ORDER ISOMORPHISMS OF B^* ALGEBRAS(1)

BY

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The self-adjoint elements of a B^* algebra A may be regarded as a partially ordered (real) vector space $H(A)$, taking as positive those elements which can be written in the form x^*x, for some x in A. From the standpoints both of mathematics and of physics, it is desirable to know the extent to which A is determined by $H(A)$. Ideally, one would like to know, given a partially ordered real vector space H, exactly which B^* algebras A have $H(A)$ isomorphic to H with respect to both linear and order structure. This is a complicated question, which we will not discuss; instead we consider the simpler question:

Given a B^* algebra A with identity e, for what other B^* algebras A_1 with identity e_1, is $H(A)$ order isomorphic(2) with $H(A_1)$ under a map taking e onto e_1?

(The restriction involving identities is necessary if the answer is to be at all simple.)

This problem has been considered by Kadison who obtained the following result [2, Theorem 10]:

THEOREM. *Let A and A_1 be weakly closed algebras of operators with identities e and e_1, respectively. Let θ be an order isomorphism(2) of $H(A)$ onto $H(A_1)$ taking e onto e_1. Then the linear extension of θ to*

$$\bar{\theta}: A \to A_1$$

*is the direct sum (2) of a *-isomorphism and a *-anti-isomorphism.*

(Kadison's hypothesis is actually that $\bar{\theta}$ is a linear isometry. It is easily seen, as in (10) below, that an order isomorphism taking the identity onto the identity is an isometry on $H(A)$; it follows from the B^* norm identity that θ is an isometry on A.)

Our purpose is to show that Kadison's result cannot be extended without change to the case where A is an arbitrary B^* algebra with identity, but that it does imply a weaker determination of multiplication by order in the more general case. This result frames itself naturally for a class of algebras slightly wider than that of B^* algebras with identity. In the terminology of Naimark [4], this class consists of the reduced symmetric rings with identity that admit a regular norm. Alternatively, the members A of this class may be characterized

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(2) Definitions are given in § 1.

by the conditions: (i) A is an algebra over the complex numbers, having involution and identity; (ii) $H(A)$ has certain properties as a partially ordered real vector space.

The properties required of $H(A)$ are shared by partially ordered real vector spaces arising in other ways. We call any such partially ordered vector space a *GM* space (because it has some of the properties of the *M* spaces of Kakutani), and in the next section note some of the properties of *GM* spaces which seem generally interesting. Most of these properties could be obtained for the space $H(A)$ by slight modifications of known results, but it seems worth noting that they are, in fact, consequences of simple assumptions about order alone.

1. **Notation, definitions, and elementary results on partially ordered vector spaces.** Throughout this paper, A will always be an algebra over the complex numbers having an involution * and identity e. X will always be a real linear space.

A linear map θ defined on A is the direct sum of a *-isomorphism and a *-anti-isomorphism if A is the direct sum of self-adjoint ideals I_1 and I_2 such that θ is a *-isomorphism on I_1 and a *-anti-isomorphism on I_2.

A *cone* in X is a subset C of X such that for every x, y in C and reals $s, t \geqq 0$, $sx + ty$ is in C. We say that x is *positive* if x is in C, and that $x \leqq y$ if $y - x$ is positive. The pair (X, C) is called a *partially ordered vector space.*

If (X,C) is a partially ordered vector space, we denote by C^p the set of all linear functionals on X that are non-negative on C. The linear space X^p is defined by $X^p = C^p - C^p$. C^p is clearly a cone in X^p; we call the partially ordered vector space (X^p, C^p) the *order dual* of (X, C).

If (X,C) and (X_1,C_1) are partially ordered vector spaces, a linear map θ of X into X_1 is *order preserving* if $\theta(C) \subseteq C_1$; θ is an *order isomorphism* if θ is one-one and $\theta(C) = \theta(X) \cap C_1$.

If $\theta:(X,C) \to (X_1,C_1)$ is order preserving, we define the *dual* map, $\theta^p(X_1^p,C_1^p) \to (X^p,C^p)$ by $(\theta^p f)(x) = f(\theta x)$. Evidently, θ^p is order preserving, and if θ is an order isomorphism of X onto X_1, then θ^p is an order isomorphism of X_1^p onto X^p.

We denote by cl$C$ the closure of C in the strongest locally convex linear topology for X.

1. Remark.

$$\begin{aligned} \text{cl}\, C &= \{x : f(x) \geqq 0, \text{ all } f \in C^p\} \\ &= \text{closure of } C \text{ in the weak } X^p \text{ topology.} \end{aligned}$$

(Let C_1 be the closure of C in the weak X^p topology, C_2 the set $\{x \in X : f(x) \geqq 0$ for all $f \in C^p\}$. The inclusion cl $C \subseteq C_1 \subseteq C_2$ is obvious. To show $C_2 \subseteq \text{cl}\, C$, suppose x is not in cl C. Then there is a convex set U containing x as an internal point and not meeting C. The fundamental separation theorem for linear spaces asserts that under these circumstances there is a nontrivial functional f and number d such that $f(y) \geqq d$ for all y in C and $f(x) < d$. Since 0 is in C, $d \leqq 0$. Since C is closed under multiplication by positive reals, f is non-negative on C. Thus, f is in C^p, $f(x) < 0$, and so x is not in C_2.)

The following remarks are easily verified, bearing in mind that θ, as a linear map, is continuous when X and X_1 are given the strongest locally convex topology.

2. Remark. If θ: $(X,C) \to (X_1,C_1)$ is order preserving, so is $\theta(X,\mathrm{cl}\, C) \to (X_1, \mathrm{cl}\, C_1)$. If θ is an order isomorphism of (X,C) onto (X_1,C_1), then it is an order isomorphism of $(X, \mathrm{cl}\, C)$ onto $(X_1, \mathrm{cl}\, C_1)$.

The *order radical* R of (X,C) is defined by

$$\begin{aligned} R &= \mathrm{cl}\, C \cap (-\mathrm{cl}\, C) \\ &= \{x : f(x) = 0 \text{ all } x \text{ in } C^p\}. \end{aligned}$$

3. Remark. Let (X,C) have order radical $\{0\}$, and let (X_1,C_1) admit a one-one order preserving map into (X,C). Then (X_1,C_1) has order radical $\{0\}$.

4. Remark. If (X^p,C^p) is the dual of the partially ordered space (X,C), then
   (a) $C^p = \mathrm{cl}\, C^p$;
   (b) the order radical of (X^p,C^p) is $\{0\}$ if and only if $X = C - C$.

5. Remark. If $X = C - C$, the canonical map κ: $(X,C) \to (X^{pp},C^{pp})$ given by

$$(\kappa x)(f) = f(x)$$

is an order preserving map with kernel R.

If $R = \{0\}$ and $C = \mathrm{cl}\, C$, then κ is an order isomorphism.

An *order unit* for (X,C) is an internal point of C; thus, e is an order unit if for each x in X there is some positive number T such that $e + tx \in C$ for all real t with $|t| \leqq T$. Observe that if (X,C) has an order unit, then $X = C - C$.

