CONTRIBUTIONS TO THE THEORY OF OPTIMAL CONTROL.[1] A GENERAL PROCEDURE FOR THE COMPUTATION OF SWITCHING MANIFOLDS

BY

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1. **Introduction.** We have given a system of differential equations of the form

$$\dot{x} = h(x) + a\varepsilon, \tag{1}$$

where the dot stands for differentiation with respect to the time t, where the unknown x is an n-vector, where h is an n-vector function of x, and where a is a constant nonzero n-vector. $h(x)$ is assumed to be of class C', at least. As for the scalar ε, this is a bounded not necessarily continuous function which is to be chosen in such a way that a solution starting with given initial conditions will be steered as quickly as possible to the origin $x = 0$. Evidently, ε can be regarded as a function of x.

Also, without essential loss of generality, we can take the bound for $|\varepsilon|$ to be 1. Otherwise we would modify the vector a by multiplying all of its components by the bound.

From the "bang-bang" principle it is known that time optimality may be achieved in a wide variety of cases by limiting ε to its extreme values $+1$ and -1. This is the case, for instance, when $h(x)$ is linear and when the system is controllable. It is also true in most cases when $h(x)$ is nonlinear, but the precise conditions are awkward to specify. We therefore merely assert that a generally significant problem is to determine how a solution may be steered to the origin by limiting the values of ε to the two values $+1$ and -1. This is known as "bang-bang control."

Thus, we can regard (1) as representing two systems of continuous differential equations, namely,

$$\dot{x} = F(x), \qquad \text{where } F(x) = h(x) + a, \tag{1a}$$

corresponding to $\varepsilon = +1$, and

$$\dot{x} = G(x), \qquad \text{where } G(x) = h(x) - a, \tag{1b}$$

corresponding to $\varepsilon = -1$. We now formulate the problem by asking how it is

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possible to steer a point x into the origin by making it move first along a solution of the system (1a) (or (1b)) and then along a solution of (1b) (or (1a)), and then, again, along a solution of (1a) (or (1b)), and so forth, until the origin is reached. The problem is to determine at what points, x, we should switch from system (1a) to (1b), or vice versa. These points are known as switching points; and point sets consisting of switching points (corresponding to all bang-bang control paths using a minimum number of switches) are known as switching manifolds, even though these point sets need not be closed manifolds in the strict technical sense, whereby each point of the set has a neighborhood whose intersection with the set is homeomorphic to a simplex of some dimensionality $\geqq 1$ and $< n$. In fact most of the switching manifolds, or at least the parts of them referred to later as "leaves," will turn out to have certain boundary points which will constitute switching manifolds of lower dimensionality. Broadly speaking, our problem is to determine equations for these switching manifolds and to develop certain inequalities which must also be satisfied by points lying on the switching manifolds. These inequalities are necessary because the switching manifolds are not completely determined by the equations. This is connected with the fact just mentioned that the switching manifolds are not closed.

It should be stated that our method of constructing the switching manifolds, as given in §3, is, in general, local. In the event that bang-bang control is time-optimal our method yields the switching manifolds for time optimal control. However, there are even some controllable linear two-dimensional systems, in which the bang-bang control is, in the large, not unique and in which the origin may be reached by a bang-bang control which is not time optimal. But invariably our procedure yields a local bang-bang control that brings points in the controllable region to the origin in a finite time.

The result in §2 is of a subsidiary nature. We understand from the referee's report that it is well known to a number of people but has apparently not been published.

We are greatly indebted to R. W. Bass for having introduced us to the problem of determining the switching manifolds of linear plants as well as for having made suggestions as to how this might be accomplished. In particular his method of working backward in time was actually adopted. He also suggested the use of certain first integrals but in a manner quite different from that which we have adopted in this paper.

