad on \quad \partial G, \qquad 1 \leqq j \leqq r. \tag{4.3}$$

If w is a function in $C^\infty(\bar{G})$ which satisfies (4.3) *only for those B_j of order less than s, then w is in the closure $V^{s,p}(G)$ of V in $H^{s,p}(G)$.*

Proof. By rearranging if necessary, we may assume that $\nu_j < s$ for $1 \leqq j < r_1$ and $\nu_j \geqq s$ for $r_1 \leqq j \leqq r$, where r_1 may equal one. Now it is easily checked that $\gamma_0, \gamma_1, \cdots, \gamma_i,\ B_{r_1}, \cdots, B_r$ form a normal set, where i is the integer such that $i < s \leqq i+1$. Hence by Lemma 4.7 there is a $u \in C^\infty(\bar{G})$ such that

$$\begin{aligned} \gamma_t u &= 0, \qquad & 0 &\leqq t \leqq i, \\ B_j u &= B_j w, \qquad & r_1 &\leqq j < r. \end{aligned}$$

Clearly $B_j u = 0$ for $1 \leqq j \leqq r_1$. Hence $w - u$ is in V while u is $H_0^{s,p}(G) \subseteq V^{s,p}(G)$. Thus $w = w - u + u$ is in $V^{s,p}(G)$ and the proof is complete.

COROLLARY 4.1. *Under the same hypotheses, let* V_1 *be the set of those* $u \in C^\infty(\bar{G})$ *which satisfy*

$$B_j u = 0 \text{ on } \partial G \text{ for those } j \text{ such that } v_j < s,$$

$$C_i u = 0 \text{ on } \partial G, \qquad 1 \leqq i \leqq t,$$

where $\{C_i\}_{i=1}^t$ *is a normal set and the order of each* C_i *is* $\geqq s$. *Then* V_1 *is dense in* $V^{s,p}(G)$.

Proof. Let $v \in C^\infty(\bar{G})$ be any function satisfying (4.3). Then there is a $u \in C^\infty(\bar{G})$ such that

$$\gamma_j u = \gamma_j, \qquad 0 \leqq j < s,$$

$$C_i u = 0, \qquad 1 \leqq i \leqq t.$$

Clearly, $B_j u = B_j v = 0$ when $v_j < s$. Thus $u \in V_1$. Moreover, by Lemma 4.8, $u - v$ is in $H_0^{s,p}(G)$ and hence there is a sequence $\{w_l\}$ of functions in $C_0^\infty(G)$ such that

$$\| w_l - (u-v) \|_{s,p} \to 0$$

as $l \to \infty$. But $u - w_l \in V_1$ and approaches v in $H^{s,p}(G)$. Since such v are dense in $V^{s,p}(G)$, the assertion follows.

LEMMA 4.10. *For all* $u,v \in C^\infty(\bar{G})$

$$\left| \int_{\partial G} uv d\sigma \right| \leqq C \| u \|_{1,p} \| v \|_{1,p'}. \tag{4.4}$$

Proof. Let γ be a first order operator which equals the outward normal derivative on ∂G. Then

$$(\gamma u, v) = -(u, \gamma v) + \int_{\partial G} u\bar{v} d\sigma$$

from which (4.4) immediately follows.

5. **Bilinear forms.** We consider bilinear integro-differential forms of order m:

$$[u,v] = \int_G \sum_{|\mu|,|\tau| \leqq m} a_{\mu\tau}(x) D^\mu u \overline{D^\tau v}\, dx, \tag{5.1}$$

where the coefficients are in $C^\infty(\bar{G})$. For any normal set $\{B_j\}_{j=1}^m$ of order $< m$, integration by parts yields

$$[u,v] = (Au, v) + \sum_{j=1}^m \int_{\partial G} F_j u\, \overline{B_j v}\, d\sigma, \tag{5.2}$$

$$[u,v] = (u, A'v) + \sum_{j=1}^m \int_{\partial G} B_j u\, \overline{F_j' v}\, d\sigma, \tag{5.3}$$

where A and the F_j and F'_j are differential operators, and A' is the formal adjoint of A (cf. [6]). A is of order $\leqq 2m$ and F_j and F'_j are each of order $\leqq 2m - \nu_j - 1$. If A is of order $2m$ and elliptic in $\bar{G}$, then the orders of F_j and F'_j are exactly $2m-\nu_j-1$.

