

ANALYTIC MEAN PERIODIC FUNCTIONS⁽¹⁾

BY

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1. **Introduction.** In accordance with Schwartz's [9] generalization of Delsarte's [2; 3] original definition, an entire function ϕ is mean periodic with respect to a complex plane measure μ of compact support if for all complex z

$$(1) \quad \int \phi(z + w) d\mu(w) = 0.$$

(Actually in Schwartz's terminology $\phi(-z)$ is mean periodic with respect to μ .) If the entire function of exponential type $f(z) = \int e^{zw} d\mu(w)$ has the series expansion $\sum_{n=0}^{\infty} a_n z^n$, and if ϕ is entire, then (1) is equivalent to the infinite order differential equation $\sum_{n=0}^{\infty} a_n \phi^{(n)}(z) = 0$. An exponential monomial $z^h e^{\zeta z}$ is a solution of (1) if and only if ζ is a zero of f of order at least $h + 1$. Let B denote the set of all such exponential monomial solutions. Among other results, Schwartz establishes an earlier corollary of Valiron [11, p. 38] to the effect that each solution of (1) can be expressed as a series of linear combinations of elements of B , the series converging uniformly on compact subsets of the plane. For (1) to have meaning for z in a region (open) R , it suffices to have ϕ analytic in a region including all points of the form $z + k$ where z is in R and k is in the smallest closed convex set containing the singularities of the Borel transform [1, p. 73] $F(z) = \int (z - w)^{-1} d\mu(w)$ of f . In particular, (1) has meaning for z in R if ϕ is analytic in the sum $R + P$ where P is the conjugate indicator diagram of f , i.e., the closed convex hull of the singularities of F . In this case, (1) is equivalent to $\int_c \phi(z + w) F(w) dw = 0$ when c is a properly chosen curve about P . For such ϕ , (1) is not in general an infinite order differential equation [4, p. 59]. We will say that a function ϕ which is analytic in $R + P$ and satisfies (1) for z in R is mean periodic with respect to f in R . The purpose of the paper is to determine conditions on f and/or on mean periodic ϕ that yield a representation for ϕ in $R + P$ as a series of linear combinations of elements of B , the convergence being uniform on compact subsets.

In order to state our results we introduce some notation: If T is a region, $A(T)$ will be the complex linear space of single-valued functions analytic in T

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with the topology of compact convergence, i.e., uniform convergence on compact subsets of T . $B(T)$ will denote the subspace of $A(T)$ generated by the elements of B . If U and V are sets of complex numbers and $\delta \geq 0$, the sum $U + V$ is the set of numbers of the form $u + v$ with u in U and v in V , while $U \oplus \delta$ is the sum of U and the disk $|z| \leq \delta$. $N(a, \delta)$ will denote the neighborhood $|z - a| < \delta$ if $\delta > 0$, and P_U will denote the convex hull of P and U .

If K is the set of all functions which are mean periodic with respect to f in R , then K is a subspace of $A(R + P)$ and contains B and $B(R + P)$. Our problem is to find sufficient conditions to insure the totality of B in K yielding $K = B(R + P)$. Briefly, such a conclusion is justified if there exists a sequence of closed contours tending to infinity on which for all p in P , $e^{pz}/f(z)$ is of minimal exponential growth, i.e., for each $\varepsilon > 0$ and all p in P , eventually $|e^{-pz}f(z)| > \exp(-\varepsilon|z|)$ for z on the contours. This is a special case of the general result to the effect that if for some $\tau \geq 0$ and all s in a set S , there is a contour sequence on which eventually $|e^{-sz}f(z)| > \exp[-(\tau + \varepsilon)|z|]$, then $K \cap A(R + P_S \oplus \tau) \subset B(R + S)$. Choosing S as P , it is always possible to take τ as the length δ of the longest diameter of P yielding the result: $K \cap A(R + P \oplus \delta) \subset B(R + P)$. In each of these cases the expansions of the mean periodic functions are given explicitly. Expansions of a mean periodic function ϕ in regions $R_1 + P$ and $R_2 + P$ whose union is not of the form $R + P$ are in general distinct. This is the case, for example, in the Fourier expansions (when μ is the difference of the Dirac measures at 0 and 1) of $\cot \pi z$ in the upper and lower half planes.