Given an order unit e for (X,C), we may define a pseudo-norm on X. Various definitions have been given. Probably the oldest is

$$\rho_1(x) = \inf\{t > 0 : -te \leqq x \leqq te\}.$$

Braunschweiger [1] has pointed out that zero is an internal point of the convex set

$$U = (C-e) \cap (e-C),$$

so that a pseudo-norm is given by the support function of U,

$$\rho_2(x) = \inf\{t > 0 : t^{-1}x \in U\}.$$

For our purposes it is convenient to use the pseudo-norm defined by

$$\rho_3(x) = \sup\{|f(x)| : f \in C^p, f(e) = 1\}.$$

A standard argument shows the following:

6. Remark. Let e be an order unit for (X,C), let $C \neq X$, and let ρ_1,ρ_2,ρ_3 be defined as above. Then for all x in X,

$$\rho_1(x) = \rho_2(x) = \rho_3(x).$$

Denote their common value by ρ_e. If u is another order unit for (X,C), then ρ_u and ρ_e are equivalent pseudo-norms. If $\rho_e(u) = \rho_u(e)$, then for all x in X, $\rho_e(x) = \rho_u(x)$.

Proof. Let $p(x)$ be the support function of $C-e$. Since $x/t \in (C-e) \cap (e-C)$ if and only if $x/t \in C-e$, $\rho_2(x) = \max(p(x),p(-x))$. But

$$p(x) = \inf\{t > 0 : x \geqq -te\},$$

and $p(-x) = \inf\{t > 0 : x \leqq te\}$. Thus by the definition of ρ_1, $\rho_1(x) = \max(p(x), p(-x))$, i.e., $\rho_1 = \rho_2$.

Clearly, if $-te \leqq x \leqq te$, $f \in C^p$, and $f(e)=1$, then $-t \leqq f(x) \leqq t$. Thus $t \geqq \rho_1(x)$ implies $t \geqq \rho_3(x)$, and so $\rho_1(x) \geqq \rho_3(x)$. To obtain the reverse inequality, consider x fixed and define on the subspace generated by x the linear functional f_0 by

$$f_0(tx) = t\,\rho_2(x).$$

We must consider two cases.

If $\rho_2(x) = p(x)$, then $f_0(tx) \leqq p(tx)$ for all t. We may then extend f_0 to a functional f defined on all of X and satisfying $f(y) \leqq p(y)$ for all y. In particular, $f(-e) \leqq p(-e) = 1$ (for if $p(-e) < 1$, $-e$ is an internal point of $C-e$, 0 is an internal point of C, and so $C = X$). We have $p(y) \leqq 1$ for y in $C-e$ and so $f(c-e) \leqq 1$ for c in C. Thus $f(tc) \leqq 1 + f(e)$ for all $c \in C$, $t > 0$, and so $f(c) \leqq 0$ for all $c \in C$. It follows that the functional $g = -f$ satisfies $g \in C^p$, $g(e) \leqq 1$, and $g(x) = p(x) = \rho_2(x)$, whence $\rho_3(x) \geqq \rho_2(x)$.

If $\rho_2(x) = p(-x)$, then $f_0(tx) \leqq p(-x)$ for all t, and we obtain an extension f of f_0 satisfying $f(y) \leqq p(-y)$ for all y. In particular, $f(e) \leqq p(-e) = 1$. Also, $f(e-c) \leqq p(c-e) \leqq 1$ for all c in C, so $f(c) \geqq f(e)-1$ for all c in C, from which $f \in C^p$. Consequently we again have $\rho_3(x) \geqq \rho_2(x)$.

We note that if u is another order unit, then for any $\varepsilon > 0$,

$$-(\rho_u(e) + \varepsilon)u \leqq e \leqq (\rho_u(e) + \varepsilon)u.$$

Thus if $-te \leqq y \leqq te$ (whence $t \geqq 0$, since $C \neq X$),

$$-(\rho_u(e) + \varepsilon)tu \leqq y \leqq (\rho_u(e) + \varepsilon)tu$$

so $t \geqq \rho_e(y)$ implies $\rho_u(e)t \geqq \rho_u(y)$, i.e., $\rho_e(y)\rho_u(e) \geqq \rho_u(y)$. The rest of the proof follows at once.

7. Remark. (a) $R = \{x : \rho_e(x) = 0\}$;

(b) If $0 \leqq x_1 \leqq x_2$, then $\rho_e(x_1) \leqq \rho_e(x_2)$;

(c) If ρ' is the pseudo-norm obtained from e and the cone cl C, then, for all x in X, $\rho'(x) = \rho(x)$.

We are interested in the case where this pseudo-norm is actually a norm; in this case we call (X,C) a *GM space*. Thus, (X,C) is a *GM* space if $R = \{0\}$ and

(X,C) has an order unit. We shall always suppose that a particular unit e is distinguished, and write $\|x\|$ for $\rho_e(x)$. Whenever we refer to a topology on X, we mean the norm topology unless we state otherwise; in particular, X^* will refer to the conjugate space of X with respect to the norm topology.

The following theorem gives the most important fact about GM spaces; it is a direct translation of a result of Takeda [8, § 2] to a slightly more general setting.

8. THEOREM (TAKEDA). *Let (X,C) be a GM space. Then for any l in X^* there are f_1, f_2 in C^p such that*

$$l = f_1 - f_2,$$
$$\|l\| = \|f_1\| + \|f_2\|.$$

Proof. Let $\Omega = \{f \in C^p : f(e) = 1\}$. Then Ω is a compact Hausdorff space in the weak X topology, since $f \in C^p$ implies $\|f\| = f(e)$. Let $C(\Omega)$ be the space of continuous, real valued functions on Ω. We order $C(\Omega)$ by taking the positive cone P to be the set of functions which are everywhere non-negative. If we define the map

$$\phi : X \to C(\Omega)$$

by $(\phi x)(f) = f(x)$, then ϕ is a linear isometry, and an order isomorphism from (X,C) into $(C(\Omega), P)$.

