MINIMAL TOPOLOGICAL SPACES(1)

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Given a set X and the lattice of all topologies on the set. We will investigate properties of the various minimal topologies on this set, namely, minimal Frechet, minimal Hausdorff, minimal completely regular, minimal normal, and minimal locally compact. A subsequent paper [1] written with R. H. Sorgenfrey will discuss minimal regular spaces.

In general, the terminology of this paper will coincide with the terminology found in [2]. Specifically, regular spaces, completely regular spaces, normal spaces, compact spaces, and locally compact spaces will be topological spaces automatically satisfying the Hausdorff separation property. Frechet spaces and T_1-spaces are identical. In the comparison of topologies, a topology $\mathscr{T}$ will be weaker (coarser) than a topology $\mathscr{T}'$ if $\mathscr{T}$ is a subfamily of $\mathscr{T}'$.

Definition 1. A filter-base $\mathscr{F}$ on a set X is said to be weaker than a filter-base $\mathscr{G}$ on X, if for each $F \in \mathscr{F}$, there exists some $G \in \mathscr{G}$ such that $G \subset F$. This relation defined on the family of filter-bases on a set X is a partial ordering.

Definition 2. A filter-base $\mathscr{F}$ on a set X is said to be equivalent to a filter-base $\mathscr{G}$ on X if $\mathscr{F}$ is weaker than $\mathscr{G}$ and $\mathscr{G}$ is weaker than $\mathscr{F}$. This relation is a genuine equivalence relation.

Definition 3. Given a topological space X. An open filter-base on X is a filter-base composed exclusively of open sets. A closed filter-base on X is a filter-base composed exclusively of closed sets.

1. Minimal Hausdorff spaces.

1.1. Definition. A topological space $(X,\mathscr{T})$ is said to be minimal Hausdorff if $\mathscr{T}$ is Hausdorff and there exists no Hausdorff topology on X strictly weaker than $\mathscr{T}$. Thus this minimality property is topological.

The following theorem is a useful characterization of minimal Hausdorff spaces as given in [2, pp. 110, 111] and [4]. Another characterization may be found in [7].

1.2. Theorem. *A necessary and sufficient condition that a Hausdorff space $(X,\mathscr{T})$ be minimal Hausdorff is that $\mathscr{T}$ satisfies property* S:

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(i) *Every open filter-base has an adherent point;*

(ii) *If an open filter-base has a unique adherent point, then it converges to this point.*

At this point, the reader may inquire whether either S(i) or S(ii) alone on a Hausdorff space will guarantee the minimality of the Hausdorff topology. This inquiry is answered in the following results.

1.3. THEOREM. *A Hausdorff space X which satisfies* S(ii) *also satisfies* S(i). *Hence such a space is minimal Hausdorff.*

Proof. Assume there exists an open filter-base $\mathcal{G}$ which has no adherent point. Take and fix some point $p \in X$. Let $\mathcal{W}$ be the filter-base of open neighborhoods of p. Let $\mathcal{H} = \{V \cup G \mid V \in \mathcal{W} \text{ and } G \in \mathcal{G}\}$.Then p is the unique adherent point of the open filter-base $\mathcal{H}$. By S(ii), $\mathcal{H}$ converges to p. Since $\mathcal{H}$ is weaker than $\mathcal{G}$, then $\mathcal{G}$ converges to p. Hence p is an adherent point of $\mathcal{G}$. This contradicts the assumpion that $\mathcal{G}$ has no adherent point.

1.4. REMARK. A Hausdorff space satisfying S(i) does not necessarily satisfy S(ii). As an example, we use a special case of a result in [2, p. 111]. Let $(X, \mathcal{T})$ be the closed interval $[0,1]$ with the natural topology. Let A be the set of rational numbers in $[0,1]$. Hence, A and $X - A$ are dense in $(X, \mathcal{T})$ and A is not an open set of $(X, \mathcal{T})$. Now define $\mathcal{T}^*$ on X in the following manner: $\mathcal{T}^*$ is the smallest topology on X such that $\mathcal{T}$ is weaker than $\mathcal{T}^*$ and A is open in $(X, \mathcal{T}^*)$. Clearly $\mathcal{T}$ is strictly weaker than the Hausdorff topology $\mathcal{T}^*$. By [2, p. 111], $\mathcal{T}^*$ also satisfies S(i).