If f is an exponential polynomial $\sum_{i=1}^n P_i(z) \exp(\omega_i z)$ where the P_i are polynomials, then (1) is a difference-differential equation, and it has been shown in [4, p. 49] that B is total in K by constructing a contour sequence on which $|e^{-pz}f(z)|$ is uniformly bounded from zero for all p in P and z on the contours. If f is of exponential type σ , then (1) is an infinite order differential equation on $A(R \oplus \sigma)$. In this case it has been shown (see [5, p. 13] and [4, p. 61]) that $K \cap A(R \oplus 2\sigma) \subset B(R)$, and, more generally, that for $\tau \geq 0$, $K \cap A(R \oplus (\sigma + \tau)) \subset B(R)$ if there is a contour sequence on which eventually $|f(z)| > \exp[-(\tau + \varepsilon)|z|]$. These results are special cases of the results of this paper. General results on entire mean periodic functions and the totality of sets of exponential polynomials in analytic function spaces are found in the papers of Schwartz [9] and Kahane [6] and in the expository article by Leont'ev [8].

In §2 the definition of mean periodicity is rephrased, and the implications of the Hahn-Banach theorem are discussed together with the relation of this problem to the study of differentiation invariant subspaces of a space $A(T)$. Also in §2, an operator having K as kernel is introduced along with contour sequences that will be used for summing series of linear combinations of elements in B . §3 contains general expansion theorems in which the expansions depend on the existence of such summing contours. The existence of such contours is considered in §4, and specific expansion theorems are obtained from the theorems

of §3. In §5 the uniqueness of expansions of mean periodic functions is examined, yielding special representations for functions that are mean periodic with respect to more than one function of exponential type.

2. **Preliminaries.** If $A^*(T)$ denotes the dual of $A(T)$, then $A^*(T)$ may be identified [7, p. 37] with the set of entire functions of exponential type whose Borel transforms have all their singularities in T . If $v \in A^*(T)$ and $g(z) = \int e^{zw} dv(w)$ has Borel transform G , then $\int \phi(w) dv(w) = (2\pi i)^{-1} \int_c \phi(w) G(w) dw$ where c is a simple closed curve in T with its interior in T and containing the singularities of G . If T contains the conjugate indicator diagram $P(g)$ of g (as is the case if T is convex), then c may be taken as a curve about $P(g)$. These observations will make it possible to write (1) in a more convenient form.

If g is entire, the spectrum $s(g)$ of g will be the set of zeros of g , each being counted according to its multiplicity. We will write $(\xi, h) \in s(g)$ to indicate that ξ is a zero of g of order at least $h + 1$. f, F, P, K , and B will be used throughout as introduced in the introduction. All curves will be rectifiable and simple closed curves positively oriented.

DEFINITION 2.1. ϕ in $A(R + P)$ is mean periodic with respect to f in the region R , if for each z in R , $\int_{c(z)} \phi(z + w) F(w) dw = 0$, where $c(z)$ is a simple closed curve containing P in its interior with $z + c(z)$ in a simply connected subregion of $R + P$.

DEFINITION 2.2. The operator $\mathcal{F} : A(R + P) \rightarrow A(R)$ is defined by $\mathcal{F}[\phi(z)] = (2\pi i)^{-1} \int_{c(z)} \phi(z + w) F(w) dw$ where $c(z)$ is as in Definition 2.1.

The kernel K of \mathcal{F} is then the set of functions which are mean periodic with respect to f in R . In a straightforward manner it can be shown that \mathcal{F} is a continuous linear operator. To establish the injectivity and continuity of \mathcal{F} it is helpful to note that if H is a compact subset of R and $H \oplus \delta \subset R$, then for all z in H , $c(z)$ may be chosen as the boundary of $P \oplus \delta$.