2. **Comments on linear plants.** Consider the so-called case of a controllable linear plant, whereby $h(x) = Ax$, A being an $n \times n$ constant matrix, and where the $n \times n$ matrix D, whose columns are the vectors, $a, Aa, A^2a, \cdots, A^{n-1}a$, is nonsingular. This definition of the controllability of a linear plant was introduced by Kalman and is designed to insure that every point in some neighborhood of the origin can be steered into the origin in the indicated manner. From this fact it is obvious

that controllability is invariant under nonsingular linear transformations of the vector x. Indeed it is easy to verify that if x is replaced by Lx, L being an $n \times n$ nonsingular constant matrix, A must be replaced by LAL^{-1}, a by La and D by LD. And, of course, LD is nonsingular, if both L and D are.

These facts make it possible to perform a preliminary normalization, so that the components of a may be assigned any special values not all zero. For instance, there is no loss of generality in assuming that the ith component of a is δ_{i1}.

We next turn to a more far-reaching reduction of the form of a controllable linear plant. We introduce a new unknown vector $y = D^{-1}x$, whose n components it will be convenient to denote by $y_0, y_1, \cdots, y_{n-1}$ (rather than by $y_1, y_2, \cdots, y_n$). Then evidently $x = Dy$ and, from the original equations of the linear plant, which we recall are

$$\dot{x} = Ax + a\varepsilon, \tag{2}$$

we find that

$$\dot{y} = D^{-1}\dot{x} = D^{-1}(Ax + a\varepsilon) = D^{-1}ADy + D^{-1}a\varepsilon.$$

Suppose that the characteristic polynomial of A is $\lambda^n - \sum_{k=0}^{n-1} p_k \lambda^k$. Note also that $x = Dy = \sum_{k=0}^{n-1} A^k a y_k$ by definition of D. Hence

$$ADy = \sum_{k=0}^{n-1} A^{k+1} a y_k = A^n a y_{n-1} + \sum_{k=0}^{n-2} A^{k+1} a y_k.$$

By the Cayley-Hamilton theorem $A^n = \sum_{k=0}^{n-1} p_k A^k$. Hence

$$ADy = \left(\sum_{k=0}^{n-1} p_k A^k\right) a y_{n-1} + \sum_{l=1}^{n-1} A^l a y_{l-1}.$$

Therefore

$$D\dot{y} = ADy + a\varepsilon = D \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_{n-1} \end{bmatrix} y_{n-1} + D \begin{bmatrix} 0 \\ y_0 \\ y_1 \\ \vdots \\ y_{n-2} \end{bmatrix} + a\varepsilon.$$

Multiplying by D^{-1}, we thus get the following equations for the linear plant when expressed in terms of $y_0, \cdots, y_{n-1}$.

$$\begin{aligned} \dot{y}_0 &= p_0 y_{n-1} + 0 + b_1 \varepsilon, \\ \dot{y}_1 &= p_1 y_{n-1} + y_0 + b_2 \varepsilon, \\ &\vdots \\ \dot{y}_k &= p_k y_{n-1} + y_{k-1} + b_{k+1} \varepsilon, \quad k = 1, 2, \cdots, n-1. \end{aligned} \tag{3}$$

Here we use $b_1, b_2, \cdots, b_n$ to represent the components of the n-vector $b = D^{-1}a$. This means that $Db = a$, so that

$$\begin{bmatrix} a_1 & \sum_i A_{1i}a_i & \sum_i A_{1i}^{(2)}a_i & \cdots \\ a_2 & \sum_i A_{2i}a_i & \sum_i A_{2i}^{(2)}a_i & \cdots \\ \vdots & \vdots & \vdots & \\ a_n & \sum_i A_{ni}a_i & \sum_i A_{ni}^{(2)}a_i & \cdots \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \tag{4}$$

where we have used $A_{ij}^{(k)}$ to represent the element in the ith row and jth column of A^k. From Cramer's rule, it is clear that (4) implies that $b_1 = 1$, while $b_2 = b_3 = \cdots = b_n = 0$. Hence, from (3), we see that any controllable linear plant can be written in the prepared form

$$\begin{aligned} \dot{y}_0 &= p_0 y_{n-1} + \varepsilon, \\ \dot{y}_k &= p_k y_{n-1} + y_{k-1}, \qquad k = 1, 2, \cdots, n-1. \end{aligned} \tag{5}$$