1.5. REMARK. Any compact space is minimal Hausdorff. In [4; 7; 8], an example due to Urysohn is given of a minimal Hausdorff space which is not compact. Since this example is necessary for later results, we will now describe this space.

Let $X = \{a_{ij}, b_{ij}, c_i, a, b \mid i = 1,2,\cdots;\ j = 1,2,\cdots\}$ where all these elements are assumed to be distinct. Define the following neighborhood systems on X:

Each a_{ij} is isolated and each b_{ij} is isolated;

$$\mathcal{B}(c_i) = \left\{ V^n(c_i) = \bigcup_{j=n}^{\infty} \{a_{ij}, b_{ij}, c_i\} \mid n = 1,2,3,\cdots \right\},$$

$$\mathcal{B}(a) = \left\{ V^n(a) = \bigcup_{j=1}^{\infty} \bigcup_{i=n}^{\infty} \{a_{ij}, a\} \mid n = 1,2,3,\cdots \right\},$$

$$\mathcal{B}(b) = \left\{ V^n(b) = \bigcup_{j=1}^{\infty} \bigcup_{i=n}^{\infty} \{b_{ij}, b\} \mid n = 1,2,3,\cdots \right\}.$$

Let us denote this topology by $\mathcal{T}$. $(X, \mathcal{T})$ is minimal Hausdorff but not compact.

Another question the reader might ask himself is whether the minimal Hausdorff property is hereditary, i.e, is a subspace of a minimal Hausdorff space necessarily minimal Hausdorff? This question is partially answered in the following theorem.

1.6. THEOREM. (i) *Any minimal Hausdorff subspace of a Hausdorff space is closed.*

(ii) *If a subspace of a minimal Hausdorff space is both open and closed, then it is minimal Hausdorff.*

Proof. (i) Let X be a Hausdorff space and let A be a minimal Hausdorff subspace. From [2, p. 110], property S(i) is a necessary and sufficient condition for a Hausdorff space to be absolutely closed, i.e., any continuous image is closed in a Hausdorff co-domain space. Since the injection map of A into X is continuous, then A is closed in X.

(ii) Now let A be an open and closed subspace of a minimal Hausdorff space X. Let $\mathscr{G}$ be an open filter-base on A with a unique adherent point $p \in A$. We wish to prove $\mathscr{G}$ converges to p in A. Since A is open in X, then $\mathscr{G}$ is an open filter-base on X. Since A is closed, then p is a unique adherent point of $\mathscr{G}$ on X. But X is minimal Hausdorff. Thus $\mathscr{G}$ converges to p on X. Since $p \in A$, then $\mathscr{G}$ also converges to p on A.

1.7. REMARKS. (i) The example of 1.5 will serve to disprove the converse of 1.6(i), namely, any closed subspace of a minimal Hausdorff space is minimal Hausdorff. Let $C = \{c_i \mid i = 1, 2, \cdots\}$. The reader will observe that C is a closed infinite subset of X whose subspace topology is discrete and hence not minimal Hausdorff.

(ii) The converse of 1.6(ii) is not always true. For example, let X be the closed interval $[0, 2]$ with the natural topology. Then X is compact and hence minimal Hausdorff. Let $A = [0, 1]$. A is compact and hence minimal Hausdorff. But A is not an open subset of X.

(iii) Finally we observe that a minimal Hausdorff space which is also regular is necessarily compact (cf. [2, p. 111]).

1.8. THEOREM. *A Hausdorff space X in which every point has a fundamental system of minimal Hausdorff neighborhoods is locally compact.*

Proof. By 1.6(i), any minimal Hausdorff subspace of X is closed. Thus every point of X has a fundamental system of closed neighborhoods. Hence X is regular. Now any subspace of a regular space is regular. Thus each point has a fundamental system of regular, minimal Hausdorff neighborhoods. By virtue of 1.7(iii), each point of X has a fundamental system of compact neighborhoods. Hence X is locally compact.

