TCHEBYCHEFF QUADRATURE ON THE INFINITE INTERVAL(1)

BY

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1. **Introduction.** Theorem A, the principal theorem of this paper can be interpreted as a result on the zeros of Faber polynomials or as a result on the theory of Tchebycheff quadrature.

1.1. The proof of Theorem A is achieved by means of two auxiliary theorems. In §2 Theorem B is stated and it is shown that it implies Theorem A. In §3 Theorem C is stated and proved. In §4 it is shown that Theorem C implies Theorem B, thus completing the proof of Theorem A. In §5 Theorem A is related to a paper on the zeros of Faber polynomials by the author [1], and a paper on Tchebycheff quadrature by Wilf [2].

1.2. A unit mass distribution on $(-\infty, \infty)$, possessing moments of all positive integer order will be said to belong to class D. If ψ, ψ_j, ψ^* are in class D, we will denote the kth moments by m_k, m_k^j, m_k^*, respectively.

THEOREM A. *There is an element ψ^* of class D which has the properties:*
(a) *the equations*

$$m_k^* = \frac{1}{n} \sum_{i=1}^{n} x_{i,n}^k, \qquad k = 1, \cdots, n, \tag{1.2.1}$$

have a real solution for infinitely many positive integers n,
(b) *the mass set of ψ^* does not lie on a finite interval.*

An element of D which satisfies (a) and (b) will be called a T_1 distribution. The set of integers for which (1.2.1) has real solutions will be called the T set of ψ^*.

2. **The first auxiliary theorem.**

2.1. Let ψ_j be an element of class D. The equations

$$m_k^j = \frac{1}{n} \sum_{i=n}^{n} x_{i,n}^k, \qquad k = 1, \cdots, n, \tag{2.1.1}$$

have a unique solution $t_{1,n}, \cdots, t_{n,n}$, up to a permutation of the first subscripts. Let

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$$F_n(z\,|\,\psi_j) = \prod_{i=1}^{n} (z - t_{i,n}) = \sum_{i=0}^{n} a_i^j z^{n-i}, \tag{2.1.2}$$

where $a_0^j = 1$. We adopt the convention that if ψ without a subscript is used, then the coefficients of the third term of (2.1.2) are written as a_i. The quantity a_i^j is a polynomial with real coefficients in the quantities $m_1^j, \cdots, m_i^j$, $i = 1, \cdots, n$.

2.2. Let ψ, ψ_1 be two elements of class D. Let

$$\| \psi - \psi_1 \|_n = \max\{|m_1 - m_1^1|, \cdots, |m_n - m_n^1|\}. \tag{2.2.1}$$

The defined quantity is called the nth order distance between the two mass distributions.

2.3. A unit mass distribution with n equal masses located at n distinct real points will be called a simple mass distribution of degree n. If the mass points of a simple mass distribution, say ψ, are $s_1, \cdots, s_n$, then

$$F_n(z\,|\,\psi) = \prod_{i=1}^{n} (z - s_i). \tag{2.3.1}$$

2.4. LEMMA A. *Let ψ be a simple mass distribution of degree n, and let ψ_1 be any element of class D. There is a number ε, $\varepsilon > 0$, called a proximity number of ψ, such that if*

$$\| \psi - \psi_1 \|_n < \varepsilon, \tag{2.4.1}$$

then the polynomial $F_n(z\,|\,\psi_1)$ has real zeros.

Proof. Let $s_1, \cdots, s_n$ be the mass points of ψ. Let σ_i be the circle $|z - s_i| = \delta_i$, $i = 1, \cdots, n$. The δ_i are chosen to be positive, and such that s_i is the only point of the set $s_1, \cdots, s_n$ inside or on σ_i. Let $E = \bigcup_{i=1}^{n} \sigma_i$, and let

$$\min_{z \in E} |F_n(z\,|\,\psi)| = \delta. \tag{2.4.2}$$

The quantity δ is positive. By Rouché's theorem, we know that if

$$|F_n(z|\,\psi_1) - F_n(z\,|\,\psi)| < \delta \tag{2.4.3}$$

for z on E, then $F_n(z\,|\,\psi_1)$ has one root inside each of the circles σ_i. By §2.1, the coefficients of $F_n(z\,|\,\psi_1)$ are real, so these roots must be real.