Consider momentarily the problem of determining conditions under which B is total in K when $R + P$ is convex (modifications in paths of integration yield a similar discussion in the nonconvex case). Then $A^*(R + P)$ may be identified with the set of entire functions g of exponential type with conjugate indicator diagrams $P(g)$ in $R + P$. If $P(g) \oplus \varepsilon \subset R + P$, one may define with g an operator \mathcal{G} mapping $A(R + P)$ into $A(N(0, \varepsilon))$ in the same way \mathcal{F} was defined with f . It is easily shown that since $z^{h-1} e^{\xi z}$ is in B whenever $z^h e^{\xi z}$ ($h \geq 1$) is in B , the functional in $A^*(R + P)$ corresponding to g annihilates B if and only if \mathcal{G} does; and that this occurs if and only if $s(f) \subset s(g)$, i.e., if and only if g/f is entire. Denoting by Δ the set of all g with g/f entire and $P(g) \subset R + P$, it follows from the Hahn-Banach theorem that $B(R + P)$ is the intersection of the null spaces of all functionals arising from g in Δ . This intersection obviously contains the intersection of the kernels $K(\mathcal{G})$ of the corresponding operators \mathcal{G} . On the other hand, the continuity of each \mathcal{G} implies that $B(R + P)$ is in each such kernel. Hence

$B(R + P) = \bigcap_{\Delta} K(\mathcal{G})$. To ask if $K \subset B(R + P)$ is to ask if each function which is mean periodic with respect to f in R is also mean periodic with respect to each g in Δ in a neighborhood of the origin. A sufficient condition insuring an affirmative reply would be that for each g in Δ , $P(g/f)$ is in R . For then, if $g/f = h$ and $P(h) \oplus \delta \subset R$, one may define an operator \mathcal{H} from $A(R)$ into $A(N(0, \delta))$ so that $\mathcal{G} = \mathcal{H}\mathcal{F}$ on $A(R + P)$ and each element in K is in $K(\mathcal{G})$. A sufficient condition in order that $P(h)$ be always in R would be that P reduce to a single point p , for then $P(g) = p + P(h)$. However, such a condition is not necessary for the totality of B in K as is illustrated in the case in which f is an exponential polynomial. If P does reduce to a point p , it follows that $e^{-pz}f(z)$ is of minimal type and there exists a sequence of circular contours tending to infinity on which eventually $|e^{-pz}f(z)| > \exp(-\varepsilon|z|)$. This is precisely the type of condition used here to insure that $K \subset B(R + P)$.

In the same way that we have asked if the differentiation invariant subspace K of $A(R + P)$ is generated by the exponential monomials in K , we may ask if a proper differentiation invariant subspace M of some $A(T)$ is the subspace generated by the set B_M of exponential monomials in M as is the case when M is finite dimensional. Considering for simplicity the case when T is convex, the fact that M is proper implies the existence of an entire function k with $P(k) \subset T$ such that if $z^h e^{\xi z}$ is in B_M , then $(\xi, h) \in s(k)$. If Δ is the set of all g with $P(g)$ in T such that $(\xi, h) \in s(g)$ if $z^h e^{\xi z} \in B_M$, then $B_M(T) = \bigcap_{\Delta} K(\mathcal{G})$. Then $M = B_M(T)$ if every element of M is mean periodic with respect to each g in Δ in some neighborhood of the origin. This related problem will not be investigated here.

If R and T are regions of the complex plane, the definition of \mathcal{F} may be extended to a map \mathcal{F}_z of $A((R + P) \times T)$ into $A(R \times T)$ by defining

$$\mathcal{F}_z[\phi(z, t)] = \frac{1}{2\pi i} \int_{c(z)} \phi(z + w, t) F(w) dw.$$

\mathcal{F}_z is a continuous linear operator when the spaces are equipped with the topology of compact convergence. $\mathcal{F}_z[\phi(z, t)]|_{z=a}$ will be denoted by $\mathcal{F}_{z=a}[\phi(z, t)]$. The Borel transform of $\phi(z, t)$ with respect to z will similarly be denoted by $L_z[\phi(z, t)]$.

