GENERAL PRODUCT MEASURES

BY

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1. **Introduction.** Our purpose is twofold. First, we desire to associate with any indexed (countable or uncountable) collection of (outer) measures free from any finiteness or σ-finiteness restrictions, an associated product space and a product measure which retains and generalizes the intuitive precepts of product measure. Secondly we wish to extend to countable products, some topological results obtained in an earlier paper *Product measures*(2) for a binary product of measures.

In the classical theory, the formation of an infinite product of measures is undertaken only when all except a finite number of the component spaces have unit(3) measure. As a first attempt to bypass this restriction, we substitute in place of the traditional covering family of measurable cylinders, the more fundamental family of rectangles having all sides measurable and for which the product of the measures of sides is finite. This product is used to gauge the measure of such a rectangle, and with this the resulting product measure faithfully agrees. There is, however, a defect in this first product measure. Under this measure, for uncountable products, our fundamental rectangular sets may not be measurable. By suitably modifying this first product measure we obtain a second one not sharing this defect, and for it, a Fubini theorem for the integrable functions under any binary decomposition of the product. The modification consists of requiring to be of measure zero each set which is contained in some union of cylinders in the product space over null sets of component subspaces. For the convenience it offers, but not of necessity, we also require to be of measure zero each cylinder in the space over some null subset of some subproduct space. Fortunately, these modifications do not disturb the measure assigned to a fundamental rectangular set.

Our second objective is obtained through an additional modification of our product measure. In finite products, much as in PM, we further require to be of measure zero each set whose characteristic function integrates iteratively to

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(2) W. W. Bledsoe and A. P. Morse, *Product measures*, Trans. Amer. Math. Soc. **79** (1955), 173–215. Hereinafter this reference is PM.

(3) We once shared with many others the belief that this was an essential restriction. We are grateful to J. Feldman for asking us a question whose answer led us to doubt, more and more, the validity of our earlier belief.

zero under each binary decomposition of the product. We extend this third version of product measure from finite to arbitrary products by means of a rather general procedure embodied in Definition 6.15.9 and Theorem 6.24 which we employ to good advantage twice in the present paper.

The topological features enjoyed by our third product measure are given in Theorem 7.7 and may be informally described as follows. Suppose that each measure in a countable product is so related to the topology on its space that (1) open sets are measurable, and relative to each set of finite measure: (2) each open set is equal in measure to the upper bound of the measures of its closed subsets; and (3) from each covering of the space by open sets a countable subfamily can be extracted which covers almost all the space. Then our associated product measure, defined free of topological considerations, is related to the product topology in this same way.

In §2 we assemble, for the convenience of the reader, our special notations and definitions which are common to the remainder of the paper.

In §3 we present the basic measure theoretic results that are needed for constructing measures or proving the measurability of given sets. In this connection we suppose the reader has a knowledge of measuie theory such as might be acquired from reading H. Hahn, *Theorie der reelen Funktionen*, Vol. 1, Berlin, 1921, pp. 424–432.

Using the theory of limits and *Runs*(4), we develop in §4, for our needs, a theory of unordered infinite numerical products rather analogous to the theory of unordered numerical summation.

In §5 we present definitions and theorems relating to product spaces, that set the scene for our treatment of product measures which follows in §6.

Topology enters our paper for the first time in §7, which concludes with the previously described Theorem 7.7.

2. Preliminary definitions and notations.

2.1. DEFINITIONS.

.1 $\mathrm{sb}\,A = \mathrm{subset}\,A = \mathrm{E}B\ (B \subset A)$ = the family of sets B such that $B \subset A$.

.2 $A \Subset B$ if and only if $A \subset B$ and $A \neq B$.

.3 $\mathrm{sp}\,A = \mathrm{superset}\,A = \mathrm{E}B\ (B \supset A)$.

.4 $\mathrm{sng}\,y = \mathrm{singleton}\,y = \mathrm{E}x\ (x = y)$.

.5 $\mathrm{fnt} = \mathrm{finite} = \mathrm{E}A$ (A is a finite set).

.6 $\mathrm{cbl} = \mathrm{countable} = \mathrm{E}A$ (A is a countable set).

.7 $\sigma\mathfrak{F} = \bigcup A \in \mathfrak{F}\, A = \mathrm{E}x\ (x \in A$ for some $A \in \mathfrak{F})$.

.8 $\pi\mathfrak{F} = \bigcap A \in \mathfrak{F}\, A = \mathrm{E}x\ (x \in A$ for each $A \in \mathfrak{F})$.

.9 $\mathrm{Join}\,\mathfrak{F} = \mathrm{E}A\ (A = \sigma\mathfrak{H}$ for some $\mathfrak{H} \subset \mathfrak{F})$.

.10 $\mathrm{Join}'\,\mathfrak{F} = \mathrm{E}A\ (A = \sigma\mathfrak{H}$ for some $\mathfrak{H} \in \mathrm{fnt} \cap \mathrm{sb}\,\mathfrak{F})$.

The reader may find it more to his taste to read statements like

(4) H. Kenyon and A. P. Morse, *Runs*, Pacific J. Math. **8** (1958), 811–824.

"$\mathfrak{H} \in \text{fnt} \cap \text{sb}\,\mathfrak{F}$" as "$\mathfrak{H}$ is a finite subset of $\mathfrak{F}$" rather than "$\mathfrak{H}$ belongs to the intersection of finite and subset $\mathfrak{F}$".

.11 $\text{Join}''\mathfrak{F} = \text{E}A\,(A = \sigma\mathfrak{H}$ for some $\mathfrak{H} \in \text{cbl} \cap \text{sb}\,\mathfrak{F})$.

.12 $\text{Meet}'\mathfrak{F} = \text{E}A\,(A = \sigma\mathfrak{F} \cap \pi\mathfrak{H}$ for some $\mathfrak{H} \in \text{fnt} \cap \text{sb}\,\mathfrak{F})$.

We should like to remind the reader that in case $\mathfrak{H}$ is the empty set, $\pi\mathfrak{H}$ is the universe and consequently $\sigma\mathfrak{F} \in \text{Meet}'\mathfrak{F}$.

.13 $\text{Meet}''\mathfrak{F} = \text{E}A\,(A = \sigma\mathfrak{F} \cap \pi\mathfrak{H}$ for some $\mathfrak{H} \in \text{cbl} \cap \text{sb}\,\mathfrak{F})$.

.14 $\text{cmpl}\,\mathfrak{F} = \text{complement}\,\mathfrak{F} = \text{E}A\,(A = \sigma\mathfrak{F} \sim B$ for some $B \in \mathfrak{F})$.

.15 $\omega =$ the set of non-negative integers.

We assume that the integer 0 and the empty set are the same and also that the integer 1 is equal to sng 0.

.16 $\text{Cr}\,xA = 1$ or 0 according as x is or is not a member of A.

.17 $\text{rct}\,AB = \text{E}x, y\,[x \in A$ and $y \in B]$.

.18 $\text{vs}\,Ax = \text{verticalsection of } A \text{ at } x = \text{E}y\,[(x, y) \in A]$.

In the interest of improving the readability of expressions like "$[f(x)](y)$" we abandon the traditional "$f(x)$" notation for a function value and substitute that defined in 2.2.1 below. We also introduce in 2.2.5 and 2.2.6 the function makers which we find so convenient.

2.2. Definitions.

.1 $.fx =$ the value of f at $x =$ the y such that $(x, y) \in f$.

Thus, if f is a function valued function (operator) then $..fxy$ is the value of the function $.fx$ at y.

.2 $\text{dmn}f = \text{E}x\,[(x, y) \in f$ for some $y]$.

.3 $\text{dmn}'f = \text{E}x \in \text{dmn}f\,(|.fx| < \infty)$.

.4 $\text{rng}f = \text{E}y\,[(x, y) \in f$ for some $x]$.

.5 $\text{fun}\,x \in AP = \text{E}x, y\,[x \in A$ and $y = P]$.

.6 $\text{fun}\,x \subset AP = \text{E}x, y\,[x \subset A$ and $y = P]$.

In .5 and .6 we allow "P" to be replaced by expressions like "σx" or "$.f(x \cap y)$" etc.

3. **Measures.** We present in this section certain well-known definitions and theorems (without proof) concerning (outer) measures. We cast these in a form convenient to our purposes.

3.1. Definitions.

.1 ϕ measures S if and only if ϕ is such a function on sb S that:

$0 \leqq .\phi A$ whenever $A \subset S$; and $.\phi A \leqq \sum B \in \mathfrak{F}\ .\phi B$ whenever $\mathfrak{F} \in \text{cbl}$ and $A \subset \sigma\mathfrak{F} \subset S$.

.2 $\text{Msr}\,S = \text{E}\phi\,(\phi$ measures $S)$.

.3 $\text{rlm}\,\phi = \text{realm}\,\phi = \sigma\,\text{dmn}\,\phi$.

.4 $\text{Measure} = \text{E}\phi\,[\phi$ measures $\text{rlm}\,\phi]$.

.5 $\text{mbl}\,\phi = \text{measurable}\,\phi = \text{E}A \in \text{dmn}\,\phi\,[\phi \in \text{Measure}$ and $.\phi T = .\phi(TA) + .\phi(T \sim A)$ whenever $T \in \text{dmn}\,\phi]$.

.6 $\mathrm{mbl}'\phi = \mathrm{mbl}\,\phi \cap \mathrm{dmn}'\phi$.

.7 $\mathrm{zr}\,\phi = \mathrm{zero}\,\phi = \mathrm{E}A\ (.\phi A = 0)$.

.8 $\mathrm{sct}\,\phi\,T = \mathrm{section}\,\phi\,T = \mathrm{fun}\,A \in \mathrm{dmn}\,\phi\ .\phi(T \cap A)$.

.9 $\mathrm{sms}\,\phi = \mathrm{submeasure}\,\phi = \mathrm{E}\psi\ [\phi \in \mathrm{Measure}$ and $\psi = \mathrm{sct}\,\phi\,T$ for some $T \in \mathrm{dmn}'\phi]$.

.10 $\mathrm{cblcvr}\,\mathfrak{H}\,A = \mathrm{countablecover}\,\mathfrak{H}\,A = \mathrm{E}\mathfrak{G} \in \mathrm{cbl} \cap \mathrm{sb}\,\mathfrak{H}\ (A \subset \sigma\mathfrak{G})$.

.11 $\mathrm{mss}\,g\,S\,\mathfrak{H} = \mathrm{fun}\,A \subset S\ \ (\inf \mathfrak{G} \in \mathrm{cblcvr}\,\mathfrak{H}\,A\ \ \Sigma B \in \mathfrak{G}\ .gB)$.

Thus, if $\phi = \mathrm{mss}\,g\,S\,\mathfrak{H}$ and $A \subset S$ then $.\phi A$ is the infimum of numbers of the form

$$\Sigma\ B \in \mathfrak{G}\ \ .gB,$$

where $\mathfrak{G}$ is a countable subfamily of $\mathfrak{H}$ which covers A.

In this connection we should like to remind the reader that the infimum of the empty set is ∞.

.12 $\mathrm{approx}\,\phi = \mathrm{approximater}\,\phi = \mathrm{E}\mathfrak{F}\ [\phi \in \mathrm{Measure}$, $\sigma\mathfrak{F} \subset \mathrm{rlm}\,\phi$ and corresponding to each $A \in \mathrm{dmn}'\phi$ and $r > 0$ there exists $C \in \mathrm{zr}\,\phi$ and $\mathfrak{G} \in \mathrm{cblcvr}\,\mathfrak{F}\ (A \sim C)$ for which

$$\Sigma\ B \in \mathfrak{G}\ \ .\phi B \leqq .\phi A\ +\ r].$$

.13 $\mathrm{bsc}\,\phi = \mathrm{basic}\,\phi = \mathrm{E}\mathfrak{F} \subset \mathrm{Join}''\mathrm{mbl}'\phi\ [\phi \in \mathrm{Measure}$ and $\phi = \mathrm{mss}\,\phi\,\mathrm{rlm}\,\phi\,\mathfrak{F}]$.

.14 $\mathrm{cnsr}\,\phi\,\mathfrak{K} = \mathrm{conservative}\,\phi\,\mathfrak{K} = \mathrm{fun}\,A \in \mathrm{dmn}\,\phi\,\inf C \in \mathfrak{K}\ .\phi\ (A \sim C)$.

.15 $\mathrm{knsr}\,\phi\,\mathfrak{K} = \mathrm{cnsr}\,\phi\,\mathrm{Join}''\mathfrak{K}$.

.16 $\mathrm{sp}'\phi A = \mathrm{E}B \subset \mathrm{rlm}\,\phi\ \ (.\phi(A \sim B) = 0)$.

3.2. THEOREM. *If g is a non-negative real-valued function, $\mathfrak{H} \subset \mathrm{dmn}\,g$, $\sigma\mathfrak{H} \subset S$, and $\phi = \mathrm{mss}\,g\,S\,\mathfrak{H}$ then*:

.1 $\phi \in \mathrm{Msr}\,S$;

.2 $.\phi A \leqq .gA$ whenever $A \in \mathfrak{H}$;

.3 *each $A \in \mathrm{dmn}'\phi$ is so contained in some member B of* $\mathrm{Meet}''\mathrm{Join}''\mathfrak{H}$ *that* $.\phi A = .\phi B$;

.4 *if $A \in \mathfrak{H}$ and $\Sigma\ B \in \mathfrak{G}\ \ .gB \geqq\ \ .gA$ whenever $\mathfrak{G} \in \mathrm{cblcvr}\,\mathfrak{H}\,A$ then $.\phi A = .gA$.*

3.3. THEOREMS.

.1 *If $\mathfrak{F} \in \mathrm{approx}\,\phi$, $A \subset \mathrm{rlm}\,\phi$, and $.\phi T = .\phi(TA) + .\phi\ (T \sim A)$ whenever $T \in \mathfrak{F}$, then $A \in \mathrm{mbl}\,\phi$.*

.2 *If $\mathfrak{F} \in \mathrm{approx}\,\phi \cap \mathrm{sb}\,\mathrm{mbl}\,\phi$ then corresponding to each $A \in \mathrm{dmn}'\phi$ there exists such a ϕ measurable set $B \in \mathrm{sp}\,A$ that $.\phi B = .\phi A$.*

3.4. THEOREM. *If $\psi \in \mathrm{Msr}\,S$, $\psi = \mathrm{mss}\,\psi\,S\mathfrak{H}$, $\sigma\mathfrak{H} \subset S$, $\sigma\mathfrak{K} \subset S$ and $\phi = \mathrm{knsr}\ \psi\,\mathfrak{K}$ then*:

.1 *corresponding to each $A \in \mathrm{dmn}'\phi$ there is such a member C of* $\mathrm{Join}''\mathfrak{K}$ *that* $.\phi A = .\psi\ (A \sim C)$;

.2 $\phi = \mathrm{mss}\,\phi S\ (\mathfrak{H} \cup \mathfrak{K}) \in \mathrm{Msr}\,S$;

.3 $.\phi A \leqq .\psi A$ *whenever* $A \subset S$;
.4 $\mathrm{mbl}\,\psi \subset \mathrm{mbl}\,\phi$;
.5 $\mathrm{Join}''\mathfrak{K} \subset \mathrm{zr}\,\phi = \bigcup A \in \mathrm{zr}\,\psi \bigcup B \in \mathrm{Join}''\mathfrak{K}\,\mathrm{sb}(A \cup B)$;
.6 $\phi = \mathrm{cnsr}\,\psi\,\mathrm{zr}\,\phi$.

3.5. THEOREM. *If* $\mathfrak{F} \in \mathrm{approx}\,\phi$, $\mathfrak{F} \subset \mathrm{mbl}'\,\phi$ *and* $\psi = \mathrm{mss}\,\phi S \mathfrak{F}$ *then* $\phi = \mathrm{cnsr}\,\psi\,\mathrm{zr}\,\phi$ *and* $\mathfrak{F} \cup \mathrm{zr}\,\phi \in \mathrm{bsc}\,\phi$.

4. **Numerical products.** In keeping with 4.2 of PM we shall assume in the present paper that for each x,

$$0 \cdot x = x \cdot 0 = 0.$$

We shall make use of *Runs*(4) especially pp. 822–823. It should be noted that in Theorem 6.9 of *Runs* it is understood that $0 \cdot \infty$ is not a real number whereas in the present paper $0 \cdot \infty = 0$.

4.1. DEFINITION. $\mathrm{clsn}'A = \mathrm{E}\alpha,\beta[\alpha \subset \beta \in \mathrm{fnt} \cap \mathrm{sb}\,A]$. Evidently $\mathrm{clsn}'A$ is a run for each A. Informally we agree that

$$\prod j \in A \ .aj$$

is the numerical product, as j traverses A, of $.aj$. More formally we accept the axiomatically definitional

4.2. THEOREMS.
.1 $\prod j \in 0 \ .aj = 1$.
.2 *If* $-\infty \leqq .ak \leqq \infty$ *then* $\prod j \in \mathrm{sng}\,k \ .aj = .ak$.
.3 *If* $A \cap B = 0$, $A \cup B \in \mathrm{fnt}$, *and* $-\infty \leqq .aj \leqq \infty$ *whenever* $j \in A \cup B$, *then*,

$$\prod j \in A \cup B = \left(\prod j \in A \ .aj\right) \cdot \left(\prod j \in B \ .aj\right).$$

.4 $\prod j \in A \ .aj = \mathrm{lm}\,\alpha\,\mathrm{clsn}'A \prod j \in \alpha \ .aj$.
.5 *If* $.aj = .bj$ *whenever* $j \in A$ *then* $\prod j \in A \ .aj = \prod j \in A \ .bj$.

From these and limit theory we infer the rest of the theorems in this section.

4.3. THEOREM. *If* $A \in \mathrm{fnt}$ *and* $-\infty \leqq .aj \leqq \infty$ *whenever* $j \in A$, *then* $-\infty \leqq \prod j \in A \ .aj \leqq \infty$.

4.4. THEOREM. *If* $A \in \mathrm{fnt}$ *and* $-\infty \leqq .aj \leqq \infty$ *and* $-\infty \leqq .bj \leqq \infty$ *whenever* $j \in A$ *then*,

$$\left(\prod j \in A \ .aj\right) \cdot \left(\prod j \in A \ .bj\right) = \prod j \in A \ (.aj \cdot .bj).$$

4.5. THEOREM. *If* $A \cap B = 0$ *and* $-\infty \leqq .aj \leqq \infty$ *whenever* $j \in A \cup B$, *and if* $-\infty \leqq p = \prod j \in A \ .aj \leqq \infty$, *and* $-\infty \leqq q = \prod j \in B \ .aj \leqq \infty$, *and* $r = \prod j \in (A \cup B) \ .aj$ *then*:

.1 *if* $|p| + |q| < \infty$ *then* $p \cdot q = r$;
.2 *if* $p \cdot q \neq 0$ *then* $p \cdot q = r$.

