RECURSIVE ENUMERABILITY AND THE JUMP OPERATOR

BY

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By degree we mean degree of recursive unsolvability as defined by Kleene and Post in [4]. Following Shoenfield [7], we say a degree $\mathbf{c}$ is recursively enumerable in a degree $\mathbf{b}$ if there is a set of degree $\mathbf{c}$ which is the range of a function of degree less than or equal to $\mathbf{b}$, and we call a degree recursively enumerable if it is recursively enumerable in $\mathbf{0}$ (i.e., if it is the degree of a recursively enumerable set). The jump operator, which takes the degree $\mathbf{d}$ to the degree $\mathbf{d}'$ (the completion of $\mathbf{d}$), was defined in [4] and has the following properties: if $\mathbf{h}$ is recursively enumerable in $\mathbf{d}$, then $\mathbf{h} \leqq \mathbf{d}'$; $\mathbf{d}' > \mathbf{d}$; and $\mathbf{d}'$ is recursively enumerable in $\mathbf{d}$. In [4] a degree $\mathbf{c}$ is said to be complete if there exists a degree $\mathbf{d}$ such that $\mathbf{d}' = \mathbf{c}$. Friedberg [1] showed that a degree $\mathbf{c}$ is complete if and only if $\mathbf{c} \geqq \mathbf{0}'$.

For any degree $\mathbf{b}$, if $\mathbf{b} \leqq \mathbf{d} \leqq \mathbf{b}'$, then $\mathbf{b}' \leqq \mathbf{d}' \leqq \mathbf{b}$ and $\mathbf{d}'$ is recursively enumerable in $\mathbf{b}'$. Shoenfield [7] proved that if $\mathbf{b}' \leqq \mathbf{c} \leqq \mathbf{b}''$ and $\mathbf{c}$ is recursively enumerable in $\mathbf{b}'$, then there is a degree $\mathbf{d}$ such that $\mathbf{b} \leqq \mathbf{d} \leqq \mathbf{b}'$ and $\mathbf{d}' = \mathbf{c}$. Thus the degrees which lie between $\mathbf{b}'$ and $\mathbf{b}''$ and are recursively enumerable in $\mathbf{b}'$ can be viewed as the completions of the degrees which lie between $\mathbf{b}$ and $\mathbf{b}'$. He also showed there is a degree greater than $\mathbf{b}$ and less than $\mathbf{b}'$ which is not recursively enumerable in $\mathbf{b}$.

Our main result below is that the degrees which lie between $\mathbf{b}'$ and $\mathbf{b}''$ and are recursively enumerable in $\mathbf{b}'$ can be viewed as the completions of the degrees which lie between $\mathbf{b}$ and $\mathbf{b}'$ and are recursively enumerable in $\mathbf{b}$. Our notation is that of [3].

THEOREM 1. *Let* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$ *be degrees such that* $\mathbf{a} \nleqq \mathbf{b}$, $\mathbf{a} \leqq \mathbf{b}' \leqq \mathbf{c}$ *and* $\mathbf{c}$ *is recursively enumerable in* $\mathbf{b}'$. *Then there exists a degree* $\mathbf{d}$ *such that* $\mathbf{a} \nleqq \mathbf{d}$, $\mathbf{b} \leqq \mathbf{d}$, $\mathbf{d}' = \mathbf{c}$ *and* $\mathbf{d}$ *is recursively enumerable in* $\mathbf{b}$.

Proof. We first prove the theorem when $\mathbf{b} = \mathbf{0}$, and then indicate the changes needed when $\mathbf{b} > \mathbf{0}$. Thus we have degrees $\mathbf{a}$ and $\mathbf{c}$ such that $\mathbf{a} > \mathbf{0}$, $\mathbf{a} \leqq \mathbf{0}' \leqq \mathbf{c}$ and $\mathbf{c}$ is recursively enumerable in $\mathbf{0}'$, and we wish find a recursively enumerable degree $\mathbf{d}$ such that $\mathbf{a} \nleqq \mathbf{d}$ and $\mathbf{d}' = \mathbf{c}$.

Let f be a function of degree less than or equal to $\mathbf{0}'$ whose range is a set C of

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degree **c**. Let **c** be the representing function of C. Let g be a recursive function whose range is a set J of degree $\mathbf{0}'$. Let j be the representing function of J. We define

$$j(s,n) = \begin{cases} 0 & \text{if } (Ek)_{k<s}(g(k)=n), \\ 1 & \text{otherwise.} \end{cases}$$

It is clear that $j(s, n)$ is a recursive function, and that for each n, $\lim_s j(s,n)$ exists and is equal to $j(n)$. Since f is recursive in j, there is a Gödel number z_1 such that

$$f(n) = \{z_1\}^j(n) = U(\mu y T_1^1(\bar{j}(y), z_1, n, y))$$

for all n. We define a recursive function $f(s,n)$ of supreme importance to our argument;

$$f(s,n) = \begin{cases} U\left(\mu y T_1^1\left(\prod_{i<y} p_i^{j(s,n)}, z_1, n, y\right)\right) & \\ \qquad \text{if} (Ey)_{y \leqq s} T_1^1\left(\prod_{i<y} p_i^{j(s,n)}, z_1, n, y\right), & \\ s+1 \qquad \text{otherwise.} & \end{cases}$$

We claim that $\lim_s f(s,n)$ exists and is equal to $f(n)$ for all n. Our claim is a consequence of the fact that $f(n) = \{z_1\}^j(n)$ and $\lim_s j(s,n) = j(n)$ for all n.

Let a be an everywhere positive function of degree **a**, and let z_2 be a Gödel number such that $\{z_2\}^j(n) = a(n)$ for all n. We define

$$a(s,n) = \begin{cases} U\left(\mu y T_1^1\left(\prod_{i<y} p_i^{j(s,n)}, z_2, n, y\right)\right) & \\ \qquad \text{if } (Ey)_{y \leqq s}\left[T_1^1\left(\prod_{i<y} p_i^{j(s,n)}, z_2, n, y\right) \& U(y) \geqq 1\right], & \\ 1 \qquad \text{otherwise.} & \end{cases}$$

The function $a(s,n)$ is recursive; for each n, $\lim_s a(s,n)$ exists and is equal to $a(n)$.

A useful property of the Gödel numbering devised by Kleene in [3] to arithmetize his formalism for recursive functions is: the Gödel number of a deduction is greater than the intuitive counterpart of any formal numeral occurring in the deduction. We will denote this fact by GND. It follows from GND that $a(s,n) = 1$ whenever $n \geqq s$.

We define two recursive functions, $t(s,n)$ and $h(s,n)$, by means of an induction on s:

$$\begin{aligned} t(s,n) &= \mu m_{m<s}(f(s,m) = n); \\ h(0,n) &= 0; \\ h(s+1,n) &= h(s,n) + \mathrm{sg}(|t(s+1,n) - t(s,n)|). \end{aligned}$$

Recall that the bounded least number operator is defined in such a way that $t(s,n) = s$ if and only if there is no $m < s$ such that $f(s,m) = n$.