In order to expand a function ϕ in K in a series of linear combinations of elements in B , we first define some linear functionals on $A(R + P)$ that will produce with B a biorthogonal system which in turn will suggest a suitable coefficient for each function in B . Then we define a sequence of contours associated with f that will enable us to group together the proper linear combinations of these exponential monomials in an attempt to produce a series that converges to ϕ .

DEFINITION 2.3. Let $\zeta_k \in s(f)$; let c_k be a circle about ζ_k containing no other zero of f inside or on itself. For each γ in region R , each natural k , and each non-negative integer h , the functional $\mathfrak{F}_{kh\gamma}$ on $A(R + P)$ is defined by

$$\mathfrak{F}_{kh\gamma}[\phi] = \frac{1}{2\pi i} \int_{c_k} \frac{(t - \zeta_k)^h}{f(t)e^{\gamma t}} \mathcal{F}_{u=\gamma} \left[e^{ut} \int_a^u \phi(s)e^{-st} ds \right] dt$$

where the paths of integration from a to u all lie in the same simply connected region (connected understood) which contains $\gamma + P$ and is in the region of analyticity of ϕ .

We will first show that \mathfrak{F}_{khy} is defined for each pair (a, S) , where S is a simply connected region containing $\gamma + P$ in the region of analyticity of ϕ and a is in S . Choose $N = N(\gamma, \epsilon)$ in R . For u in $N + P$ the bracketed expression in the right member of the definition is in $A((N + P) \times T)$ where T is the whole finite plane and the path of integration is in S . Hence \mathcal{F}_u maps this into $A(N \times T)$ and the evaluation of this image at $u = \gamma$ yields an entire function of t . Therefore $\mathfrak{F}_{khy}[\phi]$ exists for the pair (a, S) . Suppose now that (a, S) and (a', S') are two such pairs. Choose q in $\gamma + P$. Let L be a path in S from a to q , and let L' be a path in S' from a' to q . Choose $c(\gamma)$ as a curve about P with $\gamma + c(\gamma) \subset S \cap S'$. For each w on $c(\gamma)$ let $L(w)$ and $L'(w)$ be paths from a and a' to $\gamma + w$ in S and S' , respectively. Let $K(w)$ be the directed segment $[q, \gamma + w]$. The integral of $\phi(s)e^{-st}$ with respect to s over $L(w) - L'(w)$ is then equal to the integral over $L + K(w) - L' - K(w)$ or $L - L'$. Since this integral is independent of w and entire in t , it follows easily by writing \mathcal{F}_u as an integral over $c(\gamma)$ that the difference between $\mathfrak{F}_{khy}[\phi]$ with (a, S) and with (a', S') is zero.

It is easy to see that \mathfrak{F}_{khy} is a continuous linear functional since \mathcal{F}_z is continuous. The following theorem is proved in [4, p. 31].

THEOREM 2.1. *If $(\zeta_q, p) \in s(f)$, then $\mathfrak{F}_{khy}[z^p \exp(\zeta_q z)] = h! \delta_{kq} \delta_{hp}$ for all γ .*

DEFINITION 2.4. For $\tau \geq 0$ and complex number β , a sequence of closed contours $\Sigma = \{\Gamma_p\}_{p=1}^\infty$ is said to be (τ, β) associated with f if the following conditions are satisfied:

- (1) Γ_p is in the interior of Γ_{p+1} .
- (2) Γ_p passes through no zero of f .
- (3) If $r_p = \min|z|$ and $R_p = \max|z|$ for z on Γ_p , and λ_p is the length of Γ_p ; then as $p \rightarrow \infty$, $r_p \rightarrow \infty$, $\log \lambda_p = o(r_p)$, and $R_p = O(r_p)$.
- (4) For each $\epsilon > 0$ there is a $p_0 = p_0(\epsilon, \beta)$ such that if $p > p_0$, then $|f(z)\exp(-\beta z)| > \exp[-(\tau + \epsilon)r_p]$ when z is on Γ_p .