LEMMA 5.1. *In order that*

$$\| u \|_{m,2}^2 \leqq \text{const.}(\operatorname{Re}[u,u] + \| u \|_{0,2}^2) \tag{5.4}$$

hold for all $u \in C^\infty(\bar{G})$ *satisfying*

$$B_j u = 0 \text{ on } \partial G, \qquad 1 \leqq j \leqq r \tag{5.5}$$

($0 \leqq r \leqq m$), *it is necessary that A be elliptic in $\bar{G}$ and that the operators* $B_1, \cdots, B_r, F_{r+1}, \cdots, F_m$ *cover it.*

Lemma 5.1 follows from the work of Agmon [1, p. 216; 3, p. 5](5).

Next, let us consider the special case when

$$[u,u] = \sum_k \| A_k u \|_{m-m_k,2}^2,$$

where A_k is an operator of order $m_k \leqq m$.

LEMMA 5.2. *Hypotheses* (c) *and* (d) *of* §3 *are sufficient for*

$$\| u \|_{m,2}^2 \leqq \text{const.}\left(\sum_k \| A_k u \|_{m-m_k,2}^2 + \| u \|_{0,2}^2 \right)$$

to hold for all $u \in C^\infty(\bar{G})$ *satisfying* (5.5).

Lemma 5.2 was proved in [1; 17].

We shall find it convenient to employ a complete system of first order tangential operators. By this we mean a set of operators of the form

$$\tilde{D}_i = \sum_{l=1}^n \alpha_{il}(x) \frac{\partial}{\partial x_l} + \alpha_i(x), \qquad 1 \leqq i \leqq n,$$
$$\tilde{D}_0 = 1,$$

having the following properties:

(1) The coefficients α_{il}, α_i are in $C^\infty(\bar{G})$.

(2) At interior points $x \in G$ there is no real vector $\xi = (\xi_1, \cdots, \xi_n) \neq 0$ such that

$$\sum_{l=1}^n \alpha_{il}(x) \xi_l = 0, \qquad 1 \leqq i \leqq n, \tag{5.6}$$

(3) At a boundary point x^0 of G a real vector $\xi \neq 0$ satisfies (5.6) if and only if it is orthogonal to ∂G at x^0.

(5) Agmon states the theorem for formally self-adjoint value problems. This is done in order to make the conditions sufficient as well.

The construction and discussion of such systems have been carried out by Friedrichs [10] and Višik [23]. Employing them we have

LEMMA 5.3. *Hypotheses* (c), (d), *and* (e) *are sufficient for the inequality*

$$\| u \|_{m,2}^2 \leqq \text{const.} \left(\sum_{i,j} \| \tilde{D}_i^{m-m_k} A_k u \|_{0,2}^2 + \| u \|_{0,2}^2 \right)$$

to hold for all $u \in C^\infty(\bar{G})$ *satisfying* (5.5).

Proof. We consider the operator $A_{ik} = \tilde{D}_i^{m-m_k} A_k$ as an operator of order m and apply Lemma 5.2 to the A_{ik}. At interior points of G the characteristic polynomials of the A_{ik} have no nonzero real vector roots in common if and only if the same is true of the A_k. At boundary points this can happen if the A_k of order m have a normal vector root in common. However, this situation is precluded by hypothesis (e). Moreover, the complex roots of the A_k as described in hypothesis (d) are the same for the A_{ik} and hence the result follows from Lemma 5.2.

