LIMIT ULTRAPOWERS(1)

BY

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Introduction. In the theory of models of a first order predicate logic with identity we study the relationship between notions from logic and purely mathematical objects, called *relational systems*, or briefly *systems*, $\mathfrak{A} = \langle A, R_\lambda \rangle_{\lambda < \rho}$, consisting of a set A and a sequence of relations R_λ on A corresponding to the predicate symbols P_λ of the logic. In such studies some of the most useful methods are those which permit us to form systems which have certain model-theoretic properties. For example, the completeness theorem permits us to form a system which satisfies a given consistent set of sentences, and the Löwenheim-Skolem-Tarski Theorem shows that we may take that system to be of prescribed infinite cardinality.

A more recent tool is the *reduced product* construction, which is particularly simple and direct from the mathematical point of view. Speaking very roughly, a reduced product $P_D\langle \mathfrak{A}_1 : i \in I \rangle$ of the systems $\mathfrak{A}_i$, $i \in I$, is formed by first taking their direct product and then forming the quotient system determined in a certain way by the filter D on the index set I. If D is an ultrafilter we have an *ultraproduct* (or prime reduced product), and if each $\mathfrak{A}_i$ coincides with a fixed system $\mathfrak{A}$ we have a *reduced power*, or *ultrapower*, $\mathfrak{A}^I{}_D$.

Reduced products have recently been applied to obtain new, comparatively direct mathematical proofs and at the same time stronger forms of a number of theorems in model theory. An outstanding example is the proof of the compactness theorem outlined in [25]. They have also been used to obtain several completely new results in the theory of models, for example the theorem of Rabin [27] stated at the beginning of §4 of this paper. Most of these applications depend on a fundamental result of Łoś in [23, p. 105], which in one form states that any ultrapower of $\mathfrak{A}$ is elementarily equivalent to $\mathfrak{A}$, i.e. satisfies exactly the same sentences as $\mathfrak{A}$ does.

For some purposes, however, the ultrapower construction does not appear to be sufficiently general. One question, which may be used as a test problem for mathematical constructions of systems, is that of obtaining convenient mathematical criteria for elementary equivalence. This question was proposed by Tarski, and various answers are given in [7; 13; 16; 19; 26 and 35]. In the case

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of ultraproducts we have the following converse (see [16; 17]) of the result of Łoś stated above. If the generalized continuum hypothesis is assumed, then $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent if and only if $\mathfrak{A}$ and $\mathfrak{B}$ have some isomorphic ultrapowers with cardinality at most that of $2^A \cup 2^B \cup 2^\rho$. Unfortunately, it is not known whether this result holds without the generalized continuum hypothesis. In order to look at the question from another point of view, we state a slightly more general form of the theorem of Łoś. Any ultrapower $\mathfrak{B}$ of $\mathfrak{A}$ has the following property (*): for any set of new relations we adjoin to $\mathfrak{A}$, we can adjoin a set of corresponding relations to $\mathfrak{B}$ so that the resulting systems are elementarily equivalent. We shall see in Theorem 6.2 that there are systems $\mathfrak{B}$ which satisfy (*) but are not isomorphic to any ultrapower of $\mathfrak{A}$. For some purposes, it would be convenient to have a mathematical construction which would yield, up to isomorphism, exactly those systems $\mathfrak{B}$ which satisfy (*).

In this paper we shall study a generalization of the reduced power, which we shall call the *limit reduced power*. If G is a filter on $I \times I$, the limit reduced power $\mathfrak{A}^I{}_D \mid G$ is defined as the subsystem of the reduced power $\mathfrak{A}^I{}_D$ consisting of the equivalence classes of those functions $f \in A^I$ which are "almost constant", in the sense that $f(i) = f(j)$ holds throughout some member of G. The more general notion of limit reduced *product* was defined in [13], but for the sake of simplicity we shall not consider it here. Our main emphasis, in fact, will be on the special case of *limit ultrapowers*, where D is an ultrafilter on I. In §2 we shall develop some basic set-theoretic properties of limit reduced powers, but from §3 on, we shall restrict our attention to limit ultrapowers. We shall not attempt to determine to what extent the results can be generalized to limit reduced powers, or to limit ultraproducts. Our major concern will be the study of limit ultrapowers for their own sake, although we shall obtain occasional applications to other problems.

With limit ultrapowers we are able to prove the following result (Theorem 3.10) *without* the continuum hypothesis. $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent if and only if they have some isomorphic limit ultrapowers with cardinality at most that of $2^A \cup 2^B$. The notion of limit ultrapower is very stable in that there are at least two other quite different conditions which lead to the same notion as our original definition of limit ultrapower. One of these is the model-theoretic condition (*). We shall see in Theorem 3.7 that $\mathfrak{B}$ satisfies (*) if and only if $\mathfrak{B}$ is isomorphic to a limit ultrapower of $\mathfrak{A}$. The other condition is stated without proof in [14, p. 878]; it is set-theoretic in nature, and justifies our use of the adjective "limit".

As an illustration of how the ideals developed in this paper can be applied to obtain new results in the theory of models, we shall improve upon a theorem of Rabin [27]. In Theorem 4.5 we shall see that if α is not measurable (see §4), then every system of power α has a proper elementarily equivalent extension of power α if and only if α^{ω} has power α. Rabin proved this result under the additional assumptions, which we do not need, that the generalized continuum hypothesis holds and that α is less than the first inaccessible number.

In §5 we shall study briefly the notion of *strong limit ultrapower*, which was introduced in [13; 19]. We shall see that the notion of strong limit ultrapower occupies an intermediate position between those of ultrapower and of limit ultrapower. In Theorem 5.2 we show that $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent if and only if they have some isomorphic strong limit ultrapowers. This result does not require the continuum hypothesis, but on the other hand it does not provide a good bound on the cardinality.

In the last section we shall give examples to show that all of the notions of ultrapower, strong limit ultrapower, limit ultrapower, and elementary extension are essentially different from each other.

Many of our model-theoretic notions, such as relational system, elementary equivalence, and elementary class are due to Tarski (see [33; 35; 36]). The notion of elementary extension and some of its basic properties are due to Tarski and Vaught in [38]. The definition of ultraproduct, and more generally of reduced product, in the form which we shall adopt here was given by Frayne, Scott, and Tarski in [11]. For a historical discussion of the reduced product construction we refer to [29, p. 70].

Frayne, Morel, and Scott give a comprehensive treatment of the basic properties of reduced products in [9]. In §1 below we shall give a brief account of the definitions and theorems from [9] which we shall need, and we shall also introduce the necessary terminology from set theory and the theory of models.

A number of papers which contain additional material on ultraproducts and applications of ultraproducts to the theory of models can be found in the bibliography. In particular, Kochen [20] makes extensive use of strong limit ultrapowers as well as of the results in [9].

Most of the results of this paper were announced by the author in [13; 14].

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1. **Preliminaries.** We shall distinguish between *sets* and *classes*, where a set is a class which is an element of some other class. We shall always assume the axiom of choice, or equivalently the well-ordering principle.

Ordinal numbers will be denoted by the small Greek letters λ, η, ρ, ξ, ζ, and natural numbers (finite ordinal numbers) by m, n, p. We suppose that ordinal numbers have been defined so that each ordinal number coincides with the set of all smaller ordinal numbers. Thus in particular 0 is the empty set. We identify cardinal numbers with the corresponding initial ordinal numbers. The union of any set of cardinal numbers is again a cardinal number. The letters α, β, γ will be used for cardinal numbers. We denote the smallest infinite cardinal number by ω, and call sets of power $\leqq \omega$ *countable* sets. The smallest cardinal number β such that $\beta > \alpha$ is denoted by α^+.

If X and Y are any two sets, we denote the power (or cardinality) of X by $|X|$, the set of all functions on X into Y by Y^X, the cartesian product of X and Y by $X \times Y$, and the set of all elements of X which are not elements of Y by $X - Y$. Functions will be denoted by either small italic or small Greek letters, depending on the context. Let $f \in Y^X$. If $x \in X$, then $f(x)$ is the unique y such that $\langle x,y \rangle \in f$. If $X_0 \subseteq X$, then $f(X_0)$ denotes the set $\{f(x) : x \in X_0\}$. It will always be clear from the context which of the two interpretations of $f(\ )$ is called for. We shall denote by $f^{-1}(Y_0)$ the set $\{x \in X : f(x) \in Y_0\}$. The identity function $\{\langle x,x \rangle : x \in X\}$ on a set X is denoted by ι. Whenever we use the symbol ι, it will be clear from the context on which set X, ι is to be taken as the identity function. The composition $g \circ f$ of two functions $f \in Y^X$ and $g \in Z^Y$ is defined by the condition

$$(g \circ f)(x) = g(f(x)) \text{ for all } x \in X.$$

We sometimes denote a function f with domain I by $\langle X_i \rangle_{i \in I}$, where $X_i = f(i)$ for each $i \in I$. If ξ is an ordinal number, we sometimes call a function $a = \langle a_\zeta \rangle_{\zeta < \xi}$ a *ξ-termed sequence*. If X is any class, we denote the union of all members of X by $\bigcup X$, and the intersection of all members of X by $\bigcap X$.

For any set X, let $S(X)$ denote the set of all subsets of X, and let $S_\omega(X)$ denote the set of all finite subsets of X. D is said to be a *filter on* X if $D \subseteq S(X)$, $X \in D$, and for any Y_1, $Y_2 \in D$ and $Z \in S(X)$, we have $Y_1 \cap Y_2 \in D$ and $Y_1 \cup Z \in D$. Let D be a filter on X. D is *proper* if $0 \notin D$. D is *countably complete* if whenever $E \subseteq D$ and $|E| \leqq \omega$, we have $\bigcap E \in D$. D is an *ultrafilter on* X if D is proper and, for each $Y \in S(X)$, either $Y \in D$ or $X - Y \in D$. By the *filter on X generated by* a subset $E \subseteq S(X)$ we mean the filter

$$\{Y \subseteq X : \text{for some } F \in S_\omega(E), \ \bigcap F \subseteq Y\};$$

that is, the smallest filter which includes E. D is *principal* if it is the filter on X generated by some one-element subset of $S(X)$. Any principal filter is countably complete. We now state a fundamental theorem (cf. Tarski [32]) concerning the existence of ultrafilters.

ULTRAFILTER THEOREM. *For every set X and every proper filter D on X, there exists an ultrafilter on X which includes D.*

Let μ be a sequence of natural numbers whose domain is ρ; thus $\mu \in \omega^\rho$. A sequence $\mathfrak{A} = \langle A, R_\lambda \rangle_{\lambda < \rho}$ is said to be a *relational system*, or more briefly a *system, of type μ* if A is a nonempty set and R_λ is a $\mu(\lambda)$-ary relation on A for each $\lambda < \rho$. Throughout this paper we shall assume that $\mu \in \omega^\rho$ and $\mu' \in \omega^{\rho'}$, that $\mu \subseteq \mu'$, that

$$\mathfrak{A} = \langle A, R_\lambda \rangle_{\lambda < \rho}, \qquad \mathfrak{B} = \langle B, S_\lambda \rangle_{\lambda < \rho}, \qquad \mathfrak{C} = \langle C, T_\lambda \rangle_{\lambda < \rho}$$

are systems of type μ, and that $\mathbf{K}$ is a class of systems of type μ. The *complement of* $\mathbf{K}$, denoted by $\overline{\mathbf{K}}$, is the class of all systems of type μ which are not members of $\mathbf{K}$.

The function ϕ is said to be an *embedding of* $\mathfrak{A}$ *into* $\mathfrak{B}$ if $\phi \in B^A$, ϕ is one-to-one, and if for each $\lambda < \rho$ and each $a_1, \cdots, a_{\mu(\lambda)} \in A$, we have

$$R_\lambda(a_1, \cdots, a_{\mu(\lambda)}) \text{ if and only if } S_\lambda(\phi(a_1), \cdots, \phi(a_{\mu(\lambda)})).$$

$\mathfrak{A}$ is said to be *embeddable in* $\mathfrak{B}$ if there is an embedding of $\mathfrak{A}$ *into* $\mathfrak{B}$. We say that $\mathfrak{A}$ is a *subsystem of* $\mathfrak{B}$, and that $\mathfrak{B}$ is an *extension of* $\mathfrak{A}$, if the identity function ι is an embedding of $\mathfrak{A}$ into $\mathfrak{B}$. If $C \subseteq B$, then $\mathfrak{B} \upharpoonright C$ will denote the unique subsystem $\mathfrak{A}$ of $\mathfrak{B}$ such that $A = C$. If ϕ is an embedding of $\mathfrak{A}$ into $\mathfrak{B}$, we denote by $\phi(\mathfrak{A})$ the subsystem $\mathfrak{B} \upharpoonright \phi(A)$ of $\mathfrak{B}$. Clearly $\mathfrak{A} \upharpoonright A = \iota(\mathfrak{A}) = \mathfrak{A}$. ϕ is said to be an *isomorphism of* $\mathfrak{A}$ *onto* $\mathfrak{B}$, in symbols $\phi : \mathfrak{A} \cong \mathfrak{B}$, if ϕ is an embedding of $\mathfrak{A}$ into $\mathfrak{B}$ and the range of ϕ is B. We thus have $\iota : \mathfrak{A} \cong \mathfrak{A}$. $\mathfrak{A}$ and $\mathfrak{B}$ are said to be *isomorphic*, in symbols $\mathfrak{A} \cong \mathfrak{B}$, if there exists an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$.