1.9. COROLLARY. *A minimal Hausdorff space X in which every point has a fundamental system of minimal Hausdorff neighborhoods is compact.*

Proof. By 1.8, X is locally compact and hence regular. Thus by 1 7(iii), X is compact.

1.10. THEOREM. *Let $\{(X_\alpha, \mathcal{T}_\alpha)\}$ be a family of nonempty Hausdorff spaces. If the product $X = \prod X_\alpha$ is minimal Hausdorff, then each factor space is minimal Hausdorff.*

Proof. Let $\mathcal{T}$ denote the product topology on X. Assume there exists an α_0 such that $(X, \mathcal{T}_{\alpha_0})$ is not minimal Hausdorff. Then there exists a Hausdorff topology $\mathcal{T}_\theta$ on X_{α_0} such that $\mathcal{T}_\theta$ is strictly weaker than $\mathcal{T}_{\alpha_0}$. Let $\{(X_\beta, \mathcal{T}_\beta)\}$ be the family of topological spaces

$$\{(X_\alpha, \mathcal{T}_\alpha) \mid \alpha \neq \alpha_0\} \cup \{(X_{\alpha_0}, \mathcal{T}_\theta)\}.$$

Then the topology of the product $X = \prod X_\beta$ is a Hausdorff topology strictly weaker than $\mathcal{T}$. This contradicts the minimality of $\mathcal{T}$.

1.11. REMARK. The converse of the above theorem, namely, the product of minimal Hausdorff spaces is minimal Hausdorff, is neither proved nor disproved in this paper. To the writer's knowledge, this question is still unanswered.

By virtue of 1.2 and 1.3, a nonminimal Hausdorff space has at least one open filter-base with a unique adherent to which this filter-base does not converge. The following theorem gives us a method for constructing a Hausdorff topology strictly weaker than a given nonminimal Hausdorff topology. The proof of the theorem is left to the reader.

1.12. THEOREM. *Let $(X, \mathcal{T})$ be a Hausdorff space. Let $\mathcal{G}$ be an open filter-base with no adherent point (or possibly $\mathcal{G}$ has a unique adherent point to which $\mathcal{G}$ does not converge). Take and fix some point $p \in X$ (if $\mathcal{G}$ has a unique adherent point, let p be this point). Define the following family of filter-bases on X:*

$$\mathcal{B}(x) = \begin{cases} \mathcal{V}(x), \text{ the } \mathcal{T}\text{-neighborhood system of } x \text{ for } x \neq p; \\ \{V \cup G \mid V \text{ is an open } \mathcal{T}\text{-neighborhood of } x \text{ and } G \in \mathcal{G}\} \text{ for } x = p. \end{cases}$$

Then for each $x \in X$, $\mathcal{B}(x)$ determines a fundamental system of neighborhoods for a Hausdorff topology $\mathcal{T}^*$ which is strictly weaker than $\mathcal{T}$.

2. Minimal Frechet spaces.

2.1. DEFINITION. A topological space $(X, \mathcal{T})$ is said to be minimal Frechet if $\mathcal{T}$ is Frechet and there exists no Frechet topology on X strictly weaker than $\mathcal{T}$.

The following theorem reveals the structure of a minimal Frechet topology on any set.

2.2. THEOREM. If *X is any set and $\mathcal{T}$ is the family of subsets $\{A \subset X \mid X - A \text{ is finite}\} \cup \{\emptyset\}$, then X and $\mathcal{T}$ have the following properties:*

(i) *$\mathcal{T}$ defines a Frechet topology on X;*

(ii) *$\mathcal{T}$ is the weakest Frechet topology on X; consequently,*

(iii) *a Hausdorff topology on X is minimal Frechet if, and only if, X is finite.*

Proof. (i) and (ii) are well-known results (cf. [6, p. 56].) The proof of (iii) is left to the reader.

2.3. COROLLARY. *A compact space is minimal Frechet if, and only if, it is finite.*

2.4. COROLLARY. *Any subspace of a minimal Frechet space is minimal Frechet.*

2.5. COROLLARY. *A nonminimal Frechet space cannot be topologically imbedded in a minimal Frechet space.*

Since every compact space satisfies property S (cf. 1.2), then property S cannot be a characterization for a Frechet space to be minimal Frechet. However, we do have the following result.