4.6. THEOREM. *If* $-\infty \leqq .aj \leqq \infty$ *and* $-\infty \leqq .bj \leqq \infty$ *whenever* $j \in A$, *and if* $-\infty \leqq p = \prod j \in A \ .aj \leqq \infty$, $-\infty \leqq q = \prod j \in A \ .bj \leqq \infty$ *and*

$$r = \prod j \in A \ (.aj \cdot .bj)$$

then:
.1 *if* $|p| + |q| < \infty$ *then* $p \cdot q = r$;
.2 *if* $p \cdot q \neq 0$ *then* $p \cdot q = r$.

4.7. THEOREM. *If* $0 \leqq .aj \leqq 1$ *whenever* $j \in A$ *then*

$$0 \leqq \prod j \in A \ .aj = \inf \alpha \in \mathrm{fnt} \cap \mathrm{sb} A \prod j \in \alpha \ .aj \leqq 1.$$

4.8. THEOREM. *If* $1 \leqq .aj \leqq \infty$ *whenever* $j \in A$ *then*

$$1 \leqq \prod j \in A \ .aj = \sup \alpha \in \mathrm{fnt} \cap \mathrm{sb} A \prod j \in \alpha \cdot aj \leqq \infty.$$

4.9. THEOREM. $\prod j \in A 1 = 1$.

4.10. DEFINITIONS.
.1 $\mathrm{psl}\, x = \mathrm{Sup}(\mathrm{sng}\, 1 \cup \mathrm{sng}\, x)$.
.2 $\mathrm{ngl}\, x = \mathrm{Inf}(\mathrm{sng}\, 1 \cup \mathrm{sng}\, x)$.
Thus

$$\mathrm{psl}\ x = x \text{ and } \mathrm{ngl}\, x = 1 \text{ whenever } x \geqq 1,$$

$$\mathrm{ngl}\ x = x \text{ and } \mathrm{psl}\, x = 1 \text{ whenever } x \leqq 1,$$

and

$$x = \mathrm{psl}\, x \cdot \mathrm{ngl}\, x \text{ whenever } -\infty \leqq x \leqq \infty.$$

For measure theoretic purposes we feel satisfied with the
4.11. DEFINITION. $\prod + j \in A \ .aj = (\prod j \in A \ \mathrm{psl} \ .aj) \cdot (\prod j \in A \ \mathrm{ngl} \ .aj)$.

4.12. THEOREMS.
.1 *If* $.aj = .bj$ *whenever* $j \in A$ *then* $\prod + j \in A \ .aj = \prod + j \in A \ .bj$.
.2 *If* $0 \leqq .aj \leqq \infty$ *whenever* $j \in A$ *and if*

$$0 < \prod j \in A \ .aj < \infty$$

then

$$1 \leqq \prod j \in A \mathrm{psl} \ .aj < \infty.$$

.3 *If* $0 \leqq .aj \leqq \infty$ *whenever* $j \in A$ *and if*

$$0 < \prod + j \in A \ .aj < \infty$$

then

$$1 \leqq \prod j \in A \ \mathrm{psl} \ .aj < \infty.$$

.4 *If* $0 \leqq .aj \leqq \infty$ *whenever* $j \in A$ *and if*

$$\prod j \in A \ \mathrm{psl}\ .aj < \infty$$

then

$$0 \leqq \prod + j \in A\ .aj = \prod j \in A\ .aj < \infty.$$

.5 *If* $0 \leqq .aj \leqq \infty$ *whenever* $j \in A$ *and if*

$$\prod j \in A\ .aj < \infty$$

then

$$0 \leqq \prod + j \in A\ .aj = \prod j \in A\ .aj < \infty.$$

.6 *If* $0 \leqq .aj \leqq \infty$ *whenever* $j \in A$ *then*

$$0 \leqq \prod + j \in A\ .aj \leqq \infty.$$

.7 *If* $A \cap B = 0$ *and* $0 \leqq .aj \leqq \infty$ *whenever* $j \in A \cup B$ *then*

$$\left(\prod + j \in A\ .aj\right) \cdot \left(\prod + j \in B\ .aj\right) = \prod + j \in (A \cup B)\ .aj.$$

.8 *If* $0 \leqq .aj \leqq .bj \leqq \infty$ *whenever* $j \in A$ *then*

$$0 \leqq \prod + j \in A\ .aj \leqq \prod + j \in A\ .bj \leqq \infty.$$

.9 *If* $0 \leqq .aj \leqq \infty$ *and* $0 \leqq .bj \leqq \infty$ *whenever* $j \in A$, *and if* $\prod + j \in A\ .aj + \prod + j \in A\ .bj < \infty$ *then*

$$\prod + j \in A\ (.aj \cdot .bj) = \left(\prod + j \in A\ .aj\right) \cdot \left(\prod + j \in A\ .bj\right).$$

.10 *If* $0 \leqq .aj \leqq \infty$ *whenever* $j \in A$, *and if* $r > 0$ *and*

$$0 < \prod + j \in A\ .aj < \infty$$

then there exist $A' \in \mathrm{fnt} \cap \mathrm{sb}\, A$ *and* $A'' \in \mathrm{cbl} \cap \mathrm{sb}\, A$ *for which*

$$\left| \prod + j \in A\ .aj - \prod + j \in A'\ .aj \right| < r$$

and

$$.aj = 1 \text{ whenever } j \in A \sim A''.$$

5. **Product spaces.** The product of two spaces A and B is generally taken to be $\mathrm{rct}\, A\ B$. The product of a multiplicity of spaces $.Xi$, $i \in \mathrm{dmn}\, X$, however, is customarily taken to be the set of functions defined in 5.1.1 below. In the setting of this latter product space, we explore in this section the operations of forming rectangles, cylinders, projections and sections, and state, without proof, a number of orientational and useful theorems.

5.1. Definitions.

.1 $\mathrm{Pr}\, X = \mathrm{E}x[X$ is a function, x is a function, $\mathrm{dmn}\, x = \mathrm{dmn}\, X$, and $.xi \in .Xi$ whenever $i \in \mathrm{dmn}\, X]$.

.2 $\mathrm{sbmb}\, A = \mathrm{submember}\, A = \mathrm{E}y[y \subset x$ for some $x \in A]$.

.3 $(A \cup\cup B) = \mathrm{E}z\ [z = x \cup y$ for some $x \in A$ and some $y \in B]$.

.4 $(A \cap\cap B) = \mathrm{E}z\ [z = x \cap y$ for some $x \in A$ and some $y \in B]$.

.5 $\mathrm{cyl}\,AS =$ cylinder in S over $A = \mathrm{E}z \in S\ [x \subset z$ for some $x \in A]$.

.6 $\mathrm{sctn}\,Ax =$ section of A at $x = \mathrm{E}y\ [x \cap y = 0$ and $x \cup y \in A]$.

.7 $\mathrm{slice}\,A\,i = \bigcup x \in A$ sng $.xi$.

.8 $\mathrm{prj}\,AB =$ projection of A onto $B = [(A \cap\cap B) \cap B]$.

.9 Product $\mathfrak{F}\mathfrak{G} = \mathrm{E}C\ [C = (A \cup\cup B)$ for some $A \in \mathfrak{F}$ and some $B \in \mathfrak{G}]$.

5.2. THEOREMS.

.1 $(A \cup\cup B) = (B \cup\cup A)$ *and* $(A \cap\cap B) = (B \cap\cap A)$.

.2 $(A \cup\cup(B \cup\cup C)) = ((A \cup\cup B) \cup\cup C)$ *and* $(A \cap\cap(B \cap\cap C)) = ((A \cap\cap B) \cap\cap C)$.

.3 $(A \cup\cup 1) = A$ *and* $(A \cup\cup 0) = (A \cap\cap 0) = 0$.

.4 *If* $A \neq 0$ *then* $(A \cap\cap 1) = 1$.

.5 *If* $A' \subset A$ *and* $B' \subset B$ *then* $(A' \cup\cup B') \subset (A \cup\cup B)$ *and* $(A' \cap\cap B') \subset (A \cap\cap B)$.

5.3. THEOREMS.

.1 $(A \cup\cup \sigma\mathfrak{G}) = \bigcup B \in \mathfrak{G}\, (A \cup\cup B)$ *and* $(A \cap\cap \sigma\mathfrak{G}) = \bigcup B \in \mathfrak{G}\, (A \cap\cap B)$.

.2 $(A \cup\cup \pi\mathfrak{G}) \subset \bigcap B \in \mathfrak{G}\, (A \cup\cup B)$ *and* $(A \cap\cap \pi\mathfrak{G}) \subset \bigcap B \in \mathfrak{G}\, (A \cap\cap B)$.

.3 $(A \cup\cup C) \sim (B \cup\cup C) = ((A \sim B) \cup\cup C)$ *and* $(A \cap\cap C) \sim (B \cap\cap C) \subset ((A \sim B) \cap\cap C)$.

.4 $\mathrm{cyl}\,\sigma\mathfrak{G}C = \bigcup B \in \mathfrak{G}\, \mathrm{cyl}\,BC$ *and* $\mathrm{cyl}\,\pi\mathfrak{G}C \subset \bigcap B \in \mathfrak{G}\, \mathrm{cyl}\,BC$.

.5 $\mathrm{cyl}\,A\sigma\mathfrak{G} = \bigcup C \in \mathfrak{G}\, \mathrm{cyl}\,AC$ *and* $\mathrm{cyl}\,A\pi\mathfrak{G} \subset \bigcap C \in \mathfrak{G}\, \mathrm{cyl}\,AC$.

.6 $\mathrm{sctn}\,\sigma\mathfrak{G}x = \bigcup B \in \mathfrak{G}\, \mathrm{sctn}\,Bx$ *and* $\mathrm{sctn}\,\pi\mathfrak{G}x = \bigcap B \in \mathfrak{G}\, \mathrm{sctn}\,Bx$.

.7 $\mathrm{prj}\,\sigma\mathfrak{G}A = \bigcup C \in \mathfrak{G}\, \mathrm{prj}\,CA$ *and* $\mathrm{prj}\,\pi\mathfrak{G}A \subset \bigcap C \in \mathfrak{G}\, \mathrm{prj}\,CA$.

.8 $\mathrm{cyl}\,AC \sim \mathrm{cyl}\,BC \subset \mathrm{cyl}(A \sim B)C$, $\mathrm{sctn}\,Ax \sim \mathrm{sctn}\,Bx = \mathrm{sctn}(A \sim B)x$, *and* $\mathrm{prj}\,CA \sim \mathrm{prj}\,DA \subset \mathrm{prj}(C \sim D)A$.

5.4. THEOREMS.

.1 $(A \cap\cap B) \subset \mathrm{sbmb}\,A$, *and if* $B \neq 0$ *then* $A \subset \mathrm{sbmb}(A \cup\cup B)$.

.2 *If* $A \subset B$ *then* $\mathrm{sbmb}\,A \subset \mathrm{sbmb}\,B$.

.3 *If* $(A \cap\cap B) = 1$ *then* $(\mathrm{sbmb}\,A \cap\cap \mathrm{sbmb}\,B) = 1$.

.4 *If* $x' \subset x$, $y' \subset y$, $x \cap y = 0$ *and* $x' \cup y' = x \cup y$ *then* $x' = x$ *and* $y' = y$.

.5 *If* $(A \cap\cap B) = 1$, $x \in A$, $y \in B$, $x' \in A$, $y' \in B$ *and* $x' \cup y' = x \cup y$ *then* $x = x'$ *and* $y = y'$.

5.5. THEOREM. *If* $(A' \cap\cap B') = 1$ *and* $C' = (A' \cup\cup B')$ *then*:

.1 $\mathrm{cyl}\,\pi\mathfrak{G}C = \bigcap B \in \mathfrak{G}\, \mathrm{cyl}\,BC$ *whenever* $C \subset C'$ *and* $\sigma\mathfrak{G} \subset B'$;

.2 $\mathrm{prj}\,\pi\mathfrak{G}A = \bigcap C \in \mathfrak{G}\, \mathrm{prj}\,CA$ *whenever* $A \subset A'$ *and* $\sigma\mathfrak{G} \subset C'$;

.3 $\mathrm{cyl}(A \sim B)C = \mathrm{cyl}\,AC \sim \mathrm{cyl}\,BC$ *whenever* $A \subset A'$, $B \subset A'$ *and* $C \subset C'$.

5.6. THEOREMS.

.1 Pr 0 = 1.

.2 *If* X *is a function then* $[\Pr(XY) \cap \cap \Pr(X \sim Y)] = 1$ *and* $\Pr X = [\Pr(XY) \cup \cup \Pr(X \sim Y)]$.

.3 slice $0i = 0$.

.4 *If* $0 \neq A = \Pr X$ *then* slice $Ai = .Xi$ *whenever* $i \in \operatorname{dmn} X$.

5.7. THEOREMS.

.1 $\operatorname{cyl}\operatorname{cyl} A\ B\ C \subset \operatorname{cyl} A\,C$.

.2 *If* $A \subset \operatorname{sbmb} B$ *and* $B \subset \operatorname{sbmb} C$ *then* $\operatorname{cyl}\operatorname{cyl} ABC = \operatorname{cyl} AC$.

.3 $\operatorname{prj}\operatorname{prj} CBA \subset \operatorname{prj} CA$.

.4 *If* $A \subset \operatorname{sbmb} B$ *and* $B \subset \operatorname{sbmb} C$ *then* $\operatorname{prj}\operatorname{prj} CBA = \operatorname{prj} CA$.

5.8. THEOREMS.

.1 $\bigcup x \in A\ [\operatorname{sctn} B x \cup \cup \operatorname{sng} x] \subset B$.

.2 *If* $B = \operatorname{cyl} AB$ *then* $B = \bigcup x \in A\ [\operatorname{sctn} Bx \cup \cup \operatorname{sng} x]$.

.3 *If* $x \cap y = 0$ *then* $\operatorname{sctn} A(x \cup y) = \operatorname{sctn}(\operatorname{sctn} Ay)x$.

.4 *If* $(A \cap \cap B) \subset 1$ *then* $[\operatorname{sctn} Ax \cup \cup \operatorname{sctn} By] \subset \operatorname{sctn}(A \cup \cup B)(x \cup y)$.

.5 *If* $A' \subset A$, $B' \subset B$, $(A \cap \cap B) \subset 1$, $x \in \operatorname{sbmb} A$, *and* $y \in \operatorname{sbmb} B$ *then* $[\operatorname{sctn} A'x \cup \cup \operatorname{sctn} B'y] = \operatorname{sctn}(A' \cup \cup B')(x \cup y)$.

5.9. THEOREM. *If* X *is a function,* $0 \neq X$, $A \subset \Pr X = S$ *and* $\mathfrak{F} = \mathrm{E} Y \subset X\ [0 \neq Y \in \mathrm{fnt}]$ *then* $A = \bigcap Y \in \mathfrak{F} \operatorname{cyl}(\operatorname{prj} A \Pr Y) S$.

6. **Product measures.** If m is an indexed collection of measures then in 6.1.1 we call m measuretic and we define for m, in 6.1.8, our first product measure $\psi = \operatorname{cpm} m$. Our second and third product measures are defined in 6.15.11 and 6.31.2 and if ϕ is one of these, then $\phi = \operatorname{cnsr} \psi \operatorname{zr} \phi$ and we think of ϕ as being a conservative modification of ψ.

6.1. DEFINITIONS.

.1 measuretic $= \mathrm{E} m$ [m is a function and $\operatorname{rng} m \subset$ Measure].

.2 $\operatorname{spc} m = \Pr \operatorname{fun} i \in \operatorname{dmn} m \operatorname{rlm} .mi$.

.3 $\operatorname{boxer} m = \mathrm{E} X$ [$m \in$ measuretic, X is a function, $\operatorname{dmn} X = \operatorname{dmn} m$, and $.Xi \subset \operatorname{rlm} .mi$ whenever $i \in \operatorname{dmn} m$].

.4 $\operatorname{bx} m = \mathrm{E} A$ [$A = 0$ or, $A = \Pr X$ for some $X \in \operatorname{boxer} m$].

.5 $\operatorname{box} m = \mathrm{E} A \in \operatorname{bx} m$ [slice $Ai \in \operatorname{mbl} .mi$ whenever $i \in \operatorname{dmn} m$].

.6 $\operatorname{vlm} m =$ the function V on box m such that $.V0 = 0$ and $.VA = \prod + i \in \operatorname{dmn} m\ ..mi$ slice Ai whenever $0 \neq A \in \operatorname{box} m$.

.7 $\operatorname{bscbox} m = \operatorname{dmn}' \operatorname{vlm} m$.

.8 $\operatorname{cpm} m = \operatorname{mss}(\operatorname{vlm} m)\ (\operatorname{spc} m)\ (\operatorname{bscbox} m)$.

.9 $\operatorname{cp} = \operatorname{fun}\ m \in$ measuretic $\operatorname{cpm}\ m$.

.10 nilfunction $m = \mathrm{E} X \in \operatorname{boxer} m$ [$.Xi \in \operatorname{zr}\ .mi$ whenever $i \in \operatorname{dmn} m$].

.11 nilset $m =$ The family of sets of the form

$$\bigcup i \in \operatorname{dmn} m\ \ \mathrm{E} x \in \operatorname{spc} m\ \ (.xi \in .Xi),$$

where $X \in$ nilfunction m.

.12 nilcylinder m = E$A \subset$ spc m [A = cyl B spc m for some $p \subset m$ and $B \in$ zr cpm p].

6.2. DEFECTS OF cp. Suppose $\mathscr{I}$ = Et [$0 \leqq t \leqq 1$] and suppose $\mathscr{L}$ is Lebesgue measure restricted to $\mathscr{I}$. Let m = fun $t \in \mathscr{I}\mathscr{L}$ and X = fun $t \in \mathscr{I}$ {$\mathscr{I} \sim$ sng 1}. Suppose ψ = .cp m = cpm m and A = Pr X. Now $A \in$ bscbox m yet $A \notin$ mbl ψ since it is not hard to check that .$\psi A = 1$ and .ψ(spc $m \sim A$) = 1.

Fairly evident and essential is our first

6.3. THEOREM. *If* $0 \Subset p \Subset m \in$ measuretic, $q = m \sim p$, I = dmn m, S' = spc p, S'' = spc q, S = spc m, $\mathfrak{B}'$ = bscbox p, $\mathfrak{B}''$ = bscbox q, $\mathfrak{B}$ = bscbox m, V' = vlm p, V'' = vlm q, V = vlm m, $\mathfrak{N}'$ = nilset p, $\mathfrak{N}''$ = nilset q, *and* $\mathfrak{N}$ = nilset m *then*:

.1 *If* $0 \neq S$ *then* slice Si = rlm .*mi whenever* $i \in I$;

.2 $S \in$ box m;

.3 *if* $0 \neq \mathfrak{F} \subset$ bx m and X = fun $i \in I \bigcap A \in \mathfrak{F}$ slice Ai *then* $X \in$ boxer m *and* $\pi\mathfrak{F}$ = Pr $X \in$ bx m;

.4 *if* $B \in$ box m, $N \in \mathfrak{N}$, $A = B \sim N$ *then* $A \in$ box m *and* .VA = .VB;

.5 .VA = (.V' prj AS') $\cdot$ (.V'' prj AS'') *whenever* $A \in \mathfrak{B}$;

.6 $\mathfrak{B}$ = Product $\mathfrak{B}'\mathfrak{B}'' \cup$ Product box p zr $V'' \cup$ Product zr V' box q;

.7 Meet″ $\mathfrak{B} = \mathfrak{B}$;

.8 Join″ $\mathfrak{N} = \mathfrak{N}$;

.9 cyl $A'S \in \mathfrak{N}$ *whenever* $A' \in \mathfrak{N}'$;

.10 *if* $A \in \mathfrak{N}$ *then* $A = (A' \cup \cup S'') \cup (S' \cup \cup A'')$ *for some* $A' \in \mathfrak{N}'$ *and* $A'' \in \mathfrak{N}''$;

.11 cylinder $A'S \in$ nilcylinder m *whenever* $A' \in$ nilcylinder p.