6. **A stronger result for one operator.** Assume that A is an elliptic operator of even order $2q$ with coefficients in $C^\infty(\bar{G})$. Let $\{B_j\}_{j=1}^{2q}$ be a normal set of order $< 2q$ having coefficients also in $C^\infty(\bar{G})$. If A' denotes the formal adjoint of A we have by integration by parts

$$(Au,v) = (u,A'v) + \sum_{j=1}^{2q} \int_{\partial G} B_j u \overline{B_j' v} \, d\sigma, \tag{6.1}$$

where $\{B_j'\}_{j=1}^{2q}$ is a normal set and the order of B_j' is $2q - \nu_j - 1$. Let V be the set of those $u \in C^\infty(\bar{G})$ such that

$$B_j u = 0 \text{ on } \partial G, \qquad 1 \leqq j \leqq q,$$

and V' the set of those $v \in C^\infty(\bar{G})$ satisfying

$$B_j' v = 0 \text{ on } \partial G, \qquad q < j \leqq 2q.$$

We define the norm

$$| w |_{s,p}' = \operatorname*{lub}_{v \in V'} \frac{|(w,v)|}{\| v \|_{-s,p'}}.$$

When $s \geqq 0$ this is equivalent to the norm $\| w \|_{s,p}$, but not otherwise.

We shall make use of the following result proved in [20].

LEMMA 6.1. *If the set* $\{B_j\}_{j=1}^q$ *covers*(6) A, *then for every real* s *there is a constant* M_s *such that*

$$\| u \|_{s,p} \leqq M_s(| Au |_{s-2q,p}' + \| u \|_{s-2q,p})$$

for all $u \in V$.

(6) Cf. hypothesis (d) of §3. In the present case there is only one operator in the set $\{A_k.\}$

Employing the techniques of §4, we can generalize this to

THEOREM 6.1. *Under the same hypotheses, for each integer s there is a constant M_s' such that*

$$(6.2)\qquad \| u \|_{s,p} \leqq M_s'\left(| Au |'_{s-2q,p} + \sum_{j=1}^{q} \langle B_j u \rangle_{s-\nu_j-1/p,p} + \| u \|_{s-2q,p} \right)$$

for all $u \in C^\infty(\bar{G})$.

Proof. Let u_0 be any function in $C^\infty(\bar{G})$ such that

$$(6.3)\qquad B_j(u-u_0)=0 \text{ on } \partial G, \qquad 1 \leqq j \leqq q,$$

$$(6.4)\qquad B_j u_0 = 0 \text{ on } \partial G, \qquad q < j \leqq 2q.$$

Thus $u-u_0 \in V$, and by Lemma 6.1

$$\| u-u_0 \|_{s,p} \leqq M_s(| A(u-u_0) |'_{s-2q,p} + \| u-u_0 \|_{s-2q,p}).$$

Hence

$$(6.5)\quad \| u \|_{s,p} \leqq M_s(| Au |'_{s-2q,p} + \| u \|_{s-2q,p}) + \operatorname{glb} C_s(| Au_0 |'_{s-2q,p} + \| u_0 \|_{s,p}),$$

where the glb is taken over all such u_0.
First consider the case $s \geqq 2q$. Then

$$(6.6)\qquad | Au_0 |'_{s-2q,p} \leqq C \| Au_0 \|_{s-2q,p} \leqq C' \| u_0 \|_{s,p}.$$

Moreover, by Lemma 4.7,

$$(6.7)\qquad \operatorname{glb} \| u_0 \|_{s,p} \leqq \text{const.} \sum_{j=1}^{q} \langle B_j u \rangle_{s-\nu_j-1/p,p}.$$

Combining (6.5), (6.6) and (6.7) we obtain (6.2).
If $0 < s < 2q$ we note that

$$(6.8)\qquad | Au_0 |'_{s-2q,p} = \operatorname*{lub}_{v \in V'} \frac{|(Au_0, v)|}{\| v \|_{2q-s,p'}}.$$