Notice that it is easy to eliminate $y_0, y_1, \cdots, y_{n-2}$ from these equations, the result being

$$y_{n-1}^{(n)} - \sum_{k=0}^{n-1} p_k y_{n-1}^{(k)} = \varepsilon. \tag{6}$$

After (6) has once been integrated the functions $y_{n-2}, y_{n-3}, \cdots, y_0$ can be found successively without further integration from the last $n-1$ equations in the system (5).

This is a major conclusion: A *controllable* linear plant consisting of a system of n first order differential equations can *always* be expressed as a single nth order differential equation of the form (6). The converse proposition is also true. For, if (6) is given a priori, we can form the system (5), which is certainly controllable, since the matrix D pertaining to (5) may be seen by a short calculation to be merely the unit $n \times n$ matrix. Of course, this means that not every system (2) is controllable. For an example we need only to choose A so that it has a pair of equal characteristic roots *with simple elementary divisors.*

3. **General method for obtaining switching manifolds.** We now are in a position to return to the problem previously posed with regard to the linear or nonlinear plant represented by (1) or by (1a) and (1b). In considering these systems of differential equations we employ three sets of variables as follows:

The first set of variables are the components of the original n-vector x, in which we have the system (1a) in the form,

$$\dot{x} = F(x) \tag{7}$$

and the system (1b) in the form,

$$\dot{x} = G(x). \tag{8}$$

The second set of variables are the components of an n-vector y, obtained by a one-to-one transformation of class C' from x in such a manner that the system (1a) appears in the simple form

$$\dot{y}_i = \delta_{i1}, \qquad i = 1,2,\cdots,n, \tag{9}$$

while the system (1b) appears in a possibly much more complicated form such as

$$\dot{y} = K(y).$$

The third set of variables, components of an n-vector z, on the other hand, leave the system (1a) in a possibly very complicated form such as

$$\dot{z} = L(z)$$

but have the virtue of reducing the system (1b) to the simple form,

$$\dot{z}_i = \delta_{i1}, \qquad i = 1,2,\cdots,n. \tag{10}$$

It is assumed that we have equations of transformation leaving the origin invariant, and valid in a neighborhood of the origin, which enable us to pass freely from any one of these three systems of variables to either of the other two. The possibility of obtaining such transformations with the desired properties is well known, at least if $F(0) \neq 0$ and $G(0) \neq 0$, as we hereby assume. Suffice it to say that any transformation of the form $y = w(x)$, where $w_1(x) - t$, $w_2(x), \cdots, w_n(x)$ are independent first integrals, will produce the desired result. The developments of the next two sections use a particular case of such a procedure.

As a point is successfully steered into the origin, it must, after its last switching, be on the half-trajectory of (1a), or of (1b), which terminates at the origin as t monotonically increases and approaches a certain terminal value T. Of course, if the point was *originally* on either one of these half-trajectories, it can be trivially steered into the origin with no switches whatsoever. Any other point must first be steered to one or the other of these two half-trajectories before it can reach the origin and must therefore experience a switching at some point of these half-trajectories. Moreover, this switching may occur at any point of the half-trajectories depending upon the initial position. Hence these half-trajectories constitute a one-dimensional switching manifold R_1. It has two "leaves," $R_{1,1}$, the half-trajectory of system (1a), and $R_{1,2}$, the half-trajectory of system (1b). $R_1 = R_{1,1} \cup R_{1,2}$.

For the sake of brevity, we will describe in detail only $R_{1,1}$ and the leaves of switching manifolds of higher dimensionality on whose boundary $R_{1,1}$ lies. Similar considerations may be supplied by the reader for $R_{1,2}$.

