SOME GEOMETRIC CONSEQUENCES OF CONFORMAL STRUCTURE

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1. Any oriented surface immersed smoothly in E^3 may be viewed as a Riemann surface, inheriting the usual conformal structure from the Euclidean metric of the surrounding space. But a strictly convex surface, oriented so that its mean curvature is positive, has still another conformal structure imposed upon it in a natural way. It is the conformal structure obtained by using the second fundamental form as metric tensor. Only on spherical portions of a surface do the two conformal structures coincide.

In a recent paper [5], we described geometrically cases in which certain standard differential geometric correspondences are Teichmüller mappings. There, of course, we worked with the conventional conformal structure on the surfaces involved. But, standard differential geometric correspondences might in certain cases be Teichmüller mappings between the Riemann surfaces determined by using the second conformal structure described above. And, in still other cases, these correspondences might be Teichmüller mappings involving the usual conformal structure on one surface, and the second conformal structure on another.

In this paper we describe geometrically cases in which such Teichmüller mappings actually are obtained. Of special interest, perhaps, are the particular instances in which these mappings are conformal. Our results tend to parallel rather closely those obtained in [5]. Wherever possible, lemmas and theorems below have been numbered so as to indicate their correspondence to related items in that previous paper.

2. In this section we mention those properties of Teichmüller mappings which are pertinent to the exposition which follows. No attempt will be made to give background material from the theory of quasiconformal mappings which would help to describe the importance of Teichmüller mappings. A brief outline of such material can be found in [5]. For a thorough explanation of the subject matter involved, see (for instance) [1] or [2].

A quadratic differential Ω on a Riemann surface R assigns a complex valued function ϕ to the domain of each conformal parameter z on R, so that the expression

$$\phi dz^2$$

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remains invariant. A quadratic differential Ω is meromorphic on R if each such function ϕ is meromorphic in z. A quadratic differential Ω is holomorphic on R if each such function ϕ is analytic in z.

A Teichmüller mapping $f: R \to \hat{R}$ between Riemann surfaces is one which maps R homeomorphically onto $\hat{R}$ in the following way. Let $z = x + iy$ and $w = u + iv$ be conformal parameters at points of R and $\hat{R}$, respectively, in correspondence under f. Then the Beltrami equation

$$w_{\bar{z}} = k \frac{\bar{\phi}}{|\phi|} w_z$$

must be satisfied, where k is a fixed constant such that

$$0 \leqq k < 1,$$

where

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

and where $\Omega = \phi\, dz^2$ is a fixed meromorphic quadratic differential on R.

Given a Teichmüller mapping f, the constant

$$K = \frac{1 + k}{1 - k}$$

is called its dilatation, while Ω is called the defining quadratic differential of f. Note that

$$1 \leqq K < \infty.$$

The zeros and poles of Ω are called exceptional points of f. Exceptional points are, of course, isolated on R.

Teichmüller mappings are special kinds of quasiconformal homeomorphisms, and include (for $k = 0$, $K = 1$) all conformal homeomorphisms between Riemann surfaces. The inverse f^{-1} of a Teichmüller mapping f is a Teichmüller mapping with the same dilatation K as f. Moreover, the exceptional points of f^{-1} occur at the images under f of the exceptional points of f. Finally, a Teichmüller mapping composed in either order with a conformal homeomorphism is still a Teichmüller mapping with the original dilatation K.

The following fact is generally helpful in picturing the behavior of a Teichmüller mapping f (see [2, §8]). In the neighborhood of any nonexceptional point of f, a conformal parameter $z = x + iy$ may be chosen so that the assignment of $w = Kx + iy$ to $f(z)$ yields a conformal parameter on $\hat{R}$. Thus a Teichmüller mapping is, in the neighborhood of any nonexceptional point, a conformal mapping followed by an affine transformation followed by a conformal mapping.

These remarks lead directly to Lemma 1 of §3, which will be stated, therefore, without proof. We emphasize that whenever the symbols Ω or K are used below they are meant to represent the defining quadratic differential and the dilatation, respectively, of whichever Teichmüller mapping is then under discussion.

3. This section is devoted to a discussion of two lemmas. Let S be an oriented surface which is C^3 immersed in E^3. Let R_1 be the usual Riemann surface determined by using the first fundamental form on S as metric tensor. Conformal parameters $z = x + iy$ on R_1 may be introduced by using isothermal coordinates x, y on S, in terms of which

$$I = \lambda(x, y)\,(dx^2 + dy^2).$$

In case Gaussian curvature $\mathscr{K}$ and mean curvature $\mathscr{H}$ are both positive on S, let R_2 be the Riemann surface determined by using the second fundamental form on S as metric tensor. Conformal parameters $w = u + iv$ on R_2 may be introduced locally by using bisothermal coordinates u, v on S, in terms of which

$$II = \mu(u, v)\,(du^2 + dv^2).$$

The existence of isothermal and bisothermal coordinates under the conditions given is assured (see [2, § 4], for instance).