If $\mathbf{K}$ is a set of systems of type μ, then we shall denote by $\bigcup \mathbf{K}$ the system

$$\left\langle \bigcup \{A : \mathfrak{A} \in \mathbf{K}\}, \quad \bigcup \{R_\lambda : \mathfrak{A} \in \mathbf{K}\} \right\rangle_{\lambda < \rho}$$

of type μ.

Let $\mathfrak{A}' = \langle A', R'_\lambda \rangle_{\lambda < \rho'}$ be an arbitrary system of type μ'. By the *μ-reduct of* $\mathfrak{A}'$, denoted by $\mathfrak{A}' \upharpoonright \mu$, we mean the system $\langle A', R'_\lambda \rangle_{\lambda < \rho}$ of type μ.

Let us denote by $L(\mu)$ the first order predicate logic with identity symbol $=$, an ω-termed sequence of individual variables $v_0, v_1, v_2, \cdots$, and a $\mu(\lambda)$-placed predicate symbol P_λ for each $\lambda < \rho$. We shall use the symbols $\neg$, $\vee$, $\leftrightarrow$, $\exists$, $\forall$, in the usual ways to denote propositional connectives and quantifiers. $L(\mu)$ has no function variables or constants, and no predicate variables. By a sentence we mean a formula of $L(\mu)$ which has no free variables. We shall let Φ denote an arbitrary formula of $L(\mu)$.

We assume the notions of a sequence $a \in A^\omega$ *satisfying* a formula Φ *in* $\mathfrak{A}$, and of a sentence Φ *holding in* $\mathfrak{A}$, are known (see [39]). If $a \in A^\omega$ and $b \in A$, we denote by $a(n \mid b)$ the sequence obtained from a by replacing the ordered pair $\langle n, a_n \rangle$ by the ordered pair $\langle n, b \rangle$. Thus whenever a belongs to A^ω and $b \in A$, then $a(n \mid b)$ also belongs to A^ω. $\mathbf{K}$ is said to be an *elementary class*, in symbols $\mathbf{K} \in EC$, if there is a sentence Φ of $L(\mu)$ such that $\mathbf{K}$ is the set of all systems of type μ in which Φ holds. $\mathfrak{A}$ and $\mathfrak{B}$ are said to be *elementarily equivalent*, in symbols $\mathfrak{A} \equiv \mathfrak{B}$, if every sentence Φ of $L(\mu)$ which holds in $\mathfrak{A}$ also holds in $\mathfrak{B}$. The relation $\mathfrak{A}' \equiv \mathfrak{B}'$ can hold only if $\mathfrak{A}'$ and $\mathfrak{B}'$ are of the same type.

ϕ is said to be an *elementary embedding of* $\mathfrak{A}$ *into* $\mathfrak{B}$, in symbols $\phi : \mathfrak{A} \prec \mathfrak{B}$, if ϕ is an embedding of $\mathfrak{A}$ into $\mathfrak{B}$ and, for any $a \in A^{\omega}$ and any formula Φ of $L(\mu)$, a satisfies Φ in $\mathfrak{A}$ if and only if $\phi \circ a$ satisfies Φ in $\mathfrak{B}$. $\mathfrak{A}$ is said to be *elementarily embeddable in* $\mathfrak{B}$, in symbols $\mathfrak{A} \prec \mathfrak{B}$, if $\phi : \mathfrak{A} \prec \mathfrak{B}$ for some ϕ. We say that $\mathfrak{A}$ is an *elementary subsystem of* $\mathfrak{B}$, and that $\mathfrak{B}$ is an *elementary extension of* $\mathfrak{A}$, if $A \subseteq B$ and $\iota : \mathfrak{A} \prec \mathfrak{B}$. We see at once that $\phi : \mathfrak{A} \cong \mathfrak{B}$ implies $\phi : \mathfrak{A} \prec \mathfrak{B}$, and that $\phi : \mathfrak{A} \prec \mathfrak{B}$ implies both $\mathfrak{A} \equiv \mathfrak{B}$ and $\iota : \phi(\mathfrak{A}) \prec \mathfrak{B}$. Moreover, if $\mathfrak{A}'$, $\mathfrak{B}'$ are systems of type μ', then $\phi : \mathfrak{A}' \prec \mathfrak{B}'$ implies $\phi : (\mathfrak{A}' \restriction \mu) \prec (\mathfrak{B}' \restriction \mu)$. Another simple fact, pointed out in [38], is that if $\mathfrak{B}$ is an elementary subsystem of $\mathfrak{C}$ and $\mathfrak{A}$ is an arbitrary subsystem of $\mathfrak{B}$, then $\mathfrak{A}$ is an elementary subsystem of $\mathfrak{B}$ if and only if it is an elementary subsystem of $\mathfrak{C}$.

We shall now state two classical theorems in the theory of models which we shall need.

LÖWENHEIM-SKOLEM-TARSKI THEOREM. (*See* [38].) *Suppose that* A, α *are infinite and* $\alpha \geqq |\rho|$. *Then there exists a system* $\mathfrak{B}$ *such that* $\mathfrak{B} \equiv \mathfrak{A}$ *and* $|B| = \alpha$. *Moreover, if* $C \subseteq A$ *and* $|C| \leqq \alpha \leqq |A|$, *then* $\mathfrak{B}$ *may be chosen so that* $C \subseteq B$ *and* $\iota : \mathfrak{B} \prec \mathfrak{A}$.

COMPACTNESS THEOREM. (*See* [12; 24].) *Let* Σ *be any set of sentences of* $L(\mu)$. *If every finite subset of* Σ *is satisfiable, then* Σ *is satisfiable.*

Another theorem which will be important for our purposes is the following, proved in [38].

THEOREM 1.1. *Let* $\{\mathfrak{A}_n : n < \omega\}$ *be a set of systems such that* $\iota : \mathfrak{A}_n \prec \mathfrak{A}_{n+1}$ *for each n. Then* $\bigcup_{m<\omega} \mathfrak{A}_m$ *is an elementary extension of each* $\mathfrak{A}_n$.

We now introduce the notion of a *reduced power*, which was discussed in the introduction.

Let D be a filter on the set I. For any element f, $g \in A^I$, we write $f \approx_D g$ (read f is equivalent to g modulo D) if and only if $\{i \in I : f(i) = g(i)\} \in D$. The statement $f \approx_D g$ has the intuitive meaning that f and g are equal almost everywhere. It is proved in [9] that $\approx_D$ is an equivalence relation on the set A^I. For each $f \in A^I$, let $f/D = \{g : f \approx_D g\}$, the equivalence class of f with respect to the relation $\approx_D$. By the *reduced power of A modulo D* we mean the set

$$A^I{}_D = \{f/D : f \in A^I\}.$$

In [9] it is shown that, for each $\lambda < \rho$, there is a unique $\mu(\lambda)$-ary relation $R_{\lambda D}$ on $A^I{}_D$ which is defined by the condition:

For any $f_1, \cdots, f_{\mu(\lambda)} \in A^I$, $R_{\lambda D}(f_1/D, \cdots, f_{\mu(\lambda)}/D)$ *if and only if*

$$\{i \in I \ : \ R_\lambda(f_1(i), \cdots, f_{\mu(\lambda)}(i))\} \in D.$$

By the *reduced power* $\mathfrak{A}^I{}_D$ *of* $\mathfrak{A}$ *modulo* D we mean the system $\langle A^I{}_D, R_{\lambda D}\rangle_{\lambda<\rho}$ of type μ.

A reduced power $\mathfrak{A}^I{}_D$ is said to be an *ultrapower* if D is an ultrafilter on I, and is said to be a *direct power* if $D = \{I\}$.

If $\mathfrak{A}'$ is a system of type μ', it is clear that we always have $(\mathfrak{A}'^I{}_D) \restriction \mu = (\mathfrak{A}' \restriction \mu)^I{}_D$.

We shall state without proof some basic model-theoretic results concerning ultrapowers; the proofs can be found in [9]. The first theorem is essentially due to Łoś [23]. The *diagonal function* d on A into $A^I{}_D$ is defined as follows:

$$\text{for each } a \in A, \quad d(a) = \{\langle i,a\rangle : i \in I\}/D.$$

Whenever we use the symbol d it will be clear from the context which sets A, I, D determine d.

THEOREM 1.2. *If D is a proper filter on I, then d is an embedding of $\mathfrak{A}$ into $\mathfrak{A}^I{}_D$. If D is an ultrafilter, then $d : \mathfrak{A} \prec \mathfrak{A}^I{}_D$.*

THEOREM 1.3. *If A is finite and D is an ultrafilter on I, then $d : \mathfrak{A} \cong \mathfrak{A}^I{}_D$.*

THEOREM 1.4. *If A is countable and D is a countably complete ultrafilter on I then $d : \mathfrak{A} \cong \mathfrak{A}^I{}_D$.*

THEOREM 1.5. *If D is a principal ultrafilter on I, then $d : \mathfrak{A} \cong \mathfrak{A}^I{}_D$.*

THEOREM 1.6. (a) *If $\mathfrak{A} \equiv \mathfrak{B}$, then there is an ultrapower $\mathfrak{A}^I{}_D$ such that $\mathfrak{B} \prec \mathfrak{A}^I{}_D$.*

(b) *If $\phi : \mathfrak{A} \prec \mathfrak{B}$, then there is an ultrafilter $\mathfrak{A}^I{}_D$ and an elementary embedding $\psi : \mathfrak{B} \prec \mathfrak{A}^I{}_D$ such that $\psi \circ \phi = d$.*

We close this section with a theorem of a set-theoretical nature which states that a reduced power of a reduced power of $\mathfrak{A}$ is a reduced power of $\mathfrak{A}$, and an ultrapower of an ultrapower of $\mathfrak{A}$ is an ultrapower of $\mathfrak{A}$. This theorem is also proved in [9]. Let D, E be filters on I, J respectively. Define

$$D \mathbin{\overline{\times}} E = \{K \subseteq I \times J : \{j \in J : \{i \in I : \langle i,j\rangle \in K\} \in D\} \in E\}.$$

$D \mathbin{\overline{\times}} E$ is a filter on $I \times J$, and $D \mathbin{\overline{\times}} E$ is an ultrafilter if both D and E are ultrafilters.

THEOREM 1.7. $(\mathfrak{A}^I{}_D)^J{}_E \cong \mathfrak{A}^{I\times J}{}_{D \overline{\times} E}$.

2. **Limit reduced powers.** In this section we define the limit reduced power operation, which is the central notion of this paper. We shall develop some basic set-theoretic properties of limit reduced powers.

DEFINITION. Let A, I be nonempty sets, let D be a filter on I, and let G be a filter on $I \times I$. For any function $f \in A^I$, let

$$\mathrm{eq}(f) = \{\langle i,j\rangle \in I \times I : f(i) = f(j)\}.$$

We now define

$$A^I{}_D | G = \{a \in A^I{}_D : \text{for some } f \in a,\ \mathrm{eq}(f) \in G\}.$$

We refer to $A^I{}_D|G$ as a *limit reduced power of* the set A. Intuitively, $A^I{}_D|G$ is the set of equivalence classes f/D of functions $f \in A^I$ which are "almost constant".

By the *limit reduced power* $\mathfrak{A}^I{}_D|G$ of the system $\mathfrak{A}$ we mean the system $\mathfrak{A}^I{}_D|G = \mathfrak{A}^I{}_D \upharpoonright (A^I{}_D | G)$.

The reduced power operation is a special case of the limit reduced power. In fact, it is easily seen that $\mathfrak{A}^I{}_D| S(I \times I) = \mathfrak{A}^I{}_D$.

If D is an ultrafilter, $\mathfrak{A}^I{}_D | G$ is said to be a *limit ultrapower of* $\mathfrak{A}$. If $D = \{I\}$, then $\mathfrak{A}^I{}_D | G$ is said to be a *limit direct power of* $\mathfrak{A}$.

Limit reduced powers have the following important property: *If $\mathfrak{A}'$ is a system of type μ', then we always have*

$$(\mathfrak{A}'^I{}_D | G) \upharpoonright \mu = (\mathfrak{A}' \upharpoonright \mu)^I{}_D | G.$$

The diagonal function d on A into $A^I{}_D$ is also a function on A into $A^I{}_D | G$ for any filter G on $I \times I$. This is true because $I \times I \in G$ and for any constant function $f \in A^I$, $\mathrm{eq}(f) = I \times I$. In fact, the range of d is exactly the set $A^I{}_D | \{I \times I\}$. Since, when D is proper, d is an embedding of $\mathfrak{A}$ into $\mathfrak{A}^I{}_D$ and

$$\mathfrak{A}^I{}_D | \{I \times I\} = \mathfrak{A}^I{}_D \upharpoonright (A^I{}_D | \{I \times I\}),$$

d is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{A}^I{}_D|\{I \times I\}$.