We shall show in the appendix that there is a bilinear form

$$a_s(u,v) = \int_G \sum_{|\mu| \leqq s} \sum_{|\tau| \leqq 2q-s} a_{s\mu\tau} D^\mu u \overline{D^\tau v}\, dx$$

such that

$$(6.9)\qquad (Au,v) = a_s(u,v) + \sum_{j=1}^{2q} \int_{\partial G} B_j u \overline{B_j'' v}\, d\sigma$$

for all $u,v \in C^\infty(\bar{G})$, where the order of B_j'' is less than $2q - \nu_j$ and its normal(7) order is less than $2q - s$. Thus by (6.4)

$$(6.10) \qquad (Au_0, v) = a_s(u_0, v) + \sum_{j=1}^{q} \int_{\partial G} B_j u_0 \, \overline{B_j'' v} \, d\sigma.$$

Now if $\nu_j \geqq s$, then by definition

$$(6.11) \qquad \left| \int_{\partial G} B_j u_0 \, \overline{B_j'' v} \, d\sigma \right| \leqq \langle B_j u_0 \rangle_{s-\nu_j-1/p,p} \langle B_j'' v \rangle_{\nu_j+1/p-s,p'}.$$

Moreover, by Lemma 4.3,

$$(6.12) \qquad \langle B_j'' v \rangle_{\nu_j+1/p-s,p'} \leqq \| B_j'' v \|_{\nu_j+1-s,p'} \leqq C \| v \|_{2q-s,p'}$$

since the order of B_j'' is $\leqq 2q - \nu_j - 1$ and $1/p + 1/p' = 1$. If $\nu_j < s$, then

$$(6.13) \qquad \int_{\partial G} B_j u_0 \, \overline{B_j'' v} \, d\sigma = (\gamma B_j u_0, B_j'' v) + (B_j u_0, \gamma B_j'' v),$$

where γ is a first order operator which equals the normal derivative on ∂G. Neither B_j'' nor $\gamma B_j''$ has normal derivatives of order greater than $2q - s$. Hence in both expressions on the right hand side of (6.13) we can throw a sufficient number of tangential derivatives over on $\gamma B_j u_0$ and $B_j u_0$ so that only derivatives up to order $2q - s$ remain on v. Clearly, no boundary integrals are introduced by this process. Hence we have

$$(6.14) \qquad \left| \int_{\partial G} B_j u_0 \, \overline{B_j'' v} \, d\sigma \right| \leqq C \| u_0 \|_{s,p} \| v \|_{2q-s,p'} .$$

Combining (6.8), (6.10)-(6.13) we have

$$(6.15) \qquad \| u \|_{s,p} \leqq C\left(| Au |'_{s-2q,p} + \sum_{j=1}^{q} \langle B_j u \rangle_{s-\nu_j-1/p,p} + \| u \|_{s-2q,p} \right) + C \operatorname{glb} \| u_0 \|_{s,p'},$$

where the glb is taken over all $u_0 \in C_\infty(\bar{G})$ satisfying (6.3,4). By Corollary 4.1, this is the same as the glb taken over those u_0 which satisfy (6.3, 4) only for those B_j of order less than s. But then

$$(6.16) \qquad \operatorname{glb} \| u_0 \|_{s,p} \leqq C \sum_{j=1}^{q} \langle B_j u \rangle_{s-\nu_j-1/p,p}.$$

Combining (6.15) and (6.16) we obtain (6.2).

(7) I.e., in local coordinates the highest order derivative in the normal direction appearing in B_j'' is less than $2q - s$.