DEFINITION 2.5. Let S be a set of complex numbers. Then a sequence Σ of contours is (τ, S) associated with f if and only if Σ is (τ, β) associated with f for each β in S .

If Σ is (τ, S) associated with f , then the p_0 in (4) of Definition 2.4 depend on ϵ and β for $\beta \in S$. It is easy to show by a straightforward compactness argument that a single $p_0 = p_0(\epsilon)$ may be chosen for all $\beta \in S$ if S is compact.

THEOREM 2.2. *If Σ is (τ, S) associated with f and \tilde{S} is the convex hull of set S , then Σ is (τ, \tilde{S}) associated with f .*

Proof. Suppose α and β are in S and $\gamma = \alpha + t(\beta - \alpha)$ where $0 < t < 1$. Then in the half plane $\Re[(\beta - \alpha)z] \leq 0$, $|f(z)\exp(-\gamma z)| = |f(z)\exp[-t(\beta - \alpha)z - \alpha z]|$

$\geq |f(z) \exp(-\alpha z)|$. In $\mathcal{R}[(\beta - \alpha)z] > 0$,

$$|f(z) \exp(-\gamma z)| = |f(z) \exp[(1-t)(\beta - \alpha)z - \beta z]| > |f(z) \exp(-\beta z)|.$$

By considering z on $\Gamma_p \cap \{z | \mathcal{R}[(\beta - \alpha)z] \leq 0\}$ and on $\Gamma_p \cap \{z | \mathcal{R}[(\beta - \alpha)z] > 0\}$ where $\Gamma_p \in \Sigma$, the result follows from the fact that α and β are in S .

It is to be noted that r_p rather than $|z|$ is used in the lower bound in property (4) of Definition 2.3. Suppose that a sequence Σ of contours satisfies the first three conditions of that definition with $\limsup_{p \rightarrow \infty} R_p/r_p = \rho$. Suppose also that for each $\varepsilon > 0$ it is true, for large p and z on Γ_p , that $|f(z) \exp(-\beta z)| > \exp[-(\varepsilon + \tau/\rho)|z|]$. Then for p large and z on Γ_p , $|f(z) \exp(-\beta z)| > \exp[-(\varepsilon + \tau/\rho)(\rho + \varepsilon)r_p] = \exp[-(\tau + \varepsilon(\varepsilon + \rho + \tau/\rho))r_p]$, and Σ is (τ, β) associated with f . This indicates that for (τ, β) association the annuli containing the Γ_p may increase in difference of radii with p if a compensatingly better lower bound can be found for $|f(z) \exp(-\beta z)|$ on the contours.

DEFINITION 2.6. A series whose terms are linear combinations of elements of B is called an exponential series relative to f .

DEFINITION 2.7. Let $\Sigma = \{\Gamma_p\}_{p=1}^\infty$ be (τ, S) associated with f ; let region R contain γ and $\phi \in A(R + P)$; let Z_1 be the set of zeros of f inside Γ_1 and Z_p be the set of zeros inside Γ_p but outside Γ_{p-1} for $p > 1$; let $m_k + 1$ be the order of the zero ζ_k of f . The series

$$\sum_{q=1}^\infty \sum_{\zeta_k \in Z_q} \left(\sum_{h=0}^{m_k} \frac{\mathfrak{F}_{khy}[\phi]}{h!} z^h \right) e^{\zeta_k z}$$

is called the exponential series of ϕ relative to (f, Σ, γ) .

3. General expansion theorems. In preparation for the convergence theorems, we first establish some lemmas. Recall that P_S is the convex hull of P and set S ; in particular, P_0 is the convex hull of 0 and P .