If G and H are filters on $I \times I$ such that $G \subseteq H$, then it is immediate from the definition that $A^I{}_D | G \subseteq A^I{}_D | H$. In particular, we have

$$A^I{}_D|\{I \times I\} \subseteq A^I{}_D | G \subseteq A^I{}_D| S(I \times I),$$

because every filter G on $I \times I$ satisfies the formula $(I \times I) \in G \subseteq S(I \times I)$. It follows that $\mathfrak{A}^I{}_D | G$ is a subsystem of $\mathfrak{A}^I{}_D | H$ whenever $G \subseteq H$, that $\mathfrak{A}^I{}_D | \{I \times I\}$ is a subsystem of every $\mathfrak{A}^I{}_D | G$, and that $\mathfrak{A}^I{}_D| S(I \times I)$ is an extension of every $\mathfrak{A}^I{}_D | G$. Thus, whenever D is proper, d is an embedding of $\mathfrak{A}$ into $\mathfrak{A}^I{}_D | G$.

We shall say that a limit reduced power $A^I{}_D | G$ is *trivial* if $A^I{}_D | G = A^I{}_D| \{I \times I\}$. $\mathfrak{A}^I{}_D | G$ is said to be trivial if $A^I{}_D|G$ is trivial. The notion of a trivial reduced product is defined analogously. $A^I{}_D | G$ is trivial if and only if $A^I{}_D | G \subseteq A^I{}_D| \{I \times I\}$, because we always have $A^I{}_D| \{I \times I\} \subseteq A^I{}_D | G$. Thus $\mathfrak{A}^I{}_D | G$ is trivial if and only if either $d : \mathfrak{A} \cong \mathfrak{A}^I{}_D|G$ or $D = S(I)$. If $G \subseteq H$ and $\mathfrak{A}^I{}_D | H$ is trivial, then $\mathfrak{A}^I{}_D | G$ is also trivial. If A is finite and D is an ultrafilter, then by Theorem 1.3, $\mathfrak{A}^I{}_D$ is trivial, and so is $\mathfrak{A}^I{}_D | G$. Similarly if D is a principal ultrafilter, then $\mathfrak{A}^I{}_D$ is trivial by Theorem 1.5, so $\mathfrak{A}^I{}_D | G$ is trivial.

We now give a convenient criterion for a subset $B \subseteq A^I{}_D$ to be of the form $A^I{}_D | G$.

THEOREM 2.1. *Suppose A is infinite. A subset $B \subseteq A^I{}_D$ satisfies $B = A^I{}_D \mid G$ for some filter G if and only if $B \neq 0$ and for every f/D, $g/D \in B$ and every $h \in A^I$ such that* $\mathrm{eq}(f) \cap \mathrm{eq}(g) \subseteq \mathrm{eq}(h)$, *we have $h/D \in B$.*

Proof. First suppose $B = A^I{}_D \mid G$. Then $A^I{}_D \mid \{I \times I\} \subseteq B$, so $B \neq 0$. Let f/D, $g/D \in B$ and suppose $h \in A^I$ and $\mathrm{eq}(f) \cap \mathrm{eq}(g) \subseteq \mathrm{eq}(h)$. There exists $f' \in f/D$ and $g' \in g/D$ such that $\mathrm{eq}(f')$, $\mathrm{eq}(g') \in G$. Since $\mathrm{eq}(f) \cap \mathrm{eq}(g) \subseteq \mathrm{eq}(h)$, there exists $h' \in A^I$ such that $h'(i) = h(i)$ whenever $f'(i) = f(i)$ and $g'(i) = g(i)$, and also that $h'(i) = h'(j)$ whenever $f'(i) = f'(j)$ and $g'(i) = g'(j)$. Then $h' \in h/D$. Also, $\mathrm{eq}(h') \supseteq \mathrm{eq}(f') \cap \mathrm{eq}(g') \in G$. Therefore $h/D \in A^I{}_D \mid G = B$.

For the converse, suppose that whenever f/D, $g/D \in B$, $h \in A^I$, and $\mathrm{eq}(f) \cap \mathrm{eq}(g) \subseteq \mathrm{eq}(h)$, we have $h/D \in B$. Let

$$G = \{J \subseteq I \times I : \text{ for some } f \in A^I,\ f/D \in B \text{ and } \mathrm{eq}(f) \subseteq J\}.$$

Since A is infinite, it follows that for every f, $g \in A^I$ there exists $h \in A^I$ such that $\mathrm{eq}(h) = \mathrm{eq}(f) \cap \mathrm{eq}(g)$. Therefore $J, J' \in G$ implies $J \cap J' \in G$, and hence G is a filter on $I \times I$. Clearly $B \subseteq A^I{}_D \mid G$, for $f/D \in B$ implies that for some $f' \in f/D$, $\mathrm{eq}(f') \in G$. Suppose $h/D \in A^I{}_D \mid G$. Then for some $h' \in h/D$, $\mathrm{eq}(h') \in G$. That is, $\mathrm{eq}(f) \subseteq \mathrm{eq}(h')$ for some $f/D \in B$. Then $h/D = h'/D \in B$, and $A^I{}_D \mid G \subseteq B$.

The conclusion of Theorem 2.1 is true even if A is finite. However, the result when A is finite seems to require a separate proof involving induction. We shall not give this proof here, because our main emphasis in this paper is on limit ultrapowers, which become trivial when A is finite.

If J is an equivalence relation on I, then the equivalence class $\{j \in I : \langle i,j \rangle \in J\}$ of i modulo J is denoted by i/J. Any two equivalence classes i/J, j/J either coincide or are disjoint, according as $\langle i,j \rangle \in J$ or not. We also write $I/J = \{i/J : i \in I\}$. It is easily seen that if J_0, J_1 are equivalence relations on I, then so is $J_0 \cap J_1$. In fact, even an infinite intersection on equivalence relations on I is an equivalence relation on I. If J_0, J_1 are equivalence relations on I and $J_0 \subseteq J_1$, then $|I/J_0| \geqq |I/J_1|$. The largest equivalence relation on I is $I \times I$, and the smallest is $\{\langle i,i \rangle : i \in I\}$. Moreover, we have $|I/I \times I| = 1$ if $I \neq 0$, and $|I/\{\langle i,i \rangle : i \in I\}| = |I|$. Obviously, $\mathrm{eq}(f)$ is an equivalence relation on I whenever $f \in A^I$.

LEMMA 2.2. *Let G be a principal filter on $I \times I$, let K be the least equivalence relation in G, and let $J = I/K$. Then there is a filter E on J such that*:

(i) $\mathfrak{A}^J{}_E \cong \mathfrak{A}^I{}_D \mid G$; *and*

(ii) *if D is an ultrafilter, then so is E.*

Proof. If $a \in A^I{}_D$, then $a \in A^I{}_D \mid G$ if and only if there exists $f \in a$ such that $\mathrm{eq}(f) \supseteq K$. Let

$$E = \{X \subseteq J : \bigcup X \in D\}. \tag{1}$$

It is easily verified that E is a filter on J, and furthermore that E is an ultrafilter whenever D is an ultrafilter. For each $f \in A^J$, let ϕf be the unique function $g \in A^I$ such that, whenever $i \in j \in J$, we have $g(i) = f(j)$. Then

$$\phi(A^J) = \{g \in A^I : \mathrm{eq}(g) \supseteq K\}. \tag{2}$$

Also, it follows from (1) that, if f, $g \in A^J$, then

$$f/E = g/E \text{ if and only if } (\phi f)/D = (\phi g)/D. \tag{3}$$

Consequently there is a unique one-to-one function ψ on A^J into $A^I{}_D$ such that, for each $f \in A^J$, we have

$$\psi(f/E) = \phi(f)/D.$$

By (2), we have

$$\psi(A^J{}_E) = A^I{}_D \,|\, G.$$

It remains to prove that $\psi : \mathfrak{A}^J{}_E \cong \mathfrak{A}^I{}_D \,|\, G$. To see this, we observe that for each $\lambda < \rho$ and $f_1, \cdots, f_{\mu(\lambda)} \in A^J$, each of the following statements are equivalent:

$$R_{\lambda E}(f_1/E, \cdots, f_{\mu(\lambda)}/E);$$

$$\{j \in J : R_\lambda(f_1(j), \cdots, f_{\mu(\lambda)}(j))\} \in E;$$

$$\{i \in I : R_\lambda(\phi f_1(i), \cdots, \phi f_{\mu(\lambda)}(i))\} \in D;$$

$$R_{\lambda D}((\phi f_1)/D, \cdots, (\phi f_{\mu(\lambda)})/D);$$

$$R_{\lambda D}(\psi(f_1/E), \cdots, \psi(f_{\mu(\lambda)}/E)).$$

For any filter G on $I \times I$, let

$$\bar{G} = \{J \subseteq I \times I : J \supseteq J' \in G \quad \text{for some equivalence relation } J' \text{ on } I\}.$$

Then $\bar{G}$ is also a filter on $I \times I$ and $\bar{G} \subseteq G$. It is easily seen that $\bar{G} = \bar{\bar{G}}$ and that $G \subseteq H$ implies $\bar{G} \subseteq \bar{H}$. We always have $\bigcap \bar{G} \supseteq \{\langle i,i \rangle : i \in I\}$; in particular, $0 \notin \bar{G}$ and thus $S(I \times I)$ is not equal to $\overline{S(I \times I)}$.

Lemma 2.3. $\mathfrak{A}^I{}_D \,|\, G = \mathfrak{A}^I{}_D \,|\, \bar{G}$.

Proof. It is sufficient to show that for every $f \in A^I$, $\mathrm{eq}(f) \in G$ if and only if $\mathrm{eq}(f) \in \bar{G}$. If $\mathrm{eq}(f)$ belongs to $\bar{G}$, then it belongs to G because $\bar{G} \subseteq G$. Suppose $\mathrm{eq}(f) \in G$; then $\mathrm{eq}(f)$ is an equivalence relation on I. Therefore by the definition of $\bar{G}$, $\mathrm{eq}(f) \in \bar{G}$.

Theorem 2.4 below shows that a limited reduced power of a limit reduced power of $\mathfrak{A}$ is isomorphic to a limit reduced power of $\mathfrak{A}$. Also, a limit ultrapower of a limit ultrapower of $\mathfrak{A}$ is isomorphic to a limit ultrapower of $\mathfrak{A}$. The result obviously extends to finite iterations of limit reduced powers, or limit ultrapowers. Theorem 2.4 generalizes Theorem 1.7, which concerns reduced powers.

If G, H are filters on $I \times I$, $J \times J$, respectively, then $G \mathbin{\overline{\times}} H$, as defined in §1, is a filter on $(I \times I) \times (J \times J)$, rather than on $(I \times J) \times (I \times J)$. There is, however, a perfectly natural way of obtaining a filter on $(I \times J) \times (I \times J)$ from $G \mathbin{\overline{\times}} H$; if f is the one-to-one function on $(I \times I) \times (J \times J)$ onto $(I \times J) \times (I \times J)$ defined by

$$f(\langle\langle i,i'\rangle,\langle j,j'\rangle\rangle) = \langle\langle i,j\rangle,\langle i',j'\rangle\rangle,$$

then we define

$$G \otimes H = \{f(K) : K \in G \mathbin{\overline{\times}} H\}.$$

It is easily seen that $G \otimes H$ is a filter on $(I \times J) \times (I \times J)$.

THEOREM 2.4. $(\mathfrak{A}^I{}_D | G)^J{}_E | H \cong \mathfrak{A}^{I \times J}{}_{D \mathbin{\overline{\times}} E} | \bar{G} \otimes \bar{H}$.

Proof. We define the required isomorphism ϕ as follows. Let $f/E \in (A^I{}_D | G)^J{}_E | H$, and for each $j \in J$ let $f(j) = f_j/D \in A^I{}_D | G$. For each $i \in I$ let $g(i,j) = f_j(i)$. Define $\phi(f/E) = g/(D \mathbin{\overline{\times}} E)$. From the proof of the analogous result (Theorem 1.7) for reduced products (cf. [9;20]) we know that ϕ is a well-defined embedding of $(\mathfrak{A}^I{}_D | G)^J{}_E | H$ into $\mathfrak{A}^{I \times J}{}_{D \mathbin{\overline{\times}} E}$. It remains to verify that the range of ϕ is $A^{I \times J}{}_{D \mathbin{\overline{\times}} E} | \bar{G} \otimes \bar{H}$. If $f/E \in (A^I{}_D | G)^J{}_E | H$, then we may suppose that $\mathrm{eq}(f) \in \bar{H}$ and for each $j \in J$, $\mathrm{eq}(f_j) \in \bar{G}$. Therefore

$$\{\langle j_1,j_2\rangle \in J \times J : \{\langle i_1,i_2\rangle \in I \times I : f_{j_1}(i_1) = f_{j_2}(i_2)\} \in \bar{G}\} \in \bar{H}.$$

This means that $\mathrm{eq}(g) \in \bar{G} \otimes \bar{H}$, and hence

$$g/(D \mathbin{\overline{\times}} E) = \phi(f/E) \in A^{I \times J}{}_{D \mathbin{\overline{\times}} E} | \bar{G} \otimes \bar{H}.$$

Conversely, suppose $g/(D \mathbin{\overline{\times}} E) \in A^{I \times J}{}_{D \mathbin{\overline{\times}} E} | \bar{G} \otimes \bar{H}$. Then we may suppose that $\mathrm{eq}(g) \in \bar{G} \otimes \bar{H}$, and hence

$$\{\langle j_1,j_2\rangle \in J \times J : \{\langle i_1,i_2\rangle \in I \times I : g(i_1,j_1) = g(i_2,j_2)\} \in \bar{G}\} \in \bar{H}.$$

Since any member of $\bar{H}$ includes the set $\{\langle j,j\rangle : j \in J\}$, we have

$$\{\langle i_1,i_2\rangle \in I \times I : g(i_1,j) = g(i_2,j)\} \in \bar{G}$$

for every $j \in J$. Let $f_j(i) = g(i,j)$ for every $i \in I$, $j \in J$, and let $f(j) = f_j/D$ for every $j \in J$. Then $\mathrm{eq}(f_j) \in \bar{G}$ for every $j \in J$. For any $\langle j_1,j_2\rangle \in J \times J$ such that

$$\{\langle i_1,i_2\rangle \in I \times I : f_{j_1}(i_1) = f_{j_2}(i_2)\} \in \bar{G},$$

we have $f_{j_1}(i) = f_{j_2}(i)$ for every $i \in I$, and hence $f_{j_1} = f_{j_2}$. Therefore ϕ has range $A^{I \times J}{}_{D \mathbin{\overline{\times}} E} | \bar{G} \otimes \bar{H}$.

