# KERNEL CONSTRUCTIONS AND BOREL SETS

BY

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1. **Introduction.** A well-known theorem, essentially due to Cantor, states that every topological space $X$ has a "perfect kernel" $K$; $K$ is the largest dense-in-itself subset of $X$, $K$ is closed, and $X - K$ is scattered. Analogues of this situation occur frequently. Recently the author had occasion to use a "non-locally-separable kernel" [9, Lemma 7]; and further work on (nonseparable) Borel sets has led to a need for further generalizations. Hence we here study a rather inclusive class of kernel constructions[2]. The main result (Theorem 4 below) is essentially an extension of the Cantor-Bendixson theorem; roughly speaking, if $X$ (for simplicity) is metric, the complement of its "non-locally-$P$ kernel" is the union of a countable family of closed sets, each locally $P$, and having other desirable properties[3]. We also obtain (Theorem 7) a useful criterion for this kernel to be empty. The results are applied to give extensions of two well-known theorems: one, due to Banach, about sets which are locally of first category, and the other, due to Montgomery, about sets which are locally Borel. We also apply them to characterize two classes of metric spaces: those which are both absolutely $G_\delta$ and absolutely $F_\sigma$, and those of which every subspace is both absolutely $G_\delta$ and absolutely $F_\sigma$. Other applications to the theory of Borel sets will be made in a subsequent paper. Our main interest is in metric spaces, but most of the results are formulated more generally since the proofs are no simpler in the metric case.

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**Notation.** Let $P$ be a class (or "property", not necessarily topological) of topological spaces. *We shall assume throughout that $P$ is "hereditary"* in the sense that if $X \in P$ then, for every *closed* subspace $A$ of $X$, $A \in P$. If *every* subspace of ($X \in P$) is also a member of $P$, we say that $P$ is *completely hereditary*. For example, $P$ might be the class of all compact spaces, or of all absolutely Borel metric spaces, or of all spaces of (covering) dimension $\leq n$. Relevant examples

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(2) Not every kernel of importance is included, as for instance the "dimensional kernel" (see [5, p. 186] and the beginning of §4 below). A more general class of kernels is described in [4, pp. 164–166]. For a quite different method of kernel construction, see [3].

(3) The "classical" Cantor-Bendixson theorem has one further feature which cannot be extended, namely that the perfect kernel of a separable metric space $X$ can be obtained by countably many operations, i.e., in the notation of §2 below, one can take $\alpha^*$ to be countable. This feature will persist when $X$ has a countable base, but not in general (even if $X$ is metric).

of completely hereditary classes are: (1) all spaces of weight less than a given cardinal; (2) all metric spaces of dimension $\leqq n$; (3) all subspaces of measure 0 of a fixed (complete) measure space. If $\mathscr{A}$ is any class of spaces or sets, we define a completely hereditary class, denoted by $(\mathscr{A})$, as follows: $(\mathscr{A}) = \{X \mid X \subset A$ for some $A \in \mathscr{A}\}$.

A *neighborhood* of a point $x$ in a space $X$ is any set whose interior contains $x$. A neighborhood *in* $X$ is a neighborhood of some point of $X$. A space $X$ is *locally* $P$ if each $x \in X$ has a neighborhood belonging to $P$. Clearly.

(1.1) If $P$ is completely hereditary, or if $X$ is regular, and if $X$ is locally $P$, then each $x \in X$ has arbitrarily small neighborhoods belonging to $P$.

We say that $X$ is *nowhere locally* $P$ if no neighborhood in $X$ belongs to $P$. Thus, if $P$ is completely hereditary, this is equivalent to requiring that no nonempty open subset of $X$ belongs to $P$.

In a metric space $X$, with metric $\rho$, $U(x, \delta)$ denotes the neighborhood

$$\{y \mid y \in X,\ \rho(x, y) < \delta\},$$

where $x \in X$ and $\delta > 0$.

## 2. Existence and elementary properties of kernels.

THEOREM 1. *For any (hereditary) class $P$, given any space $X$ there exists a closed subset $K = K(P, X)$ of $X$ such that* (i) *$K$ is nowhere locally $P$,* (ii) *whenever $H$ is a closed subset of $X$ which is nowhere locally $P$, $H \subset K$.*

Thus $K(P, X)$ is the largest closed subset of $X$ which is nowhere locally $P$; we call it the "non-locally-$P$ kernel of $X$(4).

**Proof.** Define transfinite sequences $\{F_\alpha\}$, $\{G_\alpha\}$, of subsets of $X$ as follows: Suppose $F_\beta$, $G_\beta$ defined for all ordinals $\beta < \alpha$. Then

(a) if $\alpha = 0$, $F_\alpha = X$, $G_\alpha = \emptyset$;

(b) if $\alpha$ is a limit ordinal, $F_\alpha = \bigcap\{F_\beta \mid \beta < \alpha\}$, $G_\alpha = \emptyset$;

(c) if $\alpha = \gamma + 1$, put $G_\alpha =$ set of all $x \in F_\gamma$ such that some neighborhood of $x$ in $F_\gamma$ belongs to $P$, and define $F_\alpha = F_\gamma - G_\alpha$.

Clearly $F_0 \supset F_1 \supset \cdots$, and $G_\alpha$ is open relative to $F_\gamma$ in case (c), so that $F_\alpha$ is closed relative to $F_\gamma$. Thus in all cases $F_\alpha$ is closed in $X$. The decreasing transfinite sequence $\{F_\alpha\}$ must become ultimately constant; for some $\alpha^*$ we thus have $F_{\alpha^*} = F_{\alpha^*+1} = \cdots = K$, say(3). If some $x \in K$ has a neighborhood in $K$ which belongs to $P$, then $x \in G_{\alpha^*+1}$ and therefore $x \notin F_{\alpha^*+1} = K$, which is impossible, proving (i). If $H$ is closed and nowhere locally $P$, we show by transfinite induction that $H \subset F_\alpha$. Assume this true for all $\beta < \alpha$. In cases (a) and (b) it trivially follows that $H \subset F_\alpha$. In case (c), suppose $x \in H \cap G_\alpha$; then $x$ has a neighborhood $U$ in $F_\gamma$ such that $U \in P$. Since $H$ is a closed subset of $F_\gamma$, $U \cap H$ is a neighborhood of $x$ in $H$ and belongs to $P$, contrary to our hypothesis on $H$. Thus $H \cap G_\alpha = \emptyset$, proving $H \subset F_\alpha$. Hence $H \subset F_{\alpha^*} = K$.