In what follows we consider mappings $f : S \to \hat{S}$ which are Teichmüller mappings when viewed as maps between R_1 and $\hat{R}_2$, or R_2 and $\hat{R}_1$, or R_2 and $\hat{R}_2$. We will automatically assume Gaussian and mean curvature to be positive on a surface whenever its second conformal structure is under consideration. Lemma 1 is a direct consequence of remarks made at the close of § 2.

Lemma 1. *If $f: R_1 \to \hat{R}_2$ is a Teichmüller mapping, then near all but exceptional points on S isothermal coordinates x, y may be introduced in terms of which*

$$\begin{aligned} I &= \lambda(dx^2 + dy^2), \\ \hat{II} &= \hat{\mu}(K^2 dx^2 + dy^2), \end{aligned} \tag{1}$$

at points in correspondence under f. If $f: R_2 \to \hat{R}_1$ is a Teichmüller mapping, then near all but exceptional points on S bisothermal coordinates u, v may be introduced in terms of which

$$\begin{aligned} II &= \mu(du^2 + dv^2), \\ \hat{I} &= \hat{\lambda}(K^2 du^2 + dv^2), \end{aligned} \tag{2}$$

at points in correspondence under f. If $f: R_2 \to \hat{R}_2$ is a Teichmüller mapping, *then near all but exceptional points on S bisothermal coordinates, u, v may be introduced in terms of which*

$$
\begin{aligned}
II &= \mu(du^2 + dv^2), \\
\hat{II} &= \hat{\mu}(K^2\, du^2 + dv^2),
\end{aligned} \tag{3}
$$

at points in correspondence under f.

Before stating Lemma 2, let us clarify some terminology. A line of curvature is a curve along which Rodrigues' formula holds. An umbilic is a point at which normal curvature is independent of direction. An umbilic is either removable or irremovable in a given net of lines of curvature, depending upon whether it is a regular or a singular point of the chosen net. A removable umbilic is characterized by the fact that lines-of-curvature coordinates corresponding to the chosen net may be introduced in a neighborhood of the umbilic. Where only one net of lines of curvature exists in the neighborhood of an umbilic, reference to a chosen net is unnecessary.

It is assumed throughout this paper that a net of curves on a surface is regular on a dense subset. Thus, in particular, the closed set of irremovable umbilics in any net of lines of curvature never covers a neighborhood on the surface. We call a surface isothermal if isothermal lines-of-curvature coordinates can be introduced in some neighborhood of every point. We call a surface bisothermal if bisothermal lines-of-curvature coordinates can be introduced in some neighborhood of every point.

LEMMA 2. *Let f: $S \to \hat{S}$ preserve a net of lines of curvature. If $f: R_1 \to \hat{R}_2$ is a nonconformal Teichmüller mapping, then, except at the irremovable umbilics of the preserved net (which must be isolated), S is isothermal and $\hat{S}$ is bisothermal. The corresponding statement holds with the roles of S and $\hat{S}$ reversed in case $f: R_2 \to \hat{R}_1$ is a nonconformal Teichmüller mapping. If $f: R_2 \to \hat{R}_2$ is a nonconformal Teichmüller mapping then, except at the irremovable umbilics of the preserved net (which must be isolated), S and $\hat{S}$ are bisothermal.*

Proof of Lemma 2. Suppose $f: R_1 \to \hat{R}_2$ is a nonconformal Teichmüller mapping. Then $K > 1$, and near any nonexceptional point on S, isothermal coordinates x, y may be introduced so that (1) holds at points of S and $\hat{S}$ in correspondence under f. If we set

$$
\begin{aligned}
II &= Ldx^2 + 2Mdxdy + Ndy^2, \\
\hat{I} &= \hat{E}dx^2 + 2\hat{F}dxdy + \hat{G}dy^2,
\end{aligned}
$$

and

$$\alpha = \frac{dy}{dx},$$

then the directions of principal curvature on S satisfy

$$M\alpha^2 - (N - L)\alpha - M = 0, \tag{4}$$

while those on $\hat{S}$ satisfy

$$\hat{F}\alpha^2 - (K^2\hat{G} - \hat{E})\alpha - K^2\hat{F} = 0. \tag{5}$$

Since (1) yields $\hat{F} = M = 0$, we know that

$$F = \hat{M} = 0 \tag{6}$$

at umbilics, where first and second fundamental forms are proportional. In fact (6) holds everywhere since $K > 1$ while the common roots of (4) and (5) must correspond to mutually orthogonal directions on both S and $\hat{S}$. However, (1) and (6) together mean that x, y, are isothermal lines-of-curvature coordinates on S, and that Kx, y are bisothermal lines-of-curvature coordinates on $\hat{S}$.

Suppose $f: R_2 \to \hat{R}_1$ is a nonconformal Teichmüller mapping. Then exchanging the roles of S and $\hat{S}$ in the previous case, we obtain, in the neighborhood of any nonexceptional point, bisothermal lines-of-curvature coordinates on S, and isothermal lines-of-curvature coordinates on $\hat{S}$. A direct argument using (2) and imitating the procedure above yields the coordinates u, v on S and Ku, v on $\hat{S}$ which do the job.