Since our methods for obtaining product measures will be variable in what follows, we shall let them enter our definitions and theorems explicitly as a variable. Thus, in 6.4 and elsewhere, α may be thought of as a function which represents a method for obtaining product measures, i.e., if $m \in$ measuretic $\cap$ dmn α then $\phi = .\alpha m$ is the α associated product measure on spc m.

6.4. DEFINITIONS.

.1 approximative α = E$m \in$ measuretic [bscbox $m \in$ approx .αm].

.2 semiproductive α = E$m \in$ approximative α[..$\alpha m A$ = .vlm mA and $A \in$ mbl .αm whenever $A \in$ bscbox m;

$$\int .fz \ .\alpha m \, dz = \int\int .f(x \cup y) \ .\alpha p dx \ .\alpha(m \sim p) dy$$

whenever $0 \Subset p \Subset m$ and $-\infty \leqq \int .fz \ .\alpha m \, dz \leqq \infty$].

.3 mblproductive α = E$m \in$ semiproductive α [if $0 \Subset p \Subset m$ then Product mbl .αp mbl .$\alpha(m \sim p) \subset$ mbl .αm].

.4 productive α = E$m \in$ mblproductive α [cyl B spc $m \in$ zr .αm whenever $0 \Subset p \Subset m$ and $B \in$ zr .αp].

.5 mc H = fun $m \in$ measuretic knsr cpm m .Hm.

.6 harmony α = fun $m \in$ measuretic zr $.\alpha m$.

.7 H is α Harmonious if and only if: H is a function on measuretic; productive α = measuretic; and zr $.\alpha m \subset .Hm$,

$$\text{cyl } B \text{ spc } m \in .Hm,$$

$$0 = \int \int \mathrm{Cr}(x \cup y)A \ .\alpha p dx \ .\alpha(m \sim p) dy,$$

whenever: $m \in$ measuretic, $0 \Subset p \Subset m$, $B \in .Hp$ and $A \in .Hm$.

Thus, if α represents such a method of producing product measures that productive α = measuretic then, for any $m \in$ measuretic, if $\phi = .\alpha m$ we are assured that:

(1) members of bscbox m are ϕ measurable and the ϕ measure of such a box is its volume,

(2) the family bscbox $m \cup$ zr ϕ is ϕ basic,

(3) the Fubini equality holds for the ϕ integrable functions under any binary splitting of the product space,

(4) a rectangle of measurable sets is ϕ measurable,

(5) a cylinder over a set of underlying measure zero has ϕ measure zero.

Aided with 3.5 we infer at once from 6.4.3 and 6.4.6 the following

6.5. THEOREM. *If* productive α = measuretic *and* H = harmony α *then* H is α Harmonious *and* α = mc H.

6.6. THEOREM. *If* $m \in$ measuretic, $\psi \in$ Msr spc m, ϕ = knsr ψ nilset m, $B \in$ bscbox m, and $.\psi A = .\text{vlm } m \ A$ *whenever* $A \in$ bscbox m, *then* $.\phi B = .\text{vlm } m \ B$.

Proof. Referring to 6.3.8 and 3.4.1 we secure such a member N of nilset m that $.\phi B = .\psi(B \sim N)$. Observe (6.3.4) that $B \sim N \in$ bscbox m and that $.\text{vlm } m(B \sim N) = .\text{vlm } mB$. From these two equalities we infer $.\phi B = .\text{vlm } mB$.

Fundamental to our theory is the

6.7. THEOREM. *If* $m \in$ measuretic, $\phi \in$ Msr spc m, bscbox $m \in$ approx ϕ, nilset $m \subset$ zr ϕ *and* $.\phi A = .\text{vlm } mA$ *whenever* $A \in$ bscbox m *then* bscbox $m \subset$ mbl$'\phi$.

Proof. Let $\mathfrak{R} = \mathrm{E}A\ [A = \text{cyl } B \text{ spc } m$ for some $(i, \lambda) \in m$ and $B \in$ box sng$(i, \lambda)]$, observe that

.1 Meet$''\mathfrak{R} = \mathrm{E}A\ [A = \text{cyl } B \text{ spc } m$ for some $p \in \text{cbl} \cap \text{sb } m$ and $B \in \text{box } p]$, and divide the remainder of the proof into two parts.

PART 1. $\mathfrak{R} \subset$ mbl ϕ.

Proof. Suppose $A \in \mathfrak{R}$, $T \in$ bscbox m and check that TA and $T \sim A$ are both members of bscbox m. Also, notice that

$$.\text{vlm } m \, T = .\text{vlm } m(TA) + .\text{vlm } m(T \sim A).$$

Hence, $.\phi T = .\phi(TA) + .\phi(T \sim A)$, and employing 3.3.1 we infer $A \in$ mbl ϕ.

PART 2. bscbox$\,m \subset$ mbl$\,\phi$.

Proof. Suppose A and T are both members of bscbox$\,m$. Thus $TA \in$ bscbox$\,m$. If $.\phi(TA) = 0$ then

$$.\phi T \leqq .\phi(TA) + .\phi(T \sim A) = 0 + .\phi(T \sim A) \leqq .\phi T$$

and we conclude

$$.\phi T = .\phi(TA) + .\phi(T \sim A). \tag{1}$$

We assume below that $.\phi(TA) > 0$. Thus, $\phi(TA) = .\mathrm{vlm}\, m\ (TA)$ and employing 4.12.10 we select such countable subsets m' and m'' of m that

$$.mi(\mathrm{slice}(TA)\, i) = 1 \text{ whenever } i \in \mathrm{dmn}(m \sim m')$$

and

$$.mi(\mathrm{slice}\, T\ i) = 1 \text{ whenever } i \in \mathrm{dmn}(m \sim m'').$$

Let $p = m' \cup m''$, $B = \mathrm{cyl}(\mathrm{prj}\, A\, \mathrm{spc}\, p)\, \mathrm{spc}\, m$, and

$$N = \bigcup w \in (m \sim p)\, \mathrm{cyl}(\mathrm{prj}\, T \mathrm{spc}\, \mathrm{sng}\, w \sim \mathrm{prj}\, A\, \mathrm{spc}\, \mathrm{sng}\, w)\, \mathrm{spc}\, m.$$

Observe that

$$..mi(\mathrm{slice}\, Ti \sim \mathrm{slice}\, Ai) = 0 \text{ whenever } i \in \mathrm{dmn}(m \sim p)$$

and infer $N \in$ nilset$\,m$.

Notice also that $B \in \mathrm{Meet}''\, \mathfrak{K}$, and calculate,

$$\begin{aligned}
T \sim A &= T \sim \bigcap w \in m\ \mathrm{cyl}(\mathrm{prj}\, A\, \mathrm{spc}\, \mathrm{sng}\, w)\, \mathrm{spc}\, m \\
&= \bigcup w \in m (T \sim \mathrm{cyl}(\mathrm{prj}\, A\, \mathrm{spc}\, \mathrm{sng}\, w)\, \mathrm{spc}\, m \\
&= \bigcup w \in p (T \sim \mathrm{cyl}(\mathrm{prj}\, A\, \mathrm{spc}\, \mathrm{sng}\, w)\, \mathrm{spc}\, m) \\
&\cup \bigcup w \in (m \sim p)(T \sim \mathrm{cyl}(\mathrm{prj}\, A\, \mathrm{spc}\, \mathrm{sng}\, w)\, \mathrm{spc}\, m) \\
&\subset \bigcup w \in p (T \sim \mathrm{cyl}(\mathrm{prj}\, B\, \mathrm{spc}\, \mathrm{sng}\, w)\, \mathrm{spc}\, m) \\
&\cup \bigcup w \in (m \sim p)\, \mathrm{cyl}(\mathrm{prj}\, T \mathrm{spc}\, \mathrm{sng}\, w \sim \mathrm{prj}\, A\, \mathrm{spc}\, \mathrm{sng}\, w)\, \mathrm{spc}\, m \\
&= T \sim B \cup N.
\end{aligned}$$

Thus, using Part 1 to check that $B \in$ mbl$\,\phi$, we note

$$\begin{aligned}
.\phi T &\leqq .\phi(TA) + .\phi(T \sim A) \\
&\leqq .\phi(TB) + .\phi(T \sim B \cup N) \\
&\leqq .\phi(TB) + .\phi(T \sim B) + .\phi N \\
&= .\phi(TB) + .\phi(T \sim B) + 0 = .\phi T.
\end{aligned}$$

Aided again by 3.3.1 we conclude that $A \in$ mbl$\,\phi$.

6.8. THEOREM. *If* $m \in \text{measuretic}$, $0 \Subset p \Subset m$, $q = m \sim p$, $\mu \in \text{Msr spc}\, p$, $v \in \text{Msr spc}\, q$, $\text{nilset}\, p \subset \text{zr}\, \mu$, $\text{nilset}\, q \subset \text{zr}\, v$ *and* $N \in \text{nilset}\, m$ *then*

$$\int \int \text{Cr}(x \cup y) N \mu\, dx\, v\, dy = 0.$$

Proof. According to 6.3.10, $N = (N' \cup \cup \text{rlm}\, v) \cup (\text{rlm}\, \mu \cup \cup N'')$ for some $N' \in \text{nilset}\, p$ and $N'' \in \text{nilset}\, q$. Note that if $y \in \text{rlm}\, v \sim N''$ then $\text{sctn}\, N y = N'$ and hence,

$$\int \text{Cr}(x \cup y) N \mu\, dx = 0.$$

Since $.vN'' = 0$ we are assured that

$$\int \int \text{Cr}(x \cup y) N \mu\, dx\, v\, dy = 0.$$

6.9. THEOREM. *If* $0 \Subset p \Subset m \in \text{measuretic}$, $q = m \sim p$, $\mu \in \text{msr spc}\, p$, $v \in \text{Msr spc}\, q$, $.\mu A = .\text{vlm}\, p A$ *and* $A \in \text{mbl}\, \mu$ *whenever* $A \in \text{bscbox}\, p$, $.vB = .\text{vlm}\, qB$ *and* $B \in \text{mbl}\, v$ *whenever* $B \in \text{bscbox}\, q$, *then*

$$.\text{vlm}\, m\, C = \int \int \text{Cr}(x \cup y) C \mu\, dx\, v\, dy \ \ \textit{whenever} \ \ C \in \text{bscbox}\, m.$$

Proof. Suppose $C \in \text{bscbox}\, m$, $A = \text{prj}\, C\, \text{spc}\, p$, and $B = \text{prj}\, C\, \text{spc}\, q$. Thus, $C = (A \cup \cup B)$ and assuming first that $A \in \text{bscbox}\, p$ and $B \in \text{bscbox}\, q$ we obtain with the aid of 6.3.5 and PM 4.4, p. 182, that

$$\begin{aligned} .\text{vlm}\, m\, C &= (.\text{vlm}\, p\, A) \cdot (.\text{vlm}\, q\, B) \\ &= \left(\int \text{Cr}\, x\, A \mu\, dx\right) \cdot \left(\int \text{Cr}\, y\, B v\, dy\right) \\ &= \int \left(\int \text{Cr}\, x\, A \mu\, dx\right) \text{Cr}\, y\, B v\, dy \\ &= \int \int \text{Cr}\, x\, A\, \text{Cr}\, y\, B \mu\, dx\, v\, dy \\ &= \int \int \text{Cr}(x \cup y) C \mu\, dx\, v\, dy. \end{aligned}$$

If $A \notin \text{bscbox}\, p$ then $.vB = 0 = .\text{vlm}\, m\, C$. Also if $B \notin \text{bscbox}\, q$ then $.\mu A = 0 = .\text{vlm}\, m\, C$. In either case, clearly

$$\begin{aligned} 0 &= \int \int \text{Cr}\, x\, A\, \text{Cr}\, y\, B \mu\, dx\, v\, dy \\ &= \int \int \text{Cr}(x \cup y) C \mu\, dx\, v\, dy \end{aligned}$$

and we are assured of the desired equality.

We state the following theorem without proof. It is a one sided version of the useful 5.3 in PM (p. 189). Aside from the transfer of setting from the space $\text{rct}\, \text{rlm}\, \mu\, \text{rlm}\, v$ to the space $(\text{rlm}\, \mu \cup \cup \text{rlm}\, v)$, the proof of the present theorem is contained in that of PM 5.3.

6.10. THEOREM. *If* $m \in \text{measuretic}$, $\phi \in \text{Msr spc}\, m$, $\mathfrak{F} \in \text{bsc}\, \phi$, $0 \Subset p \Subset m$, $\mu \in \text{Msr spc}\, p$, $v \in \text{Msr spc}(m \sim p)$, $.\phi A = \int \int \text{Cr}(x \cup y) A \mu\, dx\, v\, dy$ *whenever* $A \in \mathfrak{F}$, *and* $-\infty \leqq \int .fz \phi\, dz \leqq \infty$ *then* $\int .fz \phi\, dz = \int \int .f(x \cup y) \mu\, dx\, v\, dy$.

6.11. THEOREM. *If* $0 \Subset p \Subset m \in$ measuretic, $S = \operatorname{spc} m$, $S' = \operatorname{spc} p$, $S'' = \operatorname{spc}(m \sim p)$, $\mu \in \operatorname{Msr} S'$, $v \in \operatorname{Msr} S''$,

$$g = \operatorname{fcn} B \subset S \int \int \operatorname{Cr}(x \cup y) B \mu dx v dy,$$

$\mathfrak{F}$ = Product $\operatorname{mbl}'\mu \operatorname{mbl}'v$ *and* $\psi = \operatorname{mss} g S \mathfrak{F}$ *then*:

.1 $\psi \in \operatorname{Msr} S$;

.2 Product $\operatorname{mbl}\mu \operatorname{mbl} v \subset \operatorname{mbl}\psi$;

.3 $.\psi A = .gA$ *whenever* $A \in \mathfrak{F}$.

Proof. We know .1 as a consequence of 3.2.1. For .2, suppose $A = (A' \cup \cup A'')$, $A' \in \operatorname{mbl}\mu$, $A'' \in \operatorname{mbl} v$, and let $R_1 = [(S' \sim A') \cup \cup A'']$ and $R_2 = [S' \cup \cup (S'' \sim A'')]$. Now check that

(1) $$S = A \cup R_1 \cup R_2,\ S \sim A \subset R_1 \cup R_2 \quad \text{and} \quad R_1 R_2 = 0,$$

and divide the remainder of the proof of .2 into two parts.

PART 1. If $B \in \mathfrak{F}$ then

$$.gB = .g(BA) + .g(BR_1) + .g(BR_2).$$

Proof. Suppose $B \in \mathfrak{F}$, then in view of (1) we are assured that

$$\operatorname{Cr}(x \cup y)B = \operatorname{Cr}(x \cup y)(BA) + \operatorname{Cr}(x \cup y)(BR_1) + \operatorname{Cr}(x \cup y)(BR_2)$$

whenever $x \in S'$ and $y \in S''$. Hence,

$$\begin{aligned} .gB &= \int \int \operatorname{Cr}(x \cup y) B \mu dx v dy \\ &= \int \int \{\operatorname{Cr}(x \cup y)(BA) + \operatorname{Cr}(x \cup y)(BR_1) + \operatorname{Cr}(x \cup y)(BR_2)\} \mu dx v dy \\ &= \int \{ \int \operatorname{Cr}(x \cup y)(BA)\mu dx + \int \operatorname{Cr}(x \cup y)(BR_1)\mu dx + \int \operatorname{Cr}(x \cup y)(BR_2)\mu dx\} v dy \\ &= \int \int \operatorname{Cr}(x \cup y)(BA)\mu dx v dy + \int \int \operatorname{Cr}(x \cup y)(BR_1)\mu dx v dy \\ &\qquad + \int \int \operatorname{Cr}(x \cup y)(BR_2)\mu dx v dy \\ &= .g(BA) + .g(BR_1) + .g(BR_2). \end{aligned}$$

PART 2. $A \in \operatorname{mbl}\psi$.

Proof. Suppose $T \in \operatorname{dmn}'\psi$, $r > 0$, and secure such a family $\mathfrak{G} \in \operatorname{cblcvr} \mathfrak{F} T$ that

$$\sum B \in \mathfrak{G}\ .gB \leqq .\psi T + r.$$

Using .1, (1), 3.2.2., and Part 1 we infer

$$\begin{aligned} .\psi T &\leqq .\psi(TA) + .\psi(T \sim A) \\ &\leqq .\psi(TA) + .\psi(TR_1) + .\psi(TR_2) \\ &\leqq \sum B \in \mathfrak{G}\ .\psi(BA) + \sum B \in \mathfrak{G}\ .\psi(BR_1) + \sum B \in \mathfrak{G}\ .\psi(BR_2) \\ &\leqq \sum B \in \mathfrak{G}\ .g(BA) + \sum B \in \mathfrak{G}\ .g(BR_1) + \sum B \in \mathfrak{G}\ .g(BR_2) \\ &\leqq \sum B \in \mathfrak{G}\ \{.g(BA) + .g(BR_1) + .g(BR_2)\} \\ &= \sum B \in \mathfrak{G}\ .gB \leqq .\psi T + r. \end{aligned}$$

The arbitrary nature of r assures us

$$.\psi T = .\psi(TA) + .\psi(T \sim A).$$

For .3, suppose $A \in \mathfrak{F}$, $r > 0$, and choose such a family $\mathfrak{G} \in \operatorname{cblcvr} \mathfrak{F} A$ that

$$.\psi A + r \geqq \sum B \in \mathfrak{G} \ .gB.$$

Notice that for each z,

$$0 \leqq \operatorname{Cr} z A \leqq \sum B \in \mathfrak{G} \operatorname{Cr} zB$$

and hence that

$$\begin{aligned} .\psi A + r \geqq \sum B \in \mathfrak{G} \ .gB = & \ \sum B \in \mathfrak{G} \int \int \operatorname{Cr}(x \cup y) B \, \mu dx \nu dy \\ = & \int \sum B \in \mathfrak{G} \int \operatorname{Cr}(x \cup y) B \, \mu dx \nu dy \\ = & \int \int \sum B \in \mathfrak{G} \operatorname{Cr}(x \cup y) B \, \mu dx \nu dy \\ \geqq & \int \int \operatorname{Cr}(x \cup y) A \, \mu dx \nu dy \\ = & \ .gA \geqq .\psi A. \end{aligned}$$

Since r was arbitrary we are assured of the desired equality.