2.6. THEOREM. *Let $(X,\mathscr{T})$ be a minimal Frechet space. Then $(X,\mathscr{T})$ satisfies property* S.

Proof. If X is finite, then by 2.3 $(X,\mathscr{T})$ is compact. Thus $(X,\mathscr{T})$ satisfies property S. Now suppose X is infinite. From the structure of minimal Frechet spaces (cf. 2.2), every two nonempty open sets meet. Hence every point is an adherent point of every open filter-base. Thus S(i) is satisfied and S(ii) is vacuously satisfied.

The next theorem completely resolves the question concerning the product of minimal Frechet spaces. The proof is left to the reader.

2.7. THEOREM. *For a product $X = \prod X_\alpha$ of an arbitrary family of nonempty spaces to be minimal Frechet, it is necessary and sufficient that all the spaces X_α are minimal Frechet; all but finitely many of them are singletons; and at most one of them is infinite.*

3. **Minimal completely regular spaces.**

3.1. DEFINITION. A topological space $(X,\mathscr{T})$ is said to be minimal completely regular if $\mathscr{T}$ is completely regular and there exists no completely regular topology on X strictly weaker than $\mathscr{T}$.

Since all compact spaces are minimal Hausdorff, then all compact spaces are minimal completely regular. The question as to whether there exist minimal completely regular spaces which are not compact is answered in 3.3.

But first we prove a theorem which will give us under certain conditions a method of constructing a regular topology strictly weaker than a given regular topology.

3.2. THEOREM. *Let $(X,\mathscr{T})$ be a regular space. Let $\mathscr{G}$ be an open filter-base with no adherent point (or possibly $\mathscr{G}$ has a unique adherent point to which $\mathscr{G}$ does not converge). Suppose there exists a closed filter-base $\mathscr{F}$ which is equivalent* *to $\mathscr{G}$. Take and fix some point $p \in X$ (if $\mathscr{G}$ has a unique adherent point, let p be this point). Define the following family of filter-bases on X:*

$$\mathscr{B}(x) = \begin{cases} \mathscr{V}(x), \text{ the } \mathscr{T}\text{-neighborhood system of } x \text{ for } x \neq p; \\ \{V \cup G \mid V \text{ is an open } \mathscr{T}\text{-neighborhood of } x \text{ and } G \in \mathscr{G}\} \text{ for } x = p. \end{cases}$$

Then for each $x \in X$, $\mathscr{B}(x)$ determines a fundamental system of neighborhoods for a regular topology $\mathscr{T}^*$ which is strictly weaker than $\mathscr{T}$.

Proof. By 1.12, we know $\{\mathscr{B}(x) \mid x \in X\}$ determines a Hausdorff topology $\mathscr{T}^*$ which is strictly weaker than $\mathscr{T}$. We shall now show that $\mathscr{T}^*$ is regular. Take $x \in X$ and $W \in \mathscr{B}(x)$. We wish to show there exists a closed $\mathscr{T}^*$-neighborhood of x contained in W.

CASE 1. $x \neq p$. Since $\mathscr{T}^*$ is Hausdorff, there exists some $W' \in \mathscr{V}(x)$ such that W' is disjoint from some $\mathscr{T}^*$-neighborhood U of p and $W' \subset W$. Since $(X, \mathscr{T})$ is regular, there exists a closed $\mathscr{T}$-neighborhood H of x such that $H \subset W'$. Since $U \cap W' = \emptyset$ and $H \subset W'$, *then* $U \cap H = \emptyset$. Hence H is a closed $\mathscr{T}^*$-neighborhood of x such that $H \subset W$.