Similarly, if $f: R_2 \to \hat{R}_2$ is a nonconformal Teichmüller mapping, near any nonexceptional point on S, bisothermal coordinates u, v may be chosen so that (3) holds at points of S and $\hat{S}$ in correspondence under f. If we set

$$I = Edx^2 + 2Fdxdy + Gdy^2,$$

$$\hat{I} = \hat{E}dx^2 + 2\hat{F}dxdy + \hat{G}dy^2,$$

and

$$\alpha = \frac{dy}{dx}$$

then the directions of principal curvature on $\hat{S}$ satisfy

$$F\alpha^2 - (G - E)\alpha - F = 0$$

and

$$\hat{F}\alpha^2 - (K^2\hat{G} - \hat{E})\alpha - K^2\hat{F} = 0,$$

respectively. Here, $K > 1$ and the preservation of a net of lines of curvature yield

$$F = \hat{F} = 0.$$

Thus, u, v and Ku, v are bisothermal lines-of-curvature coordinates on S and $\hat{S}$ respectively.

It remains to be shown that in all three cases covered by Lemma 2, an exceptional point p on S is an irremovable umbilic of the preserved net. We have already shown that umbilics at nonexceptional points are removable in the preserved net. Thus p has a neighborhood free of irremovable umbilics, and an index j in the preserved net of lines of curvature.

But note that the preserved net lines of curvature coincides with the net of trajectories and orthogonal trajectories of Ω on R_1 (or R_2). If working on R_1 we refer to p. 82 of [4], and if working on R_2 we refer to §4 of [6] to obtain

$$j = \frac{-m}{2},$$

where m is the order of the zero at p or minus the order of the pole at p of Ω. Thus $m \neq 0$ means that $j \neq 0$, so that p must be an irremovable umbilic of the preserved net.

The conclusions of Lemma 2 are not generally valid when f is a conformal mapping between R_1 and $\hat{R}_2$ or R_2 and $\hat{R}_1$ or R_2 and $\hat{R}_2$. Simple counterexamples in the last case are furnished by translations and rotations of arbitrary surfaces. Moreover, even if S and $\hat{S}$ are isothermal or bisothermal as required by the conclusions of Lemma 2, only in rare instances will an $f: S \to \hat{S}$ which preserves a net of lines of curvature be a Teichmüller mapping between R_1 and $\hat{R}_2$ or R_2 and $\hat{R}_1$ or R_2 and $\hat{R}_2$. Various results below may be used to illustrate this fact.

4. Our theorems will deal with mappings between surfaces which preserve a net of lines of curvature. The first such mapping considered is the standard mapping f between parallel surfaces S and $\hat{S}$ which associates with each point p on S the point on $\hat{S}$ a fixed distance $t \neq 0$ from S along the normal to S at p. It is well known that f preserves lines of curvature. Moreover, f also preserves normals if the orientation of $\hat{S}$ is the one induced upon it from S by f (see [3, p. 272]). However the orientation induced by f on $\hat{S}$ need not in general make $\hat{\mathscr{H}} > 0$ if $\hat{\mathscr{K}} > 0$ on $\hat{S}$. Thus conclusions about the nature of S may be expected when the $\hat{R}_2$ structure of $\hat{S}$ is being discussed.

A well-known theorem of Bonnet states that to each umbilic free surface $\tilde{S}$ of constant positive Gaussian curvature $\tilde{\mathscr{K}}$ there are parallel surfaces S' and S'' at distances

$$t' = \frac{1}{\sqrt{\tilde{\mathscr{K}}}}$$

and

$$t'' = -\frac{1}{\sqrt{\tilde{\mathscr{K}}}}$$

from $\tilde{S}$ and of constant mean curvature

$$\mathscr{H}' = \frac{1}{2t'}$$

and

$$\mathscr{H}'' = \frac{1}{2t''}$$

respectively. It is also well known that the standard mapping $f: R_1 \to \hat{R}_1$ between parallel surfaces S and $\hat{S}$ is conformal if and only if S and $\hat{S}$ are spherical or planar, or are a pair of umbilic free surfaces S' and S'' in the relationship described by Bonnet's Theorem. To complete the picture, we have the following.

LEMMA 3. *Let f be the standard mapping between parallel surfaces S and $\hat{S}$. If $f: R_1 \to \hat{R}_2$ is conformal, then S and $\hat{S}$ are spherical or else S is isothermal with*

$$\mathscr{H} \equiv \frac{1}{2t}, \tag{7}$$

and

$$\text{sign } t = \text{sign } \mathscr{K} = \text{sign } \mathscr{H} \neq 0, \tag{8}$$

while $\hat{S}$ is bisothermal with

$$\hat{\mathscr{K}} \equiv \frac{1}{t^2}. \tag{9}$$

The corresponding statement holds with the roles of S and $\hat{S}$ reversed and $-$ in place of t in case $f: R_2 \to \hat{R}_1$ is conformal. If $f: R_2 \to \hat{R}_2$ is conformal, then S and $\hat{S}$ must be spherical.