From (9) it is obvious that $R_{1,1}$ when expressed in terms of the y's consists of those points for which $y_1 < 0$ and $y_i = 0$, for $i = 2, 3, \cdots, n$. When we make a transformation to the z's, these conditions take some such form as $h_1^*(z) < 0$, $h_i^*(z) = 0$, for $i = 2, 3, \cdots, n$. We next write these conditions in a more suitable form, by eliminating z_1 from all but one of these n conditions; the one remaining condition is the one which expresses z_1 as a function of $z_2, \cdots, z_n$, hereafter briefly denoted by the $(n-1)$-vector $\bar{z}$. Assuming that this elimination can be effected, we obtain (in terms of the z's) conditions of the form,

$$h_1(\bar{z}) < 0, \quad z_1 = h_2(\bar{z}), \quad h_i(\bar{z}) = 0, \qquad i = 3, 4, \cdots, n, \tag{11}$$

as both necessary and sufficient that the point $z \in R_{1,1}$.

Now any point (not initially on $R_{1,1}$) being steered successfully into the origin via $R_{1,1}$ must have been proceeding along a trajectory of (1b) just before its last switching. Hence the locus of all half-trajectories of (1b) which terminate on $R_{1,1}$ must constitute a "leaf" $R_{2,1}$ of a two-dimensional switching manifold. The detailed substantiation of this statement about $R_{2,1}$ is similar to what was stated above in substantiation of the fact that $R_{1,1}$ was part of a one-dimensional switching manifold. From (10) and (11) it is clear that a point on $R_{2,1}$ is characterized by the conditions

$$h_1(\bar{z}) < 0, \quad z_1 < h_2(\bar{z}), \quad h_i(\bar{z}) = 0, \quad i = 3, 4, \cdots, n. \tag{12}$$

When we make a transformation to the y's, these conditions take some such form as $\phi_1^*(y) < 0$, $\phi_2^*(y) < 0$, $\phi_i^*(y) = 0$, $i = 3, 4, \cdots, n$. We next eliminate y_1 from all but one of these n conditions; the one remaining condition is the one which expresses y_1 as a function of $y_2, \cdots, y_n$, hereafter denoted by the $(n-1)$-vector $\bar{y}$. Assuming that this elimination can be effected, we obtain (in terms of the y's) conditions of the form,

$$\phi_1(\bar{y}) < 0, \quad \phi_2(\bar{y}) < 0, \quad y_1 = \phi_3(\bar{y}), \quad \phi_i(\bar{y}) = 0, \qquad i = 4, \cdots, n, \tag{13}$$

as both necessary and sufficient that the point $y \in R_{2,1}$.

Now any point being steered successfully into the origin via $R_{2,1}$ and $R_{1,1}$ (assuming that it did not start on $R_{2,1}$), must have been proceeding along a trajectory of (1a) just before switching onto $R_{2,1}$. Hence the locus of all half-trajectories of (1a) which terminate on $R_{2,1}$ must constitute a "leaf" $R_{3,1}$ of a three-dimensional switching manifold. From (9) and (13), it is clear that a point on $R_{3,1}$ is characterized by the conditions

$$\phi_1(\bar{y}) < 0, \quad \phi_2(\bar{y}) < 0, \quad y_1 < \phi_3(\bar{y}), \quad \phi_i(\bar{y}) = 0, \quad i = 4, \cdots, n.$$

This process may be continued by induction, yielding, for any positive integer $k < n$, a "leaf" $R_{k,1}$ of a k-dimensional switching manifold. This leaf is characterized by n conditions, k of which are inequalities and $(n-k)$ of which are equalities. These latter may be expressed by equating to 0 certain time-independent first integrals of (1a), if k is odd, and of (1b), if k is even.