We now proceed to define four recursive functions, $y(s,n,e)$, $m(s,e)$, $r(s,n,e)$ and $d(s,n)$, simultaneously by induction on s. The function $d(s,n)$ will be such that

$$0 \leqq d(s+1,n) \leqq d(s,n) \leqq 1$$

for all s and n. Thus for each n, $\lim_s d(s,n)$ will exist; furthermore, $\lim_s d(s,n)$ will be the representing function of a recursively enumerable set D. The degree of D will be the desired degree $\mathbf{d}$. At stage s of the construction we put finitely or infinitely many natural numbers in D; our main objective is to see that $\mathbf{c} \leqq \mathbf{d}'$; however, with the aid of a system of priorities, we exercise restraint when we add members to D in order to insure that $\mathbf{a} \nleqq \mathbf{d}$ and $\mathbf{d}' \leqq \mathbf{c}$.

Stage $s=0$. We set $y(0,n,e) = r(0,n,e) = 0$, $m(0,e) = e+1$ and $d(0,n) = 1$ for all n and e.

Stage $s > 0$. We define $y(s,n,e)$ for all n and e:

$$y(s,n,e) = \begin{cases} \mu y T_1^1\left(\prod_{i<y} p_i^{d(s-1,i)}, e, n, y\right) \\ \qquad \text{if } n \geqq e \,\&\, (Ey)_{y \leqq s} T_1^1\left(\prod_{i<y} p_i^{d(s-1,i)}, e, n, y\right), \\ 0 \quad \text{otherwise.} \end{cases}$$

It follows from GND that $y(s,n,e) = 0$ whenever $n \geqq s$.

We define $m(s,e)$ for all e; there are three mutually exclusive cases.

Case 1. $y(s,e,e) = 0$. We set $m(s,e) = e+1$.

Case 2. $y(s,e,e) > 0$ and there is an n such that

$$e < n < m(s-1,e) \,\&\, y(s,n,e) \neq y(s-1,n,e) \,\&\, a(s,n) \neq U(y(s,n,e)).$$

We set

$$m(s,e) = \mu n_{e<n}[y(s,n,e) \neq y(s-1,n,e) \,\&\, a(s,n) \neq U(y(s,n,e))].$$

Case 3. Otherwise.

We set

$$m(s,e) = \mu n[m(s-1,e) \leqq n < 2m(s-1,e) + s \\ \&(Et)\ (e < t \leqq n \,\&\, a(s,t) \neq U(y(s,t,e))].$$

Note that Case 3 of the definition of $m(s,e)$, the least number operator is bounded.

We define $r(s,n,e)$ and $d(s,p_e^n)$ for all n and e by means of a simultaneous induction on e. Let $e \geqq 0$ and suppose $r(s,n,i)$ and $d(s,p_i^n)$ have been defined for all $i < e$ and all n; we define $r(s,n,e)$ and $d(s,p_e^n)$ for all n as follows:

$$r(s,n,e) = \begin{cases} 0 \quad \text{if } (Ei)(Em)(Et)[i < e \leqq t \leqq n \,\&\, p_i^m < y(s,t,e) \\ \qquad\qquad \&\, d(s,p_i^m) \neq d(s-1,p_i^m)], \\ 1 \quad \text{otherwise;} \end{cases}$$

$$d(s,p_e^n) = \begin{cases} d(s-1,p_e^n) & \text{if } n \geqq h(s,e), \\ d(s-1,p_e^n) & \text{if } (Ei)(Em)[i \leqq e \,\&\, i \leqq m < m(s,i) \\ & \quad \& \, r(s,m,i) = 1 \,\&\, p_e^n < y(s,m,i)], \\ 0 & \text{otherwise.} \end{cases}$$

We conclude the construction by setting $d(s,n) = d(s-1,n)$ for all n not a power of a prime. It is readily verified by the method of [4] that each of the four functions just defined is recursive. Such a verification is possible for two reasons: each of the functions $a(s,n)$ and $h(s,n)$ is recursive; at stage $s > 0$, all quantifiers, as well as all applications of the least number operator, are bounded. For each n, let

$$d(n) = \lim_s d(s,n);$$

it is clear that $d(n) = 0$ if and only if there is an s such that $d(s,n) = 0$. Thus d is the representing function of a recursively enumerable set. Let $\mathbf{d}$ be the degree of d.

We list some remarks which will be needed in vital parts of the body of our argument:

(R1) $(s)(e)[m(s,e) > e]$;

(R2) $(s)(n)(e)[r(s,n,e) = 0 \rightarrow r(s,n+1,e) = 0]$;

(R3) $(s)(n)(e)[(y(s,n,e) = 0 \,\&\, n > e) \rightarrow m(s,e) \leqq n]$.

Remark (R1) is easily proved by induction on s if the definition of the bounded least number operator is kept in mind.

We prove remark (R3) by induction on s. We have

$$(n)(e)[(y(0,n,e) = 0 \,\&\, n > e) \rightarrow m(0,e) \leqq n].$$

Let s be such that $s > 0$ and

$$(n)(e)[(y(s-1,n,e) = 0 \,\&\, n > e) \rightarrow m(s-1,e) \leqq n].$$

Let e and n be such that

$$y(s,n,e) = 0 \qquad \text{and } n > e.$$

Then $a(s,n) \neq U(y(s,n,e))$, since $a(s,n) \geqq 1$ and $U(0) = 0$. First we suppose $n < m(s-1,e)$. Then $y(s-1,n,e) > 0$ as a consequence of the induction hypothesis. But then either Case 1 or Case 2 of the definition of $m(s,e)$ holds, and so $m(s,e) \leqq n$. Now we suppose $m(s-1,e) \leqq n$. If either Case 1 or Case 2 of the definition of $m(s,e)$ holds, then $m(s,e) \leqq m(s-1,e) \leqq n$ by remark (R1). If Case 3 holds and $n < 2m(s-1,e) + s$, then $m(s,e) \leqq n$. If Case 3 holds and $n \geqq 2m(s-1,e) + s$, then $m(s,e) \leqq 2m(s-1,e) + s \leqq n$. (Note that if Case 3 holds and

$$(t)[e < t < 2m(s-1,e) + s \rightarrow a(s,t) = U(y(s,t,e))],$$

then $m(s,e) = m(s-1,e) + s$; this last is a consequence of the definition of the bounded least number operator.)

We introduce two predicates:

$A(e)$: if the set $\{m(s,e) \mid s \geqq 0\}$ is infinite, then there is an $n \geqq e$ such that $\lim_s y(s,n,e)$ either does not exist or is equal to 0.

$B(e)$: $\lim_n d(p_e^n)$ exists and is equal to $1 - c(e)$.

We will prove $(e)A(e)$ and $(e)B(e)$ by means of a simultaneous induction on e. From $(e)A(e)$ it will follow that $\mathbf{a} \nleqq \mathbf{d}$. From $(e)B(e)$ it will follow that $\mathbf{c} \leqq \mathbf{d}'$. Fix $e^* \geqq 0$ and suppose $A(e)$ and $B(e)$ are true for all $e < e^*$. We proceed to prove $A(e^*)$ and $B(e^*)$.