(4) The proof which follows is essentially that in [4, p. 166], but is given in full as it is needed for reference later.

COROLLARY. *If further $P$ is completely hereditary, we may omit the requirement in* (ii) *that $H$ be closed. (That is, $K$ is now the largest nowhere locally $P$ subset of $X$.)*

This is clear from the proof; alternatively, one easily sees that if $H$ is nowhere locally $P$ (where $P$ is completely hereditary) then so is $\bar{H}$. The requirement of complete hereditariness is essential for this corollary, as the following example shows. Let $X$ be the unit interval, $P$ the class of all compact spaces; there is no largest nowhere locally compact subset of $X$ (as one sees by first observing that both it and its complement would have to be dense).

We shall refer to the sets $F_\alpha$, $G_\alpha$ constructed above (in the proof of Theorem 1) as $F_\alpha(P, X)$, $G_\alpha(P, X)$. When the family $P$ can be understood from the context, we abbreviate them to $F_\alpha(X)$, $G_\alpha(X)$, and abbreviate $K(P, X)$ to $K(X)$. The following properties are then easily verified by transfinite induction.

(2.1) $F_\alpha(X)$ is a closed subset of $X$, and $F_\alpha(F_\beta(X)) = F_{\beta+\alpha}(X)$.

(2.2) If $Y$ is a closed subset of $X$, $F_\alpha(Y) \subset Y \cap F_\alpha(X)$.

(2.3) If $P$ is completely hereditary and $Y \subset X$, then $F_\alpha(Y) \subset Y \cap F_\alpha(X)$.

(2.4) If either $P$ is completely hereditary or $X$ is regular, and if $Y$ is an open subset of $X$, then $F_\alpha(Y) = Y \cap F_\alpha(X)$.

From these properties the next assertions follow immediately.

(2.5) $K(X)$ is a closed subset of $X$.

(2.6) $K(K(X)) = K(X)$.

(2.7) If $Y$ is a closed subset of $X$, $K(X) \supset K(Y)$.

If $P$ is completely hereditary, then

(2.7′) $K(X) \supset K(Y)$ if $X \supset Y$.

(2.8) If $Y$ is an open subset of $X$, $K(Y) = Y \cap K(X)$ providing that either $P$ is completely hereditary or $X$ is regular. In particular, $K(X - K(X)) = \emptyset$.

The above properties of kernels are characteristic. We have:

THEOREM 2. *Suppose that a mapping $K$ (of $2^S$ in $2^S$), defined for all subspaces $X$ of a fixed space $S$, satisfies* (2.5)-(2.8). *Then there exists a hereditary family $P$ such that, for all $X \subset S$, $K(X) = K(P, X)$. If further* (2.7′) *is satisfied, $P$ may be taken to be completely hereditary.*

Take $P$ to be the family of all $X \subset S$ for which $K(X) = \emptyset$; this is hereditary, by (2.7). Write $H(X) = K(P, X)$ for each $X \subset S$; we verify that $H(X) = K(X)$. If $V = X - K(X)$, then $V$ is open in $X$, and $K(V) = \emptyset$ by (2.8); as $H(X)$ is nowhere locally $P$, we must have $V \cap H(X) = \emptyset$, proving $H(X) \subset K(X)$. To prove equality we have only to show that the closed subset $K(X)$ of $X$ is nowhere locally $P$. If this is false, there exists $A \subset S$ whose interior $U$ (in $S$) meets $K(X)$ and which is such that $K(A \cap K(X)) = \emptyset$. A straightforward computation, using (2.8) and (2.6), now shows that $U \cap K(X) = K(U \cap K(X)) \subset K(A \cap K(X)) = \emptyset$, a contradiction. Finally, if (2.7) is strengthened to (2.7′), $P$ is evidently completely hereditary.

We mention three further easily verified properties which will be used later:

(2.9) If $Y$ is a subspace of a space $X$, and if $Q$ is any hereditary class of spaces, then the class $P = \{E \mid E \cap Y \in Q\}$ is hereditary, and $K(Q,Y) = Y \cap K(P,X)$.

(2.10) If $\mathscr{A}$, $\mathscr{B}$ are families of sets such that, for each $A \in \mathscr{A}$, $A \cap X$ is a subset of some $B \in \mathscr{B}$, then $K((\mathscr{A}),X) \supset K((\mathscr{B}),X)$.

(2.11) If $\mathscr{A}_X$ denotes the family $\{A \cap X \mid A \in \mathscr{A}\}$, then $K((\mathscr{A}),X) = K((\mathscr{A}_X),X)$.

REMARKS. On taking $\mathscr{A} = \mathscr{S}$ = family of all 1-point subsets of $X$, we see that $K((\mathscr{S}),X)$ is the ordinary "perfect kernel" of $X$. Thus, from (2.10), the perfect kernel of $X$ is the largest of the kernels $K((\mathscr{B}),X)$ for which $\mathscr{B}$ covers $X$.

Again, given any closed $A \subset X$, let $P = (X - A)$ = family of all subsets of $X - A$; clearly $K(P,X) = A$ here, so that every closed subset of $X$ arises as a kernel for some completely hereditary class.

The analog of (2.8) for *closed* (instead of open) sets $U$ would be false; in fact it is false for ordinary perfect kernels. Moreover, when $P$ is not required to be completely hereditary, the requirement in (2.8) that $X$ be regular cannot be weakened to requiring $X$ merely to be Hausdorff. This is shown by the following example, based on a suggestion of the referee. Let $X$ be a Hausdorff space with a nonempty open subset $Y$ such that no neighborhood in $Y$ is closed in $X$; for instance [1a] provides such a space. Let $P$ be the family of all closed subsets of $X$. Then $K(X) = \emptyset$, but $K(Y) = Y \neq Y \cap K(X)$. It would be interesting to know whether regularity is needed for the special case $K(X - K(X)) = \emptyset$ of (2.8). The $T_1$ axiom would not suffice, as the following example shows.