The left side of (2.4.3) is less than

$$\gamma \sum_{i=0}^{n} |a_i^1 - a_i|, \tag{2.4.4}$$

for $z \in E$, where

$$\gamma = \max_{z \in E} \{1, |z|, \cdots |z|^{n-1}\} > 0. \tag{2.4.5}$$

The quantity (2.4.4) is a continuous function of m_i^1, $i = 1, \cdots, n$, and takes on the value zero when $m_i^1 = m_i$, $i = 1, \cdots, n$. Hence there is a number $\varepsilon > 0$, such that if (2.4.1) is satisfied, (2.4.4) is less than δ, and therefore (2.4.3) is satisfied. This completes the proof.

2.5. THEOREM B. *There is an element* ψ^* *of class D, and a denumerable sequence of simple mass distributions* ψ_k *of degree* r_k *and proximity numbers* ε_k, $k = 1,2,\cdots$, *where* $r_1 < r_2 < \cdots$ *such that*

$$\int_a^\infty d\psi^* > 0, \quad \textit{for all } a > 0, \tag{2.5.1}$$

and

$$\| \psi^* - \psi_k \|_{r_k} < \varepsilon_k, \qquad k = 1,2,\cdots. \tag{2.5.2}$$

2.6. Theorem B implies Theorem A. We will show that if a function ψ^* exists satisfying (2.5.1) and (2.5.2), then the same function satisfies conditions (a) and (b) of Theorem A. It is clear that (2.5.1) implies (b). Also (2.5.2) implies that (1.2.1) has real solutions for $n = r_k$, $k = 1,2,\cdots$ because by Lemma A the zeros of $F_{r_k}(z \mid \psi^*)$ are real, and by (2.1.2) they form a solution to (1.2.1) for $n = r_k$.

3. The second auxiliary theorem.

3.1. The proof of Theorem C is based on a construction. In §5 methods for generalizing the construction to arrive at a wider class of T_1 distributions is discussed.

We first define a family of sets O which will remain fixed throughout §3 and §4. The family O consists of a denumerable number of nonoverlapping intervals on the positive real axis, say $\{O_j\}$, $j = 1,2,\cdots$, having centers at u_j, $j = 1,2,\cdots$ such that $0 < u_1 < u_2 < \cdots$, and u_n tends to infinity.

3.2. THEOREM C. *Let the family of sets O be given. There exists a denumerable sequence of simple mass distributions,* ψ_k, $k = 1,2,\cdots$, *having degree* $r_k, r_1 < r_2 < \cdots$, *and proximity numbers* ε_k *such that*

$$\int_{O_j} d\psi_k = \gamma_j > 0, \qquad j = 1,\cdots,k-1, \quad k = 2,3,\cdots, \tag{3.2.1}$$

and

$$\| \psi_k - \psi_{k-1} \|_{r_{k-1}} < \min \left\{ \frac{\varepsilon_1}{2^{k-1}}, \cdots, \frac{\varepsilon_{k-1}}{2} \right\}, \quad k = 2,3,\cdots. \tag{3.2.2}$$

3.3. An element ψ of D is said to be (M,k) compatible if ψ is a simple mass distribution, if the mass points of ψ lie in the sets $O_1,\cdots,O_k$, and if there is a positive mass on O_k, all of which is located at u_k. The operation $M(k,n)$, where n is an integer and automatically greater than one, can be applied to a (M,k) compatible mass distribution, and yields a unique mass distribution we denote by

$$\psi_1 = \psi M(k,n). \tag{3.3.1}$$

The distribution ψ_1 is characterized by the following properties:

$$\psi_1(E) = \psi(E) \tag{3.3.2}$$

for all sets to the left of O_k,

$$\psi_1(u_k) = \psi(u_k)\left(\frac{n-1}{n}\right), \tag{3.3.3}$$

and

$$\psi_1(u_{k+1}) = \frac{\psi(u_k)}{n}. \tag{3.3.4}$$

3.4. Lemma B. *Let ε be an arbitrary positive number, p an arbitrary integer, and let ψ be (M,k) compatible. There exists an integer n_1 such that*

$$\| \psi_1 - \psi \|_p < \varepsilon \tag{3.4.1}$$

where

$$\psi_1 = \psi M(k,n) \tag{3.4.2}$$

and $n \geqq n_1$.

Proof. By considering the explicit expressions for m_r, m_r^1, for any positive integer r, we find that

$$m_r^1 - m_r = \frac{\psi(u_k)}{n}(u_{k+1}^r - u_k^r). \tag{3.4.3}$$

This tends to zero as n tends to infinity, so that the proof of the lemma is readily completed.

3.5. An element ψ of D is said to be (S,k) compatible if all the mass is located at a finite number of points, say masses $b_i, b_i > 0$, $i = 1, \cdots, q$ at the points $v_1, \cdots, v_q$, $v_i < v_{i+1}$, $i = 1, \cdots, q-1$, if all the mass lies on the sets $O_1, \cdots, O_k$, if there is one mass point on O_k, namely the point u_k, and if the equations

$$b_i = \alpha_i b_q, \qquad i = 1, \cdots, q-1, \tag{3.5.1}$$

are satisfied by integer values for α_i. The operation $S(k,\delta)$, δ a positive number, can be applied to an (S,k) compatible mass distribution, and will yield a unique mass distribution we denote by

$$\psi_1 = \psi S(k,\delta). \tag{3.5.2}$$

The distribution ψ_1 is defined as follows. The mass point v_i, $1 \leqq i \leqq q-1$ is replaced by α_i mass points of mass b_q, say at $v_{i,1}, \cdots, v_{i,\alpha_i}$, according to some fixed law which satisfies the condition

$$|v_i - v_{i,j}| \leqq \delta, \qquad j = 1, \cdots, \alpha_i. \tag{3.5.3}$$

We can say for definiteness that

$$v_{i,j} = v_i + \frac{j}{\alpha_i}\delta, \qquad j = 1, \cdots, \alpha_i. \tag{3.5.4}$$

The mass b_q and the mass point v_q remains unaffected.

3.6. LEMMA C. *Let ε be an arbitrary positive number, p an arbitrary positive integer, and let ψ be (S,k) compatible. There exists a number $\delta_1 > 0$ such that*

$$\| \psi_1 - \psi \|_p < \varepsilon, \tag{3.6.1}$$

where

$$\psi_1 = \psi S(k,\delta) \tag{3.6.2}$$

and $\delta \leqq \delta_1$.

Proof. Using the notations of §3.5 we find that

$$m_r^1 - m_r = \sum_{i=1}^{q-1} \left(b_q \sum_{j=1}^{\alpha_i} v_{i,j}^r - b_i v_i^r \right), \tag{3.6.3}$$

where r is an arbitrary positive integer.

Because of (3.5.3), as δ tends to zero, $v_{i,j}$ tends to v_i. Using (3.5.1) we then see that (3.6.3) tends to zero, so the proof of the lemma is readily completed.

3.7. An element ψ of D followed by a finite sequence of operations of the type being considered is said to be well defined when the following conditions are satisfied. ψ must have the type of compatibility required to perform the first operation, and after any number of operations have been performed, the resulting mass distribution must have the proper compatibility condition for the next operation, when it exists.

We note that if ψ is (M,k) compatible, then $\psi M(k,n)$ is $(S, k+1)$ compatible, so that

$$\psi M(k,n) S(k+1,\delta) \tag{3.7.1}$$

is well defined.

