ON THE SCHRÖDINGER AND HEAT EQUATIONS FOR NONNEGATIVE POTENTIALS(1)

BY

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1. **Introduction.** Consider the equation

$$\frac{1}{\sigma}\frac{\partial u(x,t)}{\partial t} = (\Delta - V(x))\,u(x,t) \tag{1}$$

x varying over Euclidean n-space, and $0 \leqq t < \infty$, with the initial condition $u(x, 0+) = f(x)$. For positive σ, this is the heat equation; for purely imaginary σ, it is Schrödinger's equation for a particle in a force field. In his dissertation, and later in a published article [6], R. Feynman indicated how one might get this solution as a limit of averages over polygonal paths. His prescription was not mathematically rigorous, however, since it involved infinite constants and integration with respect to a fictitious translation-invariant measure in an infinite product of real lines. In the case of the heat equation, Kac [11] made this precise by using Wiener measure instead. The approximating averages became finite-dimensional approximants to a Wiener integral: for sufficiently well-behaved $V \geqq 0$ and f,

$$\begin{aligned} u(x,t) &= E\left\{\exp\left(-\int_0^t V(\xi_s + x)ds\right)f(\xi_t + x)\right\} \\ &= \lim E\left\{\exp\left[-\sum_i V(\xi_{s_i} + x)\Delta s_i\right]f(\xi_t + x)\right\} \end{aligned} \tag{2}$$

where ξ_t is Brownian motion with parameter σ, starting at 0, and the limit is taken as $\max \Delta s_i \to 0$. This was developed further by Rosenblatt [16] and Ray [15]. The problem was treated for larger classes of V and for more general Markov-processes by Getoor [8; 9], Dynkin, Volkonskiĭ [17], et al. Gel'fand and Yaglom [7] indicated heuristically how the same sort of approximating finite-dimensional integrals might be used to get solutions to the Schrödinger equation. They made the error of stating that, for $\operatorname{re}\sigma \neq 0$, the limit could be expressed as an integral over path space. Cameron [3] pointed out this error, but proved rigorously (for a rather narrow class of V: required to satisfy certain analyticity assumptions) using certain other approximating expressions, that the limit existed for $\operatorname{re}\sigma > 0$.

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The case of purely imaginary σ was gotten as a boundary value of an analytic function. His approximants, incidentally, were not the same as those used by the previous authors; they corresponded to using a Simpson's rule rather than a Riemann sum to approximate $\int_0^t V(\xi_s)ds$. Recently, D. Babbitt, in his doctoral dissertation [1], noted that Feynman's program could be carried out rather effectively if one regarded t rather than σ as an analytic parameter. In this way, he defined a semigroup which, for re $\sigma > 0$, gave the solution to (1), and approximated it by an expression like (2). This worked for V satisfying a local Lipschitz condition (and, again, $\geqq 0$; or, more generally, bounded below).

In the present paper we proceed as follows. First, we construct a semigroup T_V^t which gives a solution to (1), for arbitrary positive measurable V. This was already done more generally by Getoor [8; 9], so this is merely an exposition of a special case. Next, we investigate some smoothing properties of the operators T_V^t. In §4, an infinitesimal generator for T_V^t is shown to exist. By means of the generator, T_V^ζ is defined for re $\zeta \geqq 0$. This is then approximated by Babbitt's method, but in a more general situation. In §6, it is shown that T_V^ζ may be obtained from a Green's function, whose regularity properties are investigated.

It should be added that we have just learned that E. Nelson [14] has also succeeded in constructing a semigroup and a Feynman approximation for a large class of potentials, not necessarily bounded below.

For general background and bibliographical references, we refer the reader to [2; 7; 12]. The author would like to express his gratitude to D. G. Babbitt for the opportunity of seeing his manuscript at an early stage; and to E. Nelson for a stimulating discussion, and in particular, for pointing out an error in an earlier version of Theorem 3.5.

2. **Brownian motion in k-dimensions.** Let $\mathscr{K}$ be a real Hilbert space of dimension k, the inner product being denoted by $x \cdot y$. Let Ω be the set of continuous functions $\omega : [0, \infty) \to \mathscr{K}$, and ξ_t the function from Ω to $\mathscr{K}$ defined by $\xi_t(\omega) = \omega(t)$. Let $\mathscr{F}_t$ be the smallest σ-field of subsets of Ω for which ξ_s is measurable for all s in $[0, t]$, and let $\mathscr{F}$ be the smallest σ-field containing all the $\mathscr{F}_t$.

Let

$$G_\sigma(x) = (\pi\sigma)^{-k/2} e^{-\|x\|^2/\sigma} \qquad (\sigma > 0, x \in \mathscr{K}).$$

Observe that $G_\sigma(x/\sqrt{t_0})\, d(x/\sqrt{t_0}) = G_{t_0\sigma}(x)dx$. More generally, if L_0 is a nonsingular linear transformation on $\mathscr{K}$ such that $L_0^* L_0 = t_0 I$, then

$$G_\sigma(L_0 x) d(L_0 x) = G_{t_0\sigma}(x)dx.$$

There is a unique probability measure Pr_x^σ on $\mathscr{F}$ characterized by the property that if $h_1, \cdots, h_n > 0$, and $t_j = h_1 + \cdots + h_j$, and $S_0, \cdots, S_n$ are Borel sets in $\mathscr{K}$, then $\mathrm{Pr}_x^\sigma\{\xi_0 \in S_0, \xi_{t_1} \in S_1, \cdots, \xi_{t_n} \in S_n\} = 1_{S_0}(x) \int_{S_1} \cdots \int_{S_n} G_{h_1}(x_1 - x) \cdots G_{h_n}(x_n - x_{n-1}) dx_i \cdots dx_n$.

(NOTATION. 1_S will mean the characteristic function of S. I_S will mean the operation of multiplication by 1_S.)