DEFINITION 3.1.

$$g(z, t) = \begin{cases} \frac{f(z) - f(t)}{z - t} & \text{if } z \neq t, \\ f'(z) & \text{if } z = t. \end{cases}$$

LEMMA 3.1. Let $h(\theta)$ be the indicator function of f . Then for each $\varepsilon > 0$ there is a $K = K(\varepsilon)$ such that $|g(re^{i\theta}, t)| \leq |f(t)| + K \exp[(h(\theta) + \varepsilon)r]$.

Proof. Since $\limsup_{r \rightarrow \infty} r^{-1} \log |f(re^{i\theta})| = h(\theta)$ and the limit is uniform in θ , it follows that if $|re^{i\theta} - t| \geq 1$, then there is an $M_1 = M_1(\varepsilon)$ such that $|g(re^{i\theta}, t)| < |f(t)| + M_1 \exp\{r[h(\theta) + \varepsilon]\}$. We assert that there is an $M_2 = M_2(\varepsilon)$ such that if $|re^{i\theta} - t| < 1$, then $|g(re^{i\theta}, t)| < M_2 \exp\{r[h(\theta) + \varepsilon]\}$. Since $h(\theta)$ is uniformly continuous, there is a $\delta > 0$ such that $|h(\theta_1) - h(\theta_2)| < \varepsilon/2$ when $|\theta_1 - \theta_2| < \delta$. The indicator diagram of f' is a subset of that of f and so $\limsup_{r \rightarrow \infty} r^{-1} \log |f'(re^{i\theta})| \leq h(\theta)$, the limit being uniform in θ . Hence there is an $r_1 = r_1(\varepsilon) > 1$ such that $|t| > r_1 - 1$ implies $|f'(t)| < \exp\{|t| [h(\arg t) + \varepsilon/2]\}$.

Choose $r_2 = r_2(\varepsilon)$ so that if $r > r_2$ and $|re^{i\theta} - t| < 1$, then $|\arg t - \theta| < \delta$ for some choice of $\arg t$. Let $r_0 = \max\{r_1, r_2\}$. Then if $r > r_0$ and $|re^{i\theta} - t| < 1$; (1) $|t| > r_0 - 1 \geq r_1 - 1$ so $|f'(t)| < \exp\{t|[h(\arg t) + \varepsilon/2]\}$, and (2) $r > r_2$ so $|\arg t - \theta| < \delta$ and $|h(\arg t) - h(\theta)| < \varepsilon/2$ for properly chosen $\arg t$. Together these yield $|f'(t)| < \exp\{|t|[h(\theta) + \varepsilon]\} < \exp\{(r + 1)[h(\theta) + \varepsilon]\}$ if $r > r_0$ and $|re^{i\theta} - t| < 1$. Since $|f'(t)|$ is bounded for $|t| \leq r_0 + 1$, it follows that for some $M_2 = M_2(\varepsilon)$, $|f'(t)| < M_2 \exp\{r[h(\theta) + \varepsilon]\}$ when $|re^{i\theta} - t| < 1$. Therefore if $|re^{i\theta} - t| < 1$, and $c = [t, re^{i\theta}]$, $|g(re^{i\theta}, t)| = |re^{i\theta} - t|^{-1} \left| \int_c f'(w)dw \right| \leq M_2 \exp\{r[h(\theta) + \varepsilon]\}$, the inequality obviously holding when $t = re^{i\theta}$. This establishes the assertion. Choosing K as $\max\{M_1, M_2\}$ establishes the lemma.

It is a corollary of this lemma that for any fixed t , the conjugate indicator diagram of $g(z, t)$ is a subset of P . The same is true for $z^m g(z, t)$ if m is a non-negative integer.