Proof of Lemma 3. As is usual, we use f to carry coordinates on S to corresponding points on $\hat{S}$. It is an elementary fact (see [3, p. 272]) that the coefficients of the two fundamental forms on $\hat{S}$ are given in terms of those on S by

$$\begin{aligned}
\hat{E} &= E - 2tL + t^2(2\mathscr{H}L - \mathscr{K}E),\\
\hat{G} &= G - 2tN + t^2(2\mathscr{H}N - \mathscr{K}G),\\
\hat{F} &= F - 2tM + t^2(2\mathscr{H}M - \mathscr{K}F),\\
\hat{L} &= L - t(2\mathscr{H}L - \mathscr{K}E),\\
\hat{N} &= N - t(2\mathscr{H}N - \mathscr{K}G),\\
\hat{M} &= M - t(2\mathscr{H}M - \mathscr{K}F).
\end{aligned} \tag{10}$$

Suppose $f: R_1 \to \hat{R}_2$ is conformal. The choice of isothermal coordinates on S yields

$$\hat{\mu} = \hat{L} = \hat{N} > 0, \quad \hat{M} = 0. \tag{11}$$

But (10) implies that

$$L(1 - 2t\mathscr{H}) = N(1 - 2t\mathscr{H}),$$

while

$$M(1 - 2t\mathscr{H}) = 0.$$

On the open set where $(1 - 2t\mathscr{H}) \neq 0$, S is totally umbilic and therefore spherical. (S has no planar portions since $\hat{R}_2$ structure is defined.) However, where $(1 - 2t\mathscr{H}) \equiv 0$, S is of constant mean curvature

$$\mathscr{H} = \frac{1}{2t}.$$

Since $\mathscr{H}$ is continuous, S is either entirely spherical, or else (7) holds on all of S. Further discussion is necessary only in the latter case.

A surface of constant mean curvature is isothermal except at umbilics, which must be isolated unless the surface is entirely spherical (see [4, Chapter 6]). Thus, away from isolated umbilics on S, isothermal lines-of-curvature coordinates x, y may be chosen. Moreover, use of (10) and (11) reveals that x, y are bisothermal lines-of-curvature coordinates on $\hat{S}$. Thus, not only does $E = G = \lambda$, $F = M = 0$, and $\hat{F} = \hat{M} = 0$, but

$$L = k_1\lambda, \quad N = k_2\lambda \tag{12}$$

and

$$\hat{E}\hat{k}_1 = \hat{G}\hat{k}_2 = \hat{\mu} > 0 \tag{13}$$

as well. Substitution of (12) in (10) yields

$$\begin{aligned} \hat{E} &= \lambda(1 - tk_1)^2, \\ \hat{G} &= \lambda(1 - tk_2)^2, \end{aligned} \tag{14}$$

and

$$\hat{\mu} = \lambda k_1(1 - tk_1) = \lambda k_2(1 - tk_2). \tag{15}$$

Since (7) implies that

$$\begin{aligned} (1 - tk_1) &= tk_2, \\ (1 - tk_2) &= tk_1, \end{aligned} \tag{16}$$

we obtain

$$\hat{\mu} = \lambda t\mathscr{K} > 0.$$

Thus (8) must hold on all of S. For even at the isolated umbilics on S, $\mathscr{K} \neq 0$ because of (7). On the other hand, substitution of (14) and (15) in (13), and the use of (15) yield

$$\hat{k}_1(tk_2) = k_1,$$

$$\hat{k}_2(tk_1) = k_2,$$

so that

$$\hat{\mathscr{K}} = \frac{1}{t^2}.$$

By continuity, (9) must hold on all of $\hat{S}$.

No further discussion is necessary for the case in which $f: R_2 \to \hat{R}_1$ is conformal. It is easily checked, using (10), that in the situation described by Bonnet's Theorem, the standard mappings between R_1' and $\tilde{R}_2$ and between R_1'' and $\tilde{R}_2$ are conformal, so long as S', S'' and $\tilde{S}$ are appropriately oriented. We note in passing that all surfaces of constant positive Gaussian curvature are bisothermal except at isolated irremovable umbilics (see [6]).

Suppose, finally, that $f: R_2 \to \hat{R}_2$ is conformal. Choose bisothermal coordinates on S. Then, since $L = N = \mu$, $\hat{L} = \hat{N} = \hat{\mu}$ and $M = \hat{M} = 0$, (10) yields

$$E = G, \quad F = 0.$$

Thus, all points are umbilic, and S and $\hat{S}$ must be spherical.

Note that Lemmas 4 and 5 below are of interest in the cases just discussed. Turning now to nonconformal Teichmüller mappings, we obtain the following.