So $\Pr_x^\sigma$ makes ξ_t into a temporally homogeneous Markov process starting at x, with transition function

$$\Pr_x^\sigma\{\xi_t = dy \mid \xi_{t_0} = y_0\} = G_{\sigma(t-t_0)}(y-y_0)dy.$$

When $\sigma = 1$ the superscript in $\Pr_x^\sigma$ will often be omitted, and when $x = 0$ the subscript will often be omitted. We also denote by $E_x\{\cdots\}$ the operation of integration with respect to $\Pr_x^\sigma$, and make the same conventions about omitting σ when it equals 1 and x when it equals 0.

Let us introduce some transformations on Ω:

(1) if $t > 0$, set $j_t(\omega)(s) = \omega(st)$. Thus,

$$\xi_s \cdot j_t = \xi_{st},$$

(2) if L is a continuous map: $\mathscr{H} \to \mathscr{H}$, we set $k_L(\omega)(s) = L\omega(s)$. Thus,

$$\xi_s(k_L(\omega)) = L\xi_s(\omega).$$

REMARK 2.1. If $Ly = L_0 y - x_0$, where L_0 is a linear transformation on $\mathscr{H}$ with $L_0^* L_0 = t_0^{-1} I$, then:

$$\Pr_x^\sigma \cdot k_L^{-1} = \Pr_{L^{-1}x}^{t_0\sigma}.$$

In particular:

$$\Pr^\sigma \cdot L^{-1} = \Pr_{x_0}^{t_0\sigma}.$$

Proof. This merely involves computing, for both sides of the equation, the measure of the set where $\xi_0 \in S_0$, $\xi_{h_1} \in S_1, \cdots, \xi_{h_1+\dots+h_n} \in S_n$, and using the transformation property of G_σ observed above. The details are omitted.

3. **The semigroup obtained from a potential** V. Call a complex Borel measurable function f on $\mathscr{H}$ *moderate* if $\int |f(x)| e^{-c\|x\|^2} dx < \infty$ for each $c > 0$. These functions form a translation-invariant linear space $\mathscr{M}$. Observe also that a moderate f is Lebesgue integrable on compact sets. For any $t > 0$ and positive measurable V (the value $+\infty$ being permitted), we set

$$\begin{aligned} T_V^t f(x) &= E_x\left\{\exp\left[-\int_0^t V(\xi_s)ds\right] f(\xi_t)\right\} \\ &= E\left\{\exp\left[-\int_0^t V(\xi_s + x)ds\right] f(\xi_t + x)\right\}. \end{aligned}$$

This makes sense, since

$$\left|\exp\left[-\int_0^t V(\xi_s)ds\right] f(\xi_t)\right| \leqq |f(\xi_t)|,$$

and

$$E_x\{|f(\xi_t)|\} \leqq (\pi t)^{-k/2}\int |f(y)|\, e^{-\|y-x\|^2/t}dy.$$

Now,

$$\frac{\|y-x\|^2}{t} = \frac{\|y/2-2x\|^2}{t} - \frac{3\|y\|^2}{4t} + \frac{3\|x\|^2}{t},$$

so

$$e^{-\|y-x\|^2/t} \leqq e^{-3\|y\|^2/4\ t} e^{3\|x\|^2/t},$$

and

$$E_x\{|f(\xi_t)|\} \leqq (\pi c)^{-k/2} e^{-3\|x\|^2/t} \int |f(y)|\, e^{-\|y\|^2/c}\, dy,$$

where $c = (3/4)t$.

Measurability of the integrand is not diffficult to see, by first considering V of the form 1_S, S open, and then approximating. Or see [8; 9] for proof.

THEOREM 3.1. *T_V^t is a linear transformation from $\mathscr{M}$ to $\mathscr{M}$, sending a.e. nonnegative functions to nonnegative functions. If $0 \leqq f_n \uparrow f$ a.e., then $T_V^t f_n \uparrow T_V^t f$ a.e. Finally, $T_V^{s+t} = T_V^s T_V^t$.*

Proof. All statements but the first and last are evident. To prove these:

$$\begin{aligned}
\int |T_V^t f(x)|\, e^{-c\|x\|^2}\, dx &\leqq \int E_x\{|f(\xi_t)|\} e^{-c\|x\|^2} dx \\
&= \int G_t * |f|(x) e^{-c\|x\|^2} dx \quad \text{(where $*$ is convolution)} \\
&= \left(\frac{\pi}{c}\right)^{k/2} \int |f(x)|\, G_t * G_{1/c}(x) dx \\
&= \left(\frac{\pi}{c}\right)^{k/2} \int |f(x)|\, G_{t+1/c}(x) dx < \infty.
\end{aligned}$$

So T_V^t takes $\mathscr{M}$ to $\mathscr{M}$. Finally:

$$\begin{aligned}
T_V^s T_V^t f(x) &= E_x\left\{\exp\left[-\int_0^s V(\xi_r)dr\right] T_V^t f(\xi_s)\right\} \\
&= E_x\left\{\exp\left[-\int_0^s V(\xi_r)dr\right] E_{\xi_s}\left\{\exp\left[-\int_0^t V(\xi_u)du\right] f(\xi_t)\right\}\right\} \\
&= E_x\left\{\exp\left[-\int_0^s V(\xi_r)dr\right] \exp\left[-\int_0^t V(\xi_{u+s})ds\right] f(\xi_{t+s})\right\},
\end{aligned}$$

by the strong Markov property. This can be rewritten as $T_V^{s+t} f(x)$.

Next, we examine the effect of varying V.

THEOREM 3.2. *$V \geqq W$ a.e. $\Rightarrow T_V^t f \leqq T_W^t f$ for all a.e. nonnegative f in $\mathscr{M}$. Furthermore, if $V_n \uparrow V$ a.e., then $T_{V_n}^t f \downarrow T_V^t f$ for such f.*

Proof. The only difficulty is to see that if $V = W$ a.e., then $T_V^t f = T_W^t f$. This is true because, for any set S of measure 0 in $\mathscr{X}$, we have

$$E_x\left\{\int_0^t 1_S(\xi_s)ds\right\} = \int_0^t \int_S G_x(y-x)dyds = 0,$$

so that V is unaffected by a change on a set of measure 0.