One of the purposes of this paper is to carry this procedure out in detail for the case of the linear plant of order 4 in the special case in which all eigenvalues of the matrix A vanish. In other words, the system considered can be presented in the form (5) or (6) in the special case $p_0 = p_1 = p_2 = p_3 = 0$ $(n = 4)$. This example should give a good idea of the general behavior of such systems even when the p's are not all zero, and our results obtained from a study of this simple example should approximate the results to be obtained when the p's are small. The reason for this is roughly as follows:

Our methods are based on certain transformations between the x's, y's, and z's. These transformations depend continuously upon certain systems of first integrals of (1a) and (1b), which are written down in terms of the initial value solutions of the differential systems (1a) and (1b). Now, if these systems depend continuously on certain parameters, such as the p's, it is well known that the intial value solutions likewise depend continuously on the same parameters. Hence our results will be but slightly affected by small deviations of the p's from 0.

4. **On the availability of involutory transformations in the computation of the switching manifolds.** The process described in the preceding section for developing the equations and inequalities, which characterize the control manifolds of various dimensionalities, makes repeated use of the transformation from the y's to the z's and back again. It is therefore useful to observe that, in the important linear case (cf. (2)), this transformation may be set up in such a way as to be its own inverse. In fact, the purpose of this section is to prove more generally that, *whenever the system* (1a) *is carried into the system* (1b) *by means of an involutory transformation on the x's, then it is always possible to choose the variables y and z in such a manner that they also are related to each other by means of an involutory transformation.*

We have already assumed in the preceding section that $F(0) \neq 0$. Without loss of generality we may assume further that $F_1(0)$, the first component of $F(0)$, is not zero. As in the rest of this proof we have adopted the following conventions: The first component of any n-vector v will be denoted by v_1 and its last $(n-1)$ components will be thought of as an $(n-1)$-vector, denoted by $\bar{v}$. Moreover, if $\lambda = \lambda(t, v)$, is any scalar or vector function of the scalar t and the n-vector v, we use the notation $\lambda(\alpha, \beta, \gamma)$ to denote the value of λ when t takes on the scalar value α, when v_1 takes on the scalar value β, and when $\bar{v}$ takes on the $(n-1)$-vector value γ. A similar convention is made in instances when the t and α are absent from the above statement.

We suppose that the transformation $x' = \phi(x)$ is involutory, so that $\phi(\phi(x)) = x$, and that it carries the system (1a) into the system (1b) and vice versa. Let $x = f(t, x_0)$ be the solution of (1a) such that $f(0, x_0) = x_0$ and let $x = g(t, x_0)$

be the solution of (1b) such that $g(0,x_0) = x_0$. Then $\phi(f(t,x_0))$ is a solution of (1b) and when $t = 0$ it reduces to $\phi(x_0)$. Hence, we see that

$$\phi(f(t,x_0)) \equiv g(t,\phi(x_0)). \tag{14}$$

It is clear that $f(-t,x)$ is a first integral of (1a) in the sense that when x is replaced by a solution $x(t)$ of (1a), $f(-t,x(t))$ is independent of t. Similarly $g(-t,x)$ and, hence, $\phi(g(-t,x))$, are first integrals of (1b). We next define the scalar functions τ and σ of the n-vector x in such a manner that

$$f_1(-\tau(x),x) = 0, \qquad \tau(0) = 0, \tag{15}$$

and

$$\phi_1[g(-\sigma(x),x)] = 0, \qquad \sigma(0) = 0. \tag{16}$$

Since both $f(-t,\xi)$ and $\phi[g(-t,\xi)]$ are, for any fixed n-vector ξ, solutions of $\dot{x} = -F(x)$, and since $F_1(0) \neq 0$ by hypothesis, it is obvious from the implicit function theorem that the definitions of $\tau(x)$ and $\sigma(x)$ just given are effective, at least in a neighborhood of the origin.