LEMMA D. *Let ψ be (M,k) compatible and of degree r. Let ε be an arbitrary positive number. Let $\psi_1 = \psi M(k,n)$, and $\psi_2 = \psi_1 S(k+1,\delta)$. There exist numbers $\delta_1 > 0$ and $n_1 \geqq 2$ such that for any $\delta \leqq \delta_1$, and any integer $n \geqq n_1$*

$$\psi_2 \text{ is } (M, k+1) \text{ compatible} \tag{3.7.2}$$

and

$$\| \psi_2 - \psi \|_r < \varepsilon. \tag{3.7.3}$$

Proof. Consider the set consisting of the mass points of ψ_1 and the end points of $O_1, \cdots, O_k$. Let δ_2 be the smallest distance between any pair of these points. If $\delta < \delta_2$, then

$$\psi_2(O_j) = \psi_1(O_j), \qquad j = 1, \cdots, k, \tag{3.7.4}$$

and

$$\psi_2(u_{k+1}) = \psi_1(u_{k+1}). \tag{3.7.5}$$

This means that all the mass of ψ_2 is on the sets $O_1, \cdots, O_{k+1}$. Further checking shows that it is simple and that all of its mass on O_{k+1} is concentrated at u_{k+1} and is positive. Therefore ψ_2 is $(M, k+1)$ compatible.

We next make use of the inequality

$$\| \psi_2 - \psi \|_r \leqq \| \psi_1 - \psi \|_r + \| \psi_2 - \psi_1 \|_r, \tag{3.7.6}$$

which indeed holds true for any three elements of D. By Lemma B, we can chose n_1 so that the first term on the right is less than $\varepsilon/2$ for $n \geqq n_1$. By Lemma C we can choose δ_3 so that the second term on the right is less than $\varepsilon/2$ for $\delta < \delta_3$. Then Lemma D is true for $\delta_1 = \min(\delta_2, \delta_3)$, and the above choice of n_1.

3.8. We note the following properties that hold when $n \geqq n_1, \delta \leqq \delta_1$:

$$\psi_2(O_j) = \psi(O_j), \qquad j = 1, \cdots, k-1, \tag{3.8.1}$$

$$\psi_2(O_k) = \psi(u_k) - \frac{1}{p} n, \tag{3.8.2}$$

and

$$\psi_2(u_{k+1}) = \frac{1}{pn}. \tag{3.8.3}$$

The mass distribution ψ_2 is simple and of degree pn.

3.9. Lemma E. *Let ψ_1 be the unit mass located at u_1. There exist integers $n_1, n_2, \cdots, n_i \geqq 2$, and positive numbers $\delta_1, \delta_2, \cdots$ such that*: (a)

$$\psi_1 M(1, n_1) S(2, \delta_1) \cdots M(k-1, n_{k-1}) S(k, \delta_{k-1}) \tag{3.9.1}$$

is a well-defined sequence for $k \geqq 2$, (b) *the mass distribution ψ_k defined by* (3.9.1) *is (M, k) compatible, and* (c)

$$\| \psi_k - \psi_{k-1} \|_{r_{k-1}} = \min \left\{ \frac{\varepsilon_1}{2^{k-1}}, \cdots, \frac{\varepsilon_{k-1}}{2} \right\}, \tag{3.9.2}$$

for $k \geqq 2$, where ε_j is the proximity number of ψ_j and r_j is the degree of ψ_j, $j = 1, \cdots, k-1$.

Proof. We divide the proof into two cases.

Case I. $k = 2$. Let ε_1 be a proximity number of ψ_1. We have observed that (3.9.1) is well defined for this case in §3.7. By Lemma D, n^*, δ^* exist such that ψ_2 is $(M, 2)$ compatible and such that (3.9.2) is satisfied for $p = 2$, providing $n_1 \geqq n^*$, $\delta_1 \leqq \delta^*$. Choose $n_1 = n^*$ and $\delta_1 = \delta^*$.