Finally, we consider the action of T_V^t on the various $\mathscr{L}_p$ spaces over $\mathscr{X}$ (taken with Lebesgue measure, normalized via the inner product in $\mathscr{X}$), and also on the space $\mathscr{B}$ of bounded Borel functions on $\mathscr{X}$. By $\|\cdot\|_p$ we will mean the norms in $\mathscr{L}_p = \mathscr{L}_p(\mathscr{X})$, $1 \leqq p \leqq \infty$, and by just plain $\|\cdot\|$ the norm in $\mathscr{B}$.

THEOREM 3.3. *T_V^t is a contraction on $\mathscr{L}_p$ $(1 \leqq p \leqq \infty)$ and on $\mathscr{B}$. Furthermore, if $f \in \mathscr{L}_p$ and $g \in \mathscr{L}_q$, with $1/p + 1/q = 1$, then $\int T_V^t f(x)g(x)dx = \int f(x)T_V^t g(x)dx$.*

Proof. Since $T_V^t f(x) \leqq T_0^t f(x)$ for each nonnegative f in $\mathscr{M}$, the first sentence will follow once we have it for the case $V = 0$. But this case is well known for $p = 1$ and $p = \infty$, while for other p it follows from the Riesz convexity theorem.

As for the self-adjointness: Consider $\Omega \times \Omega$. If we denote by $\tilde{\Omega}$ the subset of pairs (ω, ω') such that $\xi_0(\omega) = \xi_0(\omega')$, then $\tilde{\Omega}$ can be identified with the space of all continuous functions $\tilde{\omega}$ from the real line to $\mathscr{X}$, by letting

$$\tilde{\omega}(t) = \begin{cases} \omega(-t) & \text{if } t \leqq 0 \\ \omega'(t) & \text{if } t \geqq 0. \end{cases}$$

The measure $\Pr_x \times \Pr_x$ has its support on $\tilde{\Omega}$. We set $\tilde{\Pr}\{\tilde{\Lambda}\} = \int \Pr_x \times \Pr_x\{\tilde{\Lambda}\}dx$, for $\tilde{\Lambda}$ a measurable set in $\tilde{\Omega}$. $\tilde{\Pr}$ is, of course, an infinite measure. Define $\tilde{\xi}_t(\tilde{\omega}) = \tilde{\omega}(t)$. Then it is easy to see that the joint distributions of $\tilde{\xi}_{t_1}, \cdots, \tilde{\xi}_{t_n}$ are the same as those of $\tilde{\xi}_{t_1+h}, \cdots, \tilde{\xi}_{t_n+h}$, and of $\tilde{\xi}_{-t_1}, \cdots, \tilde{\xi}_{-t_n}$. Thus,

$$\begin{aligned}(T_V^t f, g) &= \int E_x\left\{\exp\left[-\int_0^t V(\xi_s)ds\right]f(\xi_t)\right\} g(x)dx \\ &= \tilde{E}\left\{\exp\left[-\int_0^t V(\tilde{\xi}_s)ds\right]f(\tilde{\xi}_t)g(\tilde{\xi}_0)\right\} = \tilde{E}\left\{\exp\left[-\int_0^t V(\tilde{\xi}_{t-s})ds\right]f(\tilde{\xi}_0)g(\tilde{\xi}_t)\right\} \\ &= E\left\{\exp\left[-\int_0^t V(\tilde{\xi}_u)du\right]f(\tilde{\xi}_0)g(\tilde{\xi}_t)\right\} = \int E_x\left\{\exp\left[-\int_0^t V(\xi_u)du\right]g(\xi_t)\right\}f(x)dx \\ &= (f, T_V^t g).\end{aligned}$$

REMARK. The self-adjointness has been proved by Getoor for $\mathscr{L}_2$, in [8], and could be shown generally by approximation.

THEOREM 3.4. *If f is in $\mathscr{L}_p$ and $1/q + 1/p = 1$, then $\|T_V^t f\| < C(q)t^{l(q)}\|f\|_p$, where $C(q) = \pi^{l(q)} q^{-k/2q}$ and $l(q) = (1/q - 1)k/2$. Furthermore, if S_N is the*

N-sphere, then $\| I_{S_N^\perp} T_V^t f \| \to 0$ as $N \to \infty$, uniformly in V and in t restricted to an interval $0 < t_0 < t < t_1 < \infty$.

NOTATION. $S^\perp$ is the complement of the set S.

Proof. $|T_V^t f(x)| \leqq |G_t * f(x)| \leqq \| G_t \|_q \|f\|_p$. Evaluating $\| G_t \|_q$ gives the first part.

Choose $\varepsilon > 0$. Choose M so big that $\| I_{S_M^\perp} f \|_p < \varepsilon (2 \| G_{t_0} \|_q)^{-1}$. Then $\| T_V^t I_{S_M^\perp} f \| < \varepsilon/2$. Now, if $\| y \| \leqq M$ and $\| x \| \geqq N$, then $\| y - x \| \geqq N - M$, and $|G_t(y-x)| \leqq (\pi t_0)^{-k/2} e^{-(N-M)^2/t_1}$ provided $t_0 \leqq t \leqq t_1$. Now, $I_{S_M} f$ is an $\mathscr{L}_p$ function with support of finite measure, hence an $\mathscr{L}_1$ function. Choose N so large that $|G_t(y-x)| < \varepsilon(2 \| I_{S_M} f \|_1)^{-1}$ if $\| y \| \leqq M$, $\| x \| \geqq N$, and $t_0 \leqq t \leqq t_1$. Then $|T_V^t f(x)| < \varepsilon$ if $\| x \| \geqq N$ and $t_0 \leqq t \leqq t_1$.