Finally, for the case $s \leqq 0$, we employ (6.1). We have

$$(Au_0, v) = (u_0, A'v) + \sum_{j=1}^{q} \int_{\partial G} B_j u_0 \overline{B_j' v}\, d\sigma$$

and hence

$$|(Au_0, v)| \leqq \| u_0 \|_{s,p} \| A'v \|_{-s,p'} + \sum_{j=1}^{q} \langle B_j u_0 \rangle_{s-\nu_j-1/p,p} \langle B_j' v \rangle_{\nu_j+1/p-s,p'}.$$

Now by Lemma 4.3,

$$\| A'v \|_{-s,p'} \leqq C \| v \|_{2q-s,p'}$$

and

$$\langle B_j' v \rangle_{\nu_j+1/p-s,p'} \leqq C \| v \|_{2q-s,p'}.$$

Hence

$$\| u \|_{s,p} \leqq C\left(| Au |'_{s-2q,p} + \sum_{j=1}^{q} \langle B_j u \rangle_{s-\nu_j-1/p,p} + \| u \|_{s-2q,p} \right) + C \operatorname{glb} \| u_0 \|_{s,p}.$$

However, since $C_0^\infty(G)$ is dense in $H^{s,p}(G)$,

$$\operatorname{glb} \| u_0 \|_{s,p} = 0$$

and (6.2) follows. This completes the proof.

THEOREM 6.2. *Let ν be the maximum order of the B_j, $1 \leqq j \leqq q$. Then under the same hypotheses, for each real $s \geqq \nu + 1$ there is a constant M_s' such that* (6.2) *holds for all $u \in C^\infty(\bar{G})$.*

Proof. Let N be the set of those $v \in V$ such that $Au = 0$. We know that N is finite dimensional (cf., e.g., [19]). For every $u \in C^\infty(\bar{G})$ there is an $h \in N$ such that $u - h$ is orthogonal to N. The mapping $Tu = u - h$ is unique and satisfies

$$\| Tu \|_{s,p} \leqq C\left(| Au |'_{s-2q.} + \sum_{j=1}^{q} \langle B_j u \rangle_{s-\nu_j-1/p,p} \right) \tag{6.17}$$

when s is an integer. (Here we have applied Rellich's lemma to Theorem 6.1.) One easily extends T to be a bounded map of

$$V'^{s-2q,p}(G) \times \prod_{j=1} W^{s-\nu_j-1/p,p}(\partial G) \tag{6.18}$$

into $H^{s,p}(G)$(8). We now interpolate between consecutive integers. For the first space we employ Theorem 4.1 of [20]. For the remaining spaces we note that

(8) $V'^{t,p}$ is the completion of V' with respect to the norm $|\cdot|'_{t,p}$.

when $s \geqq \nu + 1$, all of the indices are positive and we can apply Lemma 4.6 of the present paper to conclude that T is bounded from the space (6.18) into $H^{s,p}(G)$ for nonintegral values of s as well. We now merely note that since N is finite dimensional

$$\begin{aligned} \| h \|_{s,p} &\leqq c \| h \|_{s-2q,p} \leqq c(\| u \|_{s-2q,p} + \| u - h \|_{s-2q,p}) \\ &\leqq c(\| u \|_{s-2q,p} + \| u - h \|_{s,p}). \end{aligned}$$

This together with (6.17) gives the required result.

REMARK. For the Dirichlet problem, special cases of Theorems 6.1 and 6.2 were proved by Agmon [2] and Lions-Magenes [14]. For $p = 2$ and general B_j, Theorem 6.2 is included in the work of Peetre [16].

7. **Proofs of the theorems.** We consider the set of operators $\{A_k\}$, $\{B_j\}_{j=1}^r$ satisfying hypotheses (a)-(e) of §3. Consider the operator of order $2m$

$$A = \sum_{i,k} A_k'(\tilde{D}_i')^{m-m_k}\tilde{D}_i^{m-m_k}A_k,$$

where the $\tilde{D}_i$ are the tangential operators described in §5 and $\tilde{D}_i'$ is the formal adjoint of $\tilde{D}_i$. By hypotheses (c) and (e) of §3 and properties (2) and (3) of the $\tilde{D}_i$, A is elliptic in $\bar{G}$.