Case II. *The inductive step.* Assume that the numbers $n_1, \cdots, n_{p-1}, \delta_1, \cdots, \delta_{p-1}$ exist such that (3.9.1) is well defined for $k = 2, \cdots, p$, such that ψ_p is (M, p) compatible and such that (3.9.2) is satisfied for $k = 2, \cdots, p$. These assumptions imply that ψ_p is simple. Let it have proximity number ε_p and degree r_p. The proximity numbers $\varepsilon_1, \cdots, \varepsilon_{p-1}$ used in (3.9.4) will be those introduced successively in previous steps. Since ψ_p is (M, p) compatible, by Lemma D there are numbers n^*, δ^* such that

$$\psi^* = \psi_p M(p, n) S(p + 1, \delta) \tag{3.9.3}$$

is $(M, p + 1)$ compatible, and

$$\| \psi^* - \psi_p \|_{r_n} \leqq \min \left\{ \frac{\varepsilon_1}{2^p}, \cdots, \frac{\varepsilon_p}{2} \right\} \tag{3.9.4}$$

for $n \geqq n^*, \delta \leqq \delta^*$. Let $n_p = n^*, \delta_p = \delta^*$, and let ψ_{p+1} be the function defined by (3.9.3) by these values. Then for the values $n_1, \cdots, n_p, \delta_1, \cdots, \delta_p$ (a), (b) and (c) are satisfied for $k = 2, \cdots, p + 1$. This completes the proof by induction.

3.10. We list properties of the functions ψ_k of this lemma:

$$r_k = n_1, \cdots, n_{k-1}, \quad k \geqq 2. \tag{3.10.1}$$

$$\psi_k(O_j) = \frac{n_j - 1}{n_1 \cdots n_j}, \quad j = 1, \cdots, k - 1, \quad k \geqq 2. \tag{3.10.2}$$

These follow from the fact that ψ_1 has degree 1, and the properties stated in §3.8 by induction.

3.11. To complete the proof of Theorem C, we use the mass distribution Lemma E, noting that (3.10.1) implies that $r_1 < r_2 < \cdots$, that (3.10.2) implies (3.2.1), and that (3.9.2) corresponds to (3.2.2).

4. **Proof of Theorem B.**

4.1. LEMMA F. *Let ψ_i be a convergent sequence of unit mass distribution on $[0, \infty)$ which satisfies*

$$m_k^i \leqq M_k, \tag{4.1.1}$$

where M_k is a constant independent of i. The limit distribution ψ will be of class D and

$$\lim_{i \to \infty} m_k^i = m_k. \tag{4.1.2}$$

Proof. We first show that for any $\varepsilon > 0$ there is a positive number A such that

$$\int_R^S x^k d\psi_i \leqq \varepsilon \tag{4.1.3}$$

for any R, S which satisfy $A < R < S$. This follows from the inequality

$$\int_R^S x^k d\psi_i \leqq \frac{1}{A}\int_R^S x^{k+1} d\psi_i \leqq \frac{M_{k+1}}{A}. \tag{4.1.4}$$

Hence

$$\int_R^S x^k d\psi < \varepsilon \text{ for } A < R < S, \tag{4.1.5}$$

and therefore ψ has moments of all orders. We next show that for any $\varepsilon > 0$ there is an i, such that

$$\left| \int_0^\infty x^k d\psi_i - \int_0^\infty x^k d\psi \right| < \varepsilon \tag{4.1.6}$$

for $i > i_1$. Choose A so that

$$\int_A^\infty x^k d\psi_i \leqq \frac{\varepsilon}{3}, \quad \int_A^\infty x^k d\psi \leqq \frac{\varepsilon}{3}. \tag{4.1.7}$$

The left side of (4.1.6) is less than

$$\left| \int_0^A x^k d\psi_i - \int_0^A x^k d\psi \right| + \int_A^\infty x^k d\psi_i + \int_A^\infty x^k d\psi. \tag{4.1.8}$$

There is an i_1, such that for $i > i_1$, the first term is less than $\varepsilon/3$ and the proof of (4.1.6) is complete. In particular, the case $k = 0$ shows that ψ is a unit mass distribution, and since it has moments of all orders it is of class D.