LEMMA. *If V_0 is in $\mathscr{L}_p$ for some $p > k/2$, then*

$$E_x\left\{ \int_0^r V_0(\xi_u) du \right\} \leqq c(p) \| V_0 \|_p \cdot r^{k/2+1}.$$

Proof.

$$\left| E_x\left\{ \int_0^r V_0(\xi_u) du \right\} \right| = \left| \int_0^r V_0(x+y) G_u(y) dy du \right|$$

$$\leqq \int_0^r \| V_0 \|_p \| G_u \|_q du = \| V_0 \|_p \int_0^r C(q) u^{l(q)} du$$

(using the notation in the proof of the previous theorem)

$$= c(p) \| V_0 \|_p \, r^{k/2+1},$$

where $c(p) = C((1-1/p)^{-1})(1-k/2p)^{-1}$.

THEOREM 3.5. *If, for some $\bar{p} > k/2$, V is in $\mathscr{L}_{\bar{p}}$ on an open set $\mathcal{O}$, and f is in any $\mathscr{L}_p$ class, then $T_V^t f$ is continuous in $\mathcal{O}$. More precisely: for any compact subset C of $\mathcal{O}$, $\varepsilon > 0$, and $t_1 > 0$, we can choose r_0 such that $T_V^t f(x)$ differs by less than ε from the continuous function $T_0^r T_V^{t-r} f(x) = G_r * T_V^{t-r} f(x)$, for all x in C, $0 < r \leqq r_0$, and $t \geqq t_1$.*

Proof. Let $\Lambda_s = \{\omega \mid \xi_r(\omega) \in \mathcal{O}$ for $0 \leqq r \leqq s\}$. This is in $\mathscr{F}$, as is also the set $\Gamma_{s,\alpha} = \{\omega \mid \int_0^s V(\xi_r) dr < \alpha\}$. Write Φ_r^s for the function $\exp[-\int_r^s V(\xi_u) du]$ on Ω. Then

$$T_V^t f(x) = E_x\{\Phi_0^t f(\xi_t)\} = E_x\{\Lambda_s^\perp \cup \Gamma_{r,\alpha}^\perp, (\Phi_0^r - 1)\Phi_r^t f(\xi_t)\}$$

$$+ E_x\{\Lambda_s \cap \Gamma_{r,\alpha}, (\Phi_0^r - 1)\Phi_r^t f(\xi_t)\} + T_0^r T_V^{t-r} f(x).$$

We estimate the first two terms.

Let C be a compact subset of $\mathcal{O}$, d its distance from $\mathcal{O}^{\perp}$, and x any point in C. Then $\Pr_x\{\Lambda_s^{\perp}\} \leqq \Pr_0\{\|\xi_u\| \geqq d$ for some u in $[0,s]\}$. This is known to go to 0 as $s \downarrow 0$. Thus, we can choose s so small that $\Pr_x\{\Lambda_s^{\perp}\}C(q)(t_1/2)^{l(q)} < \varepsilon/3$ ($l(q)$ defined as in Theorem 3.4). We can also require that $s < t_1/2$.

Next, denote by V_0 the function $I_{\mathcal{O}}V$. Then if $r \leqq s$,

$$\Pr_x\{\Lambda_s \cap \Gamma_{\alpha,r}^{\perp}\} = \Pr_x\left\{\Lambda_s, \int_0^r V_0(\xi_u)du \geqq \alpha\right\} \leqq \Pr_x\left\{\int_0^r V_0(\xi_u)du \geqq \alpha\right\}.$$

From the previous lemma, this is dominated by $c(\bar{p})/\alpha \| V_0 \|_{\bar{p}} r^{-k/2+1}$. If α is preassigned, then by choosing r sufficiently small (and in particular, smaller than s and $t_1/2$), we can thus guarantee that

$$\Pr_x\{\Lambda_s \cap \Gamma_{\sigma,r}^{\perp}\}C(q)(t_1/2)^{l(q)} < \varepsilon/3.$$

α was at our disposal. We choose it so small that $(1 - e^{-\alpha})C(q)(t_1/2)^{l(q)} < \varepsilon/3$. Then

$$\begin{aligned} |E_x\{\Lambda_s \cap \Gamma_{r,\alpha}, (\Phi_0^r - 1)\Phi_r^t f(\xi_t)\}| &\leqq (1-e^{-\alpha})E_x\{\Phi_r^t |f|(\xi_t)\} \leqq (1 - e^{-\alpha})E_x\{T_V^{t-r}|f|(\xi_r)\} \\ &\leqq (1 - e^{-\alpha})C(q)(t - r/2)^{l(q)} \|f\|_p \leqq \varepsilon/3 \|f\|_p. \end{aligned}$$

Thus, for all $t \geqq t_1$ and x in C, $\| T_V^t f - T_0^r T_V^{t-r} f \|$ is dominated by $\varepsilon \|f\|_p$, the choice of r depending on ε, t_1 and $\| V_0 \|_{\bar{p}}$.

Summarizing some of this:

Corollary 3.1. *T_V^t takes each $\mathscr{L}_p$ class into $\mathscr{L}_p \cap \mathscr{B}_0$ (where $\mathscr{B}_0$ is $\{f$ in $\mathscr{B} | f(x) \to 0$ as $\|x\| \to \infty\}$). Furthermore, $T_V^t f$ is continuous on the open set $\{x | V$ is in $\mathscr{L}_p$ in some neighborhood of x, for some $p > k/2\}$.*

Remark 3.1. Choosing V to be $+\infty$ on the complement of some set is one way of relativizing the process to that set (see also the method used by Getoor in [9], where everything gets cut down to an open set G).