Next, consider the bilinear form

$$[u,v] = \sum_{i,k}(\tilde{D}_i^{m-m_k}A_k u, \tilde{D}_i^{m-m_k}A_k v).$$

We add, if necessary, $m-r$ operators to the set $\{B_j\}_{j=1}^r$ in such a way that the resulting set $\{B_j\}_{j=1}^m$ is normal and of order $< m$. By (5.2)

$$[u,v] = (Au,v) + \sum_{j=1}^{m}\int_{\partial G} F_j u \, \overline{B_j v}\, d\sigma, \tag{7.1}$$

where the order of F_j is $2m - \nu_j - 1$. By symmetry we also have

$$[u,v] = (u,Av) + \sum_{j=1}^{m}\int_{\partial G} B_j u \, \overline{F_j v}\, d\sigma. \tag{7.2}$$

Thus if V is the set of $u \in C^\infty(\bar{G})$ satisfying

$$B_j u = 0 \quad \text{on} \quad \partial G, \qquad 1 \leqq j \leqq r, \tag{7.3}$$

$$F_j u = 0 \quad \text{on} \quad \partial G, \qquad r < j \leqq m, \tag{7.4}$$

then V' is the same set by (7.1) and (7.2) (note that $A' = A$).

Now by Lemma 5.3

$$\| u \|_{m,2}^2 \leqq C([u,u] + \| u \|_{0,2}^2)$$

for all u satisfying (7.3). Hence by Lemma 5.1 the operators $B_1, \cdots, B_r, F_{r+1}, \cdots, F_m$ cover A. This allows us to apply Lemma 6.1 to obtain the inequality

$$\| u \|_{s,p} \leqq M_s(| Au |'_{s-2m,p} + \| u \|_{s-2m,p}) \tag{7.5}$$

holding for all $u \in V$. Now for $u \in V$ and $s \leqq m$ we have, by (7.1),

$$\begin{aligned} | Au |'_{s-2m,p} &= \operatorname*{lub}_{v \in V'} \frac{|(Au,v)|}{\| v \|_{2m-s,p'}} = \operatorname*{lub}_{v \in V} \frac{|[u,v]|}{\| v \|_{2m-s,p'}} \\ &\leqq \sum_k \operatorname*{lub}_{v \in V} \frac{|(A_k u, \sum_i (\tilde{D}'_i)^{m-m_k} \tilde{D}_i^{m-m_k} A_k v)|}{\| v \|_{2m-s,p'}} \\ &\leqq \sum_k \| A_k u \|_{s-m_k,p}, \end{aligned} \tag{7.6}$$

where we have made use of the fact that the $\tilde{D}_i$ are tangential and may be moved back and forth without introducing boundary terms. If s is not an integer, Lemma 4.3 is employed. Combining (7.5) and (7.6) we see that (1.9) holds for all $u \in V$. However, every term in (1.9) is majorized by constant $\cdot \| u \|_{m,p}$. Since the orders of the F_j are $\geqq m$, we know by Corollary 4.1 that every function satisfying only (7.3) can be approximated in $H^{m,p}(G)$ by functions in V. Hence (1.9) holds for all functions satisfying (7.3). Since $s_k \geqq s - m_k$, Theorem 3.2 is proved.

In order to prove Theorem 3.1 we employ inequality (6.2) and note that (7.6) holds for all u satisfying only (7.4). Thus (1.6) holds for all such u when s is an integer $\leqq m$. Again, since all such u are dense in $H^{m,p}(G)$, (1.6) holds for all $u \in C^\infty(\bar{G})$. This completes the proof of Theorem 3.1 for the case $s \leqq m$. The case $s > m$ is reduced to the case $s = m$ by considering the operators

$$D^\mu A_k, \qquad |\mu| \leqq s - m_k,$$

in place of the A_k. We merely note that the new operators satisfy hypotheses (a)–(d) if and only if the A_k do likewise. Moreover, in this case hypothesis (e) follows from hypothesis (c), and Corollary 3.1 is proved.