6.12. THEOREM. *If* $0 \in p \in m \in$ measuretic, $\mu \in \operatorname{Msr} \operatorname{spc} p$, $\nu \in \operatorname{Msr} \operatorname{spc}(m \sim p)$, $\operatorname{mbl}' \psi \in \operatorname{bsc} \psi$,

$$.\psi A = \int \int \operatorname{Cr}(x \cup y) A \, \mu dx \nu dy \ \textit{whenever} \ A \in \operatorname{mbl}' \psi,$$

$\mathfrak{N} \subset \operatorname{sb} \operatorname{spc} m$, $\phi = \operatorname{knsr} \psi \mathfrak{N}$ *and*

$$0 = \int \int \operatorname{Cr}(x \cup y) B \, \mu dx \nu dy \ \textit{whenever} \ B \in \mathfrak{N},$$

then:

.1 $\phi \in \operatorname{Msr} \operatorname{spc} m$;
.2 $\operatorname{mbl} \psi \subset \operatorname{mbl} \phi$;
.3 $.\phi A = .\psi A$ *whenever* $A \in \operatorname{mbl}' \psi$.

Proof. For .1 and .2 use 3.4.2 and 3.4.4. For .3, suppose $A \in \operatorname{mbl}' \psi$ and secure such a countable subfamily $\mathfrak{G}$ of $\mathfrak{N}$ that

$$.\phi A = .\psi(A \sim \sigma \mathfrak{G})$$

and such a member A' of $\operatorname{mbl} \psi \cap \operatorname{sp}(A \sim \sigma \mathfrak{G})$ that

$$.\psi A' = .\psi(A \sim \sigma \mathfrak{G}).$$

Notice that for each z,

$$\operatorname{Cr} zA' + \sum B \in \mathfrak{G} \operatorname{Cr} zB \geqq \operatorname{Cr} zA$$

and hence

$$\begin{aligned} .\phi A &= .\psi A = \int\int \mathrm{Cr}(x \cup y) A' \mu dx v dy + 0 \\ &= \int\int \mathrm{Cr}(x \cup y) A' \mu dx v dy + \sum B \in \mathfrak{G} \int\int \mathrm{Cr}(x \cup y) B \mu dx v dy \\ &= \int\int \mathrm{Cr}(x \cup y) A' \mu dx v dy + \int \sum B \in \mathfrak{G} \int \mathrm{Cr}(x \cup y) B \mu dx v dy \\ &= \int\int \{\mathrm{Cr}(x \cup y) A' + \sum B \in \mathfrak{G} \mathrm{Cr}(x \cup y) B\} \mu dx v dy \\ &\geqq \int\int \mathrm{Cr}(x \cup y) A \mu dx v dy = .\psi A \geqq .\phi A. \end{aligned}$$

6.13. THEOREM. *If* $0 \Subset p \Subset m \in$ measuretic, $\mu \in \mathrm{Msr\ spc}\, p$, $v \in \mathrm{Msr\ spc}(m \sim p)$, $\mathrm{bscbox}\, p \cup \mathrm{zr}\, \mu \in \mathrm{bsc}\, \mu$, $\mathrm{bscbox}(m \sim p) \cup \mathrm{zr}\, v \in \mathrm{bsc}\, v$, $\phi \in \mathrm{Msr\ spc}\, m$, $\mathrm{bscbox}\, m \cup \mathrm{zr}\, \phi \in \mathrm{bsc}\, \phi$, *and* $.\phi(T \cap \mathrm{cyl}\, B\, \mathrm{spc}\, m) = 0$ *whenever* $.\phi T < \infty$ *and* $B \in \mathrm{zr}\, \mu$ *or* $B \in \mathrm{zr}\, v$, *then*

$$\mathrm{Product\ mbl}\, \mu\, \mathrm{mbl}\, v \in \mathrm{mbl}\, \phi.$$

Proof. Suppose $A' \in \mathrm{mbl}\, \mu$, $A'' \in \mathrm{mbl}\, v$, $A = (A' \cup\cup A'')$, $T \in \mathrm{bscbox}\, m$, $T' = \mathrm{prj}\, T \mathrm{rlm}\, \mu$, $T'' = \mathrm{prj}\, T \mathrm{rlm}\, v$, and secure such sets $B' \in \mathrm{Meet}''\, \mathrm{Join}''\, \mathrm{bscbox}\, p$ and $B'' \in \mathrm{Meet}''\, \mathrm{Join}''\, \mathrm{bscbox}(m \sim p)$ that

$$.\mu(T'A' \sim B') = .\mu(B' \sim T'A') = 0 \tag{1}$$

and

$$.v(T''A'' \sim B'') = .v(B'' \sim T''A'') = 0. \tag{2}$$

Let $B = (B' \cup\cup B'')$ and note

$$TA \sim B = (T'A' \cup\cup (T''A'' \sim B'')) \cup ((T'A' \sim B') \cup\cup T''A'') \tag{3}$$

and

$$B \sim TA = (B' \cup\cup (B'' \sim T''A'')) \cup ((B' \sim T'A') \cup\cup B''). \tag{4}$$

From (1), (2), and the fact $.\phi(TA) < \infty$ we learn from (3) that $.\phi(TA \sim B) = 0$. Checking that $B \in \mathrm{Join}''\, \mathrm{dmn}'\, \phi$, we learn from (1), (2) and (4) that $.\phi(B \sim TA) = 0$. Since clearly $B \in \mathrm{mbl}\, \phi$ we conclude $TA \in \mathrm{mbl}\, \phi$,

$$.\phi T = .\phi(TA) + .(T \sim A),$$

and employ 3.3.1 in reaching the desired conclusion.

6.14. THEOREM. *If* H *is a* Harmonious, *and* $\alpha' = \mathrm{mc}\, H$ *then* productive α' = measuretic.

Proof. We infer the desired conclusion from Parts 1, 2, and 3 below.

PART 1. If $m \in$ measuretic, $\phi = .\alpha' m$, and $\psi = .\alpha m$ then:

.1 $\phi = \mathrm{knsr}\, \psi\ .Hm$;

.2 $\mathrm{mbl}\, \psi \subset \mathrm{mbl}\, \phi$;

.3 $.\phi A = .\psi A$ whenever $A \in \mathrm{mbl}'\, \psi$;

.4 $\int .fz \phi dz = \int .fz \psi dz$ whenever $-\infty \leqq \int .fz \psi dz \leqq \infty$.

Proof. For .1, let $\mathfrak{G} = \text{zr } .\alpha m$ and use 6.5 to check that $\psi = \text{knsr cpm } m\, \mathfrak{G}$. Thus, since $\mathfrak{G} \subset .Hm$,

$$\begin{aligned} \phi &= \text{knsr cpm } m \ .Hm \\ &= \text{knsr cpm } m \ (\mathfrak{G} \cup .Hm) \\ &= \text{knsr knsr cpm } m\, \mathfrak{G} \ .Hm \\ &= \text{knsr } \psi \ .Hm. \end{aligned}$$

For .2 and .3, assume $0 \in p \in m$, and let $\mu = .\alpha p$ and $v = .\alpha(m \sim p)$. Now, employ 6.12 taking $\mathfrak{N} = .Hm$. Finally, .4 is a direct consequence of .2 and .3.

PART 2. If $0 \in p \in m \in \text{measuretic}$, $\phi = .\alpha' m$, $\psi = .\alpha m$, $\mu = .\alpha' p$, $v = .\alpha'(m \sim p)$ then:

.5. $.Hm \cup \text{bscbox } m \in \text{bsc } \phi$;

.6 $.\phi A = \int \int \text{Cr}(x \cup y) A\, \mu dx v dy$ whenever $A \in .Hm \cup \text{bscbox } m$;

.7 $\text{cyl } B \text{ spc } m \in \text{zr } \phi$ whenever $B \in \text{zr } \mu$ or $B \in \text{zr } v$.

Proof. For .5 use Part 1 and 3.4.2. For .6, let $\xi = .\alpha p$ and $\eta = .\alpha(m \sim p)$, check that

$$.\phi A = \int \int \text{Cr}(x \cup y) A\, \xi dx \eta dy$$

whenever $A \in .Hm \cup \text{bscbox } m$, and use .4 in checking, for η almost all y,

$$\int \text{Cr}(x \cup y) A \xi\, dx = \int \text{Cr}(x \cup y) A \mu dx. \tag{1}$$

Use .4 again to learn from (1) that

$$\int \int \text{Cr}(x \cup y) A\, \xi dx \eta dy = \int \int \text{Cr}(x \cup y) A\, \mu dx v dy.$$

For .7, suppose $.\mu B = 0$ and 3.4.5 split B into $B_1 \in \text{zr } .\alpha p$ and $B_2 \subset D \in \text{Join}''\ .Hp$ so $B = B_1 \cup B_2$. Thus, $\text{cyl } B_1 \text{ spc } m \in \text{zr } .\alpha m$ and $\text{cyl } B_2 \text{ spc } m \subset \text{cyl } D \text{ spc } m \in \text{Join}''$ $.Hm \subset \text{zr } .\alpha' m$ and therefore $B \in \text{zr } \phi$. The case $B \in \text{zr } v$ is similar.

PART 3. If $0 \in p \in m \in \text{measuretic}$ then

$$\text{Product mbl } .\alpha' p \text{ mbl } .\alpha'(m \sim p) \subset \text{mbl } .\alpha' m.$$

Proof. Use .7 and 6.13.

A family of sets more general than $\text{bscbox } m$ is introduced in 6.15.5. It replaces the family of cylindrical sets of the classical theory of infinite product measures, and can be described as follows. Suppose $m \in \text{measuretic}$, $p \in \text{fnt} \cap \text{sb } m$, $A' \subset \text{spc } p$, $A'' \in \text{bscbox}(m \sim p)$, then $A = (A' \cup \cup A'')$ is one of these sets. If p is the smallest subset of m for which A can be so represented, then we call p the stand of A, $m \sim p$ the tower of A, A' the foot of A and A'' the top of A.

6.15. DEFINITIONS.

.1 $\text{stand } mA = \pi \text{E} p \in \text{fnt} \cap \text{sb } m$ $[A = (A' \cup \cup A'')$ for some $A' \subset \text{spc } p$ and some $A'' \in \text{bx}(m \sim p)]$.

.2 tower $m\,A = m \sim$ stand $m\,A$.

.3 ft $m\,A =$ prj A spc stand $m\,A$.

.4 tp $mA =$ prj A spc tower mA.

.5 frame $m = \mathrm{E}A \subset$ spc m [$m \in$ measuretic, stand $m\,A \in$ fnt, and tp $m\,A \in$ box tower mA].

.6 Vlm $\alpha\, m =$ fun $A \in$ frame m.

$$[..\alpha \text{ stand } m\,A \,\mathrm{ft}\, m\,A \cdot .\mathrm{vlm} \text{ tower } m\,A \,\mathrm{tp}\, m\,A].$$

.7 harmonil = fun $m \in$ measuretic [nilset $m \cup$ nilcylinder m].

.8 startproduction $\alpha\, m =$ mss(Vlm $\alpha\, m$) (spc m) (dmn$'$ Vlm $\alpha\, m$).

.9 production $\alpha =$ fun $m \in$ measuretic

$$\mathrm{knsr}(\mathrm{startproduction}\,\alpha\, m)\ (.\mathrm{harmonil}\, m).$$

.10 cm = mc harmonil.

.11 cnm $m =$ .cm m.

In 6.15.9 above, we have defined a general method for extending finite products of measures to infinite products. Suppose α represents a method for obtaining a product of a finite number of measures, i.e., fnt $\cap$ measuretic $\subset$ dmn α. Then $\alpha' =$ production α is the extension of that method to arbitrary products and dmn $\alpha' =$ measuretic.

Useful in 6.17 is

6.16. THEOREM. *If* $p \in$ fnt $\cap$ sb $m \subset$ mblproductive α, $q \in$ fnt $\cap$ sb m, $p \cap q = 0$, $\mu = .\alpha p$, $\nu = .\alpha q$, $\phi = .\alpha(p \cup q)$, $A \subset$ spc $p \in$ bscbox p, *and* $B \subset$ spc $q \in$ bscbox q *then*

$$.\phi(A \cup \cup B) = .\mu A \cdot .\nu B.$$

Proof. Use 3.3.2, 6.4.3 and 6.4.2.

For our purpose, we give a general version of the well known

6.17. THEOREM. *If* $m \in$ measuretic, $.\lambda$ rlm $\lambda = 1$ *for each* $\lambda \in$ rng m, fnt $\cap$ sb $m \subset$ mblproductive α, $\mathfrak{F} = \bigcup q \in$ fnt $\cap$ sb m sb spc q, *g is the function on $\mathfrak{F}$ which assigns to each $A \in \mathfrak{F}$ the value $..\alpha q A$ where q is that subset of m for which* $A \subset$ spc q, $\mathfrak{G} \in$ cbl $\cap$ sb $\mathfrak{F}$, and spc $m =$ cyl $\sigma\mathfrak{G}$ spc m *then*

$$\sum A \in \mathfrak{G}\ .gA \geqq 1.$$

Proof. First note that spc $p \in$ bscbox p and hence that $.g$ spc $p = 1$ whenever $p \in$ fnt $\cap$ sb m. Now, employ the countability of $\mathfrak{G}$ to secure such a sequence r of members of fnt $\cap$ sb m that

(1) $$0 = .r\,0 \subset .rn \subset .r(n+1) \text{ whenever } n \in \omega,$$

and

(2) $$A \subset \mathrm{sbmb\,spc}\,\sigma\, \mathrm{rng}\, r \text{ whenever } A \in \mathfrak{G}.$$

Let $Q = \operatorname{spc}\sigma \operatorname{rng} r$, $R = \operatorname{fun} n \in \omega \operatorname{spc} .rn$, and

$$b = \operatorname{fun} n \in \omega \ [.Rn \sim \operatorname{cyl} \sigma \mathfrak{G}\ .Rn].$$

Suppose $n \in \omega$ and use 5.7.1 and (2) in calculating

$$\begin{aligned}\operatorname{cyl}\ .bn\ .R(n+1) &= \operatorname{cyl}\ .Rn\ .R(n+1) \sim \operatorname{cyl}\operatorname{cyl}\sigma\mathfrak{G}\ .Rn\ .R(n+1)\\ &\supset .R(n+1) \sim \operatorname{cyl}\sigma\mathfrak{G}\ .R(n+1) = .b(n+1).\end{aligned}$$

Thus, we infer

(3) $$.b(n+1) = \operatorname{cyl}\ .bn\ .b(n+1) \text{ whenever } n \in \omega,$$

and divide the remainder of the proof into six steps.

STEP 1. If $A \in \mathfrak{F}$, $B \in \mathfrak{F}$, and $A = \operatorname{cyl} B A$ then $.gA \leqq .gB$.

Proof. Suppose $p \subset m' \in \operatorname{fnt} \cap \operatorname{sb} m$, $B \subset \operatorname{spc} p$ and $A \subset \operatorname{spc} m'$. Then either $A \subset B$ or $p \subsetneq m'$ and $A \subset \operatorname{cyl} B \operatorname{spc} m'$. For the latter alternative we employ 6.16 to infer

$$.gA \leqq .g(\operatorname{cyl} B \operatorname{spc} m') = .gB \cdot 1 = .gB$$

and hence, for either case, conclude $.gA \leqq .gB$.

STEP 2. If $A \in \mathfrak{F}$, $B \in \mathfrak{F}$, $A = \operatorname{cyl} BA$, and $x \in \operatorname{sbmb} B$ then $\operatorname{sctn} Ax \in \mathfrak{F}$, $\operatorname{sctn} Bx \in \mathfrak{F}$ and $\operatorname{sctn} Ax = \operatorname{cyl} \operatorname{sctn} B x \operatorname{sctn} Ax$.

Proof. Let p', p, and q be those subsets of m for which $x \in \operatorname{spc} p'$, $B \subset \operatorname{spc} p$ and $A \subset \operatorname{spc} q$, and notice that $p' \subset p \subset q$.

Suppose $y \in \operatorname{sctn} Ax$. Then $x \cap y = 0$ and $x \cup y \in A$. Let $z = x \cup y$. Since $A = \operatorname{cyl} BA$ there exists a member t of B which is a subset of z. Let $s = z \sim t$ and secure such $y' \in \operatorname{spc}(p \sim p')$ and $y'' \in \operatorname{spc}(q \sim p)$ that $y = y' \cup y''$. Now, $z = (x \cup y') \cup y'' = t \cup s$ and infer with the aid of 5.4.5 that $x \cup y' = t$. Thus, $y' \in \operatorname{sctn} Bx$ and we infer $y \in \operatorname{cyl} \operatorname{sctn} Bx \operatorname{sctn} Ax$. Since, $\operatorname{cyl} \operatorname{sctn} Bx \operatorname{sctn} Ax \subset \operatorname{sctn} Ax$, we conclude the desired equality. Obviously $\operatorname{sctn} Bx \subset \operatorname{spc}(p \sim p')$ and $\operatorname{sctn} Ax \subset \operatorname{spc}(q \sim p)$ and our proof is complete.

STEP 3. If $k \in \omega$, $x \in .bk$, and $\lim_{n\to\infty} .g\ (\operatorname{sctn}\ .bn x) > 0$ then there exists such an $x' \in .b(k+1) \cap \operatorname{sp} x$ that

$$\lim_{n\to\infty} .g(\operatorname{sctn}\ .bn x') > 0.$$

Proof. The choice of x' is clear when $.r(k+1) = .r(k)$. We henceforth assume $.r(k+1) \neq .rk$ and use (3), Step 2 and Step 1 in ascertaining that

(4) $$\text{if } n \in \omega \text{ and } n > k \text{ then } .g(\operatorname{sctn}\ .b(n+1)x) \leqq .g(\operatorname{sctn}\ .bn x).$$

We are now assured of the existence of such a number $s > 0$ that $.g(\operatorname{sctn}\ .bn x) > s$ whenever $n \in \omega$ and $n > k$. Let $p = \operatorname{spc}(.r(k+1) \sim .rk)$, $\mu = .\alpha p$, $\omega^* = \mathrm{E} n \in \omega(n > k+1)$, and $d = \operatorname{fun} n \in \omega^*\ \mathrm{E} t \in \operatorname{spc} p\ [.g \operatorname{sctn} .bn(x \cup t) > s/2]$.

Use (3), Step 2 and Step 1, as above, in checking $.g(\operatorname{sctn} .b(n+1)\ (x \cup t)) \leqq .g(\operatorname{sctn}\ .\mathrm{bn}(x \cup t))$ whenever $n \in \omega^*$, wherefrom we learn

(5) $$.d(n+1) \subset .dn \text{ whenever } n \in \omega^*.$$

Suppose now that $n \in \omega$, $m' = .rn \sim .rk$, $q = m' \sim p$, and secure $A \in \text{mbl } .\alpha m' \cap \text{sp sctn } .\text{bn}\, x$ for which $.gA = .g(\text{sctn } .bn\, x)$.