CASE 2. $x = p$. Since $W \in \mathscr{B}(p)$, then there existsan open $\mathscr{T}$-neighborhood V of x and $G \in \mathscr{G}$ such that $W = V \cup G$. Now $\mathscr{T}$ is regular, hence there exists a closed $\mathscr{T}$-neighborhood S of p such that $S \subset V$. Since $\mathscr{F}$ and $\mathscr{G}$ are equivalent, then there exists an $F \in \mathscr{F}$ such that $F \subset G$. Thus $S \cup F$ is a $\mathscr{T}^*$-neighborhood of p such that $S \cup F \subset V \cup G$. We will now show $S \cup F$ is closed in $(X, \mathscr{T}^*)$. Take $y \notin S \cup F$. Since $S \cup F$ is $\mathscr{T}$-closed, there exists a $U \in \mathscr{V}(y)$ such that $U \cap (S \cup F) = \emptyset$. By definition U is also a $\mathscr{T}^*$-neighborhood of y. Thus, $S \cup F$ is a closed $\mathscr{T}^*$ neighborhood of p such that $S \cup F \subset W$. Hence $\mathscr{T}^*$ is a regular topology strictly weaker than $\mathscr{T}$.

3.3. THEOREM. *All minimal completely regular spaces are compact.*

Proof. REMARK. In the proof of this theorem, the reader will observe that a technique is provided to construct a completely regular topology strictly weaker than a given noncompact completely regular topology.

Let $(X, \mathscr{T})$ be a minimal completely regular space. In order to show $(X, \mathscr{T})$ is compact, it suffices to prove that X is the same as its Čech compactification $\beta(X)$. We already know that X can be considered as a subspace of $\beta(X)$. To establish our theorem, we now offer a proof by contradiction.

Assume X is not compact. Then $\beta(X) - X$ is nonempty. Take and fix some element p of $\beta(X) - X$. Let $\mathscr{G}$ be the filter-base of open neighborhoods of p in $\beta(X)$. Let $\mathscr{G}^*$ be the trace of $\mathscr{G}$ on X. Considered as a filter-base on $\beta(X)$, $\mathscr{G}^*$ has a unique adherent point, namely p. Thus $\mathscr{G}^*$ has no adherent point on X. Now select and fix an element a in X. On X, form the following collection of fundamental systems of neighborhoods for a topology $\mathscr{T}^*$.

$$\mathscr{W}(x) = \begin{cases} \mathscr{V}(x) \text{ if } x \neq a, \\ \{V \cup G \mid V \text{ is an open } \mathscr{T}\text{-neighborhood of } a \text{ and } G \in \mathscr{G}^*\} \text{ if } x = a. \end{cases}$$

By 3.2 we see that $\mathscr{T}^*$ defines a regular topology on X such that $\mathscr{T}^*$ is strictly weaker than $\mathscr{T}$. We shall now show that $(X,\mathscr{T}^*)$ is a completely regular space.

Let Ω', Ω, and Ω^* be the families of all unit-interval-valued continuous functions defined on $\beta(X)$, $(X,\mathscr{T})$ and $(X,\mathscr{T}^*)$ respectively. Take $b \in X$ and F a closed set in $(X,\mathscr{T}^*)$ such that $b \notin F$. We wish to show there exists $f \in \Omega^*$ such that $f(b) = 1$ and $f(F) = 0$. Consider the following cases:

Case 1. $b \neq a$. Now $F \cup \{a\}$ is closed in $(X,\mathscr{T}^*)$. Since $b \notin F \cup \{a\}$, then, by regularity of $\mathscr{T}^*$, there exist disjoint open neighborhoods U and V of b and $F \cup \{a\}$ respectively. Now U is also open in $(X,\mathscr{T})$. Hence $X - U$ is closed in $(X,\mathscr{T})$. Thus there exists $f \in \Omega$ such that $f(b) = 1$ and $f(x) = 0$ when $x \in X - U$. By construction of $\mathscr{T}^*$, f is $\mathscr{T}^*$-continuous at $z \in X$ when $z \neq a$. Since V is a subset of $X - U$, then $f(V) = 0$. But V is an open neighborhood of a in $(X,\mathscr{T}^*)$. Hence, f is $\mathscr{T}^*$-continuous at a. Thus $f \in \Omega^*$.