The transformation $x \to y$ is now defined by the equations,

$$y_1 = \tau(x), \tag{17}$$

$$\bar{y} \equiv \bar{f}(-\tau(x),x). \tag{18}$$

The reader may verify that $(\partial\tau/\partial x)F(x) \equiv 1$ and that $\bar{f}(-\tau(x),x)$ is a first integral of (1a). Hence the transformation defined by (17) and (18) does indeed carry (1a) into (9) as required. Moreover we see, from (15), (18), and (17), that

$$x = f(\tau(x),0,\bar{y}) = f(y_1,0,\bar{y}), \tag{19}$$

which gives the transformation $y \to x$.

We now define the transformation $x \to z$ by the equations

$$z_1 = \sigma(x), \tag{20}$$

$$\bar{z} = \bar{\phi}[g(-\sigma(x),x)]. \tag{21}$$

Here again the reader may verify that $(\partial\sigma/\partial x)G(x) \equiv 1$ and that $\bar{\phi}[g(-\sigma(x),x)]$ is a first integral of (1b). Hence the transformation defined by (20) and (21) does indeed carry (1b) into (10) as required. Combining (16) and (21), we find that

$$\phi(0,\bar{z}) = g(-\sigma(x),x)$$

and, hence, from (20) we have $\phi(0,\bar{z}) = g(-z_1,x)$. This means that $x = g(z_1,\phi(0,\bar{z}))$. Thus, from (14), we find that

(22)
$$x=\phi[f(z_1,0,\bar{z})]$$

which gives the transformation $z \to x$.

From (19) and (22) we have

$$f(y_1,0,\bar{y})=\phi[f(z_1,0,\bar{z})],$$

which, because of the involutory character of ϕ, is equivalent to $f(z_1, 0, \bar{z}) = \phi[f(y_1,0,\bar{y})]$ and shows that the transformation $y \to z$ is itself involutory. Actually a somewhat more refined calculation along these lines yields the following explicit formulas for this transformation,

$$\begin{aligned} z_1 &= \sigma[F(y_1,0,\bar{y})], \\ \bar{z} &= \bar{F}\{-\sigma[F(y_1,0,\bar{y})],\ \phi[F(y_1,0,\bar{y})]\}, \end{aligned}$$

as well as the further information that $\tau(x)=\sigma(\phi(x))$ and $\tau(\phi(x))=\sigma(x)$.

The results of this section hold equally well for any two systems $\dot{x}=F(x)$ and $\dot{x}=G(x)$ which are transformed into each other by means of an involutory transformation and not merely when $F-G=2a$ as is the case in (1a) and (1b).

The significance of this section is that, whenever we have an involutory transformation from (1a) to (1b), we may choose variables y and z in such a manner that it is sufficient to compute one leaf only of each switching manifold. The companion leaf would then be obtained by a mere change of y's into z's or vice versa.

5. **The equation.** $x^{(\mathrm{IV})}=\mathrm{Sgn}\,[\sigma]$. The equation

(23)
$$\frac{d^4x}{dt^4} = \varepsilon, \qquad \varepsilon = \pm 1$$

is equivalent to a system of four simultaneous linear equations, namely:

(24)
$$\dot{x}_1 = \varepsilon, \quad \dot{x}_2 = x_1, \quad \dot{x}_3 = x_2, \quad \dot{x}_4 = x_3.$$

We denote the system (24) by S_ε.

It is easily seen that S_ε has the following independent first integrals

$$\begin{aligned} \varepsilon x_1 - t &= \text{const.}, \\ x_2 - \tfrac{1}{2}\varepsilon x_1^2 &= \text{const.}, \\ x_3 - \varepsilon x_2 x_1 + \tfrac{1}{3}x_1^3 &= \text{const.}, \\ x_4 - \varepsilon x_3 x_1 + \tfrac{1}{2}x_2 x_1^2 - \tfrac{1}{8}\varepsilon x_1^4 &= \text{const.} \end{aligned}$$