4. **The infinitesimal generator.** In [8] Getoor incorrectly said that T_V^t is continuous at 0 as a semigroup on $\mathscr{L}_2$ (and hence, has a densely defined infinitesimal generator). This statement was, however, corrected in [9], and even in [8] he mentioned a necessary and sufficient condition on V that T_V^t be continuous in this sense. Also, in [9], a rather stringent sufficient condition is given. The contion in [8] is just that $\lim_{t\to 0}\exp[-\int_0^t V(\xi_s)ds] = 1$ $\Pr_x$-a.e., for almost every x in $\mathscr{X}$. We shall not assume this, but rather investigate for arbitrary V the subspace on which T_V^t is continuous; or, equivalently, the closure of the domain of the infinitesimal generator of T_V^t.

Consider, for fixed x, the condition that $\lim_{t\to 0}\exp[-\int_0^t V(\xi_s)ds] = 0$ $\Pr_x^{\sigma}$-a.e. This condition is actually independent of σ. One way of seeing this is the following. Recall the map j_σ from Ω to Ω sending ω to the function whose value at

t is $\omega(t\sigma)$. Then j_σ is a 1-1 measure-preserving transformation from $(\Omega, \mathscr{F}, \mathrm{Pr})$ to $(\Omega, \mathscr{F}, \mathrm{Pr}^\sigma)$, and $\int_0^{t\sigma} V(\xi_s + x)ds = \sigma \int_0^t V(\xi_s \cdot j_\sigma + x)ds$.

Let $\Omega_x = \{\lim_{t\to 0} \exp[-\int_0^t V(\xi_s + x)ds] = 1\}$. This is exactly

$$\left\{\lim_{t\to 0} \int_0^t V(\xi_s + x)ds = 0\right\},$$

and also $\{\exists t > 0$ such that $\int_0^t V(\xi_s + x)ds < \infty\}$. The set is in $\mathscr{F}_t$ for each $t > 0$. Thus, by the zero-one law, it differs from a set in $\mathscr{F}_0$ by a set which has Pr^σ-measure 0 for each σ. Then for given x, Ω_x either has Pr^σ-measure 0 for all σ or has Pr^σ-measure 1 for all σ.

V will be called *controllable* at x if $\mathrm{Pr}\{\Omega_x\} = 1$. So if V is *not* controllable at x, then for each moderate f we have $T_V^t f(x) = 0$ for all $t > 0$.

DEFINITION. Let C_V be the set of points where V is controllable, and let I_V be the operation of multiplication by the function which is 1 on C_V and 0 elsewhere. Also, let $\mathscr{L}_p^V$ be $\mathscr{L}_p$ with respect to Lebesgue measure cut down to C_V, with corresponding norms $\| \quad \|_p^V$.

THEOREM 4.1. (a) *For each moderate f we have*

$$I_V T_V^t I_V f = T_V^t f \text{ a.e.},$$

and

$$\lim_{t\downarrow 0} T_V^t f = I_V f \text{ a.e.}$$

(b) *Furthermore, T_V^t is strongly continuous on each $\mathscr{L}_p^V$, $1 \leqq p < \infty$, and is weak $*$ continuous on $\mathscr{L}_\infty^V$. Let A_V be the infinitesimal generator (no distinction is necessary for different p, since there is agreement on functions in several different $\mathscr{L}_p^V$).*

(c) *On $\mathscr{L}_2^V$, A_V is a nonnegative self-adjoint operator, and $e^{-tA_V} = T_V^t$.*

Proof. If x is not in C_V, then $E\{\exp[-\int_0^t V(\xi_s + x)ds]f(\xi_t + x)\} = 0$ for each $t > 0$. So $I_V T_t f =_{\text{a.e.}} T_V^t f$ for moderate f. If f is actually in $\mathscr{L}_2$, then self-adjointness of T_V^t tells us that $I_V T_V^t I_V f =_{\text{a.e.}} T_V^t f$. For general moderate f, the last equality still holds, by the Lebesgue convergence theorem. The fact that $\lim_{t\downarrow 0} T_V^t f(x) = f(x)$ for a.e. x in C_V can be shown as follows: first one proves it for continuous f, by applying the Lebesgue convergence theorem; then for arbitrary moderate f by Theorem 3.1.

As for (b): since $T_V^t f$ converges a.e. to $I_V f$ as $t \downarrow 0$, and since $\| T_V^t f \|_p^V \leqq \| T_V^t f \|_p = \| T_V^t I_V f \|_p \leqq \| I_V f \|_p = \| f \|_p^V$, we have that $T_V^t f \to f$ in $\mathscr{L}_p^V$, in the weak topology, or, in the case $p = \infty$ in the weak $*$ topology. Since each T_V^t is a contraction, we get T_V^t weakly continuous at all $t \geqq 0$ if $1 \leqq p < \infty$, or weak $*$ continuous if $p = \infty$. If $1 \leqq p < \infty$, then $\mathscr{L}_p^V$ is separable, so weak continuity implies strong measurability by [10], Theorem 3.55, and therefore T_V^t is strongly continuous for $t \geqq 0$ by [10, Theorem 10.5.5].

(c) Finally, that A_V on $\mathscr{L}_2^V$ is a nonnegative self-adjoint operator can be seen as follows. T_V^t has *some* representation of the form e^{-tB} for a self-adjoint operator B (easily seen to be positive), by [13, XI, 2]. Now, the domain of A_V is the range of $\int_0^\infty T_V^t e^{-\lambda t} dt = 1/(B+\lambda)$, and $(A_V+\lambda)(1/(B+\lambda))f = f$. Thus, A_V is precisely B.

Next, two theorems which cast a little light on the question "what is C_V for given V"?