Lemma 1. *Let $y(s,n,e^*) > 0$ and $m(s,e^*) > n \geqq e^*$. Let $d(s,p_i^m) = d(s-1,p_i^m)$ for all i, m and t such that $i < e^* \leqq t \leqq n$ and $p_i^m < y(s,t,e^*)$. Then $y(s,n,e^*) = y(s+1,n,e^*)$.*

Proof. Since $y(s,n,e^*) > 0$, we have

$$y(s,n,e^*) = \mu y T_1^1\left(\prod_{i<y} p_i^{d(s-1,i)}, e^*, n, y\right).$$

We suppose $y(s+1,n,e^*) \neq y(s,n,e^*)$ and then show there is an i, an m and a t such that

$$i < e^* \leqq t \leqq n \,\&\, p_i^m < y(s,t,e^*) \,\&\, d(s,p_i^m) \neq d(s-1,p_i^m).$$

Since $y(s+1,n,e^*) \neq y(s,n,e^*)$, there must be a $j < y(s,n,e^*)$ such that $d(s,j) \neq d(s-1,j)$. Recall that $d(s,w) = d(s-1,w)$ for all w not a power of a prime. Thus there is an i' and an m' such that

$$d(s,p_{i'}^{m'}) \neq d(s-1,p_{i'}^{m'})$$

and $p_{i'}^{m'} < y(s,n,e^*)$. But then by the hypothesis of the lemma, $e^* \leqq i'$. Thus we have

$$e^* \leqq i' \,\&\, e^* \leqq n < m(s,e^*) \,\&\, p_{i'}^{m'} < y(s,n,e^*) \,\&\, d(s,p_{i'}^{m'}) \neq d(s-1,p_{i'}^{m'}).$$

It follows from the definition of $d(s,p_{i'}^{m'})$ that $r(s,n,e^*) = 0$. But this last means the desired i, m and t exist.

Lemma 2. *Let $y(s,n,e^*) > 0$ and $m(s,e^*) > n > e^*$. Let $d(s,p_i^m) = d(s-1,p_i^m)$ for all i, m and t such that $i < e^* \leqq t \leqq n$ and $p_i^m < y(s,t,e^*)$. Then $m(s+1,e^*) > n$.*

Proof. Since $m(s,e^*) > n > e^*$, it follows from remark (R3) and Case 1 of the definition of $m(s,e)$ that

$$y(s,t,e^*) > 0$$

for all t such that $e^* \leqq t \leqq n$. But then by Lemma 1,

$$y(s,t,e^*) = y(s+1,t,e^*)$$

for all t such that $e^* \leqq t \leqq n$. Suppose $m(s+1,e^*) \leqq n$. Then $m(s+1,e^*) < m(s,e^*)$, and consequently, Case 2 of the definition of $m(s+1, e^*)$ holds. This means there is a t (namely, $m(s+1, e^*)$) such that

$$e^* < t \leqq n \,\&\, y(s, t, e^*) \neq y(s, +1, t, e^*).$$

LEMMA 3. $A(e^*)$.

Proof. By the hypothesis of our theorem the function a is nonrecursive. We suppose $A(e^*)$ is false and show a is recursive. Thus the set $\{m(s, e^*) \mid s \geqq 0\}$ is infinite, and for each $n \geqq e^*$, $\lim_s y(s, n, e^*)$ exists and is positive. Let $R(n, s)$ denote the predicate

$$m(s, e^*) > n \,\&(e)(m)(t)[(p_e^m < y(s, t, e^*) \,\&\, e < e^* \leqq t \leqq n) \to d(s-1, p_e^m) = d(p_e^m)].$$

We know $B(e)$ is true for all $e < e^*$. This means $\lim_m d(p_e^m)$ exists for all $e < e^*$. For each $e < e^*$, let $g(e)$ be such that

$$(m)[m \geqq g(e) \to d(p_e^m) = d(p_e^{g(e)})].$$

We define a recursive function $z(n)$ as follows: first we require that $z(n) = 1$ for all n not a power of a prime; then we specify

$$z(p_e^m) = \begin{cases} d(p_e^{g(e)}) & \text{if } e > e^* \,\&\, m \geqq g(e), \\ d(p_e^m) & \text{if } e > e^* \,\&\, m < g(e), \\ 1 & \text{otherwise.} \end{cases}$$

The predicate $R(n, s)$ can now be rewritten as

$$m(s, e^*) > n \,\&(e)(m)(t)[(p_e^m < y(s, t, e^*) \,\&\, e < e^* \leqq t \leqq n) \to d(s-1, p_e^m) = z(p_e^m)].$$

It is clear that $R(n, s)$ is recursive, since the functions m, y and z are recursive.

Now we show $(n)(Es)R(n, s)$. Fix n. Since $\lim_s y(s, n, e^*)$ exists for all $n \geqq e^*$ there is a y such that

$$y \geqq y(s, t, e^*)$$

for all t and s such that $e^* \leqq t \leqq n$. Let s' be so large that

$$d(s-1, w) = d(w)$$

for all s and w such that $s \geqq s'$ and $w < y$. Since the set $\{m(s, e^*) \mid s \geqq 0\}$ is infinite, there is an $s \geqq s'$ such that $m(s, e^*) > n$. But then $R(n, s)$.

Let $w(n)$ denote the recursive function $\mu s R(n, s)$. Note that $w(n+1) \geqq w(n)$ for all n.

Next we prove $y(w(n), n, e^*) = \lim_s y(s, n, e^*)$ for all $n > e^*$. Fix $n > e^*$. We show by induction on s that $y(w(n), n, e^*) = y(s, n, e^*)$ for all $s \geqq w(n)$. Let s be such that $s \geqq w(n)$ and

$$y(w(n), e^*) = y(s, n, e^*) \,\&\, R(n, s).$$

Since $m(s, e^*) > n > e^*$, it follows from remark (R3) and Case 1 of the definition of $m(s, e^*)$ that $y(s, t, e^*) > 0$ for all t such that $e^* \leqq t \leqq n$. By the definition of $R(n, s)$, we have

$$d(s-1, p_e^m) = d(p_e^m)$$

for all e, m and t such that $e < e^* \leqq t \leqq n$ and $p_e^m < y(s, t, e^*)$. Recall that if $d(s-1, w) = d(w)$ then $d(s', w) = d(w)$ for all $s' \geqq s$. It follows from Lemma 1 that

$$y(s, t, e^*) = y(s+1, t, e^*)$$

for all t such that $e^* \leqq t \leqq n$. It follows from Lemma 2 that

$$m(s+1, e^*) > n.$$

But then

$$y(w(n), n, e^*) = y(s+1, n, e^*) \,\&\, R(n, s+1).$$

Thus $y(w(n), n, e^*) = y(s, n, e^*)$ for all $s \geqq w(n)$, and $\lim_s y(s, n, e^*) = y(w(n), n, e^*)$.