It is not in general true that $(\mathfrak{A}^I{}_D | G)^J{}_E | H \cong \mathfrak{A}^{I \times J}{}_{D \mathbin{\overline{\times}} E} | G \otimes H$; that is, we cannot replace $\bar{G} \otimes \bar{H}$ by $G \otimes H$ in Theorem 2.4. In fact, it is not difficult to see that

$$S(I \times I) \otimes H = G \otimes S(J \times J) = S((I \times J) \times (I \times J)),$$

and therefore

$$\mathfrak{A}^{I \times J}{}_{D \bar{\times} E} | S(I \times I) \otimes H = \mathfrak{A}^{I \times J}{}_{D \bar{\times} E} | G \otimes S(J \times J) = \mathfrak{A}^{I \times J}{}_{D \bar{\times} E}.$$

We shall conclude this section by discussing a special kind of limit reduced power which arises naturally from the notion of a topological space. This discussion has no bearing on the remainder of the paper, but it is an interesting illustration of the limit reduced power notion.

A subset $T \subseteq S(I)$ is said to be a *topology on* I if $0, I \in T$, $X \subseteq T$ implies $\bigcup X \in T$, and $X \in S_\omega(T)$ implies $\bigcap X \in T$. The members of T are called *open sets*. T is said to be a *discrete topology* if $T = S(I)$. If T, U are topologies on I, J, a function $f \in J^I$ is said to be *continuous with respect to* T *and* U if $X \in U$ implies $f^{-1}(X) \in T$.

Let T be a topology on I and D a proper filter on I. We denote by A^T the set of all functions $f \in A^I$ which are continuous with respect to T and the discrete topology on A. We define $A^T{}_D = \{f/D : f \in A^T\}$.

It is easily seen that if f, $g \in A^T$ and $\mathrm{eq}(f) \cap \mathrm{eq}(g) \subseteq \mathrm{eq}(h)$, then $h \in A^T$. Therefore by Theorem 2.1 we have $A^T{}_D = A^I{}_D | G$ for some filter G on $I \times I$. Thus the construction $A^T{}_D$ is a special case of the limit reduced power construction.

On the other hand, if T is the discrete topology on I, then $A^T{}_D = A^I{}_D$. Thus the reduced power is a special case of the construction $A^T{}_D$. Conversely, if A has at least two elements and $A^T = A^I$, then T is the discrete topology on I.

Not every limit reduced power $A^I{}_D | G$ is expressible in the form $A^T{}_D$. For example, let A and I be infinite sets and let $D = \{I\}$. Define

$$G = \{J \subseteq I \times I : \text{for some equivalence relation } J' \text{ on } I,\ J' \subseteq J \text{ and } I/J' \text{ is finite}\}.$$

Then clearly $A^I{}_D | G$ is properly included in $A^I{}_D$. However, if $A^I{}_D | G \subseteq A^T{}_D$, then T must be the discrete topology on I. Therefore $A^I{}_D | G$ is not equal to $A^T{}_D$ for any topology T on I. This shows that the limit reduced power is actually more general than the construction $A^T{}_D$. These remarks are no longer valid if both A and I have arbitrary topologies.

3. **Limit ultrapowers: model-theoretic properties.** We now leave the general discussion of limit reduced powers. For the rest of this paper we shall be concerned with the particular case of limit ultrapowers. In this section we shall obtain two model-theoretic results about limit ultrapowers. The first result is that $\mathfrak{A}^I{}_D | G$ is an elementary subsystem of $\mathfrak{A}^I{}_D | H$ whenever $G \subseteq H$. The second result is a model-theoretic characterization of limit ultrapowers in terms of elementary extensions of complete systems. For the special case $G = S(I \times I)$, Theorem 3.1 and Corollary 3.3 become theorems on ultrapowers proved in [9].

For each sequence $f = \langle f_n \rangle_{n<\omega} \in (A^I)^\omega$, let $f/D = \langle f_n/D \rangle_{n<\omega}$, and for each $i \in I$, let $f(i) = \langle f_n(i) \rangle_{n<\omega} \in A^\omega$.

THEOREM 3.1. *Let* $\mathfrak{A}^I{}_D|G$ *be a limit ultrapower of* $\mathfrak{A}$ *and let* $f \in \{g \in A^I : \mathrm{eq}(g) \in G\}^\omega$. *Then* f/D *satisfies* Φ *in* $\mathfrak{A}^I{}_D|G$ *if and only if*

$$J_\Phi = \{i \in I : f(i) \text{ satisfies } \Phi \text{ in } \mathfrak{A}\} \in D.$$

Proof. We argue by induction on the length of the formula Φ. The theorem is true for atomic formulas by the definition of limit ultrapower.

Suppose the theorem is true for Φ_1 and Φ_2. We shall prove that it is true for $\Phi_1 \wedge \Phi_2$. f/D satisfies $\Phi_1 \wedge \Phi_2$ if and only if f/D satisfies Φ_1 and f/D satisfies Φ_2. This in turn is true if and only if both the sets J_{Φ_1} and J_{Φ_2} belong to D. Moreover, since D is a filter, $J_{\Phi_1} \in D$ and $J_{\Phi_2} \in D$ if and only if $J_{\Phi_1} \cap J_{\Phi_2} \in D$. Finally, $J_{\Phi_1} \cap J_{\Phi_2} = J_{\Phi_1 \wedge \Phi_2}$, so the theorem is true for $\Phi_1 \wedge \Phi_2$.

We suppose the theorem is true for Φ and prove it is true for $\neg\Phi$ and for $\exists v_n \Phi$. The theorem is true for $\neg\Phi$ because [f/D satisfies $\neg\Phi$] if and only if [f/D does not satisfy Φ] if and only if [$J_\Phi \notin D$] if and only if [$J_{\neg\Phi} \in D$], in view of the fact that D is an ultrafilter.

To prove that the theorem is true for $\exists v_n \Phi$, we shall write a sequence of statements, separated by semicolons which are easily seen to be equivalent.

f/D satisfies $\exists v_n \Phi$;

for some $g \in A^I$, $f/D(n|g/D)$ satisfies Φ and $\mathrm{eq}(g) \in G$;

for some $g \in A^I$, $\{i \in I : f(n|g)(i)$ satisfies Φ in $\mathfrak{A}\} \in D$ and $\mathrm{eq}(g) \in G$;

$J_{\exists v_n \Phi} \in D$.

The only implication which requires explanation is the implication from the last statement to the preceding one. Suppose $J_{\exists vn\Phi} \in D$. Let $v_{m_1}, \cdots, v_{mp}$ be all the variables which occur freely in $\exists v_n \Phi$. We have $\mathrm{eq}(f_{m_1}), \cdots, \mathrm{eq}(f_{m_n}) \in G$, and since G is a filter,

$$\mathrm{eq}(f_{m_1}) \cap \cdots \cap \mathrm{eq}(f_{m_n}) \in G.$$

Using the axiom of choice, a sequence $g \in A^I$ can be found such that whenever $\langle i,j \rangle \in \mathrm{eq}(f_{m_1}) \cap \cdots \cap \mathrm{eq}(f_m)$ and $f(k)$ satisfies $\exists v_n \Phi$ in $\mathfrak{A}$, we have $g(i) = g(j)$ and $f(n|g)(k)$ satisfies Φ in $\mathfrak{A}$. Then

$$\mathrm{eq}(f_{m_1}) \cap \cdots \cap \mathrm{eq}(f_{m_n}) \subseteq \mathrm{eq}(g),$$

so $\mathrm{eq}(g) \in G$. Finally,

$$\{i \in I : f(n|g)(i) \text{ satisfies } \Phi \text{ in } \mathfrak{A}\} = J_{\exists v_n \Phi} \in D.$$

This completes the proof.

COROLLARY 3.2. *If* D *is an ultrafilter on* I *and* $G \subseteq H$, *then* $\mathfrak{A}^I{}_D|G$ *is an elementary subsystem of* $\mathfrak{A}^I{}_D|H$.

Proof. We have already remarked that $\mathfrak{A}^I{}_D|G$ is a subsystem of $\mathfrak{A}^I{}_D|H$. By Theorem 3.1, for any $f \in (A^I{}_D|G)^\omega$, the satisfaction of Φ by f/D in $\mathfrak{A}^I{}_D|G$ and in $\mathfrak{A}^I{}_D|H$ are both equivalent to the condition

$$\{i \in I : f(i) \text{ satisfies } \Phi \text{ in } \mathfrak{A}\} \in D,$$

and are therefore equivalent to each other.

COROLLARY 3.3. *If* $\mathfrak{A}^I{}_D | G$ *is a limit ultrapower of* $\mathfrak{A}$, *then* $d : \mathfrak{A} \prec \mathfrak{A}^I{}_D | G$.

Proof. $d : \mathfrak{A} \prec \mathfrak{A}^I{}_D | \{I \times I\}$. By Corollary 3.2, $\iota : \mathfrak{A}^I{}_D | \{I \times I\} \prec \mathfrak{A}^I{}_D | G$.

Limit ultrapowers have the important property that whenever we adjoin new relations to the system $\mathfrak{A}$, there is a natural way of defining new relations on $\mathfrak{A}^I{}_D | G$ such that the diagonal mapping is still an elementary embedding (see Lemma 3.5). We shall see in Theorem 3.7 below that this property is in a sense characteristic of the limit ultrapower operation. We now introduce the notion of a complete system, which was defined by Rabin in [27].

DEFINITION. $\mathfrak{A}$ is said to be *complete* if for every $n < \omega$ and every $R \subseteq A^n$, there exists $\lambda < \rho$ such that $R_\lambda = R$.

LEMMA 3.4. *Suppose* $\mathfrak{A}$ *is a complete system of type* μ. *Then*:

(i) $|2^A| \leqq |\rho|$;

(ii) *no proper extension of* $\mathfrak{A}$ *is isomorphic to* $\mathfrak{A}$;

(iii) *each elementary equivalent extension of* $\mathfrak{A}$ *is an elementary extension of* $\mathfrak{A}$;

(iv) *if* $\mathfrak{A} \equiv \mathfrak{B}$, *then* $\mathfrak{A} \prec \mathfrak{B}$.

Proof. (i) is obvious, and (ii)-(iv) follow easily from the fact that each one-element subset of A occurs among the relations $R_\lambda, \lambda < \rho$.

DEFINITION. We shall write $\phi : \mathfrak{A} \prec\prec \mathfrak{B}$ if there is a complete system $\mathfrak{A}'$ and a system $\mathfrak{B}'$ such that $\mathfrak{A}' \restriction \mu = \mathfrak{A}$, $\mathfrak{B}' \restriction \mu = \mathfrak{B}$, and $\phi : \mathfrak{A}' \prec \mathfrak{B}'$. We write $\mathfrak{A} \prec\prec \mathfrak{B}$ if we have $\phi : \mathfrak{A} \prec\prec \mathfrak{B}$ for some ϕ.

LEMMA 3.5. (i) *If* $\phi : \mathfrak{A} \prec\prec \mathfrak{B}$ *and* $\psi : \mathfrak{B} \prec\prec \mathfrak{C}$, *then* $\psi \circ \phi : \mathfrak{A} \prec\prec \mathfrak{C}$;

(ii) *if* $\phi : \mathfrak{A} \prec\prec \mathfrak{B}$, *then* $\phi : \mathfrak{A} \prec \mathfrak{B}$;

(iii) $\iota : \mathfrak{A} \prec\prec \mathfrak{A}$;

(iv) $d : \mathfrak{A} \prec\prec \mathfrak{A}^I{}_D | G$;

(v) *if* $\mathfrak{A}$ *is complete and* $\phi : \mathfrak{A} \prec \mathfrak{B}$, *then* $\phi : \mathfrak{A} \prec\prec \mathfrak{B}$.

Proof. Conditions (i)-(iii) and (v) follow at once from the definitions involved. (iv) follows from Corollary 3.3.