Let $D = \text{E}t \in \text{spc}\, p\ [.g\text{sctn}\, At > s/2]$ and check that $.\mu D = .\mu .dn$ and $D \in \text{mbl}\, \mu$. Thus,

$$\begin{aligned}
.gA &= \int \int \text{Cr}(u \cup t) A\ .\alpha q\, du \mu dt \\
&= \int .g\, \text{sctn}\, A\, t\, \mu dt \\
&= \int (\text{Cr}\, t D + \text{Cr}\, t(\text{spc}\, p \sim D))\ .g\, \text{sctn}\, A\, t\, \mu dt \\
&= \int \text{Cr}\, t D .g\, \text{sctn}\, A\, t\, \mu dt + \int \text{Cr}\, t(\text{spc}\, p \sim D)\ .g\, \text{sctn}\, A\, t\, \mu dt \\
&\leqq \int \text{Cr}\, t D\, \mu dt + \int (s/2)\, \mu dt \\
&= .\mu D + s/2.
\end{aligned}$$

Hence, $.\mu D \geqq .gA - s/2 \geqq s - s/2 = s/2$, and we infer

(6) $$.\mu\ .dn \geqq s/2 \text{ whenever } n \in \omega^*.$$

From (5) and (6) we learn $.\mu \bigcap n \in \omega^*\ .dn > 0$ and $0 \neq \bigcap n \in \omega .dn$. Since clearly, for each $n \in \omega^*$, $.dn \subset \text{sctn } .b(k+1)x$, we are assured of the existence of such a point $t \in \text{sctn} .b(k+1)x$ that $.g(\text{sctn} .bn(x \cup t)) > s/2$ whenever $n \in \omega^*$. Taking $x' = (x \cup t)$ realizes our objective.

STEP 4. $0 = \bigcap n \in \omega \text{ cyl } .bnQ$.

Proof. Use (2), 5.3.5 and 5.7.2 in checking

$$\begin{aligned}
\bigcap n \in \omega \text{cyl } .bnQ &= Q \sim \bigcup n \in \omega \text{ cyl}\, (\text{cyl}\, \sigma G \text{spc } .Rn) Q \\
&= Q \sim \text{cyl}(\bigcup n \in \omega \text{ cyl}\, \sigma G \text{spc } .Rn) Q \\
&= Q \sim \text{cyl}\, \text{cyl}\, \sigma G \bigcup n \in \omega \text{spc } .Rn\, Q \\
&= Q \sim \text{cyl}\, \sigma G Q = 0.
\end{aligned}$$

STEP 5. $\lim n \to \infty\ .g.bn = 0$.

Proof. The alternative to our assertion, in view of the (3), Step 2, Step 1 monotonic nature of the numbers $.g.bn$ for $n \in \omega$, is that

(7) $$\lim n_{n \to \infty}\ .g.bn > 0.$$

Let us tentatively assume (7) in order to reach a contradiction in (8) below.

Using Step 3, and noting $0 \in .b0 = 1$, we may inductively obtain a sequence y with the following properties: for each $n \in \omega$,

.1 $.yn \in .bn$ and $.yn \subset .y(n+1)$, and

.2 $\lim k \to \infty\ .g\ (\text{sctn } .bk\ .yn) > 0$.

Let $z = \bigcup n \in \omega\ .yn$, suppose $n \in \omega$ and notice that $.yn \in .bn$, $z \sim .yn \in \text{spc}\ (\sigma \text{ rng}\, r \sim .rn)$ and hence that $z \in \text{cyl } .bn\ Q$. Thus,

$$z \in \bigcap n \in \omega \operatorname{cyl} .bnQ \tag{8}$$

in contradiction to Step (4). We conclude therefore that $\lim n \to \infty .g .bn = 0$

STEP 6. $1 \leqq \sum A \in \mathfrak{G} .gA$.

Proof. Suppose $s > 0$, and employ Step 5 to secure such an $n \in \omega$ that

$$.g. bn < s. \tag{9}$$

Let $p = .rn$, $\mu = .\alpha p$, $S = \operatorname{spc} p$, $\mathfrak{G}' = \mathrm{E} A \in \mathfrak{G}\ [0 \neq \operatorname{cyl} A S]$ and observe that

$$S = .bn \cup \bigcup A \in \mathfrak{G}' \operatorname{cyl} A S. \tag{10}$$

Hence,

$$\begin{aligned} 1 = .\mu S &\leqq .\mu\ .bn + \sum A \in \mathfrak{G}'\ .\mu \operatorname{cyl} A S \\ &\leqq s + \sum A \in \mathfrak{G}' .gA \\ &\leqq s + \sum A \in \mathfrak{G} .gA \end{aligned}$$

and exploiting the arbitrary nature of s we infer

$$1 \leqq \sum A \in \mathfrak{G}\ .gA$$

completing both the proofs of Step 6 and our theorem.

6.18. THEOREM. *If* $m \in$ measuretic, $.\lambda \operatorname{rlm} \lambda = 1$ *for each* $\lambda \in \operatorname{rng} m$, fnt $\cap$ sb $m \subset$ mblproductive α, $\mathfrak{G} \in$ cblsbframe m, *and* $\sigma \mathfrak{G} = \operatorname{spc} m$ *then*

$$\sum C \in \mathfrak{G}\ .\mathrm{Vlm}\, \alpha m C \geqq 1.$$

Proof. Let $\mathfrak{F} = \bigcup q \in \text{fnt} \cap \text{sb}\, m \operatorname{sbspc} q$ and let g be the function on $\mathfrak{F}$ which assigns to each $A \in \mathfrak{F}$ the value $..\alpha q A$ where q is that subset of m for which $A \subset \operatorname{spc} q$. Suppose $S = \operatorname{spc} m$ and divide the proof into two parts.

PART 1. If $r > 0$ and $C \in \operatorname{frame} m$ then there is such a member A of $\mathfrak{F}$ that $C \subset \operatorname{cyl} A S$ and

$$.gA \leqq .\mathrm{Vlm}\, \alpha m C + r.$$

Proof. Let $p' = \operatorname{stand} m C$, $q' = m \sim p'$, $A' = \operatorname{ft} m C$, $B' = \operatorname{tp} m C$ and $f = .gA'$. Thus, $A' \in \mathfrak{F}$, $C \subset \operatorname{cyl} A' S$, and our conclusion is immediately inferred when $f = 0$. Suppose therefore, that $f > 0$ and employ 4.12.10 to secure such a finite subset q of q' that

$$\left| .\mathrm{vlm} q' B' - \prod + i \in \operatorname{dmn} q ..mi \operatorname{slice} B' i \right| \leqq (r/f). \tag{1}$$

Let $B = \operatorname{prj} B' \operatorname{spc} q$ and $A = (A' \cup \cup B)$. Thus $\prod + i \in \operatorname{dmn} q ..mi$ slice $B' i$ $= .\mathrm{vlm}\, q B = .gB$ and we learn from (1) after multiplication by f that

$$\left| .\mathrm{Vlm}\, \alpha m C - f \cdot .gB \right| \leqq r. \tag{2}$$

Using $p' \cup q \in \text{fnt}$ and 6.16 we readily infer that $.gA = f \cdot .gB$, and using (2) complete our proof with the observation that $C \subset \text{cyl}\, A\, S$ and $A \in \mathfrak{F}$.

PART 2. $\sum C \in \mathfrak{G}\ .\text{Vlm}\, \alpha\, m\, C \geqq 1$.

Proof. Suppose $r > 0$ and employ Part 1 to obtain such a countable subfamily $\mathfrak{H}$ of $\mathfrak{F}$ that

$$\sigma\mathfrak{G} \subset \bigcup A \in \mathfrak{H}\, \text{cyl}\, A\, S$$

and

$$\sum A \in \mathfrak{H}\ .gA \leqq \sum C \in \mathfrak{G}\ .\text{Vlm}\, \alpha\, m\, C + r.$$

Use 6.17 to infer

$$\sum A \in \mathfrak{H}\ .gA \geqq 1$$

and then conclude, in view of the arbitrary r, that

$$\sum C \in \mathfrak{G}\ .\text{Vlm}\, \alpha\, m\, C \geqq 1.$$

6.19. DEFINITIONS.

.1 weight $K\, m = \prod + i \in \text{dmn}\, m\ \ .Ki$.

.2 factorfor m = EK [K is a function, $\text{dmn}\, m \subset \text{dmn}\, K$, $0 \leqq .Ki \leqq \infty$ whenever $i \in \text{dmn}\, K$ and $0 < \text{weight}\, K\, m < \infty$].

.3 factormeasuretic $K\, m = \text{fun}\, i \in \text{dmn}\, m\ (.Ki \cdot .mi)$.

.4 responsive α = E$m \in$ mblproductive α [for each $K \in$ factorfor m, $.\alpha$ factormeasuretic $K\, m$ = (weight $K\, m$) $\cdot$ $(.\alpha m)$].

6.20. THEOREM. *If* $M \subset$ mblproductive α, *and* zr $.\alpha m$ = zr $.\alpha m'$, $m' \in M$ *whenever* m, K *and* m' *are such that* $m \in M$, $K \in$ factorfor m *and* m' = factormeasuretic $K\, m$, *then* $M \subset$ responsive α.

Proof. Suppose $m \in M$, $K \in$ factorfor m, m' = factormeasuretic $K\, m$, k = weight $K\, m$, $\phi = .\alpha m$ and $\phi' = .\alpha m'$.

STEP 1. bscbox m' = bscbox m and $.\text{vlm}\, m' A = k \cdot .\text{vlm}\, m\, A$ whenever $A \in$ bscbox m.

Proof. Use 4.12.9, 6.19.3 and 6.19.2.

STEP 2. If $.\phi A < \infty$ then $.\phi' A \leqq k \cdot .\phi A$.

Proof. Suppose $r > 0$ and secure such a countable subfamily $\mathfrak{F}$ of bscbox m that

$$.\phi(A \sim \sigma\mathfrak{F}) = 0 \text{ and } .\phi A + (r/k) \geqq \sum B \in \mathfrak{F}\ .\phi B.$$

Thus,

$$\begin{aligned} .\phi\,(A) &\leqq .\phi'(A \sim \sigma\mathfrak{F}) + .\phi'(\sigma\mathfrak{F}) \\ &\leqq 0 + \sum B \in \mathfrak{F}\ .\phi'(B) \\ &= \sum B \in \mathfrak{F}\, k \cdot .\phi(B) \\ &= k \sum B \in \mathfrak{F}\ .\phi B \\ &\leqq k \cdot .\phi A + r. \end{aligned}$$

Since r is arbitrary we infer $.\phi'(A) \leqq k \cdot .\phi A$.

STEP 3. If $.\phi' A < \infty$ then $k \cdot .\phi A \leqq .\phi' A$.

Proof. Suppose $r > 0$ and choose $\mathfrak{F} \in \text{cbl sb bscbox}\, m$ for which $.\phi'(A \sim \sigma\mathfrak{F}) = 0$ and $.\phi'(A) + kr \geqq \sum B \in \mathfrak{F}\ .\phi' B$. Then

$$\begin{aligned} .\phi A &\leqq .\phi(A \sim \sigma\mathfrak{F}) + .\phi(\sigma\mathfrak{F}) \\ &\leqq 0 + \sum B \in \mathfrak{F}\ .\phi B \\ &\leqq \sum B \in \mathfrak{F}\, k^{-1}\ .\phi' B \\ &= k^{-1} \sum B \in \mathfrak{F}\ .\phi' B \\ &\leqq k^{-1}\ .\phi' A + r. \end{aligned}$$

Thus, $k \cdot .\phi A \leqq .\phi' A$

From Steps 2 and 3 we infer

$$.\phi A = k \cdot .\phi' A \text{ whenever } A \subset \text{spc}\, m$$

and conclude $m \in \text{responsive}\ \alpha$ and therefore

$$M \subset \text{responsive}\ \alpha.$$

6.21. THEOREM. *If* $m \in \text{measuretic}$, $\text{fnt} \cap \text{sb}\, m \subset \text{responsive}\, \alpha$, $B \in \text{bscbox}\, m$, $\mathfrak{F} \in \text{cbl} \cap \text{sb frame}\, m$ *and* $B \subset \sigma\mathfrak{F}$ *then*

$$\sum C \in \mathfrak{F}\ .\text{Vlm}\, \alpha\, m\, C \geqq .\text{vlm}\, m\, B.$$

Proof. The conclusion is obvious if $.\text{vlm}\, m\, B = 0$. We therefore suppose $.\text{vlm}\, m\, B > 0$ and proceed by letting $m' = \text{fun}\, i \in \text{dmn}\, m[\text{fun}\, a \subset \text{slice}\, Bi\ ..mia]$. Suppose α' is that function on $\text{sb}\, m'$ which assigns to each $p' \subset m'$ the measure

$$\text{fun}\, A \subset (\text{prj}\, B\, \text{spc}\, p)\ (..\alpha p A),$$

where p is that subset of m for which $\text{dmn}\, p = \text{dmn}\, p'$.

Verify that

$$\text{fnt} \cap \text{sb}\, m' \subset \text{responsive}\, \alpha'. \tag{1}$$

Let $K = \text{fun}\, i \in \text{dmn}\, m\ (1/..mi\ \text{slice}\, Bi)$,

$$m'' = \text{factormeasuretic}\, K\, m',$$

and check that

$$(\text{weight}\, K\, m') \cdot .\text{vlm}\, m\, B = 1, \tag{2}$$

and

$$..m''i\, \text{rlm}\ .m''i = 1 \text{ whenever } i \in \text{dmn}\, m''. \tag{3}$$

Next let $\mathfrak{F}' = \bigcup C \in \mathfrak{F}\, \text{sng}(C \cap B)$, check that $\text{spc}\, m'' \subset \sigma\mathfrak{F}'$, and employ 6.18 to ascertain that

(4) $$\sum C' \in \mathfrak{F}' \ .\mathrm{Vlm}\, \alpha' m'' C' \geqq 1.$$

Suppose $C \in \mathfrak{F}$, $C' = C \cap B$, $p'' = \mathrm{stand}\, m'' C'$, $p' \subset m'$, $p \subset m$, $\mathrm{dmn}\, p'' = \mathrm{dmn}\, p' = \mathrm{dmn}\, p$, $A_0 = \mathrm{ft}\, m C'$ and $A_1 = \mathrm{tp}\, m C'$. Then,

$$\begin{aligned}
.\mathrm{Vlm}\, \alpha m C \geqq\ & .\mathrm{Vlm}\, \alpha m C' \\
=\ & ..\alpha p A_0 \cdot .\mathrm{vlm}(m \sim p) A_1 \\
=\ & ..\alpha' p' A_0 \cdot .\mathrm{vlm}(m' \sim p') A_1 \\
=\ & ..\alpha' p' A_0 \cdot \mathrm{weight}\, K m' \cdot .\mathrm{vlm}\, m B \cdot .\mathrm{vlm}(m' \sim p') A_1 \\
=\ & (..\alpha' p' A_0 \mathrm{weight}\, K p') \cdot (\mathrm{weight}\, K(m' \sim p') .\mathrm{vlm}(m' \sim p') A_1) \cdot .\mathrm{vlm}\, m B \\
=\ & \{(..\alpha' p'' A_0) \cdot .\mathrm{vlm}(m'' \sim p'') A_1\} \ .\mathrm{vlm}\, m B \\
=\ & .\mathrm{Vlm}\, \alpha' m'' C' \cdot .\mathrm{vlm}\, m B.
\end{aligned}$$

Hence, for each $C \in \mathfrak{F}$

(5) $$.\mathrm{Vlm}\, \alpha m C \geqq .\mathrm{Vlm}\, \alpha' m'' (C \cap B) \cdot .\mathrm{vlm}\, m B$$

and we conclude

$$\begin{aligned}
\sum C \in \mathfrak{F} \ .\mathrm{Vlm}\, \alpha m C \geqq\ & .\mathrm{vlm}\, m B \cdot \sum C \in \mathfrak{F} \ .\mathrm{Vlm}\, \alpha' m'' (C \cap B) \\
\geqq\ & .\mathrm{vlm}\, m B \cdot \sum C' \in \mathfrak{F}' \ .\mathrm{Vlm}\, \alpha' m'' C' \\
\geqq\ & .\mathrm{vlm}\, m B.
\end{aligned}$$

6.22. THEOREM. fnt $\cap$ measuretic $\subset$ responsive cp.

Proof. Let $R = \mathrm{E} n, m$ [$n \in \omega$, $m \in$ measuretic, and m contains no more than n elements].

Thus, if $n \in \omega$ then $\mathrm{vs}\, R n$ is the class of measuretic functions each of which contains n elements or less. It is evident that

(1) $$\mathrm{vs}\ R 1 \subset \text{responsive cp}.$$

Suppose $N \in \omega$ and that we know

(2) $$\mathrm{vs}\, R N \subset \text{mblproductive cp}.$$

Our proof is completed with the aid of mathematical nduction by demonstrating below that

(3) $$\mathrm{vs}\, R(N+1) \subset \text{responsive cp}.$$

Proof. Suppose $m \in \mathrm{vs}\, R(N+1)$, $0 \in p \in m$, $q = m \sim p$, $S = \mathrm{spc}\, m$, $\mu = \mathrm{cpm}\, p$, $v = \mathrm{cpm}\, q$, $\psi = \mathrm{cpm}\, m$,

$$g = \text{fun}\, B \subset S \ \int \int \text{Cr}(x \cup y)\, B\, \mu dx v dy,$$

$\mathfrak{F} = \text{Product}\, \text{mbl}'\mu\, \text{mbl}'v$ and $\phi = \text{mss}\, g\, S\, \mathfrak{F}$.