Finally, we show by means of a reductio ad absurdum that

$$a(n) = U(y(w(n), n, e^*))$$

for all $n > e^*$. It will then follow that a is recursive, since w is recursive. Fix $n > e^*$ and suppose $a(n) \neq U(y(w(n), n, e^*))$. Since $y(w(n), n, e^*) = \lim_s y(s, n, e^*)$, and since $a(n) = \lim_s a(s, n)$, there is an s^* such that for all $s \geqq s^*$,

$$a(n) = a(s, n) \neq U(y(s, n, e^*)) = U(y(w(n), n, e^*)).$$

Let $s > s^*$ and suppose $m(s-1, e^*) \leqq m(s^*, e^*) + n + e^* + 1$. If either Case 1 or Case 2 of the of the definition of $m(s, e^*)$ holds, then

$$m(s, e^*) \leqq \max(e^* + 1, m(s-1, e^*)) \leqq m(s^*, e^*) + n + e^* + 1.$$

If Case 3 holds and $n < 2m(s-1, e^*) + s$, then $m(s, e^*) \leqq n$. If Case 3 holds and $2m(s-1, e^*) + s \leqq n$, then $m(s, e^*) \leqq n$. Thus we have shown by induction on s that

$$m(s, e^*) \leqq m(s^*, e^*) + n + e^* + 1$$

for all $s \geqq s^*$. But this last is absurd, since the set $\{m(s, e^*) \mid s \geqq 0\}$ is infinite.

For each $e \geqq 0$, we say e is stable if for all $n \geqq e$, $\lim_s y(s, n, e)$ exists and is positive. Note that if e is not the Gödel number of a system of equations, then $y(s, n, e) = 0$ for all s and n, and consequently, e is not stable. It follows that there are infinitely many e which are not stable, since there are infinitely many e which are not Gödel numbers of systems of equations. We define

$$e_0 = \mu e \text{ (}e\text{ is not stable)};$$

$$e_{j+1} = \mu e \text{ (}e > e_j \text{ and } e \text{ is not stable)}.$$

Thus $e_0 < e_1 < e_2 < \cdots$ is a listing of all the e which are not stable. For each $j \geqq 0$, let n_j be the least $n \geqq e_j$ such that $\lim_s y(s, n, e_j)$ either does not exist or is equal to 0.

The most important part of our argument is contained in Lemma 4. If the proof of our theorem is a heavy meal, then the proof of Lemma 4 is the main course; furthermore, it is there that the combinatorial flavor of our reasoning is strongest.

LEMMA 4. *For each k and v, there is an $s \geqq v$ such that*

$$(j)_{j<k}[m(s, e_j) \leqq n_j \vee r(s, n_j, e_j) = 0 \vee y(s, n_j, e_j) = 0].$$

Proof. Fix k and v. We suppose there is no s with the properties required by the lemma, and then show it is possible to define an infinite, descending sequence of natural numbers.

We propose the following system of equations as a means of defining two functions, $S(t)$ and $M(t)$, simultaneously by induction:

$$S(0) = \mu s \ (s \geqq v);$$

$$M(t) = \mu j[j < k \,\&\, n_j < m(S(t), e_j)$$
$$\&\, r(S(t), n_j, e_j) = 1 \,\&\, y(S(t), n_j, e_j) > 0];$$

$$S(t+1) = \mu s(Em)[s \geqq S(t) \,\&\, m < y(S(t), n_{M(t)}, e_{M(t)})$$
$$\&\, d(s, m) \neq d(S(t) - 1, m)].$$

Clearly $S(0)$ is well defined and greater than or equal to v. Suppose $t \geqq 0$ and $S(t)$ is well defined and greater than or equal to v. Then $M(t) < k$, since we have supposed the lemma to be false. Thus

$$y(S(t), n_{M(t)}, e_{M(t)}) > 0$$

and $\lim_s y(s, n_{M(t)}, e_{M(t)})$ does not exist or is equal to 0. Then there must be an $s > S(t)$ such that

$$y(s, n_{M(t)}, e_{M(t)}) \neq y(S(t), n_{M(t)}, e_{M(t)});$$

note that $S(t) > 0$, since $y(0, n, e) = 0$ for all n and e; this means there is an $s > S(t)$ and an m such that

$$m < y(S(t), n_{M(t)}, e_{M(t)}) \,\&\, d(s-1, m) \neq d(S(t) - 1, m).$$

Then $S(t+1)$ is well defined and greater than or equal to v.

For each $t \geqq 0$, let

$$u(t) = \mu m[d(S(t+1), m) \neq d(S(t) - 1, m)].$$

Now we show $u(t) < u(t-1)$ for all $t > 0$. Fix $t > 0$. Since we have

$$u(t) < y(S(t), n_{M(t)}, e_{M(t)})$$

by definition of u, it will be sufficient to show

$$y(S(t), n_{M(t)}, e_{M(t)}) \leqq u(t-1).$$

Since $d(w) = 1$ for all w not a power of a prime, there must exist i and m such that $u(t-1) = p_i^m$. Note that $d(S(t), u(t-1)) \neq d(S(t)-1, u(t-1))$; this last follows from the definitions of $S(t)$ and $u(t-1)$. Let

$$e = e_{M(t)},\ s = S(t) \text{ and } n = n_{M(t)}.$$

First we suppose $i < e$. This means

$$i < e \leqq n \,\&\, d(s, p_i^m) \neq d(s-1, p_i^m) \,\&\, r(s,n,e) = 1,$$

since $M(t) < k$. But then it follows from the definition of $r(s,n,e)$ that $y(s,n,e) \leqq p_i^m$. Now we suppose $i \geqq e$. This means

$$e \leqq i \,\&\, e \leqq n < m(s,e) \,\&\, r(s,n,e) = 1 \,\&\, d(s, p_i^m) \neq d(s-1, p_i^m),$$

since $M(t) < k$. But then it follows from the definition of $d(s, p_i^m)$ that $y(s,n,e) \leqq p_i^m = u(t-1)$.

Lemma 5. *If $c(e^*) = 0$, then $\lim_n d(p_{e^*}^n)$ exists and is equal to* 1.

Proof. Let t be the least m such that $f(m) = e^*$. Let s' be so large that $s' > t$ and $f(s,m) = f(m)$ for all s and m such that $s \geqq s'$ and $m \leqq t$. Then $t(s,e^*) = t$ for all $s \geqq s'$, and consequently, $h(s,e^*) = h(s',e^*)$ for all $s \geqq s'$. But then

$$d(s, p_{e^*}^n) = d(s-1, p_{e^*}^n)$$

for all s and n such that $s > 0$ and $n \geqq h(s',e^*)$, since $h(s,e^*) \leqq h(s',e^*)$ for all $s \leqq s'$. It follows that $\lim_s d(s, p_{e^*}^n) = 1$ for all $n \geqq h(s',e^*)$, since $d(0,w) = 1$ for all w. Then $d(p_{e^*}^n) = 1$ for all $n \geqq h(s',e^*)$, and $\lim_n d(p_{e^*}^n) = 1$.

Lemma 6. *If $c(e^*) = 1$, then $\lim_n d(p_{e^*}^n)$ exists and is equal to* 0.

Proof. First we show that the set $\{t(s,e^*) \mid s \geqq 0\}$ is infinite. Suppose $t(s,e^*) \leqq t$ for all s. Let s' be so large that $s' > t$ and $f(s,m) = f(m)$ for all s and m such that $s \geqq s'$ and $m \leqq t$. Then $f(s', t(s',e^*)) = e^*$, since $t(s',e^*) < s'$. But

$$f(s', t(s',e^*)) = f(t(s',e^*)),$$

since $t(s',e^*) \leqq t$. But then $f(t(s',e^*)) = e^*$; this last is impossible because C is the range of f and $c(e^*) = 1$.