THEOREM 4.2. *C_V is a.e. contained in the set where V is finite.*

Proof. Let $R = \{x \mid V(x) = \infty\}$. Then, for a.e. x in R, the set R has density 1 at x. Selecting such an x:

$$E_x\left\{\frac{1}{t}\int_0^t 1_R(\xi_s)ds\right\} = \frac{1}{t}\int_0^t E_x\{1_R(\xi_s)\}ds = \frac{1}{t}\int_0^t G_R * 1_R(x)ds.$$

Now, $G_s * 1_R(x) \to 1$ as $s \to 0$, since G_s is an approximate identity and R has density 1 at x. Thus

$$\frac{1}{t}\int_0^t G_s * 1_R(x)ds \to 1.$$

As a consequence, $\Pr_x\{\int_0^t V(\xi_s)ds = \infty\} = 1$ for each $t > 0$, and x is in R.

THEOREM 4.3. *If $p > k/2$ and $V \in \mathscr{L}_p(\mathcal{O})$, $\mathcal{O}$, open $\subset K$, then $C_V \supset \mathcal{O}$ a.e.*

Proof. It is no loss of generality to assume V vanishes outside $\mathcal{O}$, since each path starting at an x in $\mathcal{O}$ stays there for a while. But then the lemma after Theorem 3.4 tells us that $E_x\{\int_0^t V(\xi_s)ds\} < \infty$, so that $\int_0^t V(\xi_s)ds < \infty \Pr_x$-a.e., and therefore x is in C_V if it is in $\mathcal{O}$.

REMARK 4.1. Operators on $\mathscr{L}_p^V$ are in an obvious 1-1 correspondence with operators B on $\mathscr{L}_p$ such that $I_V B I_V = B$. Thus, we will occasionally treat A_V as an operator on $\mathscr{L}_p$, without further comment.

REMARK 4.2. If $V = 0$, then A_V is just the negative of the usual Laplacian (on $\mathscr{L}_2$). More generally, it is shown in [8] that if C_V is almost all of $\mathscr{K}$, and M_V is the operation of multiplication by V on $\mathscr{K}$, then

$$A_V \supset (-\Delta + M_V) \mid \mathscr{D}_\Delta \cap \mathscr{D}_{M_V} \qquad \text{(where } \mathscr{D}_T \text{ is the domain of } T).$$

For example, if V is bounded, then A_V is just $-\Delta + M_V$. However, it would be of interest to have the answer to the following question, for instance. Suppose the Laplacian of f exists locally, in some sense, but the function g thereby obtained is no longer in $\mathscr{L}_2$. Suppose, however, that $-g + Vf$ is in $\mathscr{L}_2$. Is it then the case that f is in $\mathscr{D}_{A_V}$ and $A_V f = -g + Vf$? (The considerations of [9, §4], do not apply, unfortunately, because Δ is not a "local operator" in the sense used there.)

REMARK 4.3. Here is a phenomenon which was surprising at least to me. Recall that

$$T_0^t = e^{-t\Delta}f = G_t * f$$

is actually infinitely differentiable for all moderate f. Recall also that for general V, and f in $\mathscr{L}_p$, T_V^t is continuous where V is in some $\mathscr{L}_{\bar{p}}$-class (Corollary 3.1). One might therefore expect that if, say, V were bounded then $T_V^t f$ would be infinitely differentiable. However, this is far from true!

EXAMPLE. Let V be a nonnegative bounded measurable function, and let f be in $\mathscr{D}_{A_V}$ Thus $T_V^t f$ is again in $\mathscr{D}_{A_V}$, and $A_V T_V^t f = T_V{}^t A_V f$. But $\mathscr{D}_{A_V} = \mathscr{D}$ and $A_V = -\Delta + M_V$, so $-\Delta T_V^t f + V T_V^t f = A_V T_V^t f = T_V^t A_V f$. Now, $T_V^t f$ is continuous, as is also $T_V^t A_V f$, by Corollary 3.1. If $T_V^t f$ had two continuous derivatives, then a representative for $T_V^t f$ could be chosen which was continuous. But $V T_V^t f$ can be made as irregular as one likes, for example by choosing V to be unequal to any continuous function on any open set. So $T_V^t f$ cannot always be twice continuously differentiable, even if V is bounded and f itself is a C_∞ function with compact support. (However, regularity assumptions on V would presumably result in regularity for $T_V^t f$.)

5. Complexification of the semigroup, and the limiting Feynman integral.

Let Λ be the set of complex numbers with positive real part, $\bar{\Lambda}$ its closure. Let A be a fixed nonnegative self-adjoint operator on the Hilbert space $\mathscr{H}$. For any ζ in $\bar{\Lambda}$, the functional calculus defines a bounded operator $e^{-\zeta A}$. This operator is unitary if ζ is imaginary, nonnegative and self-adjoint if ζ is nonnegative, and has norm $\leqq 1$ for ζ in $\bar{\Lambda}$. The map $\zeta \to e^{-\zeta A}$ is continuous in the strong operator topology for ζ in $\bar{\Lambda}$, and satisfies $e^{-\zeta A}e^{-\zeta' A} = e^{-(\zeta+\zeta')A}$. For ζ in Λ, it is continuous in the uniform operator topology, and even holomorphic. These facts are all, at worst, straightforward applications of the functional calculus.

EXAMPLE. We can extend T_V^t, as an operator on $\mathscr{L}_2^V$, to T_V^ζ for s in $\bar{\Lambda}$, by setting $A = A_V$.

We quote, for later use, a fact about convergence of analytic functions.

FACT 5.1 (VITALI). Let $F_1, F_2, \cdots$ be a sequence of analytic functions on Λ, with values in a Banach space. Suppose the F_n are uniformly bounded in norm on each compact subset of Λ. Suppose also that they converge in norm at all points of $(0, \infty)$. Then they converge in norm on Λ, uniformly on compact subsets, to an analytic function F_∞ on Λ.

Proof. [10, p. 104, Theorem 3.14.1].

For the purposes of our first theorem, we will want, for each $t > 0$, $\Pr_x\{V(\xi_s)$ Riemann-integrable on $(0,t)\} = 1$ for a.e. x. This amounts to $\Pr_x\{V(\xi_s)$ bounded and a.e. continuous on $(0,t)\} = 1$ for a.e. x. Call such a V Riemann-approximable. Observe that if V is Reimann-approximable then C_V is almost all of E, so that T_V^t is strongly continuous at 0 in $\mathscr{L}_p$.