Let $X = \{(x,y) \mid \text{either } y \geqq 0 \text{ or } y \text{ is irrational}\} \subset R^2$, and let

$$V_n = \{(x,y) \mid x > n, y > 0\} \quad (n = 1,2,\cdots).$$

Topologize $X$ by taking a basis of neighborhoods of $(x,y)$ to consist of all sets of the form $X \cap U(x,y) \cup V_n$, where $U(x,y)$ is an ordinary plane neighborhood of $(x,y)$. Then $X$ is a $T_1$ space, with topology coarser than the subspace-of-plane topology. Let $P$ be the family of all subsets $E$ of $X$ which intersect every line $x$ = constant in a set which is closed in the ordinary plane topology. One sees without difficulty that $P$ is hereditary, that $K(P,X)$ is the subset of $X$ where $y \leqq 0$, and that $K(X - K(P,X)) = X - K(P,X) \neq \emptyset$.

THEOREM 3. *A necessary and sufficient condition that $K(P,X) = \emptyset$ is that, for each nonempty closed $A \subset X$, there is some neighborhood in $A$ belonging to $P$.*

If $K(P,X) \neq \emptyset$, the condition is violated when $A = K(P,X)$. Conversely, if $K(P,X) = \emptyset$ and a closed set $A$ violates the condition, then $A \subset K(P,X)$ and so $A = \emptyset$.

COROLLARY. *If $P$ is completely hereditary, we may omit the requirement that $A$ be closed.*

This is clear from the proof. Note, however, that if $E \subset X$ has some neighborhood belonging to $P$, it does *not* now follow that $\bar{E}$ has the same property. (Example: $P$ = class of all countable sets, $X$ = real line, $E$ = set of rational numbers.)

3. **Paracompact spaces.** It is convenient to introduce the following terminology. If a subset $E$ of a space $X$ is the intersection of an open subset of $X$ with a closed subset of $X$ (or, equivalently, is the difference between two open, or two closed, sets), $E$ will be called an "$FG$ set" (in $X$).

A family $\{E_\lambda \mid \lambda \in \Lambda\}$ of subsets of $X$ is *discrete* (in $X$) providing each $x \in X$ has a neighborhood meeting $E_\lambda$ for at most one $\lambda \in \Lambda$. Clearly, if $\{E_\lambda\}$ is discrete, then so is $\{\bar{E}_\lambda\}$.

LEMMA 1. *If $\{E_\lambda \mid \lambda \in \Lambda\}$ is a discrete system of FG subsets of a space $X$, then $\bigcup E_\lambda$ is an FG set.*

We have $E_\lambda = F_\lambda \cap G_\lambda$ where $F_\lambda$ is closed and $G_\lambda$ is open. Put $A_\lambda = \bigcup\{\bar{E}_\mu \mid \mu \neq \lambda\}$, $F = \bigcup \bar{E}_\lambda$, $G = \bigcup\{G_\lambda - A_\lambda\}$; because $\{\bar{E}_\lambda\}$ is discrete, $A_\lambda$ and $F$ are closed, while of course $G$ is open. It is easily verified that $\bigcup E_\lambda = F \cap G$.

THEOREM 4. *Let $X$ be a hereditarily paracompact space, and $P$ any hereditary class*(5). *Then $X - K(P, X)$ is expressible as the union of a countable family $\mathscr{A} = \{A_n \mid n = 1, 2, \cdots\}$ of FG sets, each of which is locally $P$, in such a way that $K((\mathscr{A}), X) = K(P, X)$.*

We first prove a special case of the theorem as a lemma.

LEMMA 2. *Theorem 4 is true when $K(P, X) = \emptyset$.*

In this case, the proof of Theorem 1 shows that $F_\alpha(X) = \emptyset$ for some ordinal $\alpha$. The proof will be by transfinite induction over $\alpha$; thus we may assume that the lemma is valid for all hereditarily paracompact spaces $X'$ for which $F_\beta(X') = \emptyset$ for some $\beta < \alpha$.

First suppose that $\alpha$ is a limit ordinal. Then $\bigcap\{F_\beta(X) \mid \beta < \alpha\} = F_\alpha(X) = \emptyset$. The open covering $\{X - F_\beta(X) \mid \beta < \alpha\}$ of $X$ has a $\sigma$-discrete refinement $\{V_{n\lambda} \mid \lambda \in \Lambda_n,\ n = 1, 2, \cdots\}$, the system $\{V_{n\lambda} \mid \lambda \in \Lambda_n\}$ being, for each $n$, a discrete system of open sets(6). Each $V_{n\lambda}$ is disjoint from some $F_\beta(X)$, and from (2.4) we have $F_\beta(V_{n\lambda}) = V_{n\lambda} \cap F_\beta(X) = \emptyset$. From the induction hypothesis we can write $V_{n\lambda} = \bigcup \mathscr{A}_{n\lambda}$ where $\mathscr{A}_{n\lambda} = \{A_{n\lambda m} \mid m = 1, 2, \cdots\}$, each $A_{n\lambda m}$ being $FG$ and locally $P$ (in $V_{n\lambda}$ and therefore also in $X$), and where $K((\mathscr{A}_{n\lambda}), V_{n\lambda}) = \emptyset$. Put $A_{nm} = \bigcup\{A_{n\lambda m} \mid \lambda \in \Lambda_n\}$; from Lemma 1 this is an $FG$ set, and it is clearly locally $P$. We have only to verify that, on taking $\mathscr{A} = \{A_{nm} \mid n, m = 1, 2, \cdots\}$, we have $K((\mathscr{A}), X) = \emptyset$. If $H$ is any nonempty closed set, $H$ meets some $V_{n\lambda}$; because

(5) That is, *every* subspace (and not merely every closed subspace) of $X$ is paracompact. Every perfectly normal paracompact space is hereditarily paracompact [2, p. 643].