The transformation

$$
\begin{aligned}
y_1 &= x_1, \\
y_2 &= x_2 - \tfrac{1}{2}x_1^2, \\
y_3 &= x_3 - x_2x_1 + \tfrac{1}{3}x_1^3, \\
y_4 &= x_4 - x_3x_1 + \tfrac{1}{2}x_2x_1^2 - \tfrac{1}{8}x_1^4,
\end{aligned} \tag{25}
$$

reduces the system S_{+1} (that is, equations (24) with $\varepsilon = +1$) to the form

$$\dot{y}_1 = 1, \quad \dot{y}_2 = 0, \quad \dot{y}_3 = 0, \quad \dot{y}_4 = 0. \tag{26}$$

The inverse of (25) is

$$
\begin{aligned}
x_1 &= y_1, \\
x_2 &= y_2 + \tfrac{1}{2}y_1^2, \\
x_3 &= y_3 + y_2y_1 + \tfrac{1}{6}y_1^3, \\
x_4 &= y_4 + y_3y_1 + \tfrac{1}{2}y_2y_1^2 + \tfrac{1}{24}y_1^4.
\end{aligned} \tag{27}
$$

On the other hand, if we introduce the variables

$$
\begin{aligned}
z_1 &= -x_1 \\
z_2 &= -(x_2 + \tfrac{1}{2}x_1^2), \\
z_3 &= -(x_3 + x_2x_1 + \tfrac{1}{3}x_1^3), \\
z_4 &= -(x_4 + x_3x_1 + \tfrac{1}{2}x_2x_1^2 + \tfrac{1}{8}x_1^4),
\end{aligned} \tag{28}
$$

the equations of S_{-1} are transformed into

$$\dot{z}_1 = 1, \quad \dot{z}_2 = 0, \quad \dot{z}_3 = 0, \quad \dot{z}_4 = 0. \tag{29}$$

The inverse of (28) is given by

$$
\begin{aligned}
x_1 &= -z_1, \\
x_2 &= -(z_2 + \tfrac{1}{2}z_1^2), \\
x_3 &= -(z_3 + z_2z_1 + \tfrac{1}{6}z_1^3), \\
x_4 &= -(z_4 + z_3z_1 + \tfrac{1}{2}z_2z_1^2 + \tfrac{1}{24}z_1^4).
\end{aligned} \tag{30}
$$

Finally, the transformations which give (y_1, y_2, y_3, y_4) in terms of (z_1, z_2, z_3, z_4) and vice versa are

$$
\begin{aligned}
y_1 &= -z_1, & z_1 &= -y_1, \\
y_2 &= -(z_2 + z_1^2), & z_2 &= -(y_2 + y_1^2), \\
y_3 &= -(z_3 + 2z_2z_1 + z_1^3), & z_3 &= -(y_3 + 2y_2y_1 + y_1^3), \\
y_4 &= -(z_4 + 2z_3z_1 + 2z_2z_1^2 + \tfrac{7}{12}z_1^4), & z_4 &= -(y_4 + 2y_3y_1 + 2y_2y_1^2 + \tfrac{7}{12}y_1^4).
\end{aligned} \tag{31}
$$

We are now in a position to compute the closed form equations of the three-dimensional switching surface R_3. In view of the preceding section it is sufficient to compute $R_{3\,1}$. In order to obtain $R_{3,1}$ we shall first compute the lower dimensional switching surfaces $R_{1,1}$ and $R_{2,1}$.

The set $R_{1,1}$ is obtained by moving backwards (with respect to time) from the origin along the (unique) solution of system S_{+1} which passes through the origin. Using (26) we characterize the set $R_{1,1}$ by the relations

$$R_{1,1} : y_1 < 0, \ y_2 = 0, \ y_3 = 0, \ y_4 = 0.$$

These relations may be expressed in terms of (z_1, z_2, z_3, z_4) as follows:

$$(32) \qquad R_{1,1} : \begin{cases} z_1 > 0, \\ z_2 + z_1^2 = 0, \\ z_3 + 2z_2 z_1 + z_1^3 = 0, \\ z_4 + 2z_3 z_1 + 2z_2 z_1^2 + \frac{7}{12} z_1^4 = 0. \end{cases}$$