THEOREM 1. *Let $f: S \to \hat{S}$ be the standard mapping between parallel surfaces S and $\hat{S}$. If $f: R_1 \to \hat{R}_2$ is a nonconformal Teichmüller mapping then the Weingarten equation*

$$(1 - tk_1)k_1 = K^2(1 - tk_2)k_2 \tag{17}$$

holds on S, and the Weingarten equation

$$(1 + t\hat{k}_2)^2\hat{k}_1 = K^2(1 + t\hat{k}_1)^2\hat{k}_2 \tag{18}$$

holds on $\hat{S}$. The corresponding statement is valid with the roles of S and $\hat{S}$ reversed, $-t$ for t, k_1 for $\hat{k}_2$, k_2 for $\hat{k}_1$, $\hat{k}_1$ for k_2 and $\hat{k}_2$ for k_1 in case $f: R_2 \to \hat{R}_1$ is a nonconformal Teichmüller mapping. If $f: R_2 \to \hat{R}_2$ is a nonconformal Teichmüller mapping then the Weingarten equations

$$(1 - tk_1) = K^2(1 - tk_2) \tag{19}$$

and

$$(1 + t\hat{k}_2) = K^2(1 + t\hat{k}_1) \tag{20}$$

hold on S and $\hat{S}$ respectively.

Proof of Theorem 1. Suppose $f: R_1 \to \hat{R}_2$ is a Teichmüller mapping with $K > 1$. Lemmas 1 and 2 guarantee that in the neighborhood of any point of S which is not an irremovable umbilic, coordinates x, y may be chosen in terms of which

$$
\begin{aligned}
I &= \lambda(dx^2 + dy^2),\\
II &= \lambda k_1 dx^2 + \lambda k_2 dy^2,\\
\hat{I} &= \frac{\hat{\mu}K^2}{\hat{k}_1}dx^2 + \frac{\hat{\mu}}{\hat{k}_2}dy^2,\\
\hat{II} &= \hat{\mu}(K^2dx^2 + dy^2).
\end{aligned} \tag{21}
$$

Since $\hat{L} = K^2\hat{N}$, (10) yields

$$k_1 - t(2\mathscr{H}k_1 - \mathscr{K}) = K^2\{k_2 - t(2\mathscr{H}k_2 - \mathscr{K})\},$$

or

$$(1 - tk_1)k_1 = K^2(1 - tk_2)k_2.$$

By continuity, (17) holds on all of S. Theorem 2 below states that all umbilics on S must be planar. Thus, no removable umbilics can occur on S, since, at all non-exceptional points of f, (10) yields

$$
\begin{aligned}
\hat{\mu} &= \lambda(1 - tk_2)k_2 > 0,\\
K^2\hat{\mu} &= \lambda(1 - tk_1)k_1 > 0.
\end{aligned} \tag{22}
$$

As to $\hat{S}$, since $\hat{k}_1\hat{E} = K^2\hat{k}_2\hat{G}$, we may use (10) to obtain

$$
\begin{aligned}
\hat{k}_1(1 - tk_1) &= k_1,\\
\hat{k}_2(1 - tk_2) &= k_2,
\end{aligned}
$$

or

$$k_1 = \frac{\hat{k}_1}{1 + t\hat{k}_1}, \quad k_2 = \frac{\hat{k}_2}{1 + t\hat{k}_2}. \tag{23}$$

Substitution of these expressions in (17) yields (18).

In case $f: R_2 \to \hat{R}_1$ is a Teichmüller mapping with $K > 1$, the roles of S and $\hat{S}$ may be reversed in the arguments above. But this involves, essentially, using the coordinates y, $-Kx$ on $\hat{S}$ (now called S), which exchanges the roles of $\hat{k}_1$ and $\hat{k}_2$ (now called k_2 and k_1) and of k_1 and k_2 (now called $\hat{k}_2$ and $\hat{k}_1$). We have maintained the original orientation on $\hat{S}$ (now called S) so that its second conformal structure is still defined. This requires the switch from t to $-t$ in order to reach S (now called $\hat{S}$) by the standard mapping between parallel surfaces.

Suppose, finally, that $f: R_2 \to \hat{R}_2$ is a Teichmüller mapping with $K > 1$. Theorem 2 below states that S must be free of umbilics. Thus Lemmas 1 and 2 yield coordinates u, v in the neighborhood of any point on S in terms of which

$$
(24)\qquad
\begin{aligned}
I &= \frac{\mu}{k_1}du^2 + \frac{\mu}{k_2}dv^2\,,\\
II &= \mu(du^2 + dv^2),\\
\hat{I} &= \frac{\hat{\mu}K^2}{\hat{k}_1}du^2 + \frac{\hat{\mu}}{\hat{k}_2}\,dv^2,\\
\hat{II} &= \hat{\mu}(K^2du^2 + dv^2).
\end{aligned}
$$

Since $\hat{L} = K^2\hat{N}$, (10) implies that

$$(1 - tk_1) = K^2(1 - tk_2),$$

and (19) holds on all of S. As to $\hat{S}$, since $\hat{k}_1\hat{E} = K^2\hat{k}_2\hat{G}$, we may use (10) and (24) to obtain (23) once more. Substitution of (23) in (19) yields (20). Note that (19) and (20) behave as they should if the roles of S and $\hat{S}$ are reversed.