We are assured by (2), 6.9, and 6.11.3 that

(4) $$.\phi T = .\text{vlm}\, m\, T \text{ whenever } T \in \text{bscbox}\, m.$$

Consequently, $\psi = \text{mss}\, \phi\, S\, \text{bscbox}\, m$ and with 3.2.4 we infer that

(5) $$.\psi T = .\phi T = .\text{vlm}\, m\, T \text{ whenever } T \in \text{bscbox}\, m.$$

We establish next that

(6) $$.\psi A \leqq .\phi A \text{ whenever } A \in \mathfrak{F}.$$

Proof. Assume $A = (A' \cup \cup A'')$, $A' \in \text{mbl}'\mu$, $A'' \in \text{mbl}'v$, and $r > 0$. Let $k = .\mu A' + .vA''$ and let t be such a number that $0 < t < r/(2k)$ and $t^2 < r/2$. Now select such families $\mathfrak{G}' \in \text{cblcvr}\, \text{bscbox}\, p\, A'$ and $\mathfrak{G}'' \in \text{cblcvr}\, \text{bscbox}\, q\, A''$ that

$$.\mu A' + t \geqq \sum B' \in \mathfrak{G}' \ .\mu B' \text{ and } .vA'' + t \geqq \sum B'' \in \mathfrak{G}'' \ .vB''.$$

Let $\mathfrak{G} = \text{Product}\, \mathfrak{G}'\mathfrak{G}''$ and use summation by partition in ascertaining

$$\begin{aligned}
.\psi A &\leqq \sum B \in \mathfrak{G} \ .\text{vlm}\, m\, B \\
&= \sum B' \in \mathfrak{G}' \ \sum B'' \in \mathfrak{G}'' \ .\text{vlm}\, m(B' \cup \cup B'') \\
&= \sum B' \in \mathfrak{G}' \ \sum B'' \in \mathfrak{G}'' \ (.\mu B' \cdot .vB'')) \\
&= \sum B' \in \mathfrak{G}' \ (.\mu B' \ \sum B'' \in \mathfrak{G}'' \ .vB'') \\
&\leqq \sum B' \in \mathfrak{G}' \ (.\mu B' \, (.vA'' + t)) \\
&= (.vA'' + t) \ \sum B' \in \mathfrak{G}' \ .\mu B' \\
&\leqq (.vA'' + t)(.\mu A' + t) = .\mu A' \cdot .vA'' + t(.\mu A' + .vA'') + t^2 \\
&\leqq .\mu A' \ .vA'' + \frac{r}{2} + \frac{r}{2} \\
&= .\phi A + r.
\end{aligned}$$

Our proof of (6) is completed by recalling the arbitrary nature of r.

Suppose $A \in \text{mbl}\, \phi$, $T \in \text{bscbox}\, m$, and $r > 0$. Note 6.11.3 and secure such members $\mathfrak{G}$ and $\mathfrak{H}$ of $\text{cblcvr}\, \mathfrak{F}(TA)$ and $\text{cblcvr}\, \mathfrak{F}(T \sim A)$, respectively, that

$$\sum B \in \mathfrak{G} \ .\phi B + r/2 \leqq .\phi(TA),$$

and

$$\sum B \in \mathfrak{H} \ .\phi B + r/2 \leqq .\phi(T \sim A).$$

Thus, using (6),

$$\begin{aligned} .\phi T &= .\phi(TA) + .\phi(T \sim A) \\ &\geqq \textstyle\sum B \in \mathfrak{G} \ .\phi B + \sum B \in \mathfrak{H} \ .\phi B + r \\ &\geqq \textstyle\sum B \in \mathfrak{G} \ .\psi B + \sum B \in \mathfrak{H} \ .\psi B + r \\ &\geqq .\psi(TA) + .\psi(T \sim A) + r \\ &\geqq .\psi T + r = .\phi T + r. \end{aligned}$$

Inferring therefrom that

$$.\psi T = .\psi(TA) + .\psi(T \sim A)$$

we conclude with 3.3.1 that $A \in \mathrm{mbl}\,\psi$.

We learn from 6.11.2 and 6.3.6 that

(7) $$\mathrm{bscbox}\, m \subset \mathrm{Product\,mbl}\,\mu \ \mathrm{mbl}\,v \subset \mathrm{mbl}\,\phi \subset \mathrm{mbl}\,\psi.$$

Our proof that $\mathrm{vs}\, R(N+1) \subset \mathrm{mblproductive\ cp}$ is completed with reference to (5), (7), and 6.10.

Suppose $m \in \mathrm{vs}\, R(N+1)$, $K \in \mathrm{factorfor}\, m$, $m' = \mathrm{factormeasuretic}\, K\, m$ and use the fact $\mathrm{bscbox}\, m = \mathrm{bscbox}\, m'$ and $\mathrm{vlm}\, m = \mathrm{k} \cdot \mathrm{vlm}\, m'$ in checking that $\mathrm{zr\, cpm}\, m' = \mathrm{zr\, cpm}\, m$.

Clearly $m' \in \mathrm{vs}\, R\ (N+1)$ and from 6.20 we infer $\mathrm{vs}\, R(N+1) \subset \mathrm{responsive}\, \alpha$ to complete our proof.

6.23. THEOREM. *If* $m \in \mathrm{measuretic}$, $A \in \mathrm{bscbox}\, m$ *then*

$$..\mathrm{cpm}\, mA = .\mathrm{vlm}\, mA.$$

Proof. In 6.21 take $\mathfrak{F} = \mathrm{bscbox}\, m$ and employ 3.2.4 and 6.22.

6.24. THEOREM. *If* $\mathrm{fnt} \cap \mathrm{measuretic} \subset \mathrm{responsive}\, \alpha$, $\alpha'' = \mathrm{startproduction}\, \alpha$, *and* $\alpha' = \mathrm{production}\ \alpha$ *then*:

.1 $.\alpha'' m = \mathrm{knsr\, cpm}\, m\, \mathrm{zr\, Vlm}\ .\alpha m$ *whenever* $m \in \mathrm{measuretic}$;

.2 $\mathrm{responsive}\, \alpha' = \mathrm{measuretic} = \mathrm{productive}\, \alpha'$;

.3 $\alpha' = \mathrm{mcharmony}\, \alpha'$.

Proof. Let $\alpha^* = \mathrm{fun}\, m \in \mathrm{measuretic\, knsr}\ .\alpha'' m \ \mathrm{nilset}\, m$.

PART 1. If $m \in \mathrm{measuretic}$, $\psi = .\alpha'' m$, and $A \in \mathrm{bscbox}\, m$ then $\psi = \mathrm{knsr\, cpm}\, m\ \mathrm{zr\, Vlm}\, \alpha\, m$ and $.\psi A = .\mathrm{vlm}\, m\, A$.

Proof. Since $\psi = \mathrm{mss\, Vlm}\, \alpha\, m\, \mathrm{spc}\, m\, \mathrm{dmn}'\, \mathrm{Vlm}\, \alpha\, m$ we may employ 6.21 and 3.2.4 to learn

(1) $$.\psi B = .\mathrm{vlm}\, m\, B \ \text{ whenever } \ B \in \mathrm{bscbox}\, m.$$

Let $\Omega = \mathrm{knsr\, cpm}\, m\, \mathrm{zr\, Vlm}\, \alpha\, m$, suppose $.\Omega T < \infty$ and secure such a countable subfamily $\mathfrak{J}$ of $\mathrm{zr\, Vlm}\, \alpha\, m$ that $.\Omega T = .\mathrm{cpm}\, m(T \sim \sigma \mathfrak{J})$. Now suppose $r > 0$ and secure $\mathfrak{D} \in \mathrm{cblcvr\, bscbox}\, m(T \sim \sigma\mathfrak{J})$ for which

$$.\mathrm{cpm}\, m(T \sim \sigma\mathfrak{J}) \geqq \sum B \in \mathfrak{D}\ .\mathrm{vlm}\, m B - r.$$

Thus,

$$\begin{aligned} .\Omega T + r &\geqq \sum B \in \mathfrak{D}\ .\mathrm{vlm}\, m B \\ &= \sum B \in \mathfrak{D}\ .\mathrm{Vlm}\, \alpha m B + 0 \\ &= \sum B \in \mathfrak{D}\ .\mathrm{Vlm}\, \alpha m B + \sum B \in \mathfrak{J}\ .\mathrm{Vlm}\, \alpha m B \\ &\geqq \sum B \in (\mathfrak{D} \cup \mathfrak{J})\ .\mathrm{Vlm}\, \alpha m B \geqq .\psi T \end{aligned}$$

and, using the fact that r is arbitrary, we conclude

(2) $$.\Omega T \geqq .\psi T \text{ whenever } .\Omega T < \infty.$$

Suppose $.\psi T > \infty$, $0 < r < \infty$ and secure such a member $\mathfrak{G}$ of cblcvr dmn$'$ Vlm $\alpha m T$ that

$$.\psi T + r \geqq \sum B \in \mathfrak{G}\ .\mathrm{Vlm}\, \alpha m B,$$

and secure such a function R on $\mathfrak{G}$ that, $0 < .RB$ whenever $B \in \mathfrak{G}$, and

$$r = \sum B \in \mathfrak{G}\ .RB.$$

Let $\mathfrak{Z} = \mathfrak{G} \cap \mathrm{zr}\,\mathrm{Vlm}\, \alpha m$ and noting mblproductive α contains fnt $\cap$ measuretic, secure such a function F on $\mathfrak{G} \sim \mathfrak{Z}$ that if $B \in \mathfrak{G} \sim \mathfrak{Z}$ then:

$$.FB \in \mathrm{cbl\, sb\, bscbox\, stand}\, m B;$$

$$..\alpha\, \mathrm{stand}\, m B(\mathrm{ft}\, m B \sim \sigma .FB) = 0;$$

$$\sum C \in .FB\ .\mathrm{vlm\, stand}\, m B C \leqq ..\alpha\, \mathrm{stand}\, m B\, \mathrm{ft}\, m B + .RB / .\mathrm{vlm\, tower}\, m B\, \mathrm{tp}\, m B$$

Consequently, if $B \in \mathfrak{G}$ then:

$$((\mathrm{ft}\, m B \sim \sigma .FB) \cup \cup\, \mathrm{tp}\, m B) \in \mathrm{frame}\, m;$$

$$.\mathrm{Vlm}\, \alpha m((\mathrm{ft}\, mB \sim \sigma .FB) \cup \cup\, \mathrm{tp}\, mB) = 0;$$

$$(C \cup \cup\, \mathrm{tp}\, m B) \in \mathrm{bscbox}\, m \text{ whenever } C \in .FB;$$

and $\sum C \in .FB\ .\mathrm{vlm}\, m(C \cup \cup\, \mathrm{tp}\, m B) \leqq .\mathrm{Vlm}\, \alpha m B + .RB$. Letting

$$\mathfrak{J} = \bigcup B \in \mathfrak{G} \bigcup C \in .FB\, \mathrm{sng}(C \cup \cup\, \mathrm{tp}\, m B)$$

and

$$\mathfrak{J}' = \mathfrak{Z} \cup \bigcup B \in \mathfrak{G}\, \mathrm{sng}((\mathrm{ft}\, m B \sim \sigma .FB) \cup \cup\, \mathrm{tp}\, m B)$$

we infer $\mathfrak{J} \in \mathrm{cbl} \cap \mathrm{sb\, bscbox}\, m$, $\mathfrak{J}' \in \mathrm{cbl} \cap \mathrm{sb\, zr\, Vlm}\, \alpha m$, $\sum B \in \mathfrak{J} .\mathrm{vlm}\, m B \leqq .\psi T + r$ and $T \subset \sigma\mathfrak{J} \cup \sigma\mathfrak{J}'$. Clearly,

$$\begin{aligned} .\Omega T &\leqq .\mathrm{cpm}\, m(T \sim \sigma\mathfrak{J}') \\ &\leqq \sum B \in \mathfrak{J}\ .\mathrm{vlm}\, m B \leqq .\psi T + r. \end{aligned}$$

Again r is arbitrary and we conclude

$$.\Omega T \leqq .\psi T \text{ whenever } .\psi T < \infty. \tag{3}$$

Taking (3) and (4) together we conclude

$$\Omega = \psi, \tag{4}$$

and our proof is complete.

PART 2. semiproductive α^* = measuretic.

Proof. Use Part 1, 6.6, 6.7, 6.8, 6.9, 6.10, after checking approximative α^* = measuretic.

PART 3. mblproductive α^* = measuretic.

Proof. Suppose $0 \in p \in m \in$ measuretic, $\mu = .\alpha^* p$, $q = m \sim p$, $v = .\alpha^* q$, $\xi = \text{cpm}\, p$, $\eta = \text{cpm}\, q$, $\phi = .\alpha^* m$, $\psi = \text{cpm}\, m$, $\mathfrak{N}' = \text{nilset}\, p$, $\mathfrak{Z}' = \text{Join}''\, \text{zr}\, \text{Vlm}\, \alpha\, p$, $\mathfrak{N}'' = \text{nilset}\, q$, $\mathfrak{Z}'' = \text{Join}''\, \text{zr}\, \text{Vlm}\, \alpha\, q$, $\mathfrak{N} = \text{nilset}\, m$ and $\mathfrak{Z} = \text{Join}''\, \text{zr}\, \text{Vlm}\, \alpha\, m$.

Let $A' \in \text{mbl}\, \mu$, $A'' \in \text{mbl}\, v$, $A = (A' \cup \cup A'')$, $T' \in \text{bscbox}\, p$, $T'' \in \text{bscbox}\, q$, and secure such sets $B' \in \text{Meet}''\, \text{Join}''\, \text{bscbox}\, p$ and $B'' \in \text{Meet}''\, \text{Join}''\, \text{bscbox}\, q$ that

$$.\mu(B' \sim T'A' \cup T'A' \sim B') = 0$$

and

$$.v(B'' \sim T''A'' \cup T''A'' \sim B'') = 0.$$

Use 3.4.5 in obtaining $Q' \in \text{zr}\, \xi$, $R' \in \mathfrak{N}'$, $S' \in \mathfrak{Z}'$, $Q'' \in \text{zr}\, \eta$, $R'' \in \mathfrak{N}''$, and $S'' \in \mathfrak{Z}''$ for which

$$B' \sim T'A' \cup T'A' \sim B' \subset Q' \cup R' \cup S'$$

and

$$B'' \sim T''A'' \cup T''A'' \sim B'' \subset Q'' \cup R'' \cup S''.$$

Let $T = (T' \cup \cup T'')$, $B = (B' \cup \cup B'')$ and note that

$$B \sim TA \cup TA \sim B \subset [(Q' \cup R' \cup S') \cup \cup (T'' \cup B'')] \cup [(T' \cup B') \cup \cup (Q'' \cup R'' \cup S'')].$$

Now,

$$(Q' \cup \cup (T'' \cup B'')] \in \text{zr}\, \psi, \quad ((T' \cup B') \cup \cup Q'') \in \text{zr}\, \psi, \quad (R' \cup \cup \text{spc}\, q) \in \mathfrak{N},$$

$(\text{spc}\, p \cup \cup R'') \in \mathfrak{N}$, $(S' \cup \cup \text{spc}\, q) \in \mathfrak{Z}$, and $(\text{spc}\, p \cup \cup S'') \in \mathfrak{Z}$, and we conclude

$$B \sim TA \cup TA \sim B \in \text{zr}\, \phi$$

and infer the ϕ measurability of TA from that of B. Consequently,

$$.\phi T = .\phi(TA) + .\phi(T \sim A)$$

and referring to 3.3.1 we learn $A \in \text{mbl}\, \phi$ to complete the proof.

PART 4. productive α' = measuretic.

Proof. STEP 1. If $m \in$ measuretic then $.\alpha' m = \text{knsr}\ .\alpha^* m\ \text{nilcylinder}\ m$.

Proof.

$$\begin{aligned} .\alpha' m &= \text{knsr } .\alpha'' m \,(\text{nilset } m \cup \text{nilcylinder } m) \\ &= \text{knsr } (\text{knsr } .\alpha'' m \text{ nilset } m) \text{ nilcylinder } m \\ &= \text{knsr } .\alpha^* m \text{ nilcylinder } m. \end{aligned}$$

STEP 2. If $m \in$ measuretic, $\psi = .\alpha^* m$, $\phi = .\alpha' m$ and $.\psi T < \infty$ then $.\phi T = .\psi T$.

Proof. Let $\mathfrak{K}$ be such a countable subfamily of nilcylinder m that $.\phi T = .\psi(T \sim \sigma\mathfrak{K})$. Suppose $B \in \mathfrak{K}$, $B = (B' \cup \cup B'')$, $p \subset m$, $B' \in \text{zr cpm } p$, $q = m \sim p$, $B'' = \text{spc}\, q$, $\mu = .\alpha^* p$, $v = .\alpha^* q$ and secure such a member T' of $\text{mbl}\,\psi \cap \text{sp}\, T$ that $.\psi T' = .\psi T$. Thus $T'B \in \text{mbl}'\psi$. Noting that for each $x \in \text{spc}\, p$ and $y \in \text{spc}\, q$, $0 \leqq \text{Cr}(x \cup y)(T'B) \leqq \text{Cr}\, x B'$, employ Part 2 in obtaining

$$\begin{aligned} 0 \leqq .\psi(T'B) &= \int\int \text{Cr}(x \cup y)(T'B)\mu dx v dy \\ &\leqq \int\int \text{Cr}\, x B' \mu dx v dy \\ &= \int 0 v dy = 0. \end{aligned}$$

Thus $.\psi(T'B) = 0$ whenever $B \in \mathfrak{K}$. Hence,

$$0 \leqq .\psi(T\sigma\mathfrak{K}) \leqq .\psi(T'\sigma\mathfrak{K}) \leqq \sum B \in \mathfrak{K} .\psi(T'B) = 0$$

and $.\psi T \leqq .\psi(T\sigma\mathfrak{K}) + .\psi(T \sim \sigma\mathfrak{K}) = 0 + .\phi T \leqq .\psi T$.

STEP 3. If $m \in$ measuretic, $\psi = .\alpha^* m$, $\phi = .\alpha' m$, and $-\infty \leqq \int .fz\psi dz \leqq \infty$ then $\int .fz\phi dz = \int .fz\psi dz$.

Proof. Use Step 2 and the 3.4.4 fact that $\text{mbl}\,\psi \subset \text{mbl}\,\phi$. (Note. Actually, in view of Step 2, it is clear that $\text{mbl}\,\phi = \text{mbl}\,\psi$.)

STEP 4. If $0 \Subset p \Subset m \in$ measuretic, and $A \in$ nilcylinder m then

$$0 = \int\int \text{Cr}(x \cup y) A .\alpha' p dx .\alpha'(m \sim p) dy.$$

Proof. Suppose $A = \text{cyl}\, A' \text{spc}\, m$, $p' \subset m$, $A' \in \text{zr cpm } p'$, $\mu = .\alpha' p$, $q = m \sim p$, $v = .\alpha' q$ and $q' = m \sim p'$.