Given any l in X^*, we obtain a functional $\bar{l}$ on the subspace $\phi(X)$ of $C(\Omega)$ by setting

$$\bar{l}(\phi x) = l(x).$$

Evidently the norm of $\bar{l}$ on $\phi(X)$ is just $\|l\|$. It follows by the Hahn-Banach Theorem that $\bar{l}$ has a linear extension $\bar{l}^-$ defined on all of $C(\Omega)$ and satisfying

$$\|\bar{l}^-\| = \|l\|.$$

The Riesz representation theorem asserts that there is a regular measure μ on Ω such that

$$\bar{l}^-(y) = \int_\Omega y d\mu, \|\bar{l}^-\| = \|\mu\|$$

where $\|\mu\|$ is the total variation of μ. The Jordan decomposition of μ yields positive measures μ_1 and μ_2 on Ω such that $\mu = \mu_1 - \mu_2$ and $\|\mu\| = \|\mu_1\| + \|\mu_2\|$. Each μ_i induces a positive functional $\bar{f}_i$ on $C(\Omega)$ satisfying $\|\bar{f}_i\| = \|\mu_i\|$. If we define f_i by $f_i = \bar{f}_i\phi$, then f_i is a positive functional on X such that

$$l = f_1 - f_2,$$
$$\|l\| = \|\mu\| = \|\mu_1\| + \|\mu_2\| = \|\bar{f}_1\| + \|\bar{f}_2\|$$
$$= \bar{f}_1(e) + \bar{f}_2(e)$$
$$= f_1(e) + f_2(e) = \|f_1\| + \|f_2\| \qquad \text{Q.E.D.}$$

9. Corollary. (a) $X^* = X^p$.

(b) *The norm topology on X is the strongest locally convex topology giving X^p as the conjugate space.*

(c) *If $f \in X^*, x \in X, 0 \leqq x \leqq e$, and $f(x) = \|f\|$, then $f \in C^p$.*

Proof. We have just shown that $X^* \subseteq X^p$. As we noted in the proof of (8), when $f \in C^p$, $\|f\| = f(e)$, so $C^p \subseteq X^*$ whence $X^p \subseteq X^*$, so (a) is proved. (b) follows from (a) and the Mackey-Arens theorem. (c) follows from the fact that we have f_i in C^p with $\|f\| = \|f_1\| + \|f_2\|$

$$\|f\| = f(x) = f_1(x) - f_2(x) \leqq f_1(x)$$
$$\leqq \|f_1\| \leqq \|f_1\| + \|f_2\| = \|f\|$$

whence

$$\|f_1\| = \|f_1\| + \|f_2\|, \qquad f_2 = 0.$$

10. Corollary. *Let (X_i, C_i), $i=1, 2$, be GM spaces with order units e_i, and let*

$$\theta: X_1 \to X_2$$

be a linear map of X_1 and X_2 taking e_1 onto e_2.

(a) *If θ is order preserving, then θ is norm reducing, and the conjugate map*

$$\theta^*: (X_2^*, C_2) \to (X_1^*, C_1^p)$$

is order preserving and norm reducing.

(b) *If θ is an order isomorphism, it is an isometry. Conversely, if θ is an isometry of X_1 onto X_2, and $C_i = \mathrm{cl} C_i$, $i = 1,2$, then θ is an order isomorphism.*

Proof. If $g \in C_2^p$ and $\|g\| \leqq 1$, then $g(e_2) \leqq 1$, so $\theta^* g(e_1) \leqq 1$. Since $\theta^* g \in C_1^p$, this means that $\|\theta^* g\| \leqq 1$. Thus θ^* is norm reducing on C_2^p. Given any l in X_2^*, we have f_1, f_2 in C_2^p such that $l = f_1 - f_2$, $\|l\| = \|f_1\| + \|f_2\|$. Consequently,

$$\|\theta^* l\| = \|\theta^* f_1 - \theta^* f_2\| \leqq \|\theta^* f_1\| + \|\theta^* f_2\|$$
$$\leqq \|f_1\| + \|f_2\| = \|l\|.$$

The rest of (a) is obvious.

Clearly, if θ is an order isomorphism it is isometry. Suppose, conversely, that θ is an isometry onto; then so is θ^*. Given g in C_2^p we have

$$\|\theta^* g\| = \|g\| = g(e_2) = \theta^* g(e_1),$$

and it follows fom (9) that $\theta^* g \in C_1^p$. If $\theta^* g \in C_1^p$ we show in the same way that $g \in C_2^p$. Thus $\theta^* C_2^p = C_1^p$. It follows by definition that if $C_i = \mathrm{cl} C_i$, $i = 1,2$, then $\theta C_1 = C_2$ and so θ is an order isomorphism.

We call (X,C) a *GL space* if

(a) $X = C - C$;

(b) X admits a norm $\| \cdot \|$ such that for any x_1, x_2 in C,

$$\| x_1 + x_2 \| = \| x_1 \| + \| x_2 \| ;$$

(c) Every positive functional on (X,C) is $\| \cdot \|$ continous.

11. THEOREM. (a) *If (X,C) is a GM space, then (X^p, C^p) is a GLspace under the usual conjugate space norm.*

(b) *If (X,C) is a GL space, then (X^p,C^p) is a GM space. In particular, $X^p = X^*$.*

(c) *If X has order radical $\{0\}$ and (X^p,C^p) is a GM space, then X is a GL space.*

Proof. (a) By definition, $X^p = C^p - C^p$. *If* $f_1, f_2 \in C^p$, then $f_1 + f_2 \in C^p$ so

$$\|f_1 + f_2\| = (f_1 + f_2)(e) = f_1(e) + f_2(e) = \|f_1\| + \|f_2\| .$$

Suppose that ξ is a positive functional on X^p; we can complete the proof of (a) by showing that ξ is bounded on the set

$$E = \{ f \in C^p : f(e) = 1 \}.$$

For, given any l in X^p, there are f_1, f_2 in C^p such that $l = f_1 - f_2$, $\| l \| = \|f_1\| + \|f_2\|$, and so

$$|\xi(l)| \leqq |\xi(f_1)| + |\xi(f_2)| \leqq \| l \| \sup \{ |\xi(f)| : f \in E \} .$$

Suppose, then, that ξ is not bounded on E. Then for each integer n there exists f_n in E with $\xi(f_n) \geqq n^2$. By (9), $X^p = X^*$, so X^p is complete; it follows that the functional f_0 defined by

$$f_0 = \sum_1^\infty (f_n/n^2)$$

is in X^p. We assert that for any N,

$$f_0 \geqq \sum_1^N (f_n/n^2) .$$

For, if not, there is some x_0 in C and positive real d such that

$$f_0(x_0) = \left[\sum_1^N (f_n(x_0)/n^2) \right] - d.$$

But for M sufficiently large,

$$\left\| f_0 - \sum_1^M (f_n/n^2) \right\| < d/2 \| x_0 \|$$

and so

$$(*) \qquad \left| f_0(x_0) - \sum_1^M (f_n(x_0)/n^2) \right| < d/2.$$

Since $x_0 \in C$ and $f_n \in E$, we have

$$\sum_{N+1}^M (f_n(x_0)/n^2) \geqq 0,$$

$$\begin{aligned} f_0(x_0) - \sum_1^M (f_n(x_0)/n^2) &= \left[f_0(x_0) - \sum_1^N (f_n(x_0)/n^2) \right] - \sum_{N+1}^M (f_n(x_0)/n^2) \\ &\leqq f_0(x_0) - \sum_1^N (f_n(x_0)/n^2) \\ &= -d, \end{aligned}$$

which contradicts (*).

We have, therefore, that

$$f_0 \geqq \sum_1^N (f_n/n^2), \qquad \text{all } N,$$

so

$$\xi(f_0) \geqq \sum_1^N (\xi(f_n)/n^2) \geqq N, \qquad \text{all } N,$$

which is impossible: thus ξ must be bounded on E.