Since the set $\{t(s,e^*) \mid s \geqq 0\}$ is infinite, it is clear that the set $\{h(s,e^*) \mid s \geqq 0\}$ is infinite.

By Lemma 3, we know $A(e)$ holds for all $e \leqq e^*$. This means that if $e \leqq e^*$ and e

is stable, then the set $\{m(s,e^*) \mid s \geqq 0\}$ is finite. If $e \leqq e^*$ and e is stable, let $m(e)$ be the greatest member of $\{m(s,e^*) \mid s \geqq 0\}$; if $e \leqq e^*$ and e is not stable, let $m(e) = n_j$, where j is such that $e = e_j$. If $e \leqq e^*$ and $e \leqq m < m(e)$, then $\lim_s y(s,m,e)$ exists. Let y be so large that

$$y \geqq y(s,m,e)$$

for all s, m and e such that $e \leqq e^*$ and $e \leqq m < m(e)$.

Fix $n > y$. We show $d(p^n_{e^*}) = 0$. It will suffice to find an s such that $d(p^n_{e^*}, s) = 0$. Let v be such that $h(v,e^*) > n$. Let k be such that if $e \leqq e^*$ and e is not stable, then $e = e_j$ for some $j < k$. By Lemma 4 there is an $s \geqq v$ such that

$$(j)_{j<k}[m(s,e_j) \leqq n_j \vee r(s,n_j,e_j) = 0 \vee y(s,n_j,e_j) = 0].$$

We will show:

$$h(s,e^*) > n;$$

$$(e)(m)[(e \leqq e^* \,\&\, e \leqq m < m(s,e)) \rightarrow (r(s,m,e) = 0 \vee p^n_{e^*} \geqq y(s,m,e))].$$

It will then follow from the definition of $d(s,p^n_{e^*})$ that $d(s,p^n_{e^*}) = 0$. We have $h(s,e^*) > n$, since $s \geqq v$ and $h(s,e^*)$ is a nondecreasing function of s. Fix e and m so that $e \leqq e^*$ and $e \leqq m < m(s,e)$. Suppose e is stable. Then $m < m(e)$, since $m(s,e) \leqq m(e)$. But then $y \geqq y(s,m,e)$, and consequently $p^n_{e^*} \geqq y(s,m,e)$, since $n > y$.

Now suppose e is not stable. Then $e = e_j$, where $j < k$, and $m(e) = n_j$. If $m < n_j$, then $m < m(e)$ and $p^n_{e^*} \geqq y(s,m,e)$. Suppose $m \geqq n_j$. Then $m(s,e_j) > n_j$. This last means that either $r(s,n_j,e) = 0$ or $y(s,n_j,e) = 0$. If $r(s,n_j,e) = 0$, then by remark (R2), $r(s,m,e) = 0$, since $m \geqq n_j$. Suppose $y(s,n_j,e) = 0$. Since $n_j \leqq m < m(s,e)$, it follows from remark (R3) that $n_j = e$. But then $y(s,e,e) = 0$, and Case 1 of the definition of $m(s,e)$ holds. It follows that $m(s,e) = e + 1, m = e$ and $y(s,m,e) = 0$.

Thus $d(p^n_{e^*}) = 0$ for all $n > y$, and $\lim_n d(p^n_{e^*})$ exists and is equal to 0.

Lemmas 5 and 6 constitute a proof of $B(e^*)$. That concludes our proof by induction of $(e)A(e)$ and $(e)B(e)$. It is now easily seen that $\mathbf{c} \leqq \mathbf{d}'$. Observe that

$$(e)(Et)[(m)_{m\geqq t}(d(p^m_e) = 1) \vee (m)_{m\geqq t}(d(p^m_e) = 0)]$$

is an immediate consequence of $(e)B(e)$. We define

$$k(e) = \mu t[(m)_{m\geqq t}(d(p^m_e) = 1) \vee (m)_{m\geqq t}(d(p^m_e) = 0)].$$

The function k has degree less than or equal to $\mathbf{d}'$, and by (e)B(e),

$$c(e) = 1 - d(p^{k(e)}_e)$$

for all e.

Lemma 7. $\mathbf{a} \nleqq \mathbf{d}$.

Proof. We suppose there is a Gödel number e such that

$$a(n) = \{e\}\ (n)$$

for all n, and then show $A(e)$ is false. First we show that $\lim_s y(s,n,e)$ exists and is positive for all $n \geqq e$. Fix $n \geqq e$; let

$$w = \mu y T_1^1(\tilde{d}(y), e, n, y).$$

Let s' be so large that $d(s,m) = d(m)$ whenever $s \geqq s'$ and $m < w$. Then

$$y(s,n,e) = w \mathbin{\&} U(w) = a(n)$$

for all $s \geqq s' + w$; $w > 0$, since 0 is not the Gödel number of a deduction.

Now we show the set $\{m(s,e) \mid s \geqq 0\}$ is infinite. We fix $m > e$ and obtain an s' such that $m(s',e) > m$. Let s be so large that $s > m$ and

$$a(s,t) = a(t) = \{e\}^d(t) = U(y(s,t,e))$$

for all t such that $e \leqq t \leqq m$. If $m(s-1,e) > m$, then $s-1$ is the desired s'. Suppose $m(s-1,e) \leqq m$. This means

$$a(s,t) = U(y(s,t,e))$$

for all t such that $e \leqq t \leqq m(s-1,e)$; in addition, $y(s,e,e) > 0$, since $a(s,e) \geqq 1$ and $U(0) = 0$. But then Case 3 of the definition of $m(s,e)$ holds. It follows that $m(s,e) > m$, since $s > m$.

Lemma 8. $\mathbf{d}' \leqq \mathbf{c}$.

Proof. We will define two functions, $E(e,n)$ and $L(e)$, simultaneously by induction on e so that each is recursive in the function $c(n)$. We will combine the definition of E and L with a proof by induction on e of the following:

$$(e)(n)\{E(e,n) = 0 \leftrightarrow [n \geqq e \mathbin{\&} (w)(Es)(s > w \mathbin{\&} m(s,e) > n)$$
$$\mathbin{\&} (m)(n \geqq m \geqq e \rightarrow (Ey)\, T_1^1(\tilde{d}(y), e, m, y))]\};$$

$$(e)(m)[m \geqq L(e) \rightarrow d(p_e^m) = d(p_e^{L(e)})].$$

It follows immediately from the above and remark (R1) that

$$(e)[E(e,e) = 0 \leftrightarrow (Ey)T_1^1(\tilde{d}(y), e, e, y)];$$

but then if E is recursive in c, we have $\mathbf{d}' \leqq \mathbf{c}$.