Let

$$Z = \text{E}y' \in \text{spc}(qp')[..\alpha^*(pp')\text{sctn} A'y' > 0].$$

Since $..\alpha^* p' A' = 0$ we are assured that $..\alpha^*(qp')Z = 0$. Thus, for some $Z_1 \in \text{zr cpm }(qp')$, $Z_2 \in \text{nilset }(qp')$ and $Z_3 \in \text{Join}''\text{ zr Vlm }\alpha(qp')$ we have $Z \subset Z_1 \cup Z_2 \cup Z_3$. Since $\text{cyl}\, Z_1 \text{spc}\, q \in \text{nilcylinder}\, q$, $\text{cyl}\, Z_2 \text{spc}\, q \in \text{nilset}\, q$, and $\text{cyl}\, Z_3 \text{spc}\, q \in \text{Join}''\text{ zr Vlm}\,\alpha\, q$ we are assured that

$$.v(\text{cyl}\, Z \,\text{spc}\, q) = 0. \tag{5}$$

Now, if $y \in \text{spc}\, q \sim \text{cyl} Z \text{spc}\, q$, $y = y' \cup y''$, $y' \in \text{spc}(qp')$, $y'' \in \text{spc}(qq')$ then

$$..\alpha^*(pp')\text{sctn}\, A'y' = 0$$

and for some $S_1 \in \text{zr cpm}(pp')$, $S_2 \in \text{nilset}(pp')$ and $S_3 \in \text{Join}''\text{ zr Vlm}\,\alpha(pp')$ we have

$$\operatorname{sctn} A'y' \subset S_1 \cup S_2 \cup S_3.$$

Since cyl S_1 spc $p \in$ nilcylinder p, cyl S_2 spc $p \in$ nilset p, and cyl S_3 spc $p \in$ Join″ zr Vlm αp we infer $0 = .\mu(\text{cyl}\operatorname{sctn} A'y' \operatorname{spc} p)$ and noting

$$\text{cyl}\operatorname{sctn} A'y' \operatorname{spc} p = \operatorname{sctn} A\, y$$

we conclude

(6) $\qquad .\mu(\operatorname{sctn} A\, y) = 0$ whenever $y \in \operatorname{spc} q \sim \text{cyl}\, Z \operatorname{spc} q$.

From (5) and (6) we infer

$$0 = \int\int \operatorname{Cr}(x \cup y) A\, \mu dx v dy.$$

STEP 5. If $p \subset m \in$ measuretic, $..\alpha' pA' = 0$, $A = \text{cyl}\, A' \operatorname{spc} m$ then $..\alpha' mA = 0$.

Proof. For some $S_1 \in$ zr cpm p, $S_2 \in$ nilset p, $S_3 \in$ zr Vlm $\alpha\, p$ and $S_4 \in$ Join″ nilcylinder p,

$$A' \subset S_1 \cup S_2 \cup S_3 \cup S_4.$$

Since cyl S_1 spc $m \in$ nilcylinder m, cyl S_2 spc $m \in$ nilset m, cyl S_3 spc $m \in$ Join″ zr Vlm$\alpha\, m$ cyl S_4 spc $m \in$ Join″ nilcylinder spc m we infer $..\alpha' m A = 0$.

STEP 6. If $0 \Subset m \Subset m \in$ measuretic, $\phi = .\alpha' m$, $\psi = .\alpha^* m$, $\mu = .\alpha' p, v = .\alpha'(m \sim p)$ then mbl′ $\psi \cup$ nilcylinder $m \in$ bsc ϕ and $.\phi A = \int\int \operatorname{Cr}(x \cup y) A\, \mu dx v dy$ whenever $A \in$ mbl′ $\psi \cup$ nilcylinder m.

Proof. Use Step 1, 3.4.2 and 3.4.4 to learn

$$\text{mbl}'\, \psi \cup \text{nilcylinder}\, m \in \operatorname{bsc} \phi$$

and then use Step 3 to learn

$$\begin{aligned} .\phi A &= .\psi A = \int\int \operatorname{Cr}(x \cup y) A\ .\alpha^* p dx\ .\alpha^*(m \sim p) dy \\ &= \int\int \operatorname{Cr}(x \cup y) A\, \mu dx v dy \end{aligned}$$

whenever $A \in$ mbl′ ψ.

Using Step 4 completes the proof.

From Step 6 we infer semiproductive α' = measuretic with the aid of 6.10. From Step 5 and 6.13 we infer productive α' = measuretic.

PART 5. responsive α' = measuretic.

Proof. We see this to be a consequence of 6.20 and the statement: If $m \in$ measuretic, $K \in$ factorfor m, m' = factormeasuretic then:

.1 zr cpm m' = zr cpm m,

.2 nilset m' = nilset m,

.3 zr Vlm $\alpha\, m'$ = zr Vlm $\alpha\, m$,

.4 nilcylinder m' = nilcylinder m.

6.25. THEOREM. responsive cm = measuretic = productive cm *and* cm = mc harmony cm.

Proof. Suppose $m \in$ measuretic, $\psi =$ .startproduction cp m, $\theta =$ cpm m, $\phi =$ .production cp m, $\phi' =$ .cm m, $\mathfrak{Z} =$ zr Vlm cp m, $\mathfrak{N} =$ nilset m and $\mathfrak{I} =$ nil-cylinder m. Then, by 6.24.1 we have

$$\psi = \text{knsr}\ \theta\, \mathfrak{Z}.$$

Also,

$$\begin{aligned}\phi &= \text{knsr}\, \psi\ (\mathfrak{N} \cup \mathfrak{I}) \\ &= \text{knsr}\, \text{knsr}\, \theta \mathfrak{Z} (\mathfrak{N} \cup \mathfrak{I}) \\ &= \text{knsr}\, \theta (\mathfrak{N} \cup \mathfrak{I} \cup \mathfrak{Z}).\end{aligned}$$

However, since each $A \subset \mathfrak{Z}$ is contained in some $B \in \mathfrak{I}$,

$$\text{knsr}\, \theta (\mathfrak{N} \cup \mathfrak{I} \cup \mathfrak{Z}) = \text{knsr}\, \theta (\mathfrak{N} \cup \mathfrak{I}),$$

and since $\phi' = \text{knsr}\, \theta (\mathfrak{N} \cap \mathfrak{I})$ we conclude

$$\phi' = \phi.$$

The desired conclusion now follows immediately from 6.22, 6.24.2 and 6.24.3.

6.26. Definition. harmon α = fun $m \in$ measuretic E$B \subset$ spc m $[0 = \int \int \text{Cr}$ $(x \cup y) B\ .\alpha p dx\ .\alpha(m \sim p) dy$ whenever $0 \Subset p \Subset m]$.

6.27. Theorem. *If* $\alpha' =$ mc harmon α *and both* responsive α *and* productive α *are equal to* measuretic *then* responsive α' = measuretic = productive α' *and* $\alpha' =$ mc harmony α'.

Proof.

Part 1. harmon α is α Harmonious.

Proof. It suffices to show that if $0 \Subset p \Subset m \in$ measuretic, $A' \in$ harmon $\alpha p'$, $A =$ cyl A' spc m then $A \in$ .harmon αm. Suppose $0 \Subset p \Subset m$, $\mu = .\alpha p$, $q = m \sim p$, $v = .\alpha q$, and $q' = m \sim p'$. Let $Z = \text{E}y' \in \text{spc}(qp')[..\alpha(pp')\text{sctn}\, A'y' > 0]$. Since $A' \in$ .harmon $\alpha p'$ we are assured that $..\alpha(qp')Z = 0$ and consequently that $.v(\text{cyl}\, Z\, \text{spc}\, q) = 0$. If $y \in \text{spc}\, q \sim \text{cyl}\, Z\, \text{spc}\, q$, $y = y' \cup y''$, $y' \in \text{spc}(qp')$, and $y'' \in \text{spc}(qq')$ then $.\ .\alpha(pp')$ sctn $A'y' = 0$. Consequently $.\mu(\text{cyl sctn}\, A'y'\ \text{spc}\, p) = 0$. Since cyl sctn $A'y'$ spc p = sctn $A\,y$ we infer, $.\mu(\text{sctn}\, A\, y) = 0$ whenever $y \in \text{spc}\, q \sim \text{cyl}\, Z$ spc q to conclude

$$0 = \int \int \text{Cr}(x \cup y) Z\, \mu dx v dy.$$

Since p was arbitrary, we infer $A \in$ .harmon αm.

Part 2. productive α' = measuretic.

Proof. Use Part 1 and 6.14.

Part 3. responsive α' = measuretic.

Proof. Suppose $m \in$ measuretic, $K \in$ factorfor m, and $m' =$ factormeasuretic Km

Clearly, .harmon $\alpha m'$ = .harmon αm and zr $.\alpha m'$ = zr $.\alpha m$. We are assured by 3.4.5 that

$$\text{zr } .\alpha' m = \bigcup A \in \text{zr } .\alpha m \bigcup B \in \text{harmon}\, \alpha m \,\text{sb}(A \cup B)$$

and

$$\text{zr } .\alpha' m' = \bigcup A \in \text{zr } .\alpha m' \bigcup B \in \text{harmon}\, \alpha m' \,\text{sb}(A \cup B),$$

and thus conclude that zr $.\alpha' m'$ = zr$.\alpha' m$. We use 6.20 to complete the proof.

Starting with $\alpha_0 = \text{cm}$ we should like to employ 6.27 to secure by induction a sequence α_i of functions for which productive α_i = measuretic and α_{i+1} = mc harmon α_i. However, since the very first, cm, is a class which is not a set, we encounter here a bit of a snag. It is indeed possible, by what strikes us as needlessly circuitous reasoning, to arrive at a definition of hms n below which does not employ our next theorem. This theorem is an instance of a modification, suitable to our needs, of the classical theorem on definition by induction.

6.28. THEOREM. *There is one and only one relation* R *such that* dmn $R = \omega$, vs $R\,0$ = cm, *and* vs $R(n+1)$ = mc harmon vs $R\,n$ whenever $n \in \omega$.

6.29. DEFINITIONS.

.1 Rhm = the relation R such that dmn $R = \omega$, vs $R\,0$ = cm, and vs $R(n+1)$ = mc harmon vs $R\,n$ whenever $n \in \omega$.

.2 hms n = vs Rhm n.

.3 hmg m = fun $A \subset$ spc m inf $n \in \omega$ ..hms $n\, m\, A$.

.4 hm = fun $m \in$ measuretic hmg m.

.5 Hrm = fun $m \in$ measuretic E$A \subset$ spc m [for each p, if $0 \Subset p \Subset m$ then $0 = \int \int \text{Cr}(x \cup y) A\, \text{hmg}\, p\, dx\, \text{hmg}(m \sim p) dy$].

There seems to be little reason to hope, in general, that hmg $m \in$ Msr spc m whenever $m \in$ measuretic. However, for $m \in$ fnt it turns out that hmg m behaves very well indeed.

6.30. THEOREMS.

.1 *If* $m \in$ fnt $\cap$ measuretic *then* hmg m = .mc Hrm m.

.2 fnt $\cap$ measuretic $\subset$ responsive *hm*.

Proof. We infer .1 and .2 from Parts 1 and 2 below.

PART 1. If $n \in \omega$ then:

.3 hms$(n+1)$ = mc harmon hms n;

.4 responsive hms n = measuretic.

Proof. Use 6.25, 6.27 and mathematical induction.

PART 2. If $0 \neq m \in$ measuretic, m has exactly n members, $k \in \omega$, and $k \geqq n-1$ then

5. hmg m = .hms $k\, m$ = .mc Hrm m.

Checking first that .5 holds when $n = 1$, we turn to mathematical induction and suppose $N \in \omega$ and that .5 holds whenever $n < N$. Suppose now that $m' \in$ measuretic and m' contains exactly N elements. Using 6.26 and the inductive hypothesis, namely that if $0 \Subset p \Subset m'$, $k \in \omega$ and $k \geqq N-1$ then $\text{hmg}\, p = .\text{hms}\,(k-1)\,p$ and $\text{hmg}(m' \sim p) = .\text{hms}(k-1)(m' \sim p)$, we infer that

$$.\text{harmon hms}\,(k-1)\,m' = .\text{Hrm}\, m',$$

and using .3, infer that

$$.\text{hms}\, k\, m' = .\text{mc}\,\text{Hrm}\, m'. \tag{1}$$

Noting that $.\text{harmon}\,\text{hms}\, j\, m' \subset .\text{harmon}\,\text{hms}(j+1)m'$ whenever $j \in \omega$ we infer from .3 that if $A \subset \text{spc}\, m'$ then $..\text{hms}(j+1)m'A \leqq ..\text{hms}\, j\, m' A$ and consequently observe that

$$..\text{hmg}\, m' A = ..\text{hms}(N-1)m'A. \tag{2}$$

With (1) and (2) we complete the inductive step and hence the proof.

6.31. Definitions.

.1 $\text{pm} = \text{production}\,\text{hm}$.

.2 $\text{prm}\, m = .\text{pm}\, m$.

6.32. Theorem. responsive pm = measuretic = productive pm *and* pm = mc harmony pm.

Proof. Use 6.30 and 6.24.

The remainder of this section is devoted to finite products and the relationship between our final product measure prm m and the fundamental product measure considered in PM. In so doing, we recast several definitions and theorems from PM in the setting of a $(\text{rlm}\,\mu \cup\cup \text{rlm}\,v)$ product space in place of the $\text{rct}\,\text{rlm}\mu\,\text{rlm}v$ space used in PM.

6.33. Definitions.

.1 $\text{Bscrct}\,\mu v = \text{Product}\,\text{mbl}'\mu\,\text{mbl}'v$.

.2 $\text{Nil}\,\mu v = \text{E}A \subset (\text{rlm}\,\mu \cup\cup \text{rlm}\,v)\ [\mu \in \text{Measure},\ v \in \text{Measure},$

$$\textstyle\int\int \text{Cr}(x \cup y) A\, \mu dx v dy = 0 = \int\int \text{Cr}(x \cup y) A\, v dy \mu dx].$$

.3 $\text{Bace}\,\mu v = \text{Bscrct}\,\mu v \cup \text{Nil}\,\mu v$.

.4 $\text{Bs}\,\mu v = \text{fun}\, A \in \text{Bace}\,\mu v \int\int \text{Cr}(x \cup y) A\, \mu dx v dy$.

.5 $\text{Mpr}\,\mu v = \text{mss}\ (\text{Bs}\,\mu v)(\text{rlm}\, u \cup\cup \text{rlm}\, v)(\text{Bace}\,\mu v)$.

Our next theorem may either be viewed as a translation of PM 5.14, p. 194, or as a consequence of 6.11, 6.12 and 6.10 with the intermediate consideration of

$$\theta = \text{mss}(\text{Bs}\,\mu v)(\text{rlm}\,\mu \cup\cup \text{rlm}\, v)(\text{Bscrct}\,\mu v).$$

6.34. Theorem. *If* $0 \Subset p \Subset m \in \text{measuretic}$, $\mu \in \text{Msr}\,\text{spc}\, p$, $v \in \text{Msr}\,\text{spc}(m \sim p)$, *and* $\psi = \text{Mpr}\,\mu v$ *then*:

.1 Product $\text{mbl}\,\mu\,\text{mbl}\,v \subset \text{mbl}\,\psi$;
.2 $\text{Nil}\,\mu v \subset \text{zr}\,\psi$;
.3 $\int .\text{fz}\psi dz = \int\int .f(x \cup y)\mu dx v dy$ *whenever* $-\infty \leqq \int .fz\psi dz \leqq \infty$;
.4 Product $\text{mbl}'\mu\,\text{mbl}'v \in \text{approx}\,\psi \cap \text{mbl}'\psi$.

We conclude this section with the

6.35. THEOREM. *If* $m \in \text{fnt} \cap \text{measuretic}$, $\phi = \text{prm}\,m$, $M = \text{E}\psi[\psi = \text{Mpr}\,\text{prm}\,p\,\text{prm}\,(m \sim p)$ *for some nonempty* $p \Subset m]$ *and* $\mathfrak{D} = \bigcap \psi \in M\,\text{dmn}'\psi$ *then*:
.1 $\phi = \text{hmg}\,m = .\text{mc}\,\text{harmon}\,\text{pm}\,m$;
.2 $\text{mbl}\,\phi = \bigcap \psi \in M\,\text{mbl}\,\psi$;
.3 $\text{dmn}'\phi = \mathfrak{D}$ *and* $.\phi A = .\psi A$ *whenever* $\psi \in M$ *and* $A \in \mathfrak{D}$.

Proof. For .1, let $\theta = \text{hmg}\,m$. In view of 6.30.2 we infer $0 \neq \text{bsc}\,\theta$ and then assert $\theta = \text{mss}\,\theta\,\text{rlm}\,\theta\,\text{dmn}'\theta$. But, $\phi = \text{mss}\,\theta\,\text{rlm}\,\theta\,\text{dmn}'\theta$ and consequently $\theta = \phi$. Employing this result in generality we learn from 6.30.1 that $\theta = \text{mc}\,\text{harmon}\,\text{pm}\,m$. We infer .3 from Parts 5 and 6 below.

PART 1. If $\psi \in M$ then $\text{zr}\,\phi \subset \text{zr}\,\psi$.

Proof. If $.\phi C = 0$ then 6.32 tells us

$$0 = \int \int \text{Cr}(x \cup y) C\,\text{prm}\,p\,dx\,\text{prm}\,(m \sim p) dy$$

whenever $0 \Subset p \Subset m$. We infer then from 6.34.2 that $.\psi C = 0$.

PART 2. If $\psi \in M$ then $\text{bscbox}\,m \in \text{approx}\,\psi \cap \text{mbl}'\psi$.

Proof. Use 6.34.4.

PART 3. If $\psi \in M$ then $\text{mbl}\,\phi \subset \text{mbl}\,\psi$.

Proof. Suppose $A \in \text{mbl}\,\phi$. In view of Part 2, it will suffice to show $TA \in \text{mbl}\,\psi$ whenever $T \in \text{bscbox}\,m$. Suppose $T \in \text{bscbox}\,m$ and secure such a member B of $\text{Meet}''\text{Join}''\text{bscbox}\,m$ that

$$.\phi(TA \sim B \cup B \sim TA) = 0.$$

We infer from Part 1 that

$$.\psi(TA \sim B \cup B \sim TA \;= 0$$

and from Part 2 that $B \in \text{mbl}\,\psi$ and conclude $TA \in \text{mbl}\,\psi$.

PART 4. If $\psi \in M$ and $A \in \text{mbl}'\phi$ then $.\phi A = .\psi A$.

Proof. Secure such a countable subfamily $\mathfrak{F}$ of $\text{bscbox}\,m$ that $.\phi(A \sim \sigma\mathfrak{F}) = 0$ and $\sum B \in \mathfrak{F}\; .\phi B < \infty$. Use Part 2 and 6.34.3 to check that $.\psi B = .\phi B$ whenever $B \in \mathfrak{F}$. Thus, by Part 1, $.\psi(A \sim \sigma\mathfrak{F}) = 0$ and

$$.\psi A \leqq .\psi(A \sim \sigma\mathfrak{F}) + \sum B \in \mathfrak{F}\; .\psi B < \infty.$$

Hence (6.34.3 and 6.30.2) both $.\psi A$ and $.\phi A$ are equal to

$$\int \int \text{Cr}(x \cup y) A\,\text{prm}\,p\,dx\,\text{prm}\,(m \sim p) dy$$

whenever p is such that $\psi = \text{Mpr}\,\text{prm}\,p\,\text{prm}\,(m \sim p)$ and $0 \Subset p \Subset m$.

PART 5. If $A \in \mathfrak{D}$ and $\psi \in M$ then $.\phi A = .\psi A$.