4.2. *Proof of Theorem* B. We now consider the sequence of mass distributions ψ_k of Theorem C. By (3.2.2), and (3.7.6) applied to arbitrary elements of D, and the inequality $\|\psi_1 - \psi_2\|_r \leqq \|\psi_1 - \psi_2\|_{r+1}$, where r is an arbitrary integer and ψ_1, ψ_2 are arbitrary elements of D, we find that

$$\|\psi_{k+p} - \psi_k\|_{r_k} \leqq \frac{\varepsilon_k}{2} + \cdots + \frac{\varepsilon_k}{2^k} < \varepsilon_k. \tag{4.2.1}$$

In particular, this means that

$$m_q^{k+p} \leqq m_q^k + \varepsilon_k, \qquad q \leqq r_k,\ p \geqq 1, \tag{4.2.2}$$

so that

$$m_q^i \leqq \max\{m_q^1, \cdots, m_q^{k-1}, m_q^k + \varepsilon_k\}. \tag{4.2.3}$$

The right-hand side is a constant independent of i, and there is an inequality for every value of q since r_k tends to infinity. Let $\psi_{k'}$ be a convergent subsequence of the ψ_k, which exists by Helly's theorem, and let ψ^* be the limit. Because of Lemma F we have

$$\|\psi^* - \psi_k\|_{r_k} < \varepsilon_k. \tag{4.2.4}$$

We now show that ψ^* satisfies the conditions of Theorem B. Indeed, (4.2.4) the same as (2.5.2), $\psi^*(O_j) = \gamma_j > 0$ by (3.10.2), and since by §3.1, u_j tends to infinity, (2.5.1) is satisfied. Thus the proof of the theorem is complete.

5. **Discussion of results.**

5.1. *Faber polynomials.* The polynomials $F_n(z|\psi)$ are the Faber polynomials of the series on the right-hand side of

$$z \exp\left(-\sum_1^\infty \frac{m_k}{kz^k}\right) = z + a_0 + \frac{a_1}{z} + \cdots, \tag{5.1.1}$$

which is obtained from the expression on the left-hand side by formal expansion. That is, they are the polynomial part of the formal nth power of the right-hand side of (5.1.1). In [1] a general theorem is given for the location of the zeros of Faber polynomials in the case

$$\limsup |a_n|^{1/n} < \infty. \tag{5.1.2}$$

If ψ is a T_1 distribution this condition is not satisfied, so that Theorem A can be interpreted as a result on the zeros of Faber polynomials for the case

$$\limsup |a_n|^{1/n} = \infty. \tag{5.1.3}$$

This example may suggest the proper formulation of a general theorem for zeros of Faber polynomials for the case (5.1.3).

5.2. *Tchebycheff quadrature.* Wilf in [2] raised the question whether T_1 distributions exist, which we have answered affirmatively. He has shown that if a T_1 distribution exists, then there must be large gaps in its T set. The question arises whether a gap condition can be devised which will discriminate between sequences of integers which are T sets of a T_1 distribution, and those sequences which are not.

5.3. The construction of §3 yields a T_1 distribution. There are several places where the construction can be generalized. The family of sets O can be on the entire real axis, the u_i need not be ordered, and we need only that $\limsup |u_i| = \infty$.

The operation $M(k,n)$ can be generalized. It takes mass from O_k to O_{k+1}. Actually, quantities of mass can be taken from $O_1, \cdots, O_k$ to $O_{k+1}, \cdots, O_{k+p}$, but with a parameter which admits a convergence property similar to Lemma B. Likewise the operation $S(k,\delta)$ can be generalized by modifying (3.5.4).

The carrying out of some such generalization would be justified if it could be shown that all T_1 distributions could be obtained by the new construction.

Bibliography

1. J. L. Ullman, *Studies in Faber polynomials*, Trans. Amer. Math. Soc. **94** (1960), 515–528.

2. H. S. Wilf, *The possibility of Tchebycheff quadrature on infinite intervals*, Proc. Nat. Acad. Sci. U. S. A. **47** (1961), 209–213.

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