Case 2. $b = a$. Consider the closure $\bar{F}$ of F in $\beta(x)$. Since F is closed in $(X,\mathscr{T}^*)$ and thus in $(X,\mathscr{T})$, we have $\bar{F} \cap X = F$. Now $a \in X - F$; hence $a \notin \bar{F}$. Also $p \notin \bar{F}$ because, if $p \in \bar{F}$, each $G \in \mathscr{G}^*$ would meet F, and so each $\mathscr{T}^*$-neighborhood of a would meet F, contradicting that F is a $\mathscr{T}^*$-closed set not containing a. Since $\beta(X)$ is regular and $\bar{F}$ is closed, there exist neighborhoods S, T of a and p respectively in $\beta(X)$ such that $\bar{S} \cap \bar{F} = \emptyset = \bar{T} \cap \bar{F}$. Also $\beta(X)$ is normal. Hence by Urysohn's Lemma, there exists $f' \in \Omega'$ such that $f' = 1$ on $\bar{S} \cup \bar{T} = \overline{S \cup T}$ and 0 on $\bar{F}$. Let $f = f' \mid X$. Thus $f(a) = 1$ and $f(F) = 0$. Since for each point $x \in X$, $x \neq a$, $\mathscr{V}(x)$ is a fundamental system of neighborhoods in $(X,\mathscr{T})$ as a subspace of $\beta(X)$, then f is continuous at x. Also f is constant on $(S \cup T) \cap X$, a $\mathscr{T}^*$-neighborhood of a. Thus f is continuous at $a = b$. Hence $f \in \Omega^*$.

Thus, we have shown $\mathscr{T}^*$ is a completely regular topology strictly weaker than $\mathscr{T}$. This contradicts the minimality of $\mathscr{T}$. Hence $(X, \mathscr{T})$ is compact.

4. **Minimal normal spaces.**

4.1. Definition. A topological space $(X,\mathscr{T})$ is said to be minimal normal if $\mathscr{T}$ is normal and there exists no normal topology on X strictly weaker than $\mathscr{T}$.

Since all compact spaces are minimal Hausdorff, then all compact spaces are minimal normal. The question as to whether there exist minimal normal spaces which are not compact is answered in the following theorem.

4.2. Theorem. *All minimal normal spaces are compact.*

Proof. Remark. In the proof of this theorem, the reader will observe that a technique is provided to construct a normal topology strictly weaker than a given noncompact normal topology.

Let $(X, \mathscr{T})$ be a minimal normal space. In order to show $(X, \mathscr{T})$ is compact, it suffices to prove that X is the same as its Čech compactification $\beta(X)$. We already know that X can be considered as a subspace of $\beta(X)$. To establish our theorem, we now offer a proof by contradiction.

Assume X is not compact. Then $\beta(X) - X$ is nonempty. Take and fix some element p of $\beta(X) - X$. Let $\mathscr{G}$ be the filter-base of open neighborhoods of p in $\beta(X)$. Now let $\mathscr{G}^*$ be the trace of $\mathscr{G}$ on X. Thus $\mathscr{G}^*$ is an open filter-base on X. Considered as a filter-base on $\beta(X)$, $\mathscr{G}^*$ has a unique adherent point, namely p. Thus $\mathscr{G}^*$ has no adherent point on X. Now select and fix an element a in X. We now construct the same topology $\mathscr{T}^*$ on X exactly as in the proof of 3.3. By 3.2 we see that $\mathscr{T}^*$ defines a regular topology on X such that $\mathscr{T}^*$ is strictly weaker than $\mathscr{T}$. We shall now show that $(X, \mathscr{T}^*)$ is a normal space. Let Q and R be two disjoint closed subsets of $(X, \mathscr{T}^*)$. We wish to find two disjoint open neighborhoods of Q and R respectively. Since $\mathscr{T}^*$ is weaker than $\mathscr{T}$, then Q and R are closed in $(X, \mathscr{T})$. Since $\mathscr{T}$ is normal, there exists disjoint open sets H' and K' in $(X, \mathscr{T})$ such that Q is a subset of H' and R is a subset of K'. We now consider the following cases:

Case 1. $a \notin Q \cup R$. Let $H = H' - \{a\}$ and $K = K' - \{a\}$. Then H and K are open in both $(X, \mathscr{T})$ and $(X, \mathscr{T}^*)$. Also H and K are disjoint neighborhoods of Q and R respectively.