Attention is called to Theorem 2 below. The conclusions there are, of course, applicable to the cases just discussed.

5. The results obtained in §4 relied heavily upon the very special nature of the standard mapping between parallel surfaces. In this section we consider a somewhat larger class of mappings, those which preserve normals and a net of lines of curvature. The conformal cases are studied first. The following comment will be of use.

LEMMA 4. *Let $f: S \to \hat{S}$ preserve normals. If $f: R_1 \to \hat{R}_2$, or $f: R_2 \to \hat{R}_1$ or $f: R_2 \to \hat{R}_2$ is conformal, then $f: S \to \hat{S}$ preserves a net of lines of curvature.*

Proof of Lemma 4. Choose a net of lines of curvature on S. In the neighborhood of any point which is not an irremovable umbilic, introduce lines of curvature coordinates x, y.

If f: $R_1 \to \hat{R}_2$ is conformal, then at points of S and $\hat{S}$ in correspondence under f, we have

$$
(25)\qquad
\begin{aligned}
I &= Edx^2 + Gdy^2,\\
II &= Ek_1dx^2 + Gk_2dy\;,\\
\hat{I} &= \hat{E}dx^2 + 2\hat{F}dxdy + \hat{G}dy^2,\\
\hat{II} &= \hat{\mu}(Edx^2 + Gdy^2).
\end{aligned}
$$

Preservation of normals implies that the coefficients of the first fundamental forms of the spherical image mappings of S and $\hat{S}$ must be equal. In particular (see [3, p. 253])

$$(26)\qquad h_{12} = 2\mathscr{H}M - \mathscr{K}F = 2\hat{\mathscr{H}}\hat{M} - \hat{\mathscr{K}}\hat{F},$$

so that

$$h_{12} = 0 = -\hat{\mathscr{K}}\hat{F}.$$

But the use of $\hat{R}_2$ structure implies that $\hat{\mathscr{K}} > 0$. Thus

$$\hat{F} = 0, \tag{27}$$

and x, y are lines-of-curvature coordinates on $\hat{S}$ as well as S. In case $f: R_2 \to \hat{R}_1$ is conformal, an interchange of the roles of S and $\hat{S}$ yields the desired result.

Suppose now that $f: R_2 \to \hat{R}_2$ is conformal. Then at points of S and $\hat{S}$ in correspondence under f we have

$$\begin{aligned} I &= E dx^2 + G dy^2, \\ II &= E k_1 dx^2 + G k_2 dy^2, \\ \hat{I} &= \hat{E} dx^2 + 2\hat{F} dx dy + \hat{G} dy^2, \\ \hat{II} &= \hat{\mu}(E k_1 dx^2 + G k_2 dy^2). \end{aligned} \tag{28}$$

Thus, (26) and (27) hold once again, and x, y are lines-of-curvature coordinates on $\hat{S}$ as well as S.

LEMMA 5. *Let $f: S \to \hat{S}$ preserve normals. If $f: R_1 \to \hat{R}_2$ is conformal, then*

$$k_1^2 \hat{k}_2 = k_2^2 \hat{k}_1 \tag{29}$$

holds at points of S and $\hat{S}$ in correspondence under f. If $f: R_2 \to \hat{R}_1$ is conformal, then

$$k_1 \hat{k}_2^2 = k_2 \hat{k}_1^2 \tag{30}$$

holds at points of S and $\hat{S}$ in correspondence under f. If $f: R_2 \to \hat{R}_2$ is conformal then

$$k_1 \hat{k}_2 = k_2 \hat{k}_1 \tag{31}$$

holds at points of S and $\hat{S}$ in correspondence under f.

Proof of Lemma 5. Choose x, y as in the proof of Lemma 4. Suppose $f: R_1 \to \hat{R}_2$ is conformal. Then (25) holds with $\hat{F} = 0$, and

$$\hat{E} = \frac{\hat{\mu} E}{\hat{k}_1}, \quad \hat{G} = \frac{\hat{\mu} G}{\hat{k}_2}.$$

But, since normals are preserved,

$$\begin{aligned} h_{11} &= 2\mathscr{H} L - \mathscr{K} E = 2\hat{\mathscr{H}}\hat{L} - \hat{\mathscr{K}}\hat{E}, \\ h_{22} &= 2\mathscr{H} N - \mathscr{K} G = 2\hat{\mathscr{H}}\hat{N} - \hat{\mathscr{K}}\hat{G}, \end{aligned} \tag{32}$$

so that

$$k_1^2 = \hat{\mu}\hat{k}_1, \quad k_2^2 = \hat{\mu}\hat{k}_2.$$

Thus, by continuity, (29) holds everywhere. In case $f: R_2 \to \hat{R}_1$ is conformal, exchanging the roles of S and $\hat{S}$, we obtain (30).