(6) See [9, p. 979].

$K((\mathscr{A}_{n\lambda}), V_{n\lambda}) = \emptyset$, there is some neighborhood in $H \cap V_{n\lambda}$ which is a subset of some $A_{n\lambda m} \subset A_{nm} \in \mathscr{A}$; and the result follows from Theorem 3.

The case $\alpha = 0$ being trivial, we may now assume $\alpha = \beta + 1$. Then each $x \in F_\beta(X)$ has an open neighborhood in $F_\beta(X)$ whose closure belongs to $P$. The resulting covering of $F_\beta(X)$ has a $\sigma$-discrete refinement $\{U_{n\lambda}\}$ by sets $U_{n\lambda}$ open in $F_\beta(X)$, and therefore $FG$ in $X$; for fixed $n$, the sets $U_{n\lambda}$ (say $\lambda \in \Lambda_n$) form a system discrete in $F_\beta(X)$ and therefore also in $X$. Put $B_n = \bigcup \{\bar{U}_{n\lambda} \mid \lambda \in \Lambda_n\}$; this is closed and locally $P$, and the system $\mathscr{B} = \{B_n \mid n = 1, 2, \cdots\}$ covers $F_\beta(X)$. Put $W = X - F_\beta(X)$; by (2.4), $F_\beta(W) = W \cap F_\beta(X) = \emptyset$, so by the induction hypothesis we have $W = \bigcup \mathscr{C}$ where $\mathscr{C} = \{C_n \mid n = 1, 2, \cdots\}$, each $C_n$ being $FG$ in $W$ (and so in $X$) and locally $P$, and where $K((\mathscr{C}), W) = \emptyset$. Put $\mathscr{A} = \mathscr{B} \cup \mathscr{C}$; we have only to verify that $K((\mathscr{A}), X) = \emptyset$. Again we show that each closed nonempty subset $H$ of $X$ has some neighborhood (in $H$) contained in some $A \in \mathscr{A}$. If $H \subset F_\beta(X)$, then $H$ meets some $U_{n\lambda}$, and $U_{n\lambda} \cap H$ is a neighborhood in $H$ contained in $B_n \in \mathscr{B}$. In the remaining case $H$ meets $W$; and, because $K((\mathscr{C}), W) = \emptyset$, $H \cap W$ has a neighborhood contained in some $C_n$, and this provides a neighborhood in $H$ contained in $C_n \in \mathscr{A}$.

To prove Theorem 4 from the lemma, put $Y = X - K(P, X)$. Then $Y$ is open; hence, from (2.8), $K(P, Y) = \emptyset$. The lemma gives $Y = \bigcup \mathscr{A}$, where $\mathscr{A} = \{A_n \mid n = 1, 2, \cdots\}$, each $A_n$ being $FG$ (in $Y$, and so in $X$) and locally $P$, and where $K((\mathscr{A}), Y) = \emptyset$. Now $Y \cap K((\mathscr{A}), X) = K((\mathscr{A}), Y) = \emptyset$, from (2.8) again; hence $K((\mathscr{A}), X) \subset K(P, X)$. On the other hand, $K(P, X)$ is closed and no neighborhood in it can be a subset of any $A_n$; thus $K(P, X) \subset K((\mathscr{A}), X)$, and the theorem is proved.

If we assume a little more about $X$ we can require more of the sets $A_n$, as the next theorem shows.

Theorem 4′. *If $X$ is paracompact and perfectly normal, and $P$ is any hereditary class, then $X - K(P, X)$ is expressible as the union of a countable family $\mathscr{A} = \{A_n \mid n = 1, 2, \cdots\}$ of locally $P$ sets, each of which is closed and expressible as the union of a discrete system of closed sets belonging to $P$, and where $K((\mathscr{A}), X) = K(P, X)$.*

To deduce Theorem 4′, let the $FG$ sets produced by Theorem 4 be denoted by $B_n$, $n = 1, 2, \cdots$; thus $B_n = C_n \cap G_n$ where $C_n$ is closed and $G_n$ is open. Using the perfect normality of $X$, we write $G_n = \bigcup \{F_{nm} \mid m = 1, 2, \cdots\}$, where $F_{nm}$ is closed and interior to $F_{n,m+1}$. Now $C_n \cap F_{nm}$ is closed and locally $P$; by paracompactness (using a $\sigma$-discrete refinement of a suitable covering) we obtain

$$C_n \cap F_{nm} = \bigcup \{A_{nmp\lambda} \mid \lambda \in \Lambda_{nmp},\ p = 1, 2, \cdots\},$$

where $\{A_{nmp\lambda} \mid \lambda \in \Lambda_{nmp}\}$ is a discrete system (in $C_n \cap F_{nm}$ and so in $X$) of closed sets, each belonging to $P$. We put $A_{nmp} = \bigcup \{A_{nmp\lambda} \mid \lambda \in \Lambda_{nmp}\}$,

$$\mathscr{A} = \{A_{nmp} \mid n, m, p = 1, 2, \cdots\},$$

and have only to verify that $K((\mathscr{A}), X) = K(P, X)$. As before, on writing $Y = X - K(P, X)$, it will be enough to prove that $K((\mathscr{A}), Y) = \emptyset$. If $H$ is a nonempty subset of $Y$, then by Theorem 4 there is some neighborhood $H \cap U$ ($U$ being open) such that $H \cap U \subset B_n$. Choose $y \in H \cap U$; then $y \in C_n \cap F_{nm}$ for some $m$; a neighborhood of $y$ in $H \cap U$ will be contained in $C_n \cap F_{n, m+1}$, and a smaller neighborhood in $H$ will thus be contained in some $A_{nmp\lambda} \subset A_{nmp} \in \mathscr{A}$, as required (Theorem 3, Corollary).