(b) Let X be a GL space, and let

$$U = \{x \in X : \| x \| = 1\}.$$

Since X is GL, U is convex. Let

$$V = \{x \in X : \| x \| \leqq 1/2\}.$$

Then V is convex, radial at the origin, and disjoint from U. A basic separation theorem says that there is a nonzero functional f on X, and a number d, such that

$$f \geqq d \text{ on } U,$$

$$f \leqq d \text{ on } V.$$

Since f is not the zero functional and V is radial at the origin, it follows that $d > 0$.

Since X is GL, every functional in X^p is $\| \cdot \|$ continous. Consequently, given l in X^p, there is a number L such that

$$| l(x) | \leqq L, \qquad \text{all } x \text{ in } U.$$

Thus

$$-L/d\, f(x) \leqq l(x) \leqq L/d\, f(x), \qquad \text{all } x \text{ in } U.$$

But for any x in C, $x/\| x \|$ is in U, so the preceding inequality holds for all x in C, and therefore says that f is an order unit for (X^p, C^p).

Since X is GL, $X = C - C$; it follows from (4) that the order radical of X is $\{0\}$, which completes the proof of (b).

(c) If (X^p, C^p) is GM, its order radical is $\{0\}$ so, by (4), $X = C - C$. If also X has order radical $\{0\}$ it follows from (4) and (5) that the canonical imbedding κ of X in X^{pp} is order preserving and has kernel zero. Thus, if $\|\cdot\|$ is the norm in X^{pp} induced by the fact that $X^{pp} = X^{p*}$ (by (9)), we can define a norm on X by

$$\|x\| = \|\kappa x\|.$$

If $x_1, x_2 \in C$, then $\kappa x_1, \kappa x_2 \in C^{pp}$ and so, since X^{pp} *is GL*,

$$\|x_1 + x_2\| = \|\kappa x_1 + \kappa x_2\| = \|\kappa x_1\| + \|\kappa x_2\| = \|x_1\| + \|x_2\|.$$

Evidently if $l \in X^p$, then

$$\begin{aligned} |l| &= \sup\{|l(x)| : x \in X, \|x\| \leqq 1\} \\ &= \sup\{|\kappa x(l)| : x \in X, \|\kappa x\| \leqq 1\} \\ &\leqq \sup\{|\xi(l)| : \xi \in X^{pp}, \|\xi\| \leqq 1\} \\ &= \|l\|, \end{aligned}$$

when $\|l\|$ is the given GM space norm, so $X^p \subseteq X^*$.

On the other hand, if $l \in X^*$, then

$$-|l|\,\|x\| \leqq l(x) \leqq |l|\,\|x\|, \text{ all } x \text{ in } X.$$

In particular, when $x \in C$, $\kappa x \in C^{pp}$ and so

$$\|x\| = \|\kappa x\| = \kappa x(e) = e(x),$$

where e is the given order unit for X^p. It follows that for x in C

$$-|l|\,e(x) \leqq l(x) \leqq |l|\,e(x)$$

so l is in X^p, and $\|l\| \leqq |l|$.

Thus $X^p = X^*$, and the norm we have defined on X has for its conjugate norm on X^p exactly the original GM space norm.

12. Remark. (X,C) is a GL space if and only if (X^p, C^p) has an order unit f which on C is zero only at the origin. (If (X,C) is GL, we may define the f in question by $f(x) = \|c_1\| - \|c_2\|$ where $x = c_1 - c_2$, $c_i \in C$. Conversely, given (X,C) with a functional f as above, the norm on X given by

$$\|x\| = \sup\{|l(x)| : l \in C^p, l \leqq f\}$$

has the desired properties.)

The results of this discussion which we shall particularly want later are as follows:

13. COROLLARY. (a) *Let* (X_i,C_i), $i = 1,2$, *be GM spaces with order units* e_i, *and let* θ *be an order isomorphism of* (X_1,C_1) *onto* (X_2,C_2) *taking* e_1 *onto* e_2. *Then* θ^{pp} *is an order isomorphism of* (X_1^{**}, C_1^{pp}) *onto* (X_2^{**}, C_2^{pp}) *taking* κe_1 *onto* κe_2.

(b) *If, in addition,* $\bar{X}_i$ *is the completion of* X_i *in the norm induced by* e_i, $\bar{C}_i$ *the closure (with respect to this norm) of* C_i *in* $\bar{X}_i$, *and* $\bar{\theta}$ *the continuous extension of* θ *to* $\bar{X}_1$, *then*

$$\bar{C}_i = \overline{\mathrm{cl}\; C_i} = \mathrm{cl}\bar{C}_i$$

and $\bar{\theta}$ *is an order isomorphism of* $(\bar{X}_1,\bar{C}_1)$ *onto* $(\bar{X}_2,\bar{C}_2)$.

Proof. (a) is just the statement that $X_i^{**} = X_i^{pp}$, which was proved in (9) and (11).

(b) Since θ is an isometry by (10), $\bar{\theta}$ will map $\bar{C}_1$ onto $\bar{C}_2$. The statement $\bar{C}_i = \overline{\mathrm{cl} C_i} = \mathrm{cl}\bar{C}_i$ follows immediately from the fact that cl C_i is the closure of C_i in the strongest locally convex topology.

2. **Applications to algebras**. Let A be an algebra over the complex numbers having an involution $*$ and identity e. Let $H(A)$ be the real linear space of self-adjoint elements of A, and let $C_0(A)$ be the cone in $H(A)$ consisting of all elements that can be expressed as a finite sum of the form $\sum x_i^* x_i$. Let $C(A) = \mathrm{cl} C_0(A)$.

We say that A is a *D algebra* if there is a *-isomorphism ϕ of A into a B^* algebra B such that

(a) $\phi(e)$ is the identity of B;

(b) every linear functional defined on $\phi(A)$ and non-negative on $\phi(C(A))$ can be extended to a positive functional on B. (We make the usual confusion between real valued functionals on $H(A)$ and complex-valued functionals on A which are real on $H(A)$.)

14. THEOREM (NAIMARK). *A is a D algebra if and only if both*

(a) $\{x \in A : f(x^*x) = 0$ *for all* f *in* $[C(A)]^p\} = \{0\}$;

(b) *for each x in A*,

$$\sup\{f(x^*x) : f \in [C(A)]^p, f(e) = 1\} < \infty.$$

This theorem follows immediately from results in [4, §§ 10 and 18]. The B^* algebra B constructed there is the Gel'fand-Naimark representation of a B^* algebra as an algebra of operators on a Hilbert space $\mathfrak{H}$. In the terminology of [4], a D algebra is a symmetric ring with identity which admits a regular norm and has reducing ideal $\{0\}$.

15. THEOREM. *A is a D algebra if and only if* $H(A)$ *is a GM space with e acting as order unit. In this case the map* ϕ *is an isometry on* $H(A)$ *when* $H(A)$ *is given the GM space norm induced by e.*

This theorem follows immediately from the results of [4, §§ 18.2 and 18.3]. That ϕ is necessarily an isometry follows from (10). Notice that each D-algebra with identity has a *unique* norm which makes it a subalgebra of a B^* algebra having the same identity; any norm referred to in connection with a D algebra will be this norm, unless otherwise stated.