THEOREM 3.6. *If* $\mathfrak{A}^I{}_D | G$ *is a nontrivial limit ultrapower of* $\mathfrak{A}$, *then there is a nontrivial ultrapower* $\mathfrak{A}^J{}_E$ *of* $\mathfrak{A}$ *such that* $\mathfrak{A}^J{}_E \prec \mathfrak{A}^I{}_D | G$, $|J| \leqq |I|$, *and* $|J| \leqq |A|$.

Proof. Since $\mathfrak{A}^I{}_D | G$ is nontrivial, there exists $f \in A^I$ such that $\mathrm{eq}(f) \in G$ but f/D does not belong to the range of d. Let H be the principal filter on $I \times I$ generated by $\mathrm{eq}(f)$. Then $H \subseteq G$, and by Corollary 3.2, $\mathfrak{A}^I{}_D | H \prec \mathfrak{A}^I{}_D | G$. Let

$\mathfrak{A}'$ be a complete system such that $\mathfrak{A}' \upharpoonright \mu = \mathfrak{A}$. By Lemma 2.2, there is an ultrapower $\mathfrak{A}'^J_E$ of $\mathfrak{A}'$ such that $J = I/\mathrm{eq}(f)$ and $\mathfrak{A}'^J_E \cong \mathfrak{A}'^I_D \mid H$. Then $\mathfrak{A}^J_E \cong \mathfrak{A}^I_D \mid H$, and consequently $\mathfrak{A}^J_E \prec \mathfrak{A}^I_D \mid G$. Moreover, $|J| \leqq |I|, |A|$. It remains to show that $\mathfrak{A}^J_E$ is nontrivial. Since f/D is not in the range of d, $\mathfrak{A}'^I_D \mid H$ is isomorphic to a proper extension of $\mathfrak{A}'$, and hence by Lemma 3.4 (ii), $\mathfrak{A}'^I_D \mid H$ cannot be isomorphic to $\mathfrak{A}'$. It follows that $\mathfrak{A}'^J_E$ is not isomorphic to $\mathfrak{A}'$. Hence d maps A properly into A^J_E, and $\mathfrak{A}^J_E$ is a nontrivial ultrapower.

Theorem 3.7. *$\mathfrak{B}$ is isomorphic to a limit ultrapower of $\mathfrak{A}$ if and only if $\mathfrak{A} \prec\prec \mathfrak{B}$.*

Proof. If $\mathfrak{B} \cong \mathfrak{A}^I_D \mid G$, then $\mathfrak{A} \prec\prec \mathfrak{B}$ follows at once from Lemma 3.5 (iv).

Assume $\mathfrak{A} \prec\prec \mathfrak{B}$. Then there is a complete system $\mathfrak{A}'$ of type μ' and a system $\mathfrak{B}'$ such that $\mathfrak{A}' \upharpoonright \mu = \mathfrak{A}$, $\mathfrak{B}' \upharpoonright \mu = \mathfrak{B}$, and $\mathfrak{A}' \equiv \mathfrak{B}'$. By Theorem 1.6, there is an ultrapower $\mathfrak{A}'^I_D$ and an elementary embedding $\phi: \mathfrak{B}' \prec \mathfrak{A}'^I_D$. Let C be the range of ϕ. Since B is nonempty, $C \neq 0$. We shall apply Theorem 2.1. If A is finite, then $C = \mathfrak{A}^I_D$, and $\mathfrak{B}' \cong \mathfrak{A}'^I_D$. Suppose A is infinite. Let f/D, $g/D \in C$, $h \in A^I$, and $\mathrm{eq}(f) \cap \mathrm{eq}(g) \subseteq \mathrm{eq}(h)$. Then there exists a function $k \in A^{A \times A}$ such that $k(a, b) = c$ whenever $f(i) = a$, $g(i) = b$, and $h(i) = c$ for some $i \in I$. Let $R \subseteq A^3$ be defined by $R(a, b, c)$ if and only if $k(a, b) = c$. Since $\mathfrak{A}'$ is complete, $R = R_\lambda$ for some $\lambda < \rho'$. The sentence

$$\forall v_0, v_1 \; \exists v_2 \; \forall v_3 [P_\lambda(v_0, v_1, v_3) \leftrightarrow v_2 = v_3]$$

holds in $\mathfrak{A}'$, and therefore holds in $\mathfrak{A}'^I_D$, $\mathfrak{B}'$, and $\mathfrak{A}'^I_D \upharpoonright C$. Since

$$\{i \in I: R_\lambda(f(i), g(i), h(i))\} = I \in D,$$

we have $R_{\lambda D}(f/D, g/D, h/D)$. Therefore, since f/D, $g/D \in C$, we must have $h/D \in C$. By Theorem 2.1, $C = A^I_D \mid G$ for some filter G. Then $\phi: \mathfrak{B}' \cong \mathfrak{A}'^I_D \mid G$, and consequently $\phi: \mathfrak{B} \cong \mathfrak{A}^I_D \mid G$. This completes the proof.

Corollary 3.8. *If ϕ is an embedding of $\mathfrak{A}$ into $\mathfrak{B}$, then the following two conditions are equivalent:*

(i) $\phi: \mathfrak{A} \prec\prec \mathfrak{B}$;

(ii) *there is a limit ultrapower $\mathfrak{A}^I_D \mid G$ and an isomorphism $\psi: \mathfrak{B} \cong \mathfrak{A}^I_D \mid G$ such that $d = \psi \circ \phi$.*

Proof. By Lemma 3.5 (iv), we have (ii) implies (i).

Assuming (i), let $\mathfrak{A}'$ be a complete system such that $\mathfrak{A}' \upharpoonright \mu = \mathfrak{A}$, and let $\mathfrak{B}'$ be such that $\mathfrak{B}' \upharpoonright \mu = \mathfrak{B}$ and $\phi: \mathfrak{A}' \prec \mathfrak{B}'$. By Lemma 3.5 (v), $\phi: \mathfrak{A}' \prec\prec \mathfrak{B}'$. Hence, by Theorem 3.7, there is a $\psi: \mathfrak{B}' \cong \mathfrak{A}'^I_D \mid G$. Thus $\psi: \mathfrak{B} \cong \mathfrak{A}^I_D \mid G$. Moreover, since every one-element subset of A occurs among the relations of $\mathfrak{A}'$, we must have $d = \psi \circ \phi$.

We now give a characterization of elementary equivalence in terms of limit ultrapowers. We shall apply the following theorem proved by Robinson in [28, p. 54].

THEOREM 3.9. *For* $i = 0, 1$, *suppose* $\mu \subseteq \mu_i \in \omega^{\rho_i}$, $\sum_i$ *is a set of sentences of* $L(\mu_i)$, *and there is a system* $\mathfrak{B}_i$ *of type* μ_i *such that* $\sum_i$ *holds in* $\mathfrak{B}_i$ *and* $\mathfrak{B}_i \upharpoonright \mu \equiv \mathfrak{A}$. *Then, for* $i = 0, 1$, *there exist systems* $\mathfrak{C}_i$ *of type* μ_i *such that* $\sum_i$ *holds in* $\mathfrak{C}_i$, $\mathfrak{C}_i \upharpoonright \mu \equiv \mathfrak{A}$, *and* $\mathfrak{C}_0 \upharpoonright \mu = \mathfrak{C}_1 \upharpoonright \mu$.

THEOREM 3.10. $\mathfrak{A} \equiv \mathfrak{B}$ *if and only if there exist limit ultrapowers* $\mathfrak{A}^I{}_D \mid G$, $\mathfrak{B}^J{}_E \mid H$ *of cardinality* $\leqq |2^A| \cup |2^B|$ *such that* $\mathfrak{A}^I{}_D \mid G \cong \mathfrak{B}^J{}_E \mid H$(2).

Proof. If $\mathfrak{A}^I{}_D \mid G \cong \mathfrak{B}^J{}_E \mid H$ then $\mathfrak{A} \equiv \mathfrak{B}$ by Corollary 3.3.

Conversely suppose $\mathfrak{A} \equiv \mathfrak{B}$. We may assume without loss of generality that A is infinite (if A is finite then $\mathfrak{A} \cong \mathfrak{B}$), and for each $\lambda < \eta < \rho$, either $R_\lambda \neq R_\eta$ or $S_\lambda \neq S_\eta$. Then $|\rho| \leqq |2^A| \cup |2^B|$, for there are at most $|2^A|$ distinct relations on A and $|2^B|$ distinct relations on B. There is a complete system $\mathfrak{A}'$ of type μ' such that $|\rho'| = |2^A|$ and $\mathfrak{A} = \mathfrak{A}' \upharpoonright \mu$. Similarly there is a complete system $\mathfrak{B}''$ of type μ'' such that $|\rho''| = |2^B|$ and $\mathfrak{B} = \mathfrak{B}'' \upharpoonright \mu$.

By Theorem 3.9 there exist systems $\mathfrak{C}'$, $\mathfrak{C}''$ such that $\mathfrak{C}' \equiv \mathfrak{A}'$, $\mathfrak{C}'' \equiv \mathfrak{B}''$, and $\mathfrak{C}' \upharpoonright \mu = \mathfrak{C}'' \upharpoonright \mu$. Since

$$|\rho'| \cup |\rho''| \leqq |2^A| \cup |2^B|,$$

it follows from the Löwenheim-Skolem-Tarski Theorem that $\mathfrak{C}'$, $\mathfrak{C}''$ may be chosen to have cardinality $\leqq |2^A| \cup |2^B|$. By Theorem 3.7 and Lemmas 3.4 and 3.5, there are limit ultrapowers $\mathfrak{A}'^I{}_D \mid G \cong \mathfrak{C}'$, $\mathfrak{B}''^J{}_E \mid H \cong \mathfrak{C}''$. Therefore

$$\mathfrak{A}^I{}_D \mid G \cong \mathfrak{C}' \upharpoonright \mu = \mathfrak{C}'' \upharpoonright \mu \cong \mathfrak{B}^J{}_E \mid H,$$

and the proof is complete.

Theorem 3.10 has the following purely mathematical consequence.

COROLLARY 3.11. *If* $\mathfrak{A}, \mathfrak{B}$ *have any isomorphic limit ultrapowers, then they have isomorphic limit ultrapowers of cardinality* $\leqq |2^A| \cup |2^B|$.

Proof. If $\mathfrak{A}, \mathfrak{B}$ have some isomorphic limit ultrapowers, then by Corollary 3.2 we have $\mathfrak{A} \equiv \mathfrak{B}$. Then by Theorem 3.10 $\mathfrak{A}$ and $\mathfrak{B}$ have isomorphic limit ultrapowers of power $\leqq |2^A| \cup |2^B|$.

Frayne, Morel, and Scott have shown in [9] that a great many model-theoretic notions can be characterized solely in terms of elementary equivalence and ultraproducts. By combining Theorem 3.10 with the results of [9], we can at once obtain characterizations of each of these notions solely in terms of ultraproducts and limit ultrapowers. We can also take advantage of the bound given

(2) This result can also be proved by the repeated application of Theorem 1.6 instead of using Theorem 3.9.

in Theorem 3.10 on the cardinalities of the limit ultrapowers to obtain characterizations of model-theoretic notions relativized to systems of cardinality at most some fixed cardinal number. We shall not carry this program through in detail but instead shall present the single case of elementary classes as an illustration.

Let α be a cardinal number. We shall denote by $\mathbf{K} \upharpoonright \alpha$ the class $\{\mathfrak{A} \in \mathbf{K} : |A| < \alpha\}$. We shall write $\mathbf{K} \in EC \upharpoonright \alpha$ if and only if $\mathbf{K} \upharpoonright \alpha = \mathbf{L} \upharpoonright \alpha$ for some $\mathbf{L} \in EC$.

Part (a) of the following theorem is proved in [9], while part (b) is implicit therein.

THEOREM 3.12. (a) $\mathbf{K} \in EC$ *if and only if both* $\mathbf{K}$ *and* $\overline{\mathbf{K}}$ *are closed under elementary equivalence and ultraproducts.*

(b) *Suppose* $\alpha > |2^{\omega \cup \rho}|$. *Then* $\mathbf{K} \in EC \upharpoonright \alpha$ *if and only if* $\mathbf{K} \upharpoonright \alpha = \{\mathfrak{A} : |A| < \alpha$ *and* $\mathfrak{A} \equiv \mathfrak{B}$ *for some* $\mathfrak{B} \in \mathbf{K} \upharpoonright \alpha\}$ *and both* $\mathbf{K} \upharpoonright \alpha$ *and* $\overline{\mathbf{K}} \upharpoonright \alpha$ *are closed under ultraproducts of power* $< \alpha$.

By combining Theorems 3.10 and 3.12 we obtain the following characterization of elementary classes.

COROLLARY 3.13. (a) $\mathbf{K} \in EC$ *if and only if both* $\mathbf{K}$ *and* $\overline{\mathbf{K}}$ *are closed under isomorphisms, ultraproducts, and limit ultrapowers.*

(b) *Suppose* $\alpha > \omega \cup \rho$ *and that* $\beta < \alpha$ *implies* $|2^\beta| < \alpha$(3). *Then* $\mathbf{K} \in EC \upharpoonright \alpha$ *if and only if both* $\mathbf{K} \upharpoonright \alpha$ *and* $\overline{\mathbf{K}} \upharpoonright \alpha$ *are closed under isomorphisms, ultraproducts of cardinality* $< \alpha$, *and limit ultrapowers of cardinality* $< \alpha$.