Case 2. $a \in Q \cup R$. Assume $a \in Q$. Then $a \notin R$. Since $(X, \mathscr{T}^*)$ is regular, then there exist disjoint open sets H'' and K'' such that $a \in H''$ and $R \subset K''$. Clearly H'' and K'' are open in $(X, \mathscr{T})$. Now let $H = H' \cup H''$ and $K = K' \cap K''$. Thus we see that H and K are disjoint open neighborhoods of Q and R respectively in $(X, \mathscr{T})$. Since H'' is an open neighborhood of a in $(X, \mathscr{T}^*)$, then H is an open set also in $(X, \mathscr{T}^*)$. Since $a \notin K$, then K is an open set also in $(X, \mathscr{T}^*)$.

Thus $(X, \mathscr{T}^*)$ is a normal space. Moreover, $\mathscr{T}^*$ is strictly weaker than $\mathscr{T}$. This contradicts the minimality of $\mathscr{T}$. Hence $(X, \mathscr{T})$ is compact.

5. Minimal locally compact spaces.

5.1. Definition. A topological space $(X, \mathscr{T})$ is minimal locally compact if $\mathscr{T}$ is locally compact and there exists no locally compact topology strictly weaker than $\mathscr{T}$.

Since all compact topologies are minimal Hausdorff, then all compact topologies are minimal locally compact. The question as to whether there exist minimal locally compact topologies which are not compact is answered negatively as a corollary of the following theorem.

5.2. Theorem. *Let $(X, \mathscr{T})$ be a locally compact topology which is not compact. Then $\mathscr{T}$ is stronger than some compact topology $\mathscr{T}^*$ defined on X.*

Proof. Remark. In the proof of this theorem, the reader will observe that a technique is provided to construct a compact topology weaker than a given locally compact topology which is not compact. Also the reader is reminded that "locally compact" implies the Hausdorff separation axiom.

Let $X' = X \cup \{p\}$ where $p \notin X$; let $\mathscr{T}'$ be the Alexandroff compactification of $\mathscr{T}$ in X' at the point p. Let $\mathscr{G}$ be the filter-base of open neighborhoods of p in $(X', \mathscr{T}')$. Let $\mathscr{G}^*$ be the trace of $\mathscr{G}$ on X. Clearly $\mathscr{G}^*$ has no adherent point in $(X, \mathscr{T})$. Now select and fix an element $a \in X$. On X, again construct the same topology $\mathscr{T}^*$ exactly as in the proof of 3.3. By 1.12 we see that $\mathscr{T}^*$ defines a Hausdorff topology on X such that $\mathscr{T}^*$ is strictly weaker than $\mathscr{T}$. We shall now show that $(X, \mathscr{T}^*)$ is a compact space. Let $\mathscr{F}$ be a filter-base on $(X, \mathscr{T}^*)$. It suffices to prove that $\mathscr{F}$ has an adherent point on $(X, \mathscr{T}^*)$. If $\mathscr{F}$ has an adherent point on $(X, \mathscr{T})$, then $\mathscr{F}$ will have an adherent point on $(X, \mathscr{T}^*)$ since $\mathscr{T}^*$ is weaker than $\mathscr{T}$. On the other hand, if $\mathscr{F}$ has no adherent point on $(X, \mathscr{T}^*)$, then p will be an adherent point of $\mathscr{F}$ on $(X', \mathscr{T}')$. Thus for all $G \in \mathscr{G}$ and for all $F \in \mathscr{F}$, $G \cap F \neq \emptyset$. But $\mathscr{F}$ is a family of subsets of X. Hence for all $H \in \mathscr{G}^*$ and for all $F \in \mathscr{F}$, $H \cap F \neq \emptyset$. Thus a is an adherent point of $\mathscr{F}$ in $(X, \mathscr{T}^*)$. Hence $(X, \mathscr{T}^*)$ is a compact space.

5.3. COROLLARY. *The minimal locally compact spaces are precisely the compact spaces.*

We finally remark that some other properties one might investigate in the study of minimal spaces are the following:

(i) Can an arbitrary Hausdorff (regular) space be topologically embedded in some minimal Hausdorff (minimal regular) space?

(ii) Is the topological product of minimal Hausdorff (minimal regular) spaces necessarily minimal Hausdorff (minimal regular)?

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