Two D algebras A_1 and A_2 will be called *order isomorphic* under a map θ if θ is a linear map of A_1 onto A_2 which is an order isomorphism of $(H(A_1),C(A_1))$ onto $(H(A_2),C(A_2))$. It is clear that the linear extension to A_1 of an order isomorphism of $H(A_1)$ onto $H(A_2)$ is an order isomorphism of A_1 onto A_2.

We note the following facts: if A is a Banach algebra under a norm making the involution continuous, then e is an order unit for $(H(A),C(A))$ — see, e.g., [4, § 10.4]. If A has a faithful *-representation in a D algebra A_1 taking e onto the identity of A_1, then the order radical of $(H(A),C(A))$ is $\{0\}$; this follows at once from Remark 3. Consequently, any A^* algebra (in the sense of [5]) is a D algebra: for such an algebra is a Banach algebra with continuous involution [5, p. 187] and has an auxiliary norm satisfying the B^* identity; the completion of the algebra in the auxiliary norm provides a faithful representation in a B^* algebra.

As specific examples, we note the following:

(i) The convolution algebra $M(G)$ of bounded Radon measures on a locally compact group G. This is a Banach algebra with identity and continuous involution $(\mu^*(S) = \overline{\mu(S^{-1})})$ under the total variation norm. On the other hand [7, pp. 47–48], it has a faithful *-representation in the algebra of bounded operators on $L^2(G)$.

(ii) The algebra $R(G)$ of L^1 functions on G, with an identity adjoined if G is not discrete, and with convolution as multiplication. $R(G)$ may be regarded as a closed, self-adjoint subalgebra of $M(G)$ containing the identity, and so is again an A^* algebra.

A little may be said about the relation between order isomorphisms between L^1 algebras and order isomorphisms between R algebras: let G_1 and G_2 be non-discrete and θ a linear map of $L^1(G_1)$ onto $L^1(G_2)$. Then the linear extension of θ to $R(G_1)$ is an order isomorphism of $R(G_1)$ onto $R(G_2)$ if and only if θ^p maps the continuous (L^1 norm) positive functionals on $L^1(G_2)$ isometrically (L^∞ norm) onto the continuous positive functionals on $L^1(G_1)$. This follows from the fact that an arbitrary functional f on $R(G)$ is positive if and only if f is a continuous positive functional on $L^1(G)$ and $f(e) \geqq \|f\|_1$, the norm of f restricted to $L^1(G)$. (See [3, 26H and 31G].)

(iii) If G is compact, we may form the algebra $R^p(G)$, $p>1$, consisting of $L^p(G)$, with an identity adjoined (we assume G not discrete since, if discrete, it is finite and $L^p(G) = L^1(G)$) and convolution multiplication. We may regard $R^p(G)$ both as a self-adjoint subalgebra of $R(G)$ and as a Banach algebra with continuous involution under the norm induced by the L^p norm via the left regular representation of $R^p(G)$ on $L^p(G)$.

It is possible to construct examples of D algebras which are not A^* algebras — which are not, in fact, complete in any norm. However, the author does not know of any such example possessing intrinsic interest.

We now outline a possible method for constructing a D algebra order isomorphic with a given D algebra A; we shall see later that every order isomorphism between algebras which preserves the identity must be the composition of a *-isomorphism with a map of the type we are about to construct.

To carry out this construction, we must have in A (a D algebra with identity e) self-adjoint ideals I_1 and I_2 satisfying

$$I_1 I_2 = 0, \tag{*}$$

$$xy - yx \in I_1 \oplus I_2 \text{ for every } x,y \in A. \tag{**}$$

The quotient spaces A/I_i may be given multiplications and involutions such that the natural maps

$$\pi_i \colon A \to A/I_i$$

are *-homomorphisms; then $\pi_i(e)$ is the identity for A/I_i. A/I_i will also be a D algebra; this follows from (14) and the fact that every positive functional f on A/I_i induces a positive functional $f(\pi)$ on A.

Let B be the linear space $A/I_1 \oplus A/I_2$. B is again a D algebra under the multiplication

$$(\pi_1(x_1),\ \pi_2(y_1))(\pi_1(x_2),\ \pi_2(y_2)) = (\pi_1(x_1x_2),\ \pi_2(y_1y_2))$$

and involution

$$(\pi_1(x),\ \pi_2(y))^* = (\pi_1(x^*),\ \pi_2(y^*)).$$

It is easy to show that the completion of B with respect to the norm

$$\| (\pi_1(x), \pi_2(y) \| = \max \{ \| \pi_1(x) \| , \| \pi_2(y) \| \}$$

gives the desired B^* algebra.

We have a natural map π of A onto a subalgebra of B, given by

$$\pi(x) = (\pi_1(x),\ \pi_2(x)).$$

π is clearly a *-homomorphism and indeed, since I_1 and I_2 are disjoint, a *-isomorphism. Since π is a linear isomorphism, it is a homeomorphism between A and $\pi(A)$ when A is given the strongest locally convex topology and $\pi(A)$ the relative topology induced by the strongest locally convex topology on B. Consequently, $\pi(C(A)) = \pi(A) \cap C(B)$ and so π is an order isomorphism and an isometry.

We form another D algebra B^T from $A/I_1 \oplus A/I_2$, using the previous definition of involution and the multiplication law

$$(\pi_1(x_1), \pi_2(y_1))(\pi_1(x_2), \pi_2(y_2)) = (\pi_1(x_1x_2), \pi_2(y_2y_1)).$$

It is easy to verify that this is indeed a multiplication consistent with the involution, and that the positive cone in B^T is identical with the positive cone in B. The map π is therefore an order isomorphism of A into B^T. Moreover, $\pi(A)$ is a sub-algebra of B^T—for given $x,y \in A$, the B^T product of $\pi(x)$ and $\pi(y)$ is

$$\pi(x)\pi(y) = (\pi_1(xy), \pi_2(yx)).$$

But according to (**), there exist $w_1 \in I_1$ and $w_2 \in I_2$ such that

$$xy - yx = w_1 + w_2$$

so

$$xy - w_1 = yx + w_2.$$

But

$$(\pi_1(xy), \pi_2(yx)) = (\pi_1(xy - w_1), \pi_2(yx + w_2))$$

so $\pi(x)\pi(y)$ is in $\pi(A)$.

We have thus found a subalgebra, $\pi(A)$, of B^T which is order isomorphic with A. We shall show later that π need not be the direct sum of a *-isomorphism and a *-anti-isomorphism.