THEOREM 5.1. *Suppose V is Riemann-approximable. Let τ be a finite sequence of positive numbers: $\tau = (\tau_1, \cdots, \tau_{n(\tau)})$, with $\sum \tau_j = 1$. Let $|\tau| = \max_j \tau_j$. For ζ in Λ, let $T^{\zeta}_{V,\tau} = \prod_j e^{-\zeta\tau_j V} e^{\zeta\tau_j\Delta}$, everything operating on $\mathscr{L}_2$. This is clearly holomorphic on Λ, strongly continuous on $\bar{\Lambda}$. Then $\lim_{|\tau|\to 0} T^{\zeta}_{V,\tau}$ exists in the strong operator topology, and uniformly for ζ in any compact subset of Λ, and equals T^{ζ}_V. Finally: if ϕ is any integrable function on the real line, then*

$$\lim_{|\tau|\to 0} \int_{-\infty}^{+\infty} (T^{is}_{V,\tau} f, g)\phi(s)ds = \int_{-\infty}^{+\infty} (T^{is}_V f, g)\phi(s)ds,$$

for all f, g in $\mathscr{L}_2$ (where T^{is}_V is the strongly continuous extension of T^{ζ}_V to the imaginary axis).

REMARK 5.1. The fact of convergence was proved by D. Babbitt [1], under the added assumption that V satisfied a local Lipschitz condition. The proof of the present generalization is just a simplification of Babbitt's proof.

Proof of theorem. Consider the sum $\sum_j V(\xi_{(\tau_1+\dots+\tau_j)t}(\omega))\tau_j t$. This is a Riemann sum for the integral $\int_0^t V(\xi_s(\omega))ds$, using the partition $(\tau_1 t, \cdots, \tau_{n(\tau)} t)$. Thus, $\sum_j V(\xi_{(\tau_1+\dots+\tau_j)t}(\omega))\tau_j t$ converges to $\int_0^t V(\xi_s(\omega))ds$ as $|\tau| \to 0$, for $\Pr_x$–almost every ω. Now let f be in $\mathscr{M}$. Then, since the functions within $E_x\{\dots\}$ are all bounded in norm by $|f(\xi_t)|$, and converge $\Pr_x$–a.e., we have

$$\lim_{|\tau|\to 0} E_x\left\{\exp\left[-\sum_j V(\xi_{(\tau_1+\dots+\tau_j)t})\tau_j t\right] f(\xi_t)\right\} = T^t_V f(x)$$

for each x. Also

$$\left|E_x\left\{\exp\left[-\sum_j V(\xi_{(\tau_1+\dots+\tau_j)t})\tau_j t\right] f(\xi_t)\right\} - T^t_V f(x)\right|^2 \leqq 2E_x\{|f(\xi_t)|\}^2.$$

Thus, if f is in $\mathscr{L}_2$, then $\lim_{|\tau|\to 0}\|T^t_{V,\tau} f - T^t_V f\|_2 = 0$, i.e. $T^t_{V,\tau}$ converges strongly to T^t_V. Now we can apply Fact 4.1 to get the existence of a holomorphic limit T^{ζ}_V, ζ in Λ, which must agree with $e^{-\zeta AV}$ on $\bar{\Lambda}$ since it agrees for $\zeta > 0$.

The fact that $\lim_{|\tau|\to 0} \int_{-\infty}^{\infty}(T^{is}_{V,\tau}f,g)\phi(s)ds = \int_{-\infty}^{\infty}(T^{is}_V f,g)\phi(s)ds$ for all ϕ in $\mathscr{L}_1$ is a consequence of the fact that $(T^{\zeta}_{V,\tau}f,g)$ and $(T^{\zeta}_V f,g)$ are bounded holomorphic functions and $(T^{\zeta}_{V,\tau}f,g) \to (T^{\zeta}_V f,g)$ on Λ. This can be seen as follows. Let $P_t(s) = (1/\pi)(t/(t^2+s^2))$. P_t is an approximate identity, so that $P_t * \phi \to \phi$ in $\mathscr{L}_1(-\infty,\infty)$. If Ψ is any bounded analytic function in the right half plane, then $\Psi(t+is) = \int P_t(s-s')\,\Psi(is')ds'$. Now let $\Psi_\tau(\zeta) = (T^{\zeta}_{V,\tau}f,g)$, and $\Psi(\zeta) = (T^{\zeta}_V f,g)$. Notice that $\int P_t * \psi(s)\phi(s)ds = \int \psi(s)P_t * \phi(s)ds$. Thus: $\int(\Psi_\tau(is) - \Psi(is))\phi(s)ds = \int(\Psi_\tau(t+is) - \Psi(t+is))\phi(s)ds + \int(\Psi_\tau(is) - \Psi(is))(\phi(s) - P_t * \phi(s))ds$. The second term has absolute value $\leqq$ const. $\int|\phi(s) - P_t * \phi(s)|ds$. By choosing t small, this can be made arbitrarily small (for fixed ϕ). The first term can then be made small by choosing $|\tau|$ small, since $|\Psi_\tau(t+is) - \Psi_\tau(t+is)|$ stays bounded by $2\|f\|\ \|g\|$, and converges to 0 for each s.

REMARK 5.2. Observe that one point which came out in the proof was that for each x at which V is Riemann-approximable,

$$E_x\{\exp[-\sum_j V(\xi_{(\tau_1+\dots+\tau_j)t})\tau_j t]f(\xi_t)\}$$

converges to $T_V^t f(x)$, for each f in $\mathscr{M}$. (Note: Riemann-approximabity at x has not been defined, but it should be obvious what is meant.)

What sort of V are Riemann-approximable? A large class is the following. It permits arbitrarily bad infinities on a set of capacity zero.