4. **On a theorem of Rabin.** In this section we shall apply the results of §3 to obtain an improvement of the following theorem of Rabin [27]. Let $|A| > 1$.

RABIN'S THEOREM. *If* $|A^\omega| = |A|$, *then* $\mathfrak{A}$ *has a proper elementary extension of power* $|A|$. *Conversely, if* $\mathfrak{A}$ *is complete,* $|A^\omega| > |A|$, *and we make the two additional hypotheses*:

(R1) *the generalized continuum hypothesis*;

(R2) $|A|$ *is accessible from* ω(4);

then $\mathfrak{A}$ *does not have a proper elementary extension of power* $|A|$.

We shall show that Rabin's Theorem is still valid when the hypothesis (R1) is removed and (R2) is considerably weakened. Our proof will be essentially different from the original proof given by Rabin. We shall begin with two theorems which are set-theoretical in nature and concern cardinalities of ultrapowers.

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(3) Cardinals α such that $\beta < \alpha$ imply $|2^\beta| < \alpha$ are sometimes called strong limit cardinals. It is well known that there are arbitrarily large strong limit cardinals. For example, on page 9 of [31], all the cardinals $\aleph_{\pi(\xi)}$ with ξ a limit ordinal are obviously strong limit cardinals.

(4) A cardinal α is said to be (weakly) accessible from ω if, whenever $\omega < \beta \leqq \alpha$, there is a $\gamma < \beta$ such that either $\beta \leqq |2^\gamma|$ or $\beta < |\beta^\gamma|$. For details see [34].

THEOREM 4.1. *Let A, I, and J be infinite sets. Let D be an ultrafilter on I such that for some function ϕ on I into $S_\omega(J)$ and each $j \in J$,*

$$\hat{\jmath} = \{i \in I : j \in \phi(i)\} \in D.$$

Then $|A^I{}_D| \geqq |A^J|$ (5).

Proof. Since A is infinite, there is a sequence $\langle A_n \rangle_{n<\omega}$ of disjoint sets $A_n \subseteq A$ such that $|A_n| = |A|$ for each $n < \omega$. For each $n < \omega$, let ψ_n be a one-to-one function on A^n into A_n. Let $\langle j_\xi \rangle_{\xi < |J|}$ be a well ordering of the set J. If

$$\xi(1) < \xi(2) < \cdots < \xi(n) < |J| \text{ and } s = \{j_{\xi(1)}, \cdots, j_{\xi(n)}\} \in S_\omega(J),$$

and if $f \in A^J$, then let

$$f^*(s) = \langle f(j_{\xi(1)}), \cdots, f(j_{\xi(n)}) \rangle.$$

Thus $f^*(s) \in A^n$. Define the function θ of A^J into A^I as follows:

For each $f \in A^J$ and $i \in I$, $(\theta f)(i) = \psi_{|\phi(i)|}(f^*(\phi(i)))$.

Suppose $f, g \in A^J$ and $f \neq g$. Then $f(j) \neq g(j)$ for some $j \in J$. It follows that for every $i \in \hat{\jmath}$, we have $f^*(\phi(i)) \neq g^*(\phi(i))$, and therefore $(\theta f)(i) \neq (\theta g)(i)$. Since $\hat{\jmath} \in D$, we have $\theta f/D \neq \theta g/D$. Consequently the function θ' on A^J into $A^I{}_D$ defined by $\theta' f = \theta f/D$ is one-to-one, and $|A^J| \leqq |A^I{}_D|$.

COROLLARY 4.2. *For every ultrafilter D on I which is not countably complete, and every infinite set A,* $|A^I{}_D| \geqq |A^\omega|$.

Proof. Since D is not countably complete, there exists a sequence $\langle x_n \rangle_{n<\omega}$ of disjoint subsets of I such that $\bigcup_{n<\omega} x_n = I$ but $x_n \notin D$ for each $n < \omega$. Let ϕ be the function on I into $S_\omega(\omega)$ such that $\phi(i) = \{0, \cdots, n\}$ whenever $i \in x_n$. For each $n < \omega$, we have, in the notation of Theorem 4.1, that $\hat{n} = \bigcup_{n \leqq m < \omega} x_m$, and since $x_0 \cup \cdots \cup x_{n-1} \notin D$, we conclude that $\hat{n} \in D$. It then follows from Theorem 4.1 that $|A^I{}_D| \geqq |A^\omega|$.

We shall say that a cardinal α is *nonmeasurable* if every countably complete ultrafilter on a set of power α is principal. In the language of measure theory, this says that every countably additive two-valued measure on the field of all subsets of a set of power α is trivial in the sense that there is a one-element set

(5) Assume the hypotheses of Theorem 4.1. The weaker result that $|A^I{}_D| \geqq |2^\omega|$, and also the fact that A has arbitrarily large ultrapowers, were stated in [8]. The result that $|A^I{}_D| > |A|$ if $|A|$ is confinal with ω was stated in [14]. D. Monk proved that $|A^I{}_D| \geqq |2^I|$. Theorem 4.1 is an improvement of all these results, and our method of proof was motivated by Monk's argument. Monk has pointed out that his proof was in turn motivated by the proof of a related result by Kochen in [20, p. 14]. A subsequent generalization of Theorem 4.1 has been obtained by Chang and is stated in [9].

which has positive measure[6]. It is known that every cardinal number which is accessible from ω is nonmeasurable (see [39]). Moreover, according to some recent results (see [37]), the condition that α is nonmeasurable is much weaker than hypothesis (R2), unless every cardinal number is accessible from ω. That is, the class of nonmeasurable cardinals not only contains all cardinals which are accessible from ω, but also extends far into the heirarchy of inaccessible cardinals. In fact, it has recently been shown by Scott in [30] that the axiom of constructibility implies that every cardinal is nonmeasurable.

The following simple lemma is known from the literature (see [39]).

LEMMA 4.3. *If the cardinal number α is nonmeasurable, then so is any cardinal $\beta < \alpha$.*

Proof. Suppose β has a nonprincipal countably complete ultrafilter E. Let $D = \{J \subseteq \alpha : J \cap \beta \in E\}$. Then D is clearly a nonprincipal countably complete ultrafilter on α, contradicting the assumption that α is nonmeasurable. Hence β is nonmeasurable.

THEOREM 4.4. *Let A be an infinite set and suppose $|A|$ is nonmeasurable. If $\mathfrak{A}^I{}_D | G$ is a nontrivial limit ultrapower of $\mathfrak{A}$, then $|A^I{}_D | G| \geqq |A^\omega|$.*

Proof. By Theorem 3.6, there is a nontrivial ultrapower $\mathfrak{A}^J{}_E$ of $\mathfrak{A}$ such that $\mathfrak{A}^J{}_E \prec \mathfrak{A}^I{}_D | G$ and $|J| \leqq |A|$. By 4.3, $|J|$ is nonmeasurable. Since $\mathfrak{A}^J{}_E$ is nontrivial, E is nonprincipal, and therefore E is not countably complete. By 4.2, $|A^J{}_E| \geqq |A^\omega|$, and therefore $|A^I{}_D | G| \geqq |A^\omega|$.

THEOREM 4.5. *Suppose $\mathfrak{A}$ is a complete system and $|A|$ is nonmeasurable. Then $\mathfrak{A}$ has a proper elementary extension of cardinality $|A|$ if and only if $|A| = |A^\omega|$.*

Proof. If A is finite, then $\mathfrak{A}$ has no proper elementary extension and $|A| < |A^\omega|$.

If $|A| = |A^\omega|$, then $\mathfrak{A}$ has a proper elementary extension of power $|A|$ by the first part of Rabin's Theorem.

Suppose A is infinite and $|A| < |A^\omega|$. By Corollary 3.8 and Lemma 3.5 (v), any proper elementary extension $\mathfrak{B}$ of $\mathfrak{A}$ is isomorphic to a nontrivial limit ultrapower of $\mathfrak{A}$. It follows from Theorem 4.4 that $|B| \geqq |A^\omega|$, and hence also $|B| > |A|$.

(6) In [39] and elsewhere, the term measurable cardinal has been used in a different sense than we use it here; namely, a cardinal α is said to be measurable if there is a countably additive real valued nontrival measure on the field of all subsets of a set of power α. However, in [39] it is proved that the continuum hypothesis implies that a cardinal α is measurable in our sense if and only if it is measurable in the sense of [39]. In [30] the cardinals which are measurable in our sense are called two-valued measurable.

Corollary 4.6. *Suppose $\mathfrak{A}$ is a complete system and $|A|$ is nonmeasurable. If $\mathfrak{B}$ is a proper elementary extension of $\mathfrak{A}$, then $|B| \geqq |A^\omega|$.*

Proof. Since $\mathfrak{B}$ is isomorphic to a proper elementary extension of $\mathfrak{A}$, $\mathfrak{A}$ must be infinite. By Corollary 3.8 and Lemma 3.5 (v), $\mathfrak{B}$ is isomorphic to a nontrivial limit ultrapower of $\mathfrak{A}$. Then $|B| \geqq |A^\omega|$ by Theorem 4.4.

The situation has not been completely cleared up in the case that $\mathfrak{A}$ is measurable. However, the following theorem provides an example showing that, if there are any measurable cardinals, then we cannot simply remove the hypothesis "$|A|$ is nonmeasurable" from Theorem 4.5.

Theorem 4.7. *Suppose α is a measurable cardinal, $\beta_n = |\beta_n{}^\alpha|$ and $\beta_n < \beta_{n+1}$ for each $n < \omega$, and $\beta = \bigcup_{n<\omega}\beta_n$. Then β is a cardinal number such that $\beta < |\beta^\omega|$ but every system $\mathfrak{A}$ of cardinality β has a proper elementary extension of cardinality β.*

Proof. Since β is a union of cardinal numbers, β itself is a cardinal number. The inequality $\beta < |\beta^\omega|$ follows, e.g., by König's Theorem (see [21]). We also have $\alpha < \beta$, because $\alpha < |2^\alpha| \leqq |\beta_2{}^\alpha| = \beta_2 < \beta$.

We may suppose without loss of generality that $A = \beta$. Let D be a nonprincipal countably complete ultrafilter on α. Consider the ultrapower $\mathfrak{A}^\alpha{}_D$. By Theorem 1.2, $d : \mathfrak{A} \prec \mathfrak{A}^\alpha{}_D$. Moreover, we have $d(A) \neq A^\alpha{}_D$, because it follows from the inequality $\alpha \leqq \beta$ that there is a one-to-one function $f \in A^\alpha$, and since D is nonprincipal we have $f/D \notin d(A)$. Therefore $\mathfrak{A}^\alpha{}_D$ is isomorphic to a proper elementary extension of $\mathfrak{A}$.

It remains to prove that $|A^\alpha{}_D| \leqq \beta$. For each $n < \omega$, we have $|(\beta_n)^\alpha{}_D| = \beta_n$, because $|\beta_n{}^\alpha| = \beta_n$. Consider any $f \in A^\alpha$, and for each $n < \omega$ let

$$X_n = \{\xi < \alpha : f(\xi) \in \beta_n\}.$$

Since D is a countably complete ultrafilter and $\bigcup_{n<\omega} X_n = \alpha$, we must have $X_n \in D$ for some $n < \omega$. It follows that $f/D = g/D$ for some $g \in \beta_n{}^\alpha$. Therefore

$$|A^\alpha{}_D| \leqq |\bigcup\{(\beta_n)^\alpha{}_D : n < \omega\}|.$$

However, the right hand side of this inequality is equal to β, because $|(\beta_n)^\alpha{}_D| = \beta_n$ for each $n < \omega$ and $\beta = \bigcup_{n<\omega}\beta_n$. Hence $|A^\alpha{}_D| \leqq \beta$, and the proof is complete.

The following lemma is proved (much more generally) in [39].

Lemma 4.8. *If α is the smallest measurable cardinal, then $|\alpha^\omega| = \alpha$.*

A cardinal α is said to be *confinal with ω* if there is a sequence $\langle \gamma_n \rangle_{n<\omega}$ of cardinals such that $\gamma_n < \gamma_{n+1}$ for each n, and $\alpha = \bigcup_{n<\omega}\gamma_n$.

The next lemma is proved (much more generally) in [31, p. 7].

Lemma 4.9. *If α is not confinal with ω, then $|\alpha^\omega| = \bigcup_{\beta<\alpha}|\beta^\omega|$.*

If we assume that the generalized continuum hypothesis holds for all *measurable* cardinals, then we shall always be able to answer the question of which complete systems $\mathfrak{A}$ have proper elementary extensions of power $|A|$. Theorem 4.5 gives the answer, without any continuum hypothesis, in case $|A|$ is nonmeasurable. The following theorem provides the answer in case $|A|$ is measurable.