Suppose $f : R_2 \to \hat{R}_2$ is conformal. Then (28) holds with $\hat{F} = 0$, so that

$$\hat{E} = \frac{\hat{\mu}Ek_1}{\hat{k}_1}, \quad \hat{G} = \frac{\hat{\mu}Gk_2}{\hat{k}_2}.$$

Since normals are preserved, (32) yields

$$k_1 = \hat{\mu}\hat{k}_1, \quad k_2 = \hat{\mu}\hat{k}_2.$$

Thus, by continuity, (31) holds everywhere.

The equation (31) is also obtained, incidentally, if we apply Theorem 1 of [5], assuming that $f: S \to \hat{S}$ preserves normals between strictly convex surfaces, while $f: R_1 \to \hat{R}_1$ is conformal. This coincidence is easily explained. For the computations above reveal that when $f: S \to \hat{S}$ preserves normals and $f: R_2 \to \hat{R}_2$ is conformal, then (in the notation of (28)) $\hat{I} = \hat{\mu}^2 I$. In fact, the following can be said.

Remark. If $f: S \to \hat{S}$ preserves normals and $\mathscr{K}$, $\mathscr{H}$, $\hat{\mathscr{K}}$ and $\hat{\mathscr{H}} > 0$, then $f: R_1 \to \hat{R}_1$ is conformal if and only if $f: R_2 \to \hat{R}_2$ is conformal.

Turning now to nonconformal Teichmüller mappings, we will obtain, just as in Lemma 5, joint Weierstrass conditions $W(k_1, k_2; \hat{k}_1, \hat{k}_2) = 0$ relating the principal curvatures at points of S and $\hat{S}$ in correspondence under f.

Theorem 2. *Let $f: S \to \hat{S}$ preserve a net of lines of curvature, and normals. If $f: R_1 \to \hat{R}_2$ is a nonconformal Teichmüller mapping, then all umbilics on S are planar, and*

$$k_1^2\hat{k}_2 = K^2 k_2^2 \hat{k}_1 \tag{33}$$

holds at points of S and $\hat{S}$ in correspondence under f. If $f: R_2 \to \hat{R}_1$ is a nonconformal Teichmüller mapping, then all umbilics on $\hat{S}$ are planar, and

$$k_1\hat{k}_2^2 = K^2 k_2 \hat{k}_1^2 \tag{34}$$

holds at points of S and $\hat{S}$ in correspondence under f. If $f: R_2 \to \hat{R}_2$ is a nonconformal Teichmüller mapping, then S and $\hat{S}$ are free of umbilics, while

$$k_1\hat{k}_2 = K^2 \hat{k}_1 k_2 \tag{35}$$

holds at points of S and $\hat{S}$ in correspondence under f.

Proof of Theorem 2. Suppose $f: R_1 \to \hat{R}_2$ is a Teichmüller mapping and $K > 1$. Then Lemmas 1 and 2 guarantee that in the neighborhood of any point of S which is not an irremovable umbilic of the preserved net, coordinates x, y may be chosen in terms of which

$$\begin{aligned} I &= \lambda(dx^2 + dy^2), \\ II &= \lambda k_1 dx^2 + \lambda k_2 dy^2, \\ \hat{I} &= \frac{\hat{\mu}K^2}{\hat{k}_1} dx^2 + \frac{\hat{\mu}}{\hat{k}_2} dy^2, \\ \hat{II} &= \hat{\mu}(K^2 dx^2 + dy^2). \end{aligned}$$

But normals are preserved. Thus, recalling (26) and (32), we obtain

$$\begin{aligned} \lambda k_1^2 &= \hat{\mu}K^2\hat{k}_1, \\ \lambda k_2^2 &= \hat{\mu}\hat{k}_2. \end{aligned}$$

By continuity, therefore, (33) holds everywhere.

In case $f : R_2 \to \hat{R}_1$ is a nonconformal Teichmüller mapping, the claims of Theorem 2 can be verified by exchanging the roles of S and $\hat{S}$ in the discussion just completed. This involves the switch from k_1 to $\hat{k}_2$, k_2 to $\hat{k}_1$, $\hat{k}_1$ to k_2 and $\hat{k}_2$ to k_1.

Suppose $f : R_2 \to \hat{R}_2$ is a Teichmüller mapping with $K > 1$. Then Lemmas 1 and 2 guarantee that in the neighborhood of any point on S which is not an irremovable umbilic of the preserved net, coordinates u, v may be chosen in terms of which

$$\begin{aligned} I &= \frac{\mu}{k_1} du^2 + \frac{\mu}{k_2} dv^2, \\ II &= \mu(du^2 + dv^2), \\ \hat{I} &= \frac{\hat{\mu}K^2}{\hat{k}_1} du^2 + \frac{\hat{\mu}}{\hat{k}_2} dv^2, \\ \hat{II} &= \hat{\mu}(K^2 du^2 + dv^2). \end{aligned} \tag{36}$$

But, since normals are preserved, we may use (26) and (32) to obtain

$$\begin{aligned} \mu k_1 &= \hat{\mu}K^2\hat{k}_1, \\ \mu k_2 &= \hat{\mu}\hat{k}_2. \end{aligned} \tag{37}$$

By continuity, therefore, (34) holds everywhere.