There is an important special case in which the requirement that $K((\mathscr{A}), X) = K(P, X)$ is fulfilled automatically:

THEOREM 5. *If $X$ is a complete metric space and $\mathscr{A} = \{A_n \mid n = 1, 2, \cdots\}$ is any countable family of closed locally $P$ sets whose union contains $X - K(P, X)$, then $K((\mathscr{A}), X) = K(P, X)$.*

Write $K(P, X) = K, K((\mathscr{A}), X) = L$; trivially $K \subset L$. If there is a point $x \in L - K$, it has an open neighborhood $U$ such that $\bar{U} \cap K = \emptyset$. Then (with the notation $\mathrm{Cl}(Y)$ for $\bar{Y}$) we have $\mathrm{Cl}(U \cap L) \subset \bigcup A_n$, so by Baire's theorem there exists a nonempty set $W \cap \mathrm{Cl}(U \cap L)$, where $W$ is open, such that $W \cap \mathrm{Cl}(U \cap L) \subset A_n$ for some $n$. Then $W \cap U \cap L$ is a neighborhood in $L$ which is a subset of $A_n$, contradicting $L = K((\mathscr{A},) X)$.

As the proof shows, the assumption (in Theorem 5) that $X$ is complete metric could be weakened to the assumption that every nonempty closed subset of $X$ is of the second category (in itself). (This is a genuine weakening; see [5, p. 423] for an example.) This weaker property can in fact be characterized in terms of kernels, as follows.

THEOREM 6. *If $X$ is any perfectly normal space, the following statements are equivalent:*

(i) *Every nonempty closed subset of $X$ is of the second category* (*in itself*).

(ii) *Every nonempty $G_\delta$ subset of $X$ is of the second category* (*in itself*).

(iii) *Whenever $\mathscr{A}$ is a countable family of closed sets whose union is $X$, $K((\mathscr{A}), X) = \emptyset$.*

The implication (ii) $\Rightarrow$ (i) is trivial; and (i) $\Rightarrow$ (iii) by the same argument as that used to prove Theorem 5. To prove (iii) $\Rightarrow$ (ii), let $H$ be a nonempty $G_\delta$ set, and suppose $H \subset \bigcup A_n$ $(n = 1, 2, \cdots)$ where each $A_n$ is closed (in $X$) and nowhere dense in $H$. We also have $X - H = \bigcup B_n$ $(n = 1, 2, \cdots)$ where $B_n$ is closed in $X$. Let $\mathscr{A}$ be the family of all sets $A_n$ and $B_n$; clearly $K((\mathscr{A}), X) \supset H \neq \emptyset$.

REMARK. The spaces of which every nonempty $F_\sigma$ subset is of second category will be considered later (Theorem 11).

THEOREM 7. *If $X$ is a complete metric space* (*or, more generally, is perfectly* *normal and paracompact and satisfies condition* (i) *of Theorem* 6), *then* $K(P, X) = \emptyset$ *if and only if* $X$ *is the union of a countable family of closed locally* $P$ *sets.*

From Theorems 4, 5 and 6.

4. **The theorems of Banach and Montgomery.** There are a number of rather obvious applications of Theorem 4, for instance the result (obtained by taking $P =$ class of metrizable spaces of covering dimension $\leqq n$) that every metric space $X$ can be decomposed into a closed set $K$, in which every neighborhood has dimension $> n$, and the complementary open set, which has dimension $\leqq n$. Thus $K$ is similar to, but not identical with, the "dimensional kernel" of $X$. The situation here is somewhat trivial, because here $K = F_1$. Another illustration is the Cantor-Bendixson theorem expressing every separable metric space as the union of a perfect (i.e., closed and dense-in-itself) set $K$ and a countable scattered set (obtained by taking $P=(\mathscr{S})$, the class of 1-point sets). An application which lies deeper is the following extension (Theorem 8 below) of a well-known theorem of Banach [1], which we first derive (in slightly generalized form) as a lemma.

LEMMA 3. *If* $X$ *is a subset of a hereditarily paracompact space* $Y$, *and each* $x \in X$ *has a neighborhood in* $X$ *which is of first category in* $Y$, *then* $X$ *is of first category in* $Y$.

The hypothesis gives, for each $x \in X$, an open set $U(x)$ in $Y$ such that $U(x) \cap X$ is of first category in $Y$. Let $G = \bigcup\{U(x) \mid x \in X\}$; $G$ is open and paracompact, so the covering $\{U(x)\}$ of $G$ has a $\sigma$-discrete open refinement

$$\{V_{n\lambda} \mid \lambda \in \Lambda_n,\ n = 1, 2, \cdots\},$$

the system $\{V_{n\lambda} \mid \lambda \in \Lambda_n\}$ being, for each $n$, discrete in $G$. Each set $X \cap V_{n\lambda}$ is of first category in $Y$, so expressible as $\bigcup\{A_{n\lambda m} \mid m = 1, 2, \cdots\}$, where no $\bar{A}_{n\lambda m}$ (closure in $Y$) contains any neighborhood in $Y$. Write $B_{nm} = \bigcup\{A_{n\lambda m} \mid \lambda \in \Lambda_n\}$. It is easily verified that $\bar{B}_{nm}$ also contains no neighborhood in $Y$ (else it would have to contain a neighborhood in $G$); and $X = \bigcup\{B_{nm} \mid n, m = 1, 2, \cdots\}$ of first category in $Y$.

COROLLARY. *If* $Y$ *is hereditarily paracompact and* $X$ *is the set of points of* $Y$ *which have first category neighborhoods in* $Y$, *then* $X$ *is of first category in* $Y$.

THEOREM 8. *Let* $\mathscr{A}$ *be any family of subsets of a hereditarily paracompact space* $Y$; *let* $\bigcup \mathscr{A} = X$. *A necessary and sufficient condition that* $X$ *be of first category in* $Y$ *is that* $K((\mathscr{A}), X)$ *and each* $A \in \mathscr{A}$ *be of first category in* $Y$.

The necessity is trivial. To prove sufficiency, apply Theorem 4 to $X$, with $P = (\mathscr{A})$; we obtain $X - K((\mathscr{A}), X) = \bigcup\{B_n \mid n = 1, 2, \cdots\}$ where each point of each $B_n$ has a neighborhood in $B_n$ which is contained in some $A \in \mathscr{A}$, and which is therefore of first category in $Y$. By Lemma 3, $B_n$ is of first category in $Y$; and $X$, as the union of the countably many first category sets $B_n$ and $K((\mathscr{A}), X)$, is also of first category in $Y$.