Fix $e \geqq 0$. Our induction hypothesis has two parts:

(1, e) for each $i < e$ and each n, $E(i,n)$ has been defined and

$$E(i,n) = 0 \leftrightarrow [n \geqq i \mathbin{\&} (w)(Es)(s > w \mathbin{\&} m(s,i) > n)$$
$$\mathbin{\&} (m)(n \geqq m \geqq i \rightarrow (Ey)T_1^1(\tilde{d}(y), i, m, y))];$$

(2, e) for each $i < e$, $L(i)$ has been defined and

$$(m)[m \geqq L(i) \rightarrow d(p_i^m) = d(p_i^{L(i)})].$$

We proceed to define $E(e,n)$ for all n, verify $(1, e+1)$, define $L(e)$ and verify $(2, e+1)$.

Let $(1, e+1, n)$ denote the following predicate: for each $t < n$, $E(e,t)$ has been defined and

$$E(e,t) = 0 \leftrightarrow [\,t \geqq e \,\&\, (w)(Es)(s > w \,\&\, m(s,e) > t)$$
$$\&\, (m)(t \geqq m \geqq e \rightarrow (Ey)T_1^1(\tilde{d}(y), e, m, y))].$$

To verify $(1, e+1)$, it suffices to prove $(1, e+1, n)$ for all n. We define $E(e,n)$ and prove $(1, e+1, n)$ for all n by means of an induction on n. First we set $E(e,t) = 1$ for all $t < e$. Then it is clear that $(1, e+1, t)$ holds for all $t \leqq e$. Now we fix $n \geqq e$ and suppose $(1, e+1, n)$ holds. We define $E(e,n)$ and then prove $(1, e+1, n+1)$. The definition of $E(e,n)$ has two cases.

Case 1. $(Et)(e \leqq t < n \,\&\, E(e,t) \neq 0)$. We set $E(e,n) = 1$.

Case 2. Otherwise. It follows from $(1, e+1, n)$ that

$$(m)(n > m \geqq e \rightarrow (Ey)T_1^1(\tilde{d}(y), e, m, y)).$$

For each m such that $n > m \geqq e$, let

$$y(m) = \mu y T_1^1(\tilde{d}(y), e, m, y).$$

Let y^* be the largest member of $\{y(m) \mid n > m \geqq e\} \cup \{0\}$. Let

$$s^* = \mu s[(i)(i < y^* \rightarrow d(s-1, i) = d(i)) \,\&\, s > y^*].$$

Recall that for each $s > 0$ and i, if $d(s-1, i) = d(i)$, then $d(s', i) = d(i)$ for all $s' \geqq s$. It follows from the definition of $y(s,m,e)$ and the fact that 0 is not the Gödel number of a deduction that

$$(s)(m)[(s \geqq s^* \,\&\, n > m \geqq e) \rightarrow y(s,m,e) = y(m) > 0].$$

We define

$$E(e,n) = \begin{cases} 0 & \text{if } (Es)\{y(s,n,e) > 0 \,\&\, n < m(s,e) \,\&\, s > s^* \,\& \\ & \qquad (i)_{i<e}[(m)(m < L(i) \rightarrow d(s-1, p_i^m) = d(p_i^m)) \\ & \qquad \&\, (m)(y(s,n,e) > m \geqq L(i) \rightarrow d(s-1, p_i^m) \\ & \qquad\qquad = d(p_i^{L(i)}))]\}, \\ 1 & \text{otherwise.} \end{cases}$$

To verify $(1, e+1, n+1)$, it suffices to prove

$$E(e,n) = 0 \leftrightarrow [\,n \geqq e \,\&\, (w)(Es)(s > w \,\&\, m(s,e) > n)$$
$$\&\, (m)(n \geqq m \geqq e \rightarrow (Ey)T_1^1(\tilde{d}(y), e, m, y))].$$

Suppose $E(e,n) = 0$. Then Case 2 of the definition of $E(e,n)$ must hold. Let s be the natural number whose existence is required by the fact $E(e,n) = 0$; thus

$$y(s,n,e) > 0 \,\&\, n < m(s,e) \,\&\, s > s^*.$$

It is our aim now to prove

$$y(s,n,e) = y(s',n,e) \,\&\, n < m(s',e)$$

for all $s' \geqq s$ by means of an induction on s'. Fix $s' > s$ and suppose

$$y(s,n,e) = y(s'-1,n,e) \,\&\, n < m(s'-1,e).$$

Suppose for the sake of a reductio ad absurdum that there is a w such that $d(s'-1,w) \neq d(s'-2,w)$ and $w < y(s'-1,n,e)$. Then there must be an i and an m such that (0) and (1) are true:

$$(0) \qquad d(s'-1,p_i^m) \neq d(s'-2,p_i^m) \,\&\, p_i^m < y(s'-1,n,e) = y(s,n,e);$$

$$(1) \qquad (i')_{i'<i}(m')[d(s'-1,p_{i'}^{m'}) = d(s'-2,p_{i'}^{m'}) \vee p_{i'}^{m'} \geqq y(s'-1,n,e)].$$

If $i < e$, then it follows from the second half of (0), the definition of s and $(2,e)$ that $d(s-1,p_i^m)=d(p_i^m)$; but this last contradicts the first half of (0), since $s' > s$. Thus $i \geqq e$. Since $e \leqq n < m(s'-1,e)$, it follows from (0) and the definition of $d(s'-1,p_i^m)$ that $r(s'-1,n,e) = 0$. This last means there is an i', an m' and a t such that

$$i' < e \leqq t \leqq n \,\&\, p_{i'}^{m'} < y(s'-1,t,e) \,\&\, d(s'-1,p_{i'}^{m'}) \neq d(s'-2,p_{i'}^{m'}).$$

If $t = n$, this last contradicts (1), since $i' < e \leqq i$. Thus $t < n$, and

$$p_{i'}^{m'} < y(s'-1,t,e) = y(s^*,t,e) \leqq y^*,$$

since $s' > s > s^*$ and $e \leqq t < n$. But this is absurd, since

$$d(s'-1,p_{i'}^{m'}) = d(s'-2,p_{i'}^{m'}) = d(s-1,p_{i'}^{m'}) = d(p_{i'}^{m'})$$

is a consequence of the fact that $s' > s > s^*$ and $p_{i'}^{m'} < y^*$.

Since $d(s'-1,w) = d(s'-2,w)$ for all $w < y(s'-1,n,e)$, and since $y(s'-1,n,e) = y(s,n,e) > 0$, it must be that

$$y(s',n,e) = y(s'-1,n,e) = y(s,n,e).$$

Then we have

$$(m)[n \geqq m \geqq e \rightarrow y(s',n,e) = y(s'-1,n,e) > 0],$$

since $s' > s > s^*$. It follows that either Case 2 or Case 3 of the definition of $m(s',e)$ holds. If Case 2 of the definition of $m(s',e)$ holds, then it is clear $n < m(s',e)$. If Case 3 holds, then $n < m(s',e)$ because $n < m(s'-1,e)$.