8. **Coerciveness for bilinear forms.** We now show how one can extend the inequalities in §3 to include general bilinear forms (cf. §5). Consider such a form $[u,v]$ of order m and assume that

$$\| u \|^2_{m,2} \leqq \text{const.}(\operatorname{Re}[u,u] + \| u \|^2_{0,2}) \tag{8.1}$$

for all $u \in C^\infty(\bar{G})$ satisfying

$$B_j u = 0 \quad \text{on} \quad \partial G, \qquad 1 \leqq j \leqq r. \tag{8.2}$$

Then by Lemma 5.1, A is elliptic and the boundary operators $B_1, \cdots, B_r$, $F_{r+1}, \cdots, F_m$ cover it, where A and the F_j are given by (5.2). Thus we can apply Lemma 6.1 to obtain the inequality

$$\| u \|_{s,p} \leqq M_s(| Au |'_{s-2m,p} + \| u \|_{s-2m,p}) \tag{8.3}$$

for all $u \in C^\infty(\bar{G})$ satisfying (8.2) and

$$F_j u = 0 \quad \text{on} \quad \partial G, \qquad r < j \leqq m, \tag{8.4}$$

where

$$| Au |'_{s-2m,p} = \operatorname*{lub}_{v \in V'} \frac{|(Au,v)|}{\| v \|_{2m-s\, p'}} \tag{8.5}$$

and V' is the set of those $u \in C_\infty(\bar{G})$ satisfying (8.2) and

$$F'_j u = 0 \quad \text{on} \quad \partial G, \qquad r < j \leqq m \tag{8.6}$$

(cf. (5.3) and (6.1)). But by (5.2), $(Au, v) = [u, v]$ whenever u satisfies (8.4) and v satisfies (8.2). Setting

$$[u]_{s-m,p} = \operatorname*{lub}_{v \in V'} \frac{|[u,v]|}{\| v \|_{2m-s.p'}},$$

we have

THEOREM 8.1. *If inequality* (8.1) *holds for all u satisfying* (8.2), *then for each real $s \leqq m$*

$$\| u \|_{s,p} \leqq C([u]_{s-m,p} + \| u \|_{s-m,p}) \tag{8.7}$$

for all such u.

Proof. From (8.3) and (8.5) we see that inequality (8.7) holds for all u satisfying (8.2) *and* (8.4). However, since the order of each F_j is $\geqq m$ and $[u]_{s-m,p}$ is majorized by $\| u \|_{m,p}$ when $s \leqq m$; functions satisfying only (8.2) can be approximated by those satisfying both (8.2) and (8.4). This completes the proof.

Employing Theorem 6.1 in place of Lemma 6.1 we have

THEOREM 8.2. *Under the same hypotheses, when s is an integer $\leqq m$*

$$\| u \|_{s,p} \leqq C\left([u]_{s-m,p} + \sum_{j=1}^{r} \langle B_j u \rangle_{s-v_j-1/p,p} + \| u \|_{s-m,p} \right) \tag{8.8}$$

for all $u \in C^\infty(\bar{G})$.

In order to complete our discussion, we note that Agmon [1] has given necessary and sufficient conditions for (8.1) to hold for u satisfying (8.2). Set

$$\tilde{L}(x,\xi) = \operatorname{Re} \sum_{|\mu| = |\tau| = m} a_{\mu\tau}(x) \xi^{\mu+\tau},$$

and at each boundary point x^0 let $\sigma(x^0)$ denote the interior unit normal vector. Then the conditions are

(i) $\tilde{L}(x,\xi) > 0$ in $\bar{G}$ for real $\xi \neq 0$.