The results in [4] previously mentioned show that if A is a D algebra, then A has a Gel'fand-Naimark representation as a dense subalgebra of a norm-closed algebra B of bounded operators on a Hilbert space $\mathfrak{H}$. We denote by $\tilde{A}$ the weak closure of B (and so, of the image of A in B) in $\mathscr{B}(\mathfrak{H})$. Then $H(\tilde{A})$ is ordered by $C(\tilde{A})$. We have seen that $H(A^{**})$ is ordered by $C(A)^{pp}$. It turns out that the ordered spaces $(H(\tilde{A}), C(\tilde{A}))$ and $(H(A^{**}), C(A)^{pp})$ are the same. This is a simple reformulation of a theorem of Sherman and Takeda [6;8].

16. THEOREM (SHERMAN-TAKEDA). *There is a linear isometry β of $\tilde{A}$ onto A^{**} which is an order isomorphism of $(H(\tilde{A}), C(\tilde{A}))$ onto $(H(A^{**}), C(A)^{pp})$. If κ is the canonical embedding of A in A^{**}, and ϕ the Gel'fand-Naimark embedding of A in $\tilde{A}$, then $\beta\phi = \kappa$.*

If I is an ideal in the D algebra A, we denote by I^a the set of x in A such that $xy = yx = 0$ for all y in I. If I is a right ideal, I^a is a left ideal; if I is self-adjont, so is I^a; I^a is closed in any topology making multiplication continuous in each variable separately.

17. THEOREM. *Let A_i, $i = 1,2$, be D algebras with identity e_i, and let θ be an order isomorphism of A_1 onto A_2 taking e_1 onto e_2. Then there exist self-adjoint ideals I_i in A_i such that*

(a) $I_i^{aa} = I_i$;

(b) $\theta I_1 = I_2$, $\theta I_1^a = I_2^a$;

(c) *for every x, y in A_i,*

$$[x,y] = xy - yx \in I_i \oplus I_i^a \ ;$$

(d) *the natural extension of* θ *to*

$$\bar{\theta}: A_1/I_1 \oplus A_1/I_1^a \to A_2/I_2 \oplus A_2/I_2^a$$

*is an algebraic *-isomorphism on* A_1/I_1 *and a *-anti-isomorphism on* A_1/I_1^a

Proof. We denote by κ_i the canonical imbedding of A_i in A_i^{**} and by ϕ_i the Gel'fand-Naimark imbedding of A_i in $\tilde{A}_i$. Consider the map

$$\theta^{**}: A_1^{**} \to A_2^{**}$$

induced by θ; it is the linear extension of the map

$$\theta^{pp}: H(A_1)^{pp} \to H(A_2)^{pp},$$

which is, by (13), an order isomorphism. Further, $\theta^{**}\kappa_1(A_1) = \kappa_2\theta(A_1) = \kappa_2(A_2)$, and on A_1,

$$\kappa_2^{-1}\theta^{**}\kappa_1 = \theta.$$

By the Sherman-Takeda Theorem we have order isomorphisms $\beta_i: \tilde{A}_i \to A_i^{**}$. Consequently, $\psi = \beta_2^{-1}\theta^{**}\beta_1$ is an order isomorphism of $\tilde{A}_1$ onto $\tilde{A}_2$, taking the identity, ϕe_1 of $\tilde{A}_1$ onto the identity ϕe_2 of $\tilde{A}_2$. On A_1 we have

$$\phi_2^{-1}\psi\phi_1 = \kappa_2^{-1}\theta^{**}\kappa_1 = \theta.$$

Now the $\tilde{A}_i$ are weakly closed algebras of operators, so it follows from Kadison's theorem that the order isomorphism ψ is the direct sum of a *-isomorphism and a *-anti-isomorphism. Thus, there is an central (self-adjoint) projection p in $\tilde{A}_1$ such that ψ is an isomorphism on $p\tilde{A}_1$, and an anti-isomorphism on $(e_1 - p)\tilde{A}_1$. Define the sets I_i by

$$I_1 = \{x \in A_1 : p\phi_1 x = 0\},$$

$$I_2 = \{x \in A_2 : (\psi p)(\phi_2 x) = 0\}.$$

It is clear that ψp is a central projection in $\tilde{A}_2$, and that the I_i are self-adjoint ideals satisfying (a) and (b).

To obtain (c), observe that, given $x, y \in A_1$,

$$\psi(\phi_1(xy)) = \psi(p)(\psi(\phi_1 x)\psi(\phi_1 y)) + \psi(e_1 - p)(\psi(\phi_1 y)\psi(\phi_1 x)),$$

$$\psi(\phi_1(yx)) = \psi(p)(\psi(\phi_1 y)\psi(\phi_1 x)) + \psi(e_1 - p)(\psi(\phi_1 x)\psi(\phi_1 y))$$

so

$$\psi(\phi_1(xy - yx)) = \psi(2p - \phi e_1)(\psi(\phi_1 x)\psi(\phi_1 y) - \psi(\phi_1 y)\psi(\phi_1 x)).$$

Since $\psi\phi_1(A_1) = \phi_2 A_2$, a subalgebra of $\tilde{A}_2$, we have

$$\psi(\phi_1 x)\psi(\phi_1 y) - \psi(\phi_1 y)\psi(\phi_1 x) \in \phi_2 A_2.$$

But also $\psi(\phi_1(xy-yx)) \in \phi_2 A_2$, and so

$$\psi(p)(\psi(\phi_1 x)\psi(\phi_1 y) - \psi(\phi_1 y)\psi(\phi_1 x)) \in \phi_2 A_2;$$

but this implies

$$\psi(p)(\psi(\phi_1 x)\psi(\phi_1 y) - \psi(\phi_1 y)\psi(\phi_1 x)) \in \phi_2 I_2^a.$$

Thus there is an element u_2 in I_1^a such that

$$\psi(p)(\psi(\phi_1 x)\psi(\phi_1 y) - \psi(\phi_1 y)\psi(\phi_1 x)) = \phi_2 \theta u_2.$$

Similarly, there is u_1 in I_1 such that

$$\psi(e_1 - p)(\psi(\phi_1 x)\psi(\phi_1 y) - \psi(\phi_1 y)\psi(\phi_1 x)) = \phi_2 \theta u_1$$

so

$$\psi\phi_1(xy - yx) = \phi_2\theta(u_2 - u_1),$$

$$\phi_2^{-1}\psi\phi_1(xy - xy) = \theta(u_2 - u_1),$$

$$\theta(xy - yx) = \theta(u_2 - u_1).$$

Since θ is one-one, we have

$$xy - yx = u_2 - u_1 \in I_1 \oplus I_1^a,$$

and (c) in established.

If π_i is the natural map of A_i onto $A_i/I_i \oplus A_i/I_i^a$, then the map $\bar\theta$ of (d) is given by

$$\bar\theta = \pi_2\phi_2^{-1}\theta\phi_1\pi_1^{-1}.$$

Clearly $\bar\theta$ has the desired property, and the proof is complete.

Evidently θ is a *-isomorphism on B^T, the algebra formed from $A_1/I_1 \oplus A_1/I_1^a$ by interchanging right and left multiplication on the second summand, and so every order isomorphism can be written as a composition of a*-isomorphism and a map of A onto B^T of the sort previously described.