Proof. Let a and b be such functions on M that if $\theta \in M$ then:

$$.a\theta \in \text{mbl}\, \theta \cap \text{sp}\, A \text{ and } .\theta.a\theta = .\theta A; \tag{1}$$

$$.b\theta \in \text{Meet}'' \text{ Join}'' \text{ bscbox}\, m; \tag{2}$$

$$.\theta(.a\theta \sim .b\theta \cup .b\theta \sim .a\theta) = 0. \tag{3}$$

Take $A' = \bigcap \theta \in M \ .a\theta$ and $B = \bigcup \theta \in M \ .b\theta$. Clearly $A \subset A'$, $.\phi B = .\theta B < \infty$ and,

$$.\theta(A' \sim B) = 0 \text{ whenever } \theta \in M.$$

Thus, $A' \sim B \in .\text{harmon}\, \text{pm}\, m$ and in view of .1, $.\phi(A' \sim B) = 0$ and we infer

$$.\phi A \leqq .\phi(A' \sim B) + .\phi B < \infty.$$

Now secure $C \in \text{mbl}\, \phi \cap \text{sp}\, A$ and $B' \in \text{Meet}'' \text{Join}''$ bscbox m for which $.\phi A = .\phi C$ and $.\phi(C \sim B' \cup B' \sim C) = 0$, and note, in conclusion, that

$$\begin{aligned} .\phi A &= .\phi B' = .\psi B' \leqq .\psi(AB') + .\psi(A \sim B') = .\psi(AB') + 0 \\ &\leqq .\psi A \leqq .\psi C = .\phi C = .\phi A. \end{aligned}$$

PART 6. $\text{dmn}'\, \phi = \mathfrak{D}$.

Proof. Observe that if $A \in \text{dmn}'\, \phi$ then $.\phi A = .\phi A'$ for some $A' \in \text{mbl}\, \phi \cap \text{sp} A$. Now use Part 4 to infer $\text{dmn}'\, \phi \subset \mathfrak{D}$ and Part 5 to complete the proof.

Conclude our proof by inferring .2 from Part 3 and

$$\text{If } A \in \bigcap \psi \in M\, \text{mbl}\, \psi \text{ then } A \in \text{mbl}\, \phi. \tag{4}$$

Proof. If $T \in \text{bscbox}\, m$ then $.\psi T = .\psi(TA) + \psi(T \sim A) < \infty$ whenever $\psi \in M$. Thus, using Part 5 we infer $.\phi T = .\phi(TA) + .\phi(T \sim A)$ and employ 3.3.1 to conclude, since T was arbitrary, that $A \in \text{mbl}\, \phi$.

7. Topological measures.

7.1. DEFINITIONS.

.1 Topology = $\text{E}\mathfrak{B}\,[\text{Join}\, \mathfrak{B} \cup \text{Meet}'\, \mathfrak{B} \subset \mathfrak{B}]$.

.2 Closed $\mathfrak{B}$ = $\text{cmpl}\, \mathfrak{B}$.

.3 topologetic = $\text{E}t$ [t is a function and $\text{rng}\, t \subset$ Topology].

.4 $\text{Spc}\, t = \text{Pr}\, \text{fun}\, i \in \text{dmn}\, t\, \sigma.ti$.

.5 opnbox t = $\text{Pr}\, t$.

.6 topologicalbase t = $\text{E}A$ [$A = \text{cylPr}\, X\, \text{Spc}\, t$ for some $t' \in \text{fnt} \cap \text{sb}\, t$ and $X \in \text{opnbox}\, t'$].

.7 $\text{tpr}\, t$ = Join topologicalbase t.

.8 opencylinder t = $\text{E}A$ [$A = \text{cyl}\, B\, \text{Spc}\, t$ for some $t' \in \text{fnt} \cap \text{sb}\, t$ and $B \in \text{tpr}\, t'$].

.9 $(\mathfrak{U} \boxtimes \mathfrak{B})$ = Join Product $\mathfrak{U}\mathfrak{B}$.

7.2. THEOREMS. *If* $t \in$ topologetic, $\mathfrak{V} = \text{tpr}\, t$, $t' \subset t$, $\mathfrak{V}' = \text{tpr}\, t'$, *and* $\mathfrak{V}'' = \text{tpr}(t \sim t')$ *then*:

.1 $\mathfrak{V} \in$ Topology *and* $\sigma\mathfrak{V} = \text{Spc}\, t$;

.2 Product $\mathfrak{V}'\mathfrak{V}'' \subset \mathfrak{V}$;

.3 Product Closed $\mathfrak{V}'$ Closed $\mathfrak{V}'' \subset$ Closed $\mathfrak{V}$;

.4 *If* $A \in \mathfrak{V}$ *and* $B \in$ Closed $\mathfrak{V}$ *then* $\text{prj}\, A\sigma\mathfrak{V}' \in \mathfrak{V}'$, *and* $\text{prj}\, B\sigma\mathfrak{V}' \in$ Closed $\mathfrak{V}'$;

.5 $\mathfrak{V} = (\mathfrak{V}' \boxtimes \mathfrak{V}'')$.

7.3. THEOREM. *If* $t \in \text{cbl} \cap$ topologetic *then* $\text{tpr}\, t = \text{Join}''$ opencylinder t.

In 7.4 below, .1 is equivalent to PM 6.2 p. 195, .2 and .3 are, respectively, reproductions of PM 6.4.1 and 6.4.2 p. 196, .4, .5, and .6 are the coordinatewise extensions for measuretic functions of .1, .2, and .3.

7.4. DEFINITIONS.

.1 Core $\mathfrak{V} = \text{E}\phi \in \text{Msr}\sigma\mathfrak{V}$ [$\mathfrak{V} \in$ Topology, $\mathfrak{V} \subset \text{mbl}\,\phi$, and

$$\inf B \in (\text{Closed}\mathfrak{V} \cap \text{sb}\, A)\ .\psi(A \sim B) = 0$$

whenever $\psi \in \text{sms}\,\phi$ and $A \in \mathfrak{V}$].

.2 Lind $\mathfrak{V} = \text{E}\phi \in \text{Msr}\,\sigma\mathfrak{V}$ [$\mathfrak{V} \in$ Topology, corresponding to each $\psi \in \text{sms}\,\phi$ and each $\mathfrak{F} \subset \mathfrak{V}$ for which $\sigma\mathfrak{F} = \sigma\mathfrak{V}$ there is a countable subfamily $\mathfrak{G}$ of $\mathfrak{F}$ for which

$$.\psi(\sigma\mathfrak{V} \sim \sigma\mathfrak{G}) = 0].$$

.3 Clin $\mathfrak{V}$ = Core $\mathfrak{V} \cap$ Lind $\mathfrak{V}$.

.4 core $t = \text{E}m \in$ measuretic [$t \in$ topologetic, dmn m = dmn t, and $.mi \in$ Core $.ti$ whenever $i \in$ dmn t].

.5 lind $t = \text{E}m \in$ measuretic [$t \in$ topologetic, dmn m = dmn t, and $.mi \in$ Lind $.ti$ whenever $i \in$ dmn t].

.6 clin t = core $t \cap$ lind t.

As an immediate consequence of PM 7.7, p. 209, is the

7.5. THEOREM. *If* X *is a function* $0 \Subset Y \Subset X$, $S = \text{Pr}\, X$, $S' = \text{Pr}\, Y$, $S'' = \text{Pr}(X \sim Y)$, $\mathfrak{V}' \in$ Topology, $\sigma\mathfrak{V}' = S'$, $\mathfrak{V}'' \in$ Topology, $\sigma\mathfrak{V}'' = S''$, $\mathfrak{V} = (\mathfrak{V}' \boxtimes \mathfrak{V}'')$, $\mu \in$ Clin $\mathfrak{V}'$, $v \in$ Clin $\mathfrak{V}''$, *and* $\phi = \text{Mpr}\,\mu v$ *then* $\phi \in$ Clin $\mathfrak{V}$.

7.6. THEOREM. *If* $m \in \text{fnt} \cap \text{clin}\, t$ *then* prm $m \in$ Clin tpr t.

Proof. We employ mathematical induction on the number of elements in m. Since the result is immediately obtainable when m has exactly one element, we suppose the result known whenever m contains less than n elements and proceed to examine an $m \in \text{clin}\, t$ which contains exactly n elements. Let $\phi = \text{prm}\, m$, $\mathfrak{V} = \text{tpr}\, t$, and $M = \text{E}\psi$ [$\psi = \text{Mpr}\,\text{prm}\, p\, \text{prm}(m \sim p)$ for some nonempty $p \Subset m$]. From the inductive hypothesis, 7.5 and 7.2.5 we infer that

.1 $\psi \in$ Clin $\mathfrak{V}$ whenever $\psi \in M$.

Referring to 6.35.2 we deduce immediately from .1 that

.2 $\mathfrak{B} \subset \mathrm{mbl}\,\phi$.

We also learn from 6.35.3 that if $\theta \in \mathrm{sms}\,\phi$ then $\theta \in \mathrm{sms}\,\psi$ for each $\psi \in M$, and this with .1 assures us of the desired conclusion, namely

.3 $\phi \in \mathrm{Core}\,\mathfrak{B} \cap \mathrm{Lind}\,\mathfrak{B}$.

7.7. THEOREM. *If* $m \in \mathrm{cbl} \cap \mathrm{clin}\,t$ *then* $\mathrm{prm}\,m \in \mathrm{Clin}\,\mathrm{tpr}\,t$.

Proof. Suppose $\phi = \mathrm{prm}\,m$, $S = \mathrm{spc}\,m$ and $\mathfrak{B} = \mathrm{tpr}\,t$. We know from 7.3 that

$$\mathfrak{B} = \mathrm{Join}''\,\mathrm{opencylinder}\,t. \tag{1}$$

Employing 7.6 and 6.32 to learn opencylinder $t \subset \mathrm{mbl}\,\phi$, we infer

$$\mathfrak{B} \subset \mathrm{mbl}\,\phi. \tag{2}$$

Suppose $.\phi T < \infty$, $\psi = \mathrm{sct}\,\phi\,T$ and let $\mathfrak{H}$ be such a countable subfamily of bscbox$\,m$ that

$$.\phi(\sigma\mathfrak{H}) < \infty \text{ and } .\phi(T \sim \sigma\mathfrak{H}) = 0. \tag{3}$$

To complete our proof we infer $\phi \in \mathrm{Clin}\mathfrak{B}$ from Parts 1 and 2 below.

PART 1. If $A \in \mathfrak{B}$ then $\inf C \in \mathrm{Closed}\,\mathfrak{B} \cap \mathrm{sb}\,A\ .\psi(A \sim C) = 0$.

Proof. Let $\mathfrak{K}$ be such a countable subfamily of opencylinder t that $A = \sigma\mathfrak{K}$. Suppose $r > 0$ and noting (3) let $\mathfrak{K}'$ be such a finite subfamily of $\mathfrak{K}$ that

$$.\phi(\sigma\mathfrak{H} \cap (A \sim \sigma\mathfrak{K}')) \leqq r/3. \tag{4}$$

We secure next a finite subfamily $\mathfrak{H}'$ of $\mathfrak{H}$ for which

$$.\phi(\sigma\mathfrak{H} \sim \sigma\mathfrak{H}') \leqq r/3. \tag{5}$$

Let $B = \sigma\mathfrak{K}'$ and note that $B \in \mathrm{opencylinder}\,t$. Thus for some $p \in \mathrm{fnt} \cap \mathrm{sb}\,m$, $u \subset t$, and B' we have $\mathrm{dmn}\,p = \mathrm{dmn}\,u$, $B = \mathrm{cyl}\,B'S$ and $B' \in \mathrm{tpr}\,u$. Let $\mathfrak{B}' = \mathrm{tpr}\,u$, $\mu = \mathrm{prm}\,p$, $v = \mathrm{prm}\,(m \sim p)$, $D' = \mathrm{prj}\,\sigma\mathfrak{H}'\,\mathrm{rlm}\,\mu$, $D'' = \mathrm{prj}\,\sigma\mathfrak{H}'\,\mathrm{rlm}\,v$, $D = (D' \cup\cup D'')$ and check that $D' \in \mathrm{dmn}'\mu$, $D'' \in \mathrm{dmn}'v$, and $\sigma\mathfrak{H}' \subset D$. Suppose for the moment that $.vD'' > 0$ and use the 7.6 fact that $\mu \in \mathrm{Core}\,\mathfrak{B}'$ to obtain $C' \in \mathrm{Closed}\,\mathfrak{B}'\,\mathrm{sb}\,B'$ for which

$$.\mu(D' \cap (B' \sim C')) \leqq (r/3 \cdot .vD''). \tag{6}$$

If $.vD'' = 0$ take $C' = 0$. In either case we infer

$$.\phi((D' \cap (B' \sim C')) \cup\cup D'') \leqq r/3. \tag{7}$$

Let $C = \mathrm{cyl}\,C'S$ and notice

$$.\phi(D \cap (B \sim C)) \leqq r/3. \tag{8}$$

Hence, using (4), (5) and (8) we obtain

$$
\begin{aligned}
.\psi(A \sim C) &= .\phi(T \cap (A \sim C)) \\
&\leqq .\phi(\sigma\mathfrak{H} \cap (A \sim C)) \\
&\leqq .\phi(\sigma\mathfrak{H} \cap ((A \sim B) \cup (B \sim C))) \\
&\leqq .\phi(\sigma\mathfrak{H} \cap (A \sim B)) + .\phi(\sigma\mathfrak{H} \cap (B \sim C)) \\
&\leqq r/3 + .\phi((\sigma\mathfrak{H}' \cup \sigma\mathfrak{H} \sim \sigma\mathfrak{H}') \cap (B \sim C)) \\
&\leqq r/3 + .\phi(\sigma\mathfrak{H}' \cap (B \sim C)) + .\phi(\sigma\mathfrak{H} \sim \sigma\mathfrak{H}') \\
&\leqq r/3 + .\phi(D \cap (B \sim C)) + r/3 \\
&\leqq r/3 + r/3 + r/3 = r
\end{aligned}
$$

and recalling the arbitrary nature of r we infer the desired conclusion.

PART 2. If $\mathfrak{F} \subset \mathfrak{B}$ and $\sigma\mathfrak{F} = S$ then there is such a countable subfamily $\mathfrak{G}$ of $\mathfrak{F}$ that $.\psi(S \sim \sigma\mathfrak{G}) = 0$.

Proof. The conclusion is inferred from Step 2 below.

STEP 1. If $p \in \text{fnt} \cap \text{sb}\, m$ then there is such a countable subfamily $\mathfrak{G}$ of $\mathfrak{F}$ that

$$.\psi(S \sim \bigcup A \in \mathfrak{G}\ \text{cyl}\,(\text{prj}\, A\, \text{spc}\, p)S) = 0.$$

Proof. Suppose $\mu = \text{prm}\, p$, $u \subset t$, $\text{dmn}\, u = \text{dmn}\, p$, $\mathfrak{B}' = \text{tpr}\, u$, $S' = \text{spc}\, p$, $\mathfrak{F}' = \bigcup A \in \mathfrak{F}\, \text{sng}\, \text{prj}\, AS'$ and $\mathfrak{H}' = \bigcup B \in \mathfrak{H}\, \text{sng}\, \text{prj}\, BS'$. Since $\sigma\mathfrak{F} = S$ we are assured that $\sigma\mathfrak{F}' = S'$. Using 7.2.4 we learn $\mathfrak{F}' \subset \mathfrak{B}'$. Noting that $\mu \in \text{Clin}\, \mathfrak{B}'$ and $\mathfrak{H}' \subset \text{dmn}'\mu$ we can and do select such a function w on $\mathfrak{H}'$ that for each $A \in \mathfrak{H}'$, $.wA \in \text{cbl} \cap \text{sb}\, \mathfrak{F}'$ and $.\mu(A \sim \sigma.wA) = 0$. We let $\mathfrak{G}' = \bigcup A \in \mathfrak{H}'.wA$ and check that $\mathfrak{G}' \in \text{cbl} \cap \text{sb}\, \mathfrak{F}'$ and

$$
\begin{aligned}
0 \leqq .\mu(\sigma\mathfrak{H}' \sim \sigma\mathfrak{G}') &= .\mu(\bigcup A \in \mathfrak{H}'(A \sim \sigma\mathfrak{G}')) \\
&\leqq .\mu(\bigcup A \in \mathfrak{H}'(A \sim \sigma.wA)) \\
&\leqq \sum A \in \mathfrak{H}'\ .\mu(A \sim \sigma.wA) = 0.
\end{aligned}
$$

Hence, taking $\mathfrak{G} = EA \in \mathfrak{F}(\text{prj}\, AS' \in \mathfrak{G}')$ and $B = \bigcup A \in \mathfrak{G}\, \text{cyl}\, \text{prj}\, AS'S$ we have

$$
\begin{aligned}
0 \leqq .\psi(S \sim B) &= .\phi(T \sim B) \\
&\leqq .\phi((\sigma\mathfrak{H} \cup T \sim \sigma\mathfrak{H}) \sim B) \\
&\leqq .\phi(\sigma\mathfrak{H} \sim B) + .\phi(T \sim \sigma\mathfrak{H}) \\
&\leqq .\phi(\text{cyl}\, \sigma\mathfrak{H}'S \sim B) + 0 \\
&= .\phi(\text{cyl}\, \sigma\mathfrak{H}'S \sim \text{cyl}\, \sigma\mathfrak{G}'S) \\
&= .\phi(\text{cyl}(\sigma\mathfrak{H}' \sim \sigma\mathfrak{G}')S) = 0
\end{aligned}
$$

and our proof is complete.

STEP 2. There is such a countable subfamily $\mathfrak{G}$ of $\mathfrak{F}$ that $.\psi(S \sim \sigma\mathfrak{F}) = 0$.

Proof. Let $P = \text{fnt} \cap \text{sb}\, m$, note that $P \in \text{cbl}$, and using Step 1 secure a function f on P for which $.fp \in \text{cbl} \cap \text{sb}\,\mathfrak{F}$ and $.\psi(S \sim \bigcup A \in .fp\, \text{cyl}\, \text{prj}\, A\, \text{spc}\, p\, S) = 0$ whenever $p \in P$. Taking $\mathfrak{G} = \bigcup p \in P\ .fp$ we employ 5.9 to learn

$$\sigma\mathfrak{G} = \bigcap p \in P \text{ cyl prj } \sigma\mathfrak{G} \text{ spc } p\,S.$$

Hence,

$$S \sim \sigma\mathfrak{G} = \bigcup p \in P(S \sim \text{cyl}\,\text{prj}\,\sigma\mathfrak{G}\,\text{spc}\,pS)$$

and we infer

$$\begin{aligned} 0 \leqq .\psi(S \sim \sigma\mathfrak{G}) &\leqq \sum p \in P\ .\psi(S \sim \text{cyl}\,\text{prj}\,\sigma\mathfrak{G}\,\text{spc}\,pS) \\ &\leqq \sum p \in P\ .\psi(S \sim \text{cyl}\,\text{prj}\,\sigma.fp\,\text{spc}\,pS) \\ &= \sum p \in P\ .\psi(S \sim \bigcup A \in .fp\,\text{cyl}\,\text{prj}\,A\,\text{spc}\,pS) \\ &= 0. \end{aligned}$$

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