REMARK. Theorem 8 includes Lemma 3 as a special case; for when $X$ is the union of a family $\mathscr{A}$ of sets, each *open* in $X$, we clearly have $K((\mathscr{A}), X) = \emptyset$.

COROLLARY. *A necessary and sufficient condition that a subset $X$ of a hereditarily paracompact space $Y$ be of first category in $Y$, is that, for each nonempty (relatively) closed subset $A$ of $X$, there exists a neighborhood in $A$ which is of first category in $Y$.*

Again only the sufficiency needs proof. Let $P$ = family of all first category subsets of $Y$. By Theorem 3, the condition implies that $K(P, X) = \emptyset$. Hence, by Theorem 4, $X = \bigcup \mathscr{A}$ where $\mathscr{A} = \{A_n \mid n = 1, 2, \cdots\}$, and where $K((\mathscr{A}), X) = \emptyset$ and each $A_n$ is locally of first category in $Y$. By Lemma 3, $A_n$ is of first category in $Y$; hence, by Theorem 8, so is $X$.

As another application of the theory we give a similar extension of a well-known theorem of Montgomery [6, p. 527]. Again the first step is to derive a slight extension of Montgomery's theorem as a lemma.

LEMMA 4. *If $X$ is a subset of a perfectly normal paracompact space $Y$, and each $x \in X$ has a neighborhood in $X$ which is a Borel subset of $Y$ of additive class $\xi$ (multiplicative class $\xi \geqq 1$), then $X$ is a Borel subset of $Y$ of the same class*(7).

(Note that $\xi$ must be independent of $x \in X$ here.)

Assume that the lemma is valid for all smaller $\xi$ than the given one, and suppose first that $\xi > 1$. There are two cases to consider:

(i) Additive class $\xi$. Each $x \in X$ has an open neighborhood $U(x)$ in $Y$ such that $U(x) \cap X$ is of (additive) class $\xi$. Without loss we may replace $Y$ by $\bigcup\{U(x) \mid x \in X\}$; let $\{V_\lambda \mid \lambda \in \Lambda\}$ be a locally finite open refinement of the covering $\{U(x) \mid x \in X\}$ of $Y$. Each $V_\lambda \cap X$ is then of additive class $\xi$ in $Y$, and so is expressible as $\bigcup\{F_{\lambda n} \mid n = 1, 2, \cdots\}$ where $F_{\lambda n}$ is of multiplicative class $\eta_n$, $0 < \eta_n < \xi$, and where $F_{\lambda 1} \subset F_{\lambda 2} \subset \cdots$. By repeating the sets $F_{\lambda n}$, if necessary, we may arrange that $\eta_n$ is independent of $\lambda$. Put $H_n = \bigcup\{F_{\lambda n} \mid \lambda \in \Lambda\}$; from the local finiteness of $\{V_\lambda\}$, $H_n$ is locally of multiplicative class $\eta_n$ in $Y$. Thus, by the hypothesis of induction, $H_n$ is of multiplicative class $\eta_n$ in $Y$, and $X = \bigcup H_n$, of additive class $\xi$, as required.

(ii) Multiplicative class $\xi$. We define $U(x)$, $V_\lambda$ as before and now have $V_\lambda \cap X = \bigcap\{G_{\lambda n} \mid n = 1, 2, \cdots\}$ where $V_\lambda \supset G_{\lambda 1} \supset G_{\lambda 2} \supset \cdots$ and $G_{\lambda n}$ is of additive class $\eta_n$ in $Y$, where $0 < \eta_n < \xi$, and where we may suppose $\eta_n$ independent of $\lambda$, as before. Put $H_n = \bigcup\{G_{\lambda n} \mid \lambda \in \Lambda\}$; $H_n$ is locally, and therefore globally, of additive class $\eta_n$. Now the fact that $\{V_\lambda\}$ is locally finite gives $X = \bigcap H_n$, of multiplicative class $\xi$.

(7) Here $\xi$ denotes a countable ordinal. For the notation and elementary properties of Borel sets, see [5]. Note that, if $\eta < \xi$, each Borel set of (additive or multiplicative) class $\eta$ is of both additive and multiplicative class $\xi$ (because of perfect normality).

All that remains is to check the (trivial) case of additive class 0 (i.e., open sets) and the cases $\xi = 1$, where one sees that the above arguments still apply with slight modifications. In fact, for multiplicative class 1 (the $G_\delta$ case) the assumptions on $Y$ can be weakened from perfect normality and paracompactness; it suffices that $Y$ be hereditarily pointwise paracompact.

THEOREM 9. *Let $\xi$ be a countable ordinal, let $\mathscr{A}$ be any family of subsets of a perfectly normal paracompact space $Y$, such that each $A \in \mathscr{A}$ is Borel of additive class $\xi$ (multiplicative class $\xi > 0$) in $Y$, and let $\bigcup \mathscr{A} = X$. A necessary and sufficient condition for $X$ to be Borel of additive class $\xi$ (multiplicative class $\xi > 0$) in $Y$ is that $K((\mathscr{A}), X)$ be Borel in $Y$ of this class.*

Again, Theorem 9 contains the lemma as a special case, for when the sets $A \in \mathscr{A}$ are open in $X$, $K((\mathscr{A}), X) = \emptyset$.

Only the sufficiency needs proof; and, the case of additive class 0 being trivial, we assume $\xi > 0$ throughout. For additive class $\xi$, the theorem follows from Theorem 4 by an argument very similar to the proof of Theorem 8; unfortunately the case of multiplicative class $\xi$ does not seem to follow in this way. Hence we give a direct argument, which applies to both cases, and which is similar to the proof of Theorem 4([8]).

It is enough to prove that $X - K((\mathscr{A}), X)$ is Borel of the required class. We replace $X$ by $X - K((\mathscr{A}), X)$ and $\mathscr{A}$ by $\{A - K((\mathscr{A}), X) \mid A \in \mathscr{A}\}$; in view of (2.8) and (2.11) this means that we may without loss assume $K((\mathscr{A}), X) = \emptyset$. Thus we have $F_\alpha(X) = \emptyset$ for some ordinal $\alpha$. We use transfinite induction over $\alpha$, keeping $\xi$ fixed, and assume the theorem known for all spaces $X'$ for which $F_\beta(X') = \emptyset$ for some $\beta < \alpha$.