THEOREM 5.2. *Let D be the closed set of x for which V is essentially unbounded in every neighborhood of x. Suppose D forms a set of capacity 0. Suppose also the points of discontinuity of V form a set of measure 0. Then V is Riemann-approximable.*

Proof. By changing V on a set of measure 0 in $D^\perp$, we can assume that V is actually locally bounded in $D^\perp$. Namely, let $C_n \uparrow D$, C_n compact, and replace V on $C_n - C_{n-1}$ by $V \wedge \| 1_{C_n} V\|$. This will not introduce any new discontinuities. Then $\Pr_x\{\xi_s$ lies in D for some $s\} = 0$, for a set D of capacity 0. See, for example, [4]. Thus, for $\Pr_x$–a.e. ω, the range of $\xi_s(\omega)$, $0 \leqq s \leqq t$, is a compact subset of $D^\perp$, and so $s \to V(\xi_s(\omega))$ is bounded on $[0,t]$. Also, for $\Pr_x$–a.e. ω, the set of s for which $\xi_s(\omega)$ lies in the set of discontinities of V has Lebesgue measure 0. Thus, for $\Pr_x$–a.e. ω, $\xi_s(\omega)$ is Riemann-integrable for $0 < s \leqq t$.

6. **The Green's function.** Recall that if $f \in \mathscr{L}_p$ then $\| T_V^t f\|_\infty \leqq C(q)t^{(1/q-1)k/2}$. So, for $1 \leqq p < \infty$, $T_V^t f(x) = \int k_x^t(y)f(y)dy$, where k_x is an equivalence class of Lebesgue measurable functions, and $\| k_x^t \|_q \leqq C(q)t^{(1/q-1)k/2}$. For $f \geqq 0$ in $\mathscr{L}_\infty$ choose $f_n \in \mathscr{L}_2$, $f_n \uparrow f$. Then $T_V^t f_n(x) \uparrow T_V^t f(x)$, so that

$$\int k_x^t(y)f(y)dy = \lim_{n\to\infty} \int k_x^t f_n(y)dy = \lim_{n\to\infty} T_V^t f_n(x) = T_V^t f(x),$$

so again we have $\int k_x^t(y)f(y)dy = T_V^t f(x)$, $\| k_x^t \|_1 \leqq C(1)$, independent of t.

We introduce a canonical version of k_x^t.

LEMMA 6.1. *$\int k_x^r(z)k_y^s(z)dz$ is, for each x, equal to $k_x^t(y)$ for almost every y, provided $r + s = t$. Further, it is independent on the choice of r and s.*

Proof. $k_x^s(y)$ can be chosen a jointly Borel measurable function of x and y, since the map $x \to k_x^s$ is a measurable map from $\mathscr{X}$ to, for example, $\mathscr{L}_2$. Furthermore,

$$\int g(x)k_x^s(y)f(y)dy = \int g(x)T_V^s f(x)dx = \int T_V^s g(y)f(y)dy,$$

since T_V^s is self-adjoint. So

$$\int\left(\int k_x^r(z)k_z^s(y)dz\right)f(y)dy = \int\int k_x^r(z)k_z^s(y)f(y)dydz = \int k_x^r(z)T_V^s f(z)dz = T_V^r T_V^s f(x)$$

$$= \int k_x^t(y)f(y)dy.$$

To show independence of r and s, we choose r, s, r', s', with $r + s = r' + s'$. Assume $r < r'$. Then, from what has been shown,

$$\begin{aligned}\int k_x^{r'}(z)k_y^{s'}(z)dz &= \int k_x^{r+(r'-r)}(z)k_y^{s'}(z)dz \\ &= \int\int k_x^{r}(q)k_w^{r'-r}(z)k_z^{s'}(y)dwdz \\ &= \int\int k_x^{r}(w)k_z^{s-s'}(w)k_z^{s'}(y)dwdz \\ &= \int k_x^{r}(w)k_w^{s}(y)dw,\end{aligned}$$

which completes the proof.

Now it makes sense to define $K_V^t(x,y) = \int k_x^r(z)k_y^s(z)dz$, since it is independent of r and s, provided $r + s = t$.

Thus we have

REMARK 6.1. (a) There is a function $K_V^t(x,y)$ such that $T_V^t f(x) = \int K_V^t(x,y)f(y)dy$ for f in any $\mathscr{L}_p$-class. K_V^t is symmetric. Further, $K_V^{s+t}(x,z) = \int K_V(x,y)K_V(y,z)dy$.

(b) The last property, together with the fact that $T_V^t f(x) = \int K_V^t(x,y)f(y)dy$ for enough f, uniquely determine K_V^t. (I use the label "remark" rather than "theorem" in order to avoid being precise about the word "enough".)

Properties of T_V^t easily translate into properties of K_V^t. For example:

THEOREM 6.1. *$x \to K_V^t(x, \cdot)$ is continuous into all $\mathscr{L}_p$, and K_V^t is jointly continuous in x and y, on the open set where V is locally integrable.*

Proof. The first statement is an immediate consequence of the definition and of Theorem 3.5. As for the second part: if $x_n \to x$ and $y_n \to y$, and V is locally integrable at x and y, then

$$\begin{aligned}&\left|\int K_V^r(x_n,z)K_V^s(y_n,z)dz - \int K_V^r(x,z)K_V^s(y,z)dz\right| \\ &\qquad \leqq \|K_V^r(x_n,\cdot) - K_V^r(x,\cdot)\|_2 \|K_V^s(y_n,\cdot)\|_2 + \|K_V^r(x,\cdot)\|_2 \|K_V^s(y_n,\cdot) - K_V^s(y_n,\cdot)\|_2.\end{aligned}$$

But $\|K_V^s(y_n,\cdot)\|_2$ and $\|K_V^r(x_n,\cdot)\|_2$ stay bounded, while the other factors go to zero.

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