THEOREM 4.10. *Assume $\alpha^+ = |2^\alpha|$ holds for every measurable cardinal number α. Then every system $\mathfrak{A}$ such that $|A|$ is measurable has a proper elementary extension of power $|A|$.*

Proof. Let α be the smallest measurable cardinal, and let $\beta = |A|$. If $\beta = |\beta^\omega|$, then $\mathfrak{A}$ has a proper elementary extension of power $|A|$ by the first half of Rabin's Theorem. Suppose $\beta < |\beta^\omega|$. By Lemma 4.8, $\beta \neq \alpha$, and hence $\alpha < \beta$. From Lemma 4.3 we see that $\gamma^+ = |2^\gamma|$ holds for every $\gamma \geqq \alpha$. For each $\gamma < \beta$, we then have

$$|\gamma^\omega| \leqq |(\gamma \cup \alpha)^\omega| \leqq |(\gamma \cup \alpha)^{(\gamma\cup\alpha)}| = (\gamma \cup \alpha)^+ \leqq \beta.$$

Therefore $\bigcup_{\gamma<\beta} |\gamma^\omega| \leqq \beta$, and since $\beta < |\beta^\omega|$, it follows that

$$|\beta^\omega| > \bigcup_{\gamma<\beta} |\gamma^\omega|.$$

By Lemma 4.9 we conclude that β is confinal with ω. Let $\langle \gamma_n \rangle_{n<\omega}$ be a sequence of cardinals such that $\gamma_n < \gamma_{n+1}$ for each n and $\beta = \bigcup_{n<\omega} \gamma_n$. Then for some $m < \omega$ we have $\alpha < \gamma_m$. For each $n < \omega$, define $\beta_n = (\gamma_{m+n})^+$. Then all of the hypotheses of Theorem 4.7 are satisfied, and therefore $\mathfrak{A}$ has a proper elementary extension of power β.

5. **Strong limit ultrapowers.** $\mathfrak{B}$ is said to be a *strong extension of* $\mathfrak{A}$ if there is an ultrapower $\mathfrak{A}^I{}_D$ and an isomorphism ϕ of $\mathfrak{A}^I{}_D$ onto $\mathfrak{B}$ such that $\phi \circ d$ is the identity function on A. It follows that any strong extension of $\mathfrak{A}$ is an extension of $\mathfrak{A}$. By Theorem 1.7, any strong extension of a strong extension of $\mathfrak{A}$ is itself a strong extension of $\mathfrak{A}$.

$\mathfrak{B}$ is said to be a *strong limit ultrapower of* $\mathfrak{A}$ if there is a sequence $\langle \mathfrak{A}_n \rangle_{n<\omega}$ such that $\mathfrak{A}_0 = \mathfrak{A}$, each $\mathfrak{A}_{n+1}$ is a strong extension of $\mathfrak{A}_n$, and $\mathfrak{B} = \bigcup \{\mathfrak{A}_n : n < \omega\}$(7).

THEOREM 5.1. *Any strong limit ultrapower of $\mathfrak{A}$ is isomorphic to a limit ultrapower of $\mathfrak{A}$.*

Proof. In view of Theorem 3.7, it suffices to prove that if $\mathfrak{B}$ is a strong limit ultrapower of $\mathfrak{A}$, then $\mathfrak{A} \prec\prec \mathfrak{B}$. Consider a sequence $\langle \mathfrak{A}_n \rangle_{n<\omega}$ such that $\mathfrak{A}_0 = \mathfrak{A}$, each $\mathfrak{A}_{n+1}$ is a strong extension of $\mathfrak{A}_n$, and $\mathfrak{B} = \bigcup_{n<\omega} \mathfrak{A}_n$. For each $n < \omega$, choose J_n, E_n, θ_n such that

$$\theta_n : \mathfrak{A}_n{}^{J_n}{}_{E_n} \cong \mathfrak{A}_{n+1} \text{ and } \theta_n \circ d = \iota.$$

(7) The author is indebted to T. Frayne for suggesting this elegant form of the definition of strong limit ultrapower, which considerably simplifies an earlier formulation.

Let $\mathfrak{A}'_0$ be a complete system such that $\mathfrak{A}'_0 \restriction \mu = A$. For each $n < \omega$, let $\mathfrak{A}'_{n+1}$ be the system defined by the condition

$$\theta_n : \mathfrak{A}_n'^{J_n}{}_{E_n} \cong \mathfrak{A}'_{n+1},$$

and let $\mathfrak{B}' = \bigcup_{n<\omega} \mathfrak{A}'_n$. Then clearly $\mathfrak{B}' \restriction \mu = \mathfrak{B}$. By Theorem 1.2, we have, since $\theta \circ d = \iota$, that $\iota : \mathfrak{A}'_n \prec \mathfrak{A}'_{n+1}$ for each n. Thus by Theorem 1.1 it follows that $\iota : \mathfrak{A}'_0 \prec \mathfrak{B}'$, and therefore $\mathfrak{A} \prec\prec \mathfrak{B}$.

One of the main reasons for considering strong limit ultrapowers is the following theorem, which is another characterization of elementary equivalence analogous to Theorem 3.10.

THEOREM 5.2(8). *$\mathfrak{A} \equiv \mathfrak{B}$ if and only if there is a strong limit ultrapower of $\mathfrak{A}$ which is isomorphic to a strong limit ultrapower of $\mathfrak{B}$.*

Proof. If $\mathfrak{A}'$ is a strong limit ultrapower of $\mathfrak{A}$, $\mathfrak{B}'$ is a strong limit ultrapower of $\mathfrak{B}$, and $\mathfrak{A}' \cong \mathfrak{B}'$, then by Theorem 5.1 and Corollary 3.3, $\mathfrak{A} \equiv \mathfrak{B}$.

Now suppose $\mathfrak{A} \equiv \mathfrak{B}$. By Theorem 1.6 (a) there is an ultrapower $\mathfrak{A}^I{}_D$ and an elementary embedding $\theta : \mathfrak{B} \prec \mathfrak{A}^I{}_D$. It follows that there is a strong extension $\mathfrak{A}_1$ of $\mathfrak{A}$ and an elementary embedding $\psi_0 : \mathfrak{B} \prec \mathfrak{A}_1$. By Theorem 1.6 (b) there is an ultrapower $\mathfrak{B}^J{}_E$ and an elementary embedding $\theta' : \mathfrak{A}_1 \prec \mathfrak{B}^J{}_E$ such that $\theta' \circ \psi_0 = d$. It follows that there is a strong extension $\mathfrak{B}_1$ of $\mathfrak{B}$ and an elementary embedding $\phi_0 : \mathfrak{A}_1 \prec \mathfrak{B}_1$ such that $\phi_0 \circ \psi_0 = \iota$. By repeatedly applying Theorem 1.6 (b) in this way we obtain a commutative diagram

$$\begin{tikzcd}
\mathfrak{A} = \mathfrak{A}_0 \arrow[r, "\iota"] & \mathfrak{A}_1 \arrow[r, "\iota"] \arrow[d, "\phi_0"] & \mathfrak{A}_2 \cdots, \arrow[d, "\phi_1"] \\
\mathfrak{B} = \mathfrak{B}_0 \arrow[r, "\iota"] \arrow[ur, "\psi_0"] & \mathfrak{B}_1 \arrow[r, "\iota"] \arrow[ur, "\psi_1"] & \mathfrak{B}_2 \cdots,
\end{tikzcd}$$

such that for each $n < \omega$, $\mathfrak{A}_{n+1}$ and $\mathfrak{B}_{n+1}$ are strong extensions of $\mathfrak{A}_n$ and $\mathfrak{B}_n$ respectively, and $\phi_n : \mathfrak{A}_{n+1} \prec \mathfrak{B}_{n+1}$ and $\psi_n : \mathfrak{B}_n \prec \mathfrak{A}_{n+1}$. It follows that $\bigcup_{n<\omega} \phi_n$ is an isomorphism of $\mathfrak{A}_\omega = \bigcup_{n<\omega} \mathfrak{A}_n$ onto $\mathfrak{B}_\omega = \bigcup_{n<\omega} \mathfrak{B}_n$; moreover, $\mathfrak{A}_\omega$ and $\mathfrak{B}_\omega$ are strong limit ultrapowers of $\mathfrak{A}$ and $\mathfrak{B}$. This completes the proof.

In Theorem 5.2 we do not obtain as small a bound on the cardinality of the strong limit ultrapowers as we did for the limit ultrapowers in Theorem 3.10.

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(8) The statement of this theorem as originally given by the author was rather complicated, due to an awkward definition of strong limit ultrapower. However, Frayne and Kochen soon discovered more elegant formulations. Kochen's formulation involves the notion of direct limits of relational systems (see [19; 20]). We adopt here the formulation suggested by Frayne, which was also used in [13].

The proofs of Theorems 3.10 and 5.2 both depend ultimately on Theorem 1.6, but are otherwise independent of each other. As we have already mentioned, the results of [9] permit us to obtain characterizations of many model-theoretic notions whenever we have a characterization, such as Theorem 5.2, of elementary equivalence. This program is carried out by Kochen in [20], who goes considerably beyond what is indicated in [9], and also improves Theorem 5.2.

However, it can be seen from the proof of Theorem 5.2 that, if $\mathfrak{A} \equiv \mathfrak{B}$, then $\mathfrak{A}$ and $\mathfrak{B}$ have strong limit ultrapowers which are isomorphic and have cardinality at most

$$\left|2^{A \cup B}\right| \cup \left|2^{2^{A \cup B}}\right| \cup \left|2^{2^{2^{A \cup B}}}\right| \cup \cdots .$$

6. **Complementary examples.** For any system $\mathfrak{A}$, we may consider the following classes of systems:

$$\mathbf{K}_0(\mathfrak{A}) = \{\mathfrak{B} : \mathfrak{B} \cong \mathfrak{A}\};$$

$$\mathbf{K}_1(\mathfrak{A}) = \{\mathfrak{B} : \mathfrak{B} \text{ is isomorphic to an ultrapower of } \mathfrak{A}\};$$

$$\mathbf{K}_2(\mathfrak{A}) = \{\mathfrak{B} : \mathfrak{B} \text{ is isomorphic to a strong limit ultrapower of } \mathfrak{A}\};$$

$$\mathbf{K}_3(\mathfrak{A}) = \{\mathfrak{B} : \mathfrak{A} \prec\prec \mathfrak{B}\};$$

$$\mathbf{K}_4(\mathfrak{A}) = \{\mathfrak{B} : \mathfrak{A} \prec \mathfrak{B}\}.$$

Recall that by Theorem 3.7, $\mathbf{K}_3(\mathfrak{A})$ is just the class of all systems $\mathfrak{B}$ which are isomorphic to a limit ultrapower of $\mathfrak{A}$. For each system $\mathfrak{A}$, we have the inclusions:

$$\mathbf{K}_0(\mathfrak{A}) \subseteq \mathbf{K}_1(\mathfrak{A}) \subseteq \mathbf{K}_2(\mathfrak{A}) \subseteq \mathbf{K}_3(\mathfrak{A}) \subseteq \mathbf{K}_4(\mathfrak{A}).$$

$\mathbf{K}_0(\mathfrak{A}) \subseteq \mathbf{K}_1(\mathfrak{A})$ because $\mathfrak{A} \cong \mathfrak{A}^I{}_D$ whenever D is a principal ultrafilter, by Theorem 1.5. $\mathbf{K}_1(\mathfrak{A}) \subseteq \mathbf{K}_2(\mathfrak{A})$ because if $\mathfrak{B} \in \mathbf{K}_1(\mathfrak{A})$, then $\mathfrak{B} \cong \mathfrak{B}'$ for some strong extension $\mathfrak{B}'$ of $\mathfrak{A}$, and since $\mathfrak{B}' = \mathfrak{A} \cup \bigcup_{1<n<\omega} \mathfrak{B}'$, $\mathfrak{B}'$ is a strong limit ultrapower of $\mathfrak{A}$, so $\mathfrak{B} \in \mathbf{K}_2(\mathfrak{A})$. Theorem 5.1 tells us that $\mathbf{K}_2(\mathfrak{A}) \subseteq \mathbf{K}_3(\mathfrak{A})$. Finally, Lemma 3.5 (ii) states that $\mathbf{K}_3(\mathfrak{A}) \subseteq \mathbf{K}_4(\mathfrak{A})$.

In this section we shall give examples showing that for a particular system $\mathfrak{A}$ all these inclusions are proper. Thus the notions of ultrapower, strong limit ultrapower, limit ultrapower, and elementary extension are all essentially different from each other.

Let $\mathfrak{A}_0 = \langle A,R \rangle$ be the particular relational system in which A is the set of rational numbers and R is the usual order relation on them. Thus $\mathfrak{A}_0$ is of type $\langle 2 \rangle$.

THEOREM 6.1. $\mathbf{K}_0(\mathfrak{A}_0) \neq \mathbf{K}_1(\mathfrak{A}_0)$(9).

Proof. Let D be a nonprincipal ultrafilter on ω, and let $\mathfrak{B} = \mathfrak{A}_0{}^\omega{}_D$. Then $\mathfrak{B} \in \mathbf{K}_1(\mathfrak{A}_0)$. By Theorem 4.1, $|B| \geqq |A^\omega|$. Since $|A| = \omega < |A^\omega|$, $|B| \neq |A|$ and hence $\mathfrak{B} \notin \mathbf{K}_0(\mathfrak{A}_0)$.