Note now that umbilics are nonplanar on a strictly convex surface. Thus, since $K > 1$, (33), (34) and (35) justify the claims about umbilics made by Theorem 2.

A closing comment can be made about the last case covered by Theorem 2 The substitution of (37) into (36) reveals that

$$\hat{I} = \frac{\hat{\mu}^2}{\mu}\left(\frac{K^4}{k_1}du^2 + \frac{1}{k_2}dv^2\right).$$

Let g be the mapping which sends the point with coordinates x, y on S to the point with coordinates K^2u, v on $\hat{S}$.

REMARK. If $f: S \to \hat{S}$ preserves lines of curvature and normals, while $f: R_2 \to \hat{R}_2$ is a nonconformal Teichmüller mapping, then $g: R_1 \to \hat{R}_1$ is conformal.

6. The spherical image mapping of a surface onto the unit sphere, and the identity mapping of a surface onto itself are particular examples of the kind of of mapping discussed in §5. The results which follow are in fact just corollaries of Lemma 5 and Theorem 2.

THEOREM 3. *Let f be the spherical image mapping of S onto $\hat{S}$, part of the unit sphere. If $f: R_1 \to \hat{R}_2$ is a Teichmüller mapping, then*

$$k_1 = \pm Kk_2. \tag{38}$$

If $f: R_2 \to \hat{R}_1$ or $f: R_2 \to \hat{R}_2$ is a Teichmüller mapping, then

$$k_1 = K^2k_2. \tag{39}$$

Moreover, in all three cases, S is both isothermal and (if $\mathscr{H}, \mathscr{K} > 0$) bisothermal. Of course, S has no umbilics in the latter cases unless $K = 1$.

Proof of Theorem 3. Suppose $f: R_1 \to \hat{R}_2$ is a Teichmüller mapping. Then, since $\hat{S}$ is the unit sphere, $f: R_1 \to \hat{R}_1$ is a Teichmüller mapping with the same dilatation K. Theorem 3 of [5] yields (38). Naturally, this conclusion can be reached even more quickly by using (29) and (33) above. Similarly, $f: R_2 \to \hat{R}_1$ is a Teichmüller mapping if and only if $f: R_2 \to \hat{R}_2$ is a Teichmüller mapping. Using either (30) and (34), or (31) and (35), we obtain (39).

Suppose, moreover, that $\mathscr{H}$ and $\mathscr{K}$ are positive. Then when $f: R_1 \to \hat{R}_2$ is a Teichmüller mapping, $k_1 = Kk_2$. It is easily checked that here if x, y are isothermal lines-of-curvature coordinates on S, then $x\sqrt{K}$, y are bisothermal lines-of-curvature coordinates on S. Similarly if $f: R_2 \to \hat{R}_1$ (or $f: R_2 \to \hat{R}_2$) is a Teichmüller mapping, then bisothermal lines-of-curvature coordinates u, v on S yield isothermal lines-of-curvature coordinates u/K, v on $\hat{S}$.

THEOREM 4. *If the identity mapping $f: S \to S$ is a Teichmüller mapping $f: R_1 \to R_2$, then S is both isothermal and bisothermal while*

$$k_1 = K^2k_2. \tag{40}$$

Proof of Theorem 4. Using (29) and (33) we obtain (40). The isothermal lines-of-curvature coordinates x, y on S yield bisothermal lines-of-curvature coordinates Kx, y on S.

REMARK. Let $\mathscr{K}$, $\mathscr{H} > 0$ on S. Then the following statements are equivalent.

1. The spherical image mapping of S is a Teichmüller mapping from R_1.
2. The spherical image mapping of S is a Teichmüller mapping from R_2.
3. The identity mapping of S is a Teichmüller mapping between R_1 and R_2.

BIBLIOGRAPHY

1. L. Ahlfors, *On quasiconformal mappings*, J. Analyse Math. **3** (1954), 1–58.

2. L. Bers, *Quasiconformal mappings and Teichmüller's theorem*, pp. 89–119, Conference on Analytic Functions, Princeton, N. J., 1960.

3. L. Eisenhart, *An introduction to differential geometry*, Princeton Univ. Press, Princeton, N. J., 1947.

4. H. Hopf, *Lectures on differential geometry in the large*, Mimeographed notes, Stanford University, Stanford, Calif., 1956.

5. T. Klotz, *The geometry of extremal quasiconformal mappings*, Michigan Math. J. **9** (1962), 129–136.

6. ———, *Some uses of the second conformal structure on strictly convex surfaces*, Proc. Amer. Math. Soc. (to appear).

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