First suppose that $\alpha$ is a limit ordinal. Then $\bigcap \{F_\beta(X) \mid \beta < \alpha\} = F_\alpha(X) = \emptyset$, so that each $x \in X$ has an open neighborhood $U(x)$ in $X$ which is disjoint from $F_\beta(X)$ for some $\beta < \alpha$. From (2.4), $F_\beta(U(x)) = \emptyset$. Also $U(x) = \bigcup \{A \cap U(x) \mid A \in \mathscr{A}\}$, where each $A \cap U(x)$ is Borel of class $\xi$ in $Y$ (because it is open in $A$ and $\xi > 0$). From the induction hypothesis it follows that $U(x)$ is Borel of class $\xi$ in $Y$. By Lemma 4, so is $X$.

The case $\alpha = 0$ being trivial ($X = \emptyset$), the only remaining case is $\alpha = \beta + 1$. Here $F_{\beta+1}(X) = \emptyset$, so $F_\beta(X) = G_{\beta+1}(X)$, and hence each $x \in F_\beta(X)$ has an open neighborhood $V(x)$ in $X$ such that $V(x) \cap F_\beta(X) \subset$ some $A(x) \in \mathscr{A}$. Because $V(x)$ is open and $F_\beta(X)$ is closed (in $X$), $V(x) \cap F_\beta(X) \cap A(x)$—that is, $V(x) \cap F_\beta(X)$—is Borel of class $\xi$ in $Y$. Hence again Lemma 4 shows that $F_\beta(X)$ is Borel of class $\xi$ in $Y$. Write $W = X - F_\beta(X)$; by (2.4), we have $F_\beta(W) = \emptyset$. Since $W = \bigcup \{W \cap A \mid A \in \mathscr{A}\}$, where $W \cap A$ is Borel of class $\xi$ in $Y$ (for $W$ is open in $X$), the induction hypothesis shows that $W$ is Borel of class $\xi$ in $Y$. Hence so is $W \cup F_\beta(X) = X$.

(8) It would be possible to obtain a common generalization of Theorems 4 and 9, but its formulation would apparently have to be very unwieldy.

COROLLARY. *A necessary and sufficient condition that a subspace $X$ of a perfectly normal paracompact space $Y$ be Borel of additive class $\xi$ (multiplicative class $\xi > 0$) is that, for each nonempty (relatively) closed subset $A$ of $X$, there exists a neighborhood in $A$ which is Borel of additive class $\xi$ (multiplicative class $\xi > 0$) in $Y$.*

The deduction is essentially the same as that of the corollary to Theorem 8. We remark that in Theorem 9, its corollary, and Lemma 4, the assertions fail for multiplicative class 0 (closed sets). Of course, closed sets can be regarded as being both of additive and multiplicative class 1 (i.e., both $F_\sigma$ and $G_\delta$).

5. **$F_\sigma$-and-$G_\delta$ sets.** As is well known, a (not necessarily separable) metric space $X$ is an absolute $G_\delta$ (i.e., $G_\delta$ in every metrizable space containing $X$) if and only if $X$ can be given a complete metric. It was shown in [10] that $X$ is an absolute $F_\sigma$ if and only if it is $\sigma$-locally-compact (i.e., expressible as the union of a countable family of locally compact sets)(9). Thus every locally compact space is both an absolute $F_\sigma$ and an absolute $G_\delta$. We shall now characterize these "absolute $F_\sigma$ and $G_\delta$" sets, showing that they are not far from being locally compact.

THEOREM 10. *The following statements about a metric space $X$ are equivalent.*

(1) *$X$ is both an absolute $F_\sigma$ and an absolute $G_\delta$.*

(2) *The non-locally-compact kernel of $X$ is empty.*

(3) *Each nonempty closed subset of $X$ is locally compact at at least one point.*

(4) *$X$ is expressible as the union of a countable family $\mathscr{A} = \{A_n \mid n = 1,2,\cdots\}$ of closed locally compact sets $A_n$ such that $A_1 \subset A_2 \subset \cdots$ and $K((\mathscr{A}), X) = \emptyset$.*

(5) *$X$ is expressible as the union of an arbitrary family $\mathscr{A}$ of sets, each of which is both an absolute $F_\sigma$ and an absolute $G_\delta$, such that $K((\mathscr{A}), X) = \emptyset$.*

First we show (1) $\Rightarrow$ (4). Assume (1); then $X$, as an absolute $F_\sigma$, is [10] expressible as $\bigcup\{B_p \mid p = 1,2,\cdots\}$ where $B_p$ is locally compact and hence open in $\bar{B}_p$. Thus $B_p = \bigcup\{C_{pq} \mid q = 1,2,\cdots\}$ where $C_{pq}$ is closed and therefore also locally compact. The union of two closed locally compact sets is locally compact; hence $A_n = \bigcup\{C_{pq} \mid p + q \leqq n\}$ is closed and locally compact, and clearly $A_1 \subset A_2 \subset \cdots$ and $\bigcup A_n = X$. $X$, as an absolute $G_\delta$, may be supposed complete; hence, by Theorem 6, $K((\mathscr{A}), X) = \emptyset$.

Next, (1) $\Rightarrow$ (2). Assume (1); then (4) follows, and $X$ is the union of a countable family of closed locally-$P$ sets, where $P$ is the class of all compact spaces. As before, $X$ may be supposed complete; hence, by Theorem 7, $K(P, X) = \emptyset$.

The equivalence of (2) and (3) follows from Theorem 3.

(9) The referee points out that it follows that a metric space $X$ will be $F_\sigma$ in every *Hausdorff* space in which it is imbedded, if and only if it is a *separable* absolute $F_\sigma$, i.e., is $\sigma$-compact. It is easy to see that an arbitrary $T_1$ space is $F_\sigma$ in every $T_1$ space in which it is imbedded, if and only if it is countable. The empty set is the only $T_i$ space ($i = 1, 2, 3, 4$) which is $G_\delta$ in every $T_i$ space in which it is embedded.