(ii) For every fixed boundary point x^0 and every real vector T orthogonal to $\sigma(x^0)$

$$\operatorname{Re} \int_0^\infty \sum_{|\mu|=|\tau|=m} a_{\mu\tau}(x^0)\left(T - i\sigma(x^0)\,\frac{d}{dt}\right)^\mu f(t) \times \overline{\left(T - i\sigma(x^0)\frac{d}{dt}\right)^\tau f(t)}\; dt > 0$$

for all functions $f(t) \not\equiv 0$ in $C_0^\infty(-\infty,\infty)$ which satisfy

$$\tilde{L}\left(x^0, T - i\sigma(x^0)\,\frac{d}{dt}\right) f = 0, \qquad t > 0,$$

$$Q_j\left(x^0, T - i\sigma(x^0)\,\frac{d}{dt}\right) f = 0, \qquad t = 0, \quad 1 \leqq j \leqq r,$$

where $Q_j(x,\xi)$ is the characteristic polynomial of B_j.

Combining this with Theorems 8.1 and 8.2 we have

THEOREM 8.3. *Under hypotheses* (i) *and* (ii) *inequality* (8.7) *holds for all u satisfying* (8.2) *whenever s is real and $\leqq m$. Inequality* (8.8) *holds for all u when s is an integer.*

APPENDIX

We now give a proof of (6.9). We write A in the form

$$A = \sum_{i=0}^{m} \Gamma_i \gamma_i,$$

where $m = 2q$, γ_i is an operator of order i which equals the normal derivative of order i on ∂G and Γ_i is an operator of order $\leqq m-i$ involving only derivatives in tangential directions. Then

$$\begin{aligned} (Au, v) &= \sum_{i=0}^{s} (\gamma_i u, \Gamma_i' v) + \sum_{i=s+1}^{m} (\gamma_i u, \Gamma_i' v) \\ &= \sum_{i=0}^{s} (\gamma_i u, \Gamma_i' v) + \Big(\gamma_s u, \sum_{i=s+1}^{m} \gamma_{i-s} \Gamma_i' v\Big) \\ &\quad + \sum_{i=s+1}^{m} \sum_{j=s}^{i} \int_{\partial G} \gamma_{j-1} u, \gamma_{i-j}\, \overline{\Gamma_i' v}\; d\sigma, \end{aligned} \tag{A.1}$$

where we have absorbed the alternating signs in the operators. By rearrangement, if necessary, we may assume that the order of B_j is $j-1$. Then by (4.2)

$$\gamma_{j-1} = \sum_{t=1}^{j} \Lambda_{jt} B_t, \qquad 1 \leqq j \leqq m$$

and hence the boundary terms in (A.1) become

$$\sum_{i=s+1}^{m} \sum_{j=s}^{i} \sum_{t=1}^{j} \int_{\partial G} \Lambda_{jt} B_t u \overline{\gamma_{i-j} \Gamma_i' v}\, d\sigma = \sum_{t=1}^{m} \int_{\partial G} B_t u \overline{B_t'' v}\, d\sigma$$

where

$$B_t'' = \sum_{j=\max(s+1,t)}^{m} \; \sum_{i=\max(s+1,j)}^{m} \Lambda_{jt}' \gamma_{i-j} \Gamma_i'.$$

One easily checks that the order of B_t'' is $\leqq m-t$ and that the highest order of the γ_i occurring in it is less than $\min(m-t+1,\ m-s)$.

Added in Proof. If we define $W^{0,p}(\partial G)$ in a suitable way (e.g., $W^{0,p}(\partial G) = [W^{-1,p}(\partial G),\, W^{1,p}(\partial G);\, \delta(\theta)]$) we can show that Theorems 3.1, 6.1 and 8.2 hold for all real values of s. Details will be given in a forthcoming publication.

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