Let $\mathfrak{B} = \langle B,S \rangle \equiv \mathfrak{A}_0$. We shall say that $\mathfrak{B}$ is *confinal with* ω if there is a sequence $\langle b_n \rangle_{n<\omega} \in B^\omega$ such that for each $b \in B$ there is an m for which $S(b,b_m)$. In particular, $\mathfrak{A}_0$ is confinal with ω.

(9) This is a known result for any infinite system $\mathfrak{A}$ (see [8]). We include this theorem here for comparison with the results which follow.

THEOREM 6.2. $\mathbf{K}_1(\mathfrak{A}_0) \neq \mathbf{K}_2(\mathfrak{A}_0)$.

Proof. We shall first prove that if $\mathfrak{B} \in \mathbf{K}_1(\mathfrak{A}_0)$, then either $\mathfrak{B} \cong \mathfrak{A}_0$ or $\mathfrak{B}$ is not confinal with ω. Suppose $\mathfrak{B} \cong \mathfrak{A}_0$ fails. Then $\mathfrak{B} \cong \mathfrak{A}_0{}^I{}_D$ where $\mathfrak{A}_0{}^I{}_D$ is a nontrivial ultrapower. Let $f_0/D,\ f_1/D,\ f_2/D, \cdots \in A^I{}_D$. Since $|A| = \omega$ and $\mathfrak{A}_0{}^I{}_D$ is nontrivial, D is not countably complete, by Theorem 1.4. Hence we can choose $X_0, X_1, X_2, \cdots \in D$ such that $X_{n+1} \subseteq X_n$ for each n and $\bigcap_{n<\omega} X_n = 0$. There exists a function $g \in A^I$ such that for each $n < \omega$ and $i \in X_n$, we have

$$R(f_0(i),g(i)),\ R(f_1(i),g(i)), \cdots, R(f_n(i),g(i)).$$

It follows that for each $n < \omega$,

$$\{i \in I : R(f_n(i),g(i))\} \supseteq X_n \in D,$$

and hence $R_D(f_n/D,g/D)$. Therefore $\mathfrak{A}_0{}^I{}_D$ and $\mathfrak{B}$ are not confinal with ω.

We shall now construct a strong limit ultrapower of $\mathfrak{A}_0$ which is not isomorphic to $\mathfrak{A}_0$ but is confinal with ω. We first show that any system $\mathfrak{A}' = \langle A', R' \rangle$, $\mathfrak{A}' \equiv \mathfrak{A}_0$, has a strong extension $\mathfrak{A}'' = \langle A'', R'' \rangle$ such that A' has an upper bound b in $\mathfrak{A}''$, that is, $R''(a,b)$ for every $a \in A'$. Let D be an ultrafilter on A' such that

$$\{b \in A' : R'(a,b)\} \in D \text{ for each } a \in A'.$$

Let $g \in A'^{A'}$ be the identity function on A', and for each $a \in A'$ let $f_a \in A'^{A'}$ be the constant function at a. Then for each $a \in A'$ we have

$$\{b \in A' : R'(f_a(b),g(b))\} = \{b \in A' : R'(a,b)\} \in D,$$

and therefore $R'_D(da, g/D)$. Let $\mathfrak{A}''$ be an extension of $\mathfrak{A}'$ for which there is an isomorphism ϕ of $\mathfrak{A}'^{A'}{}_D$ onto $\mathfrak{A}''$ such that $\phi \circ d$ is the identity. Then $\mathfrak{A}''$ is a strong extension of $\mathfrak{A}'$ and $\phi(g/D)$ is an upper bound of A' in $\mathfrak{A}''$.

We may construct a sequence $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \cdots$ such that, for each $n < \omega$, $\mathfrak{A}_{n+1}$ is a strong extension of $\mathfrak{A}_n$ and A_n has an upper bound b_n in $\mathfrak{A}_{n+1}$. Let

$$\mathfrak{B} = \bigcup_{n<\omega} \mathfrak{A}_n = \langle B, S \rangle.$$

Then $\mathfrak{B} \in \mathbf{K}_2(\mathfrak{A}_0)$. However, for every $b \in B$ there is an m such that $b \in A_m$ and thus $S(b,b_m)$. Hence $\mathfrak{B}$ is confinal with ω. Since A has an upper bound in $\mathfrak{A}_1$ but not in $\mathfrak{A}_0$, $\mathfrak{A}_1$ is a proper strong extension of $\mathfrak{A}_0$, and by Theorem 4.1, $|B| \geqq |A_1| > \omega$. Therefore $\mathfrak{B} \cong \mathfrak{A}_0$ fails, and $\mathfrak{B} \notin \mathbf{K}_1(\mathfrak{A}_0)$.

THEOREM 6.3. *If* $\omega^+ = |2^\omega|$, *then* $\mathbf{K}_2(\mathfrak{A}_0) \neq \mathbf{K}_3(\mathfrak{A}_0)$.

Proof. We first construct a system $\mathfrak{B} \in \mathbf{K}_3(\mathfrak{A}_0)$ such that $\mathfrak{B}$ is confinal with ω and $|B| = \omega^+$. Let $\mathfrak{A}'_0$ be a complete system whose $\langle 2 \rangle$-reduct is $\mathfrak{A}_0$, and which has ω^+ relations (this is possible because we are assuming the continuum hypothesis). Using the Compactness Theorem and the Löwenheim-Skolem-Tarski theorem, we can easily obtain a sequence $\mathfrak{A}'_0, \mathfrak{A}'_1, \mathfrak{A}'_2, \cdots$ such that, for each $n < \omega$,

$\mathfrak{A}'_{n+1}$ is an elementary extension of $\mathfrak{A}'_n$, A'_n has an upper bound b_n in the $\langle 2\rangle$-reduct $\mathfrak{A}_{n+1}$ of $\mathfrak{A}'_{n+1}$, and $|A'_{n+1}| = \omega^+$. Let $\mathfrak{B}' = \bigcup_{n<\omega}\mathfrak{A}'_n$ and let $\mathfrak{B}$ be the $\langle 2\rangle$-reduct of $\mathfrak{B}'$. For each $b \in B$ there is an m such that $b \in A_m$ and hence $S(b,b_m)$. Therefore $\mathfrak{B}$ is confinal with ω. By Theorem 1.1, $\mathfrak{B}'$ is an elementary extension of $\mathfrak{A}'_0$. Consequently we have $\mathfrak{A}_0 \prec\prec \mathfrak{B}$, and $\mathfrak{B} \in \mathbf{K}_3(\mathfrak{A}_0)$. Finally, $|B| = \omega^+$ because $|A_{n+1}| = \omega^+$ for each n.

We now show that $\mathfrak{B} \notin \mathbf{K}_2(\mathfrak{A}_0)$. Suppose on the contrary that $\mathfrak{B} \in \mathbf{K}_2(\mathfrak{A}_0)$. Then there is a strong limit ultrapower $\mathfrak{C} = \langle C,T\rangle$ of $\mathfrak{A}_0$ such that $\mathfrak{B} \cong \mathfrak{C}$. There is a sequence $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \cdots$ such that $\mathfrak{C} = \bigcup_{n<\omega}\mathfrak{A}_n$ and, for each n, $\mathfrak{A}_{n+1}$ is a strong extension of $\mathfrak{A}_n$. Since $|C| = \omega^+$, we have $|A_p| = \omega^+$ for some $p < \omega$. We shall show that, whenever $p \leqq m < \omega$, A_p is not bounded above in $\mathfrak{A}_m$.

Suppose $p \leqq m < \omega$ and b is an upper bound of A_p in $\mathfrak{A}_m$. Since $\mathfrak{A}_m$ is a strong extension of $\mathfrak{A}_p$, there is an ultrapower $(\mathfrak{A}_p)^I{}_D$ and an isomorphism ϕ of $(\mathfrak{A}_p)^I{}_D$ onto $\mathfrak{A}_m$ such that $\phi \circ d$ is the identity function on A_p. For some $g \in (A_p)^I$, we have $\phi(g/D) = b$. $\mathfrak{A}_p \in \mathbf{K}_1(\mathfrak{A}_0) - \mathbf{K}_0(\mathfrak{A}_0)$, so by the first paragraph of the proof of Theorem 6.2, $\mathfrak{A}_p$ is not confinal with ω. It follows that there is an $a \in (A_p)^{\omega^+}$ such that the range of a is not bounded above in $\mathfrak{A}_p$ and $R_p(a_\zeta,a_\xi)$ whenever $\zeta < \xi < \omega^+$; that is, the range of a is an unbounded well-ordered subset of A_p of type ω^+. For each $\xi < \omega^+$ let $X_\xi = \{i \in I : R_p(a_\xi, g(i))\}$. Since g/D is an upper bound of $d(A_p)$ in $\mathfrak{A}_m$, we have $X_\xi \in D$ for each $\xi < \omega^+$. Moreover, $\bigcap_{\xi<\omega^+} X_\xi = 0$, and $X_\xi \subseteq X_\zeta$ whenever $\zeta < \xi < \omega^+$. Since $|A_m| = |(A_p)^I{}_D| = \omega^+$, we may let $(A_p)^I{}_D = \{f_\xi/D : \xi < \omega^+\}$. For each $i \in I$ there is a $\zeta < \omega^+$ such that $i \notin X_\xi$ whenever $\zeta \leqq \xi < \omega^+$. Also, $|\zeta| = \omega < |A_p|$, so $A_p - \{f_\eta(i) : \eta < \zeta\} \neq 0$. We may therefore choose a function $h \in (A_p)^I$ such that, for each $\xi < \omega^+$ and each $i \in X_\xi$, we have $h(i) \neq f_\xi(i)$. Then $h/D \neq f_\xi/D$ for each $\xi < \omega^+$, contradicting $h/D \in (A_p)^I{}_D$. This contradiction shows that A_p is not bounded above in $\mathfrak{A}_m$.

Since $\mathfrak{C} = \bigcup_{p\leqq m<\omega}\mathfrak{A}_m$, it follows that A_p is not bounded above in $\mathfrak{C}$. $\mathfrak{C}$ is confinal with ω because $\mathfrak{B}$ is. Hence there is a set $\{b_n : n < \omega\} \subseteq C$ which is not bounded above in $\mathfrak{C}$. For each $n < \omega$, there is a $b'_n \in A_p$ such that $T(b_n, b'_n)$, for otherwise b_n would be an upper bound of A_p in $\mathfrak{C}$. Then $\{b'_n : n < \omega\}$ is not bounded above in $\mathfrak{C}$. Moreover, $\{b'_n : n < \omega\} \subseteq A_p$ and is not bounded above in $\mathfrak{A}_p$. However this contradicts the fact that $\mathfrak{A}_p$ is not confinal with ω. Hence our assumption that $\mathfrak{B} \in \mathbf{K}_2(\mathfrak{A}_0)$ is incorrect, and we must conclude that $\mathfrak{B} \notin \mathbf{K}_2(\mathfrak{A}_0)$.

THEOREM 6.4. $\mathbf{K}_3(\mathfrak{A}_0) \neq \mathbf{K}_4(\mathfrak{A}_0)$.

Proof. It is well known that any dense simply ordered system $\mathfrak{B}$ is isomorphic to an elementary extension of $\mathfrak{A}_0$ and thus belongs to $\mathbf{K}_4(\mathfrak{A}_0)$. This follows, for example, from Cantor's result that every countable dense simply ordered system is isomorphic to $\mathfrak{A}_0$, and by the Löwenheim-Skolem-Tarski Theorem $\mathfrak{B}$ has a countable elementary subsystem; see [1; 38]. It can also be proved by eliminating quantifiers; see [22].

Now let $\mathfrak{B} = \langle B,S \rangle$ be a dense simply ordered system of type $\eta_0 \cdot \omega_1$, that is, $\mathfrak{B}$ is isomorphic to the system $\langle A \times \omega_1, R' \rangle$ where $R'(\langle a,\xi \rangle, \langle b,\eta \rangle)$ if and only if either $\xi < \eta$, or $\xi = \eta$ and $R(a,b)$. Then $\mathfrak{B} \in \mathbf{K}_4(\mathfrak{A}_0)$.

$\mathfrak{B}$ has the property that, for any $b \in B$,

$$|\{b' \in B : S(b',b)\}| = \omega. \tag{1}$$

Suppose $\mathfrak{B} \in \mathbf{K}_3(\mathfrak{A}_0)$. Since $|B| > |A|$, $\mathfrak{B}$ is isomorphic to a nontrivial limit ultrapower of $\mathfrak{A}_0$. By Theorem 3.6, there is a nontrivial ultrapower $\mathfrak{A}_1 = {\mathfrak{A}_0}^I{}_D$ which has an elementary embedding ϕ into $\mathfrak{B}$. Let $a \in A_0$ and let $C = \{a' \in A : R(a',a)\}$. Then C is infinite and

$$|C^I{}_D| \geqq |\omega^\omega| > \omega.$$

Since there is a natural one-to-one correspondence between $C^I{}_D$ and $\{a_1 \in A_1 : R_1(a_1,da)\}$, we have

$$|\{a_1 \in A_1 : R_1(a_1, da)\}| > \omega.$$

Then

$$|\{b' \in B : S(b',(da))\}| > \omega,$$

which contradicts (1). We conclude that $\mathfrak{B} \notin \mathbf{K}_3(\mathfrak{A}_0)$.

The question whether Theorem 6.3 can be proved without the continuum hypothesis remains open.

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