Thus we have shown that

$$y(s',n,e) = y(s,n,e) > 0 \,\&\, n < m(s',e)$$

for all $s' \geqq s$. It follows immediately that $(Ey)T_1^1(\tilde{d}(y),e,n,y)$ and

$$(w)(Es)(s > w \,\&\, m(s,e) > n).$$

Note that

$$(m)(n > m \geqq e \rightarrow (Ey)T_1^1(\tilde{d}(y), e, m, y))$$

is a consequence of the fact that Case 2 of the definition of $E(e,n)$ holds. That completes the first half of the verification of $(1, e+1, n+1)$; in order to verify the second half, we suppose

$$(w)(Es)(s > w \,\&\, m(s,e) > n) \,\&\, (m)(n \geqq m \geqq e \rightarrow (Ey)T_1^1(\tilde{d}(y), e, m, y)),$$

and then show $E(e,n) = 0$. It follows from $(1, e+1, n)$ that Case 2 of the definition of $E(e,n)$ holds. Let v be so large that $v > L(i)$ for all $i < e$. Let $z = \mu y T_1^1(\tilde{d}(y), e, n, y)$. Let w be so large that $w > z$ and

$$d(w-1, t) = d(t)$$

for all $t < p_e^{z+v}$. Let s be such that $s > w + s^*$ and $m(s,e) > n$. It follows easily from $(2, e)$ that s has the properties required to conclude $E(e,n) = 0$; note that $y(s,n,e) = z > 0$.

The definition of $L(e)$ has two cases:

Case 1. $c(e) = 0$. Then by $B(e)$, $\lim_m d(p_e^m) = 1$. We set

$$L(e) = \mu t(s)(m)[m \geqq t \rightarrow d(s, p_e^m) = 1].$$

Case 2. $c(e) = 1$. It is a consequence of $(1, e+1)$ and of the definition of $y(s,n,i)$ that for each $i \leqq e$ and each n,

$$E(i,n) = 0 \rightarrow [(Es)(m(s,i) > n) \,\&\, \lim_s y(s,n,i) \text{ exists and is positive}].$$

For each $i \leqq e$, it follows from $A(i)$ that there is a t such that $t \geqq i$ and $E(i,t) = 1$. For each $i \leqq e$, let

$$t_i = \mu t(E(i,t) = 1 \,\&\, t \geqq i).$$

It follows from $(1, e+1)$ that for each $i \leqq e$, t_i satisfies either (2) or (3):

(2) $(Ew)(s)(s > w \rightarrow m(s,i) \leqq t_i)$;

(3) $\lim_s y(s, t_i, i)$ does not exist or is equal to 0.

Note that since $E(i,t) = 0$ whenever $t_i > t \geqq i$, it follows from $(1, e+1)$ that

$$(m)(t_i > m \geqq i \rightarrow \lim_s y(s,m,i) \text{ exists and is positive}).$$

But then

$$(Ey)(i)(m)(s)[(i \leqq e \,\&\, t_i > m \geqq i \,\&\, s \geqq 0) \rightarrow y(s,m,i) \leqq y];$$

let $L(e)$ be the least such y. We now verify $(2, e+1)$. What follows is similar to the proof of Lemma 6. Fix $n \geqq L(e)$. We must show $d(p_e^n) = \lim_m d(p_e^m)$. If Case 1 of the definition of $L(e)$ holds, there is nothing to prove. Suppose Case 2 of the definition of $L(e)$ holds. Then $c(e) = 1$, and by $B(e)$, $\lim_m d(p_e^m) = 0$. In order to show $d(p_e^n) = 0$, it suffices to find an s such that $d(s, p_e^n) = 0$. Let k be such that if $i \leqq e$ and i is not stable, then $i = e_j$ for some $j < k$. Let w be so large that for all $i \leqq e$, if (2) holds, then

$$(s)(s > w \rightarrow m(s,i) \leqq t_i).$$

By the same argument as in Lemma 6, there is a $v > w$ such that

$$(s)(s \geqq v \rightarrow h(s,e) > n).$$

By Lemma 4, there is an $s \geqq v$ such that

$$(j)_{j<k}[m(s,e_j) \leqq n_j \vee r(s,n_j,e_j) = 0 \vee y(s,n_j,e_j) = 0].$$

If we can show for each i and m that

$$(i \leqq e \,\&\, i \leqq m < m(s,i)) \rightarrow (r(s,m,i) = 0 \vee p_e^n \geqq y(s,m,i)),$$

then it will be clear that $d(s,p_e^n) = 0$. Fix i and m so that $i \leqq e$ and $i \leqq m < m(s,i)$. Suppose (2) holds. Then $m(s,i) \leqq t_i$, since $s \geqq v > w$. But then $t_i > m \geqq i$, and consequently,

$$p_e^n > n \geqq L(e) \geqq y(s,m,i).$$

Now suppose (3) holds. Then i is not stable, and there is a $j < k$ such that $e_j = i$; in addition, $n_j = t_i$. If $m < n_j = t_i$, then $p_e^n \geqq y(s,m,i)$, since $n \geqq L(e)$. Suppose $m \geqq n_j$. Then $m(s,i) > n_j$. This last means either

$$r(s,n_j,e_j) = 0 \qquad \text{or} \qquad y(s,n_j,e_j) = 0.$$

If $r(s,n_j,i) = 0$, then by remark (R2), $r(s,m,i) = 0$, since $m \geqq n_j$. Suppose $y(s,n_j,i) = 0$. Since $n_j \leqq m < m(s,i)$, it follows from remark (R3) that $n_j = i$. But then $y(s,i,i) = 0$, $m(s,i) = i + 1$, $m = i$ and $y(s,m,i) = 0 < p_e^n$.

Inspection of the definitions of E and L readily reveals they are recursive in c. We make some informal remarks to indicate how to write equations defining E and L recursively in c. Fix $e \geqq 0$ and $n \geqq e$, and consider the definition of $E(e,n)$. The choice between Case 1 and Case 2 can be made effectively once the values of $E(e,t)(t < n)$ are known. Suppose Case 2 holds. The values of y^* and s^* are defined by means of a predicate recursive in d. Then the value of $E(e,n)$ is found from the values of $d(p_i^m)$ ($m \leqq L(i)$ and $i < e$) by means of a predicate of degree less than or equal to $\mathbf{0}'$. Thus the value of $E(e,n)$ can be expressed in terms of the values of $E(e,t)(t < n)$ and $L(i)$ ($i < e$) with the aid of a predicate of degree less than or equal to $\mathbf{d} \cup \mathbf{0}'$. Similarly, the value of $L(e)$ can be expressed in terms of the values of $E(i,n)$ ($i \leqq e$ and $n \geqq 0$) with the aid of a predicate of degree $\mathbf{c} \cup \mathbf{0}'$. Then $\mathbf{d}' \leqq \mathbf{c}$, since $\mathbf{d}'$ is the degree of the function $E(e,e)$, and since $\mathbf{d} \cup \mathbf{0}' \cup \mathbf{c} = \mathbf{c}$.

When $\mathbf{b} > \mathbf{0}$, the changes needed in the above argument are largely notational. The notion of recursiveness is replaced throughout by the notion of recursiveness in a function of degree $\mathbf{b}$. The arguments contained in Lemmas 1–8 are retained unaltered save for relativization to a function of degree $\mathbf{b}$. The functions $a(s,n)$, $f(s,n)$ and $d(s,n)$ are now recursive in a function of degree $\mathbf{b}$. We set $d(s,2\cdot 3^n) = b(n)$ for all s and all $n > 0$, where b has degree $\mathbf{b}$, in order to insure $\mathbf{b} \leqq \mathbf{d}$. The value of $E(e,n)$ is now obtained from the values of $E(e,t)$ $(t < n)$ and $L(i)(i < e)$ with the aid of a predicate of degree less than or equal to $\mathbf{b}' \cup \mathbf{d}$.