The hypotheses of (17) may be weakened a little:

18. COROLLARY. (a) *The conclusion of* (17) *remains true if A_2 is assumed only to be an algebra over the complex numbers with involution and identity.*

(b) *If A_2 is assumed to be a D algebra, θ need only be assumed to be a map of $C(A_1)$ onto $C(A_2)$ satisfying*

$$\theta(x+y) = \theta(x) + \theta(y) \quad \textit{all } x, y \in C(A_1),$$

$$\theta(tx) = t\theta(x), \quad \textit{all } x \in C(A_1), \textit{ real } t \geqq 0,$$

$$\theta(e_1) = e_2.$$

(c) *Part* (b) *remains true if "$C(A_1)$" is replaced by "$C_0(A_1)$" and "$C(A_2)$", by "$C_0(A_2)$".*

Proof. (a) follows since the remaining hypotheses of (17) imply that $H(A_2)$ is a GM space with e_2 acting as order unit, so by (15), that A_2 actually is a D algebra.

(b) follows since, if the A_i are D algebras, $H(A_i) = C(A_i) - C(A_i)$, and θ thus has a unique linear extension to $H(A_1)$, and thence to A_1, satisfying the requirements of (17).

(c) follows from (13b).

19. COROLLARY. *Let A be a D algebra with identity e. Then A admits an order isomorphism onto itself, leaving e fixed, which is neither a *-isomorphism nor a *-anti-isomorphism if and only if A contains a self-adjoint ideal I satisfying*

(i) $\{0\} \neq I \neq A$;

(ii) $I = I^{aa}$;

(iii) $[x,y] \in I \oplus I^a$ *for all x,y in A.*

If the order isomorphism constructed from I and I^a as in the discussion preceding (17) *is the direct sum of a *-isomorphism and a *-anti-isomorphism, then either*

(iv) $I \oplus I^a = A$ *or*

(v) *at least one of I, I^a contains a nontrivial commutative direct summand.*

Proof. The proof of (17) shows that the existence of an order isomorphism as described above implies the existence of a self-adjoint ideal I satisfying (i)-(iii). The discussion preceding (17), with $I_1 = I$, $I_2 = I^a$, shows that such an I produces an order isomorphism of the sort described.

If θ, the order isomorphism constructed from I and I^a, is the direct sum of a *-isomorphism and a *-anti-isomorphism, there is a central projection p in A such that θ is a *-isomorphism on Ap and a *-anti-isomorphism on $A(e-p)$. It follows that θ is at once an isomorphism and an anti-isomorphism on Ip, and so, since θ is one-one, that Ip is commutative—and clearly a direct summand of I. Similarly, $I^a(e-p)$ is a commutative direct summand of I^a.

Suppose first that $Ip = I$, so $A(e-p) \subseteq I^a$. If $I^a(e-p) = \{0\}$, then $p = e$ and θ is an isomorphism. If $I^a(e-p) = I^a$, then $I^a \subseteq A(e-p)$, so $I^a = A(e-p)$ and $A = I \oplus I^a$.

Suppose, then, that $Ip = \{0\}$, so $Ap \subseteq I^a$. If $I^a(e-p) = \{0\}$, then $I^a \subseteq Ap$ and $A = I \oplus I^a$. If $I^a(e-p) = I^a$, then $Ap \subseteq I^a(e-p)$, so $Ap = \{0\}$, $p = 0$, and θ is an anti-isomorphism.

Thus, under the hypotheses above, at least one of Ip, $I^a(e-p)$ is nontrivial.

3. **Examples.** The last corollary shows us at once how to construct an order isomorphism which is not a direct sum of a *-isomorphism and a *-anti-isomorphism: choose I_1 and I_2 to be B^* algebras without identity each of which either is simple or has center $\{0\}$. Let A be the direct sum of I_1 and I_2, with an identity adjoined. A is easily seen to be a B^* algebra (norming $I_1 \oplus I_2$ by $\|(x,y)\| = \max\{\|x\|_1, \|y\|_2\}$, and A by its regular representation on $I_1 \oplus I_2$), but the order isomorphism con-

structed from I_1 and I_2 cannot be a direct sum of a *-isomorphism and a *-anti-isomorphism.

There is a class of D algebras for which all order isomorphisms leaving the identity fixed are of this sort:

20. REMARK. Let A be a D algebra consisting of annihilator algebra A_0 [4, §25] with an identity e adjoined. Then every order isomorphism of A onto itself which leaves e fixed is on A_0 the direct sum of a *-isomorphism and a *-anti-isomorphism.

Proof. Given such an order isomorphism, let I be the ideal of (17) and let $J = I \oplus I^a$. Then $A_0 \cap J$ is a norm-closed left ideal in A_0 and so, since A_0 is an annihilator algebra, either $A_0 \cap J = A_0$ (the desired conclusion) or there is a $z \neq 0$ in A_0 such that $(A_0 \cap J)z = 0$. The latter is impossible, for, since $A_0 J \subseteq A_0 \cap J$, it would imply $zz^*k^2zz^* = 0$ for every self-adjoint k in J; thus that $zz^*k = kzz^* = 0$ for every such k. But J is self-adjoint, so this would mean $zz^*J = Jzz^* = 0$. In particular it would mean $zz^*I = Izz^* = 0$ and so that $zz^* \in I^a$. Since $I^a \subseteq J$, this would imply $zz^* = 0$, so $z = 0$, a contradiction.

One might suppose from the preceding remarks that the source of "bad" order isomorphisms was the adjunction of an identity. We now give an example showing that this is not the case—i.e., we exhibit a B^* algebra A with identity and an order isomorphism of A onto itself which leaves the identity fixed and which is not a direct sum of isomorphism and anti-isomorphism on any ideal of deficiency one.

Let $\mathfrak{H}$ be the Hilbert space of sequences $\{\xi_n\}$ of complex numbers such that $\sum_1^\infty |\xi_n|^2 < \infty$, with the usual inner product

$$(\{\xi_n\}, \{\eta_n\}) = \sum_1^\infty \xi_n \bar{\eta}_n.$$

Let $\mathscr{B}(\mathfrak{H})$ be the algebra of all bounded linear operators on $\mathfrak{H}$, e the identity of $\mathscr{B}(\mathfrak{H})$, and u the shift two places to the left:

$$u(\{\xi_1,\xi_2,\xi_3,\cdots\}) = \{\xi_3,\xi_4,\cdots\}.$$

Then u^* acts on $\mathfrak{H}$ by

$$u^*(\{\xi_1,\xi_2,\cdots\}) = \{0,0,\xi_1,\xi_2,\cdots\}.$$

Let $\mathfrak{M}$ be the (closed) linear subspace of $\mathfrak{H}$ consisting of all sequences $\{\xi_n\}$ such that ξ_n is zero for all odd n. Let A_0 be the subalgebra of $\mathscr{B}(\mathfrak{H})$ generated by e, u, u^* and the set of all compact operators in $\mathscr{B}(\mathfrak{H})$ which are reduced by $\mathfrak{M}$. Let A be the norm closure of A_0.