Elimination of z_1 between the first two equations in (32) yields

$$z_3 + z_2 z_1 = 0,$$

whence $R_{1,1}$ may be characterized by

$$(33) \qquad R_{1\,1} : \begin{cases} \dfrac{z_3}{z_2} < 0, \\ z_3 + z_2 z_1 = 0, \\ z_3^2 + z_2^3 = 0, \\ 12z_4 + 7z_2^2 = 0. \end{cases}$$

It is now easy to compute $R_{2,1}$. One simply solves (29) in negative time starting on $R_{1,1}$ (equation 33). The result is:

$$R_{2,1} : \begin{cases} \dfrac{z_3}{z_2} < 0, \\ z_1 < -\dfrac{z_3}{z_2}, \\ z_3^2 + z_2^3 = 0, \\ 12z_4 + 7z_2^2 = 0, \end{cases}$$

or, in terms of (y_1, y_2, y_3, y_4),

$$
(34) \qquad R_{2,1}\ :\ \begin{cases} \dfrac{y_3 + 2y_2y_1 + y_1^3}{y_2 + y_1^2} < 0, \\[2ex] y_1 > \dfrac{y_3 + 2y_2y_1 + y_1^3}{y_2 + y_1^2}, \\[2ex] y_2y_1^4 + 2y_3y_1^3 + y_2^2y_1^2 + 4y_3y_2y_1 + (y_3^2 - y_2^3) = 0, \\[1ex] 10y_2y_1^2 + 24y_3y_1 + (12y_4 - 7y_2^2) = 0. \end{cases}
$$

We may use the last two equations of (34) to obtain two linear equations in y_1. The algebra is somewhat tedious but straightforward. One gets

$$
(35) \qquad Ay_1 + B = 0,
$$

$$
(36) \qquad Cy_1 + D = 0,
$$

where

$$
\begin{aligned}
A &= 12y_3\{-15y_2^3 + 140y_2y_4 - 96y_3^2\}, \\
B &= 836y_2^2y_3^2 + 95y_2^5 - 1440y_4y_2^3 + 720y_2y_4^2 - 576y_4y_3^2, \\
C &= -6340y_3^2y_2^3 + 23{,}040y_2y_3^2y_4 + 7200y_2^4y_4 - 3600y_4^2y_2^2 - 475y_2^6 - 13{,}824y_3^4, \\
D &= -6960y_2^3y_3y_4 + 10{,}080y_2y_3y_4^2 - 6912y_3^3y_4 + 630y_2^5y_3 + 4032y_2^2y_3^3.
\end{aligned}
$$

Equations (35) and (36) are two simultaneous linear equations in y_1. Hence the determinant of their coefficients vanishes identically. That is

$$
AD - BC = 0.
$$

Let

$$
E = \frac{y_3 - 2y_2\dfrac{B}{A} - \left(\dfrac{B}{A}\right)^3}{y_2 + \left(\dfrac{B}{A}\right)^2}.
$$

The equations of $R_{2,1}$ now become

$$
R_{2,1}\ :\ \begin{cases} E < 0, \\[1ex] E < -\dfrac{B}{A}, \\[1ex] Ay_1 + B = 0, \\[1ex] AD - BC = 0. \end{cases}
$$

We note that the expression $AD - BC$ does not contain y_1; neither does E.

The equations of $R_{3,1}$ are now immediate. Using (26), one has

$$R_{3,1} \ : \ \begin{cases} E < \min\left(0, -\dfrac{B}{A}\right), \\ y_1 < -\dfrac{B}{A}, \\ AD - BC = 0. \end{cases} \tag{37}$$

The equations of $R_{3,2}$ are identical in form with (37). One simply replaces y_i by z_i throughout. These equations may also be written in terms of the original state variables (x_1, x_2, x_3, x_4) by using (25) and (28).

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