$(2) \Rightarrow (4)$. By Theorem 5, $X$ is the union of a countable family

$$\mathscr{B} = \{B_n \mid n = 1, 2, \cdots\}$$

of closed locally compact sets such that $K((\mathscr{B}), X) = K(P, X) = \emptyset$. Let $A_n = \bigcup \{B_m \mid m \leqq n\}$; property (2.10) gives $K((\mathscr{A}), X) \subset K((\mathscr{B}), X) = \emptyset$, and (4) follows.

Trivially $(4) \Rightarrow (5)$. Finally, $(5) \Rightarrow (1)$ by Theorem 9. These implications together establish the theorem.

Finally we characterize those (metric) spaces all of whose subsets are of the kind described in Theorem 10; they turn out to coincide with a very well-known class of spaces (the scattered spaces) and also to coincide with the spaces all of whose subsets are of the kind described in Theorem 6.

THEOREM 11. *The following statements about a metric space $X$ are equivalent*(10).

(1) *Every subset of $X$ is absolutely both $F_\sigma$ and $G_\delta$.*

(2) *$X$ is an absolute $G_\delta$ and every subset of $X$ is an absolute Borel set.*

(3) *$X$ is an absolute $G_\delta$ and is $\sigma$-discrete.*

(4) *$X$ is the union of a countable family $\mathscr{A}$ of closed sets $A_n$ $(n = 1, 2, \cdots)$ such that $A_1 \subset A_2 \subset \cdots$, $A_n$ is discrete, and $K((\mathscr{A}), X) = \emptyset$.*

(5) *$X$ is the union of a family $\mathscr{A}$ of sets such that, for all $A \in \mathscr{A}$, each subset of $A$ is absolutely both $F_\sigma$ and $G_\delta$, and such that $K((\mathscr{A}), X) = \emptyset$.*

(6) *The perfect kernel of $X$ is empty.*

(7) *$X$ contains no subset homeomorphic to the space of rational numbers.*

(8) *Every nonempty subset of $X$ has an isolated point.*

(9) *Every nonempty closed subset of $X$ has an isolated point.*

(10) *Every nonempty subset of $X$ is of the second category (in itself).*

(11) *Every nonempty $F_\sigma$ subset of $X$ is of the second category (in itself).*

**Proof.** Trivially $(1) \Rightarrow (2)$. In proving $(2) \Rightarrow (6)$, we may assume $X$ complete; if its perfect kernel $K$ is not $\emptyset$, a well-known theorem of W. H. Young shows that $K$ has a subset homeomorphic to the Cantor set; this contains a non-Borel set, contradicting (2). Next, $(6) \Rightarrow (4)$; for, if (6) holds, Theorem 4′ enables us to write $X = \bigcup B_n$ $(n = 1, 2, \cdots)$ where $B_n$ is closed and discrete and where $K((\mathscr{B}), X) = \emptyset$, $\mathscr{B}$ denoting $\{B_n \mid n = 1, 2, \cdots\}$. We take $A_n = B_1 \cup \cdots \cup B_n$ and use (2.10). To prove $(4) \Rightarrow (3)$, we observe that if (4) holds then, by the corollary to Theorem 9, $X$ is an absolute $G_\delta$. If (3) holds, every subset of $X$ is $\sigma$-discrete and therefore [10] an absolute $F_\sigma$. Its complement is likewise $F_\sigma$ in $X$; so every subset of $X$ is $G_\delta$ in the absolute $G_\delta$ set $X$, and is therefore itself absolutely $G_\delta$. Thus $(3) \Rightarrow (1)$, and the first four statements are equivalent to (6).

(10) See [7] for other equivalent properties. The equivalence of (6), (7), (8), (9) is, of course, well known.

The implication (4) $\Rightarrow$ (5) is trivial; and (5) $\Rightarrow$ (1) by Theorem 9, in view of (2.7′). The equivalence of (6), (8) and (9) is a well-known special case of Theorem 3 and its corollary. Obviously (6) $\Rightarrow$ (7); conversely, if the perfect kernel $K$ of $X$ is not empty, one easily constructs (by a straightforward iteration) a countable dense-in-itself subset $R$ of $K$; and $R$ is homeomorphic to the space of rational numbers([11]). Thus (7) $\Rightarrow$ (6); and the first nine statements are equivalent.

Trivially (8) $\Rightarrow$ (10) $\Rightarrow$ (11); we complete the proof of the theorem by proving (11) $\Rightarrow$ (6). Suppose the perfect kernel $K$ of $X$ is not empty. For each $n$ ($= 1,2,\cdots$) let $A_n$ be a maximal subset of $K$ each two (distinct) points of which have distance $\geqq 1/n$. Then $A_n$ is discrete and closed (in $K$ and so in $X$). Let $B = \bigcup A_n$; $B$ is dense in $K$, and clearly $B$ is a nonempty $F_\sigma$ subset of $X$. We show that $B$ is *not* of second category (in itself) by showing that no $A_n$ contains any neighborhood in $B$. In fact, given $x \in A_n$ and $\varepsilon > 0$, then (because $K$ is dense-in-itself) there exists $y \in K$ such that $0 < \rho(x, y) < \min(\varepsilon, 1/n)$, and there exists $z \in B$ so close to $y$ that $0 < \rho(x,z) < \min(\varepsilon, 1/n)$ also. Thus $z \notin A_n$, since otherwise two distinct points of $A_n$ have distance $< 1/n$. But $z \in B \cap U(x,\varepsilon)$, proving that $A_n$ contains no neighborhood in $B$. This completes the proof of the theorem.

We remark that if we alter (11) by replacing "$F_\sigma$" by "$G_\delta$", we obtain the spaces of Theorem 6, which may be "pathological". It can be shown that if we delete from (2) and (3) the requirement that $X$ be an absolute $G_\delta$, they remain equivalent; the author hopes to publish the proof elsewhere.

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([11]) See [8].