Observe that u and u^* are reduced by $\mathfrak{M}$, so every element of A is reduced by $\mathfrak{M}$. We define two (closed) left ideals in A by

$$I_1 = \{x \in A : x(\mathfrak{M}) = 0\},$$

$$I_2 = \{x \in A : x(\mathfrak{M}^\perp) = 0\}.$$

Since every element of A is reduced by $\mathfrak{M}$, I_1 and I_2 are actually self-adjoint ideals in A; clearly $I_1 I_2 = 0$.

21. LEMMA. *For any x, y in A, the commutator*

$$[x,y] = xy - yx$$

is in $I_1 \oplus I_2$.

Proof. Since I_1 and I_2 are closed, it is enough to prove the lemma with A replaced by A_0. If either x or y is compact, then $[x,y]$ will also be compact, and is reduced by $\mathfrak{M}$. In this case $[x,y]$ can be written as the sum of two compact operators, one vanishing on $\mathfrak{M}$ and the other on $\mathfrak{M}^{\perp}$. Since A_0 contains all such operators, this shows that $[x,y] \in I_1 \oplus I_2$. The remaining case is that in which neither x nor y is compact—therefore, since the compact operators are a self-adjoint ideal in A_0, where x and y are both in the sub-algebra of A generated by u, u^*, and e. We appeal to the general

REMARK. Let R be a ring, I a two-sided ideal in R. For any subset S of R, denote by $[S]$ the ring generated by S, and S' the set of commutators $[x,y]$ for which $x,y \in S$. Then $S' \subseteq I$ implies $[S]' \subseteq I$. This remark follows at once from the identities

$$[xy,z] = x[y,z] + [x,z]y,$$

$$[xy,zw] = x[y,z]w + [x,z]yw + zx[y,w] + z[x,w]y.$$

To prove the lemma we note that $I_1 \oplus I_2$ is a two-sided ideal in A_0 and verify directly that if S is the set consisting of u, u^*, and e, then $S' \subseteq I_1 \oplus I_2$.

We may thus form the map π: $A \to B^T$ as in the last section.

22. LEMMA. *Let δ_n be the element of $\mathfrak{H}$ having 1 at the nth place and zero elsewhere.*

Given any x in A and real $\varepsilon > 0$, there is a number N such that for all $r,s \geqq N$ and $k \geqq 0$,

$$|(x\delta_{r+k}, \delta_{s+k}) - (x\delta_r, \delta_s)| < \varepsilon.$$

Proof. Again, it is enough to show this for x in A_0. If x is compact, it is easy to see that for N sufficiently large,

$$|(x\delta_{r+k}, \delta_{s+k})| < \varepsilon/2 \text{ and } |(x\delta_r, \delta_s)| < \varepsilon/2$$

whenever r, s, and k are as in the statement of the lemma.

Thus the problem reduces to the case where x is in the subalgebra generated by u, u^*, and e. The desired conclusion then follows at once from the easily seen facts that

(i) $uu^* = e$, so any product of u's and u^*'s can be written in the form $(u^*)^\alpha u^\beta$;

(ii) for $n \geqq \beta$, $(u^*)^\alpha u^\beta \delta_n = \delta_{n-\beta+\alpha}$.

23. Theorem. *There cannot exist disjoint, two sided ideals J_1, J_2 in A such that*

(a) *$J_1 \oplus J_2$ is of deficiency at most one in A;*

(b) *for $x,y \in J_1$, $\pi(xy) = \pi(y)\pi(x)$; for $x,y \in J_2$, $\pi(xy) = \pi(x)\pi(y)$.*

Proof. We first show that for J_i as described, then whenever $x \in J_i$, $[x,y] \in I_i$ for all y in $J_1 \oplus J_2$. Since the J_i are disjoint, it is enough to show this for $y \in J_i$. In the case $i = 1$ we have

$$\pi(xy) = \pi(y)\pi(x),$$

i.e.,

$$(\pi_1(xy), \pi_2(xy)) = (\pi_1(yx), \pi_2(xy))$$

so

$$\pi_1(xy - yx) = 0, \qquad xy - yx \in I_1.$$

The case $i = 2$ is done in the same way. Next, if the J_i are as described, then

(*) For each x in J_1 there is a number λ such that

$$x\delta_{2s+1} = \lambda\delta_{2s+1}, \qquad s = 1,2,\cdots.$$

For each x in J_2 there is a number μ such that

$$x\delta_{2s} = \mu\delta_{2s}, \qquad s = 1,2,3,\cdots.$$

We prove only the first part of (*); the second is done in the same way. By the definition of $\mathfrak{H}$ there are numbers α_{ij} such that

$$x\delta_i = \sum_{j=1}^{\infty} \alpha_{ij}\delta_j.$$

Given any integer $s > 0$, define the operator y by

$$y\delta_i = \begin{cases} 0, & i \neq 1, \\ \delta_{2s+1}, & i = 1. \end{cases}$$

y is an operator of finite rank, reduced by $\mathfrak{M}$, and so y is in A. Note further that $y + \lambda e$ is regular for $\lambda \neq 0$, in fact, $(y+\lambda)^{-1} = e/\lambda - y/\lambda_2$. Thus $y + \lambda e \notin J_1 \oplus J_2$ for $\lambda \neq 0$ unless $J_1 \oplus J_2 = A$. It follows that if $J_1 \oplus J_2$ is of deficiency at most one, $y \in J_1 \oplus J_2$.

Therefore $xy - yx$ is in I_1, *i.e.*,

$$xy\delta_{2i+1} = yx\delta_{2i+1}.$$

In particular,

$$xy\delta_1 = yx\delta_1,$$

$$\sum_{j=1}^{\infty} \alpha_{2s+1,j}\, \delta_j = \alpha_{11}\delta_{2s+1},$$

$$\alpha_{2s+1,j} = \begin{cases} 0, & j \neq 2s+1, \\ \alpha_{11}, & j = 2s+1, \end{cases}$$

which is the desired result.

We now show:

For each x in $J_1 \oplus J_2$ and $\varepsilon > 0$, there is a number N such that for every $r, s \geqq N$, $r \neq s$,

$$|(x\delta_r, \delta_s)| < \varepsilon.$$

Clearly it is enough to show this separately for $x \in J_1$, and for $x \in J_2$. Since the proof is essentially the same for either case, we suppose $x \in J_1$. By (22), there is an N such that for $r, s \geqq N$ and $k \geqq 0$,

$$|(x\delta_{r+k}, \delta_{s+k}) - (x\delta_r, \delta_s)| < \varepsilon.$$

For suitable choice of k, depending on r, $r + k$ is odd. For such k, (*) and the fact that x is in J_1 imply

$$(x\delta_{r+k}, \delta_{s+k}) = 0 \qquad \text{for } r \neq s;$$

This gives the desired result.

The promised contradiction is obtained by noting that for any $s \geqq 0$,

$$(u\delta_{s+2}, \delta_s) = 1,$$

so $u \notin J_1 \oplus J_2$.

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