COROLLARY 1. *If* $\mathbf{b}$ *and* $\mathbf{c}$ *are degrees, then the following conditions are equivalent*:

(i) $\mathbf{b}' \leqq \mathbf{c} \leqq \mathbf{b}''$ *and* $\mathbf{c}$ *is recursively enumerable in* $\mathbf{b}'$;

(ii) *there is a* $\mathbf{d}$ *such that* $\mathbf{b} \leqq \mathbf{d} \leqq \mathbf{b}'$ *and* $\mathbf{d}' = \mathbf{c}$;

(iii) *there is a* $\mathbf{d}$ *such that* $\mathbf{b} \leqq \mathbf{d} \leqq \mathbf{b}'$, $\mathbf{d}$ *is recursively enumerable in* $\mathbf{b}$ *and* $\mathbf{d}' = \mathbf{c}$.

For each degree $\mathbf{b}$, let $R_{\mathbf{b}}$ denote the set of all degrees greater than or equal to $\mathbf{b}$, recursively enumerable in $\mathbf{b}$ and less than or equal to $\mathbf{b}'$. Let j denote the jump operator. Then Corollary 1 tells us that the order-preserving map

$$j: R_{\mathbf{b}} \to R_{\mathbf{b}'}$$

is onto. It also follows from Theorem 1 that any element of $R_{\mathbf{b}'}$ greater than $\mathbf{b}'$ is the image of more than one element of $R_{\mathbf{b}}$; Friedberg (result unpublished) has shown that $\mathbf{b}'$ does not have a unique pre-image in $R_{\mathbf{b}}$. We do not know if $R_{\mathbf{b}}$ and $R_{\mathbf{b}'}$ are order-isomorphic, but we conjecture that they are. We can show (announced in [6] for $\mathbf{b} = \mathbf{0}$): for any degree $\mathbf{b}$, $R_{\mathbf{b}}$ is a universal, countable partial ordering.

COROLLARY 2. *There exists a recursively enumerable degree* $\mathbf{d}$ *such that* $\mathbf{d} < \mathbf{0}' < \mathbf{0}'' = \mathbf{d}'$.

Proof. Let $\mathbf{b} = \mathbf{0}$, $\mathbf{c} = \mathbf{0}''$ and $\mathbf{a} = \mathbf{0}'$, and apply Theorem 1 to obtain $\mathbf{d}$. Then $\mathbf{d}$ is recursively enumerable, $\mathbf{0}' \nleqq \mathbf{d}$ and $\mathbf{d}' = \mathbf{0}''$.

Note that Corollary 2 provides still another solution to Post's problem.

COROLLARY 3. *For each degree* $\mathbf{b}$ *and each natural number* n, *there is a degree* $\mathbf{d}$ *recursively enumerable in* $\mathbf{b}$ *such that*

$$\mathbf{b} < \mathbf{d} < \mathbf{b}' < \mathbf{d}' < \mathbf{b}'' < \cdots < \mathbf{b}^{(n)} < \mathbf{d}^{(n)} < \mathbf{b}^{(n+1)}.$$

Proof. We know from [2] that there exists a degree $\mathbf{g}$ such that $\mathbf{g}$ is recursively enumerable in $\mathbf{b}^{(n)}$ and $\mathbf{b}^{(n)} < \mathbf{g} < \mathbf{b}^{(n+1)}$. By Theorem 1, there is a degree $\mathbf{h}_1$ such that $\mathbf{h}_1$ is recursively enumerable in $\mathbf{b}^{(n-1)}$, $\mathbf{b}^{(n-1)} < \mathbf{h}_1 < \mathbf{b}^{(n)}$ and $\mathbf{h}_1' = \mathbf{g}$. By making $n-1$ further applications of Theorem 1, we obtain degrees $\mathbf{h}_2, \mathbf{h}_3, \cdots, \mathbf{h}_n$ such that for $2 \leqq i \leqq n$, $\mathbf{h}_i$ is recursively enumerable in $\mathbf{b}^{(n-i)}$,

$$\mathbf{b}^{(n-i)} < \mathbf{h}_i < \mathbf{b}^{(n-i+1)} \qquad \text{and} \qquad \mathbf{h}_i' = \mathbf{h}_{i-1}.$$

Let $\mathbf{d} = \mathbf{h}_n$. Then $\mathbf{d}$ is recursively enumerable in $\mathbf{b}$, and for all $i \leqq n$, $\mathbf{b}^{(i)} < \mathbf{d}^{(i)} < \mathbf{b}^{(i+1)}$.

Corollary 3 improves a result of Shoenfield [7]; he showed that for each degree $\mathbf{b}$, there is a degree $\mathbf{d}$ such that $\mathbf{b} < \mathbf{d} < \mathbf{b}' < \mathbf{d} < \mathbf{b}''$. We do not know if for any degree $\mathbf{b}$ there exists a degree $\mathbf{d}$ such that for all $n \geqq 0$,

$$\mathbf{b}^{(n)} < \mathbf{d}^{(n)} < \mathbf{b}^{(n+1)};$$

if for some $\mathbf{b}$, such a $\mathbf{d}$ exists, then by Theorem 1, $\mathbf{d}$ can be given the additional property of recursive enumerability in $\mathbf{b}$.

Theorem 1 can be extended without any radical alteration of its proof. For example, we can show: if $\mathbf{g}$ is a recursively enumerable degree such that $\mathbf{g}' < \mathbf{0}''$, then there is a recursively enumerable degree $\mathbf{d}$ such that

$$\mathbf{g} < \mathbf{d} < \mathbf{0}' < \mathbf{0}'' = \mathbf{d}'.$$

We say a sequence $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \cdots$ of degrees is simultaneously recursively enumerable if there is a sequence $A_0, A_1, A_2, \cdots$ of simultaneously recursively enumerable sets such that $\mathbf{a}_i$ is the degree of A_i for all i. Using the method underlying the proof of Theorem 1, we can show: if $\mathbf{a}_0 < \mathbf{a}_1 < \mathbf{a}_2 < \cdots$ is an infinite, ascending sequence of simultaneously recursively enumerable degrees, then there exists a recursively enumerable degree $\mathbf{d}$ such that

$$\mathbf{a}_0 < \mathbf{a}_1 < \mathbf{a}_2 < \cdots < \mathbf{d} < \mathbf{0}'.$$

We end with a conjecture: the upper semi-lattice of recursively enumerable degrees is dense (i.e., if $\mathbf{b}$ and $\mathbf{c}$ are recursively enumerable degrees such that $\mathbf{b} < \mathbf{c}$, then there exists a recursively enumerable degree $\mathbf{d}$ such that $\mathbf{b} < \mathbf{d} < \mathbf{c}$). The only evidence we have to offer in favor of this conjecture is contained in the results we announced above and the result of Muchnik [5] that there is no minimal, nonzero, recursively enumerable degree(2).

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(2) *Added in proof*. Our conjucture is true; we will give a proof elsewhere.