LEMMA 3.2. *If m is a non-negative integer, then for z in the complement of P_0 and for all t ,*

$$\mathcal{F}_{u=0} \left[e^{ut} \int_0^u \frac{m! e^{-st}}{(z-s)^{m+1}} ds \right] = L_z[g(z, t)].$$

Proof. Let U and V denote respectively the left and right members of the identity. Let t be fixed. For each $\delta > 0$, the bracketed function in U is analytic in the pair (u, z) for u in the interior of $P_0 \oplus \delta$ and z in the complement of $P_0 \oplus \delta$. Since $P_0 \oplus \delta \supset N(0, \delta/2) + P$, the image of this function under \mathcal{F}_u when evaluated at 0 is analytic in z for z in the complement of $P_0 \oplus \delta$. Since δ may be chosen arbitrarily small, U is analytic in z for z in the complement of P_0 . V is also analytic in the complement of P_0 since the conjugate indicator diagram of $z^m g(z, t)$ is a subset of P , which is in turn a subset of P_0 . To establish the identity it will suffice to show that it holds for real $z > \sigma$ where f is of type σ .

For $z > \sigma$, $V = \int_0^\infty x^m g(x, t) e^{-zx} dx$ where the integral converges absolutely since $g(x, t)$ is of exponential type σ in x for each t . Let c be the circle $|w| = (\sigma + z)/2$. A straightforward calculation shows that

$$g(x, t) = \frac{1}{2\pi i} \int_c e^{wt} F(w) \int_0^w e^{(x-t)s} ds dw.$$

Substituting this expression for $g(x, t)$ in the last expression for V , we obtain

$$\begin{aligned} V &= \frac{1}{2\pi i} \int_c e^{wt} F(w) \int_0^\infty x^m e^{-zx} \int_0^w e^{(x-t)s} ds dx dw \\ &= \frac{1}{2\pi i} \int_c e^{wt} F(w) \int_0^w e^{-st} \int_0^\infty x^m e^{(s-z)x} dx ds dw. \end{aligned}$$

The first change in the order of integration is justified by observing that for each w on c , the integral with respect to s is of exponential type less than z in x yielding the absolute convergence of the first integral with respect to x . The second change

follows from the fact that the last integral with respect to x converges absolutely for each s in $[0, w]$ since $\Re(z - s) > 0$. Noting that the last integral with respect to x is $m!/(z-s)^{m+1}$, it follows that this last expression for V is equal to U .

THEOREM 3.1. *If Σ is $(\tau, 0)$ associated with f , γ is in region R , and $\phi \in K \cap A(R + P_0 \oplus \tau)$, then the exponential series of ϕ relative to (f, Σ, γ) converges to ϕ in $A(R)$.*

Proof. The first part of the proof will be devoted to establishing the uniform convergence of the series to ϕ in a neighborhood of γ . Then convergence will be established elsewhere in R by showing that the series relative to (f, Σ, γ) and (f, Σ, γ') are the same if $\gamma' \in R$ and $\gamma' \neq \gamma$.

Let $\limsup_{p \rightarrow \infty} R_p/r_p = \rho$ and $\varepsilon = h/(\rho + 7)$. Convergence will be established in $N = N(\gamma, \varepsilon)$. Using the notation of Definition 2.7, let $\Sigma = \{\Gamma_p\}$ and choose a to be in $R + P_0 \oplus \tau$ with the paths of integration from a to u in the same simply connected subregion of $R + P_0 \oplus \tau$ containing $\gamma + P$. Let $S_p(z)$ denote the p th partial sum of the series. In $S_p(z)$ the sum of the integrals over the c_k may be replaced with an integral over Γ_p . The upper limits on the sums over h may be changed from m_k to infinity because the terms so introduced are zero by Cauchy's theorem. Then,

$$S_p(z) = \frac{1}{2\pi i} \int_{\Gamma_p} \frac{e^{(z-\gamma)t}}{f(t)} \mathcal{F}_{u=\gamma} \left[e^{ut} \int_a^u \phi(s) e^{-st} ds \right] dt.$$

Since the $\mathfrak{F}_{kh\gamma}[\phi]$ are independent of the pair consisting of a and the simply connected region containing $\gamma + P$, so is $S_p(z)$. Since $N + P_0$ is a convex and simply connected subregion of $R + P_0 \oplus \tau$ containing $\gamma + P$, and since $z \in N + P_0$ when $z \in N$, we may write z for a and integrate with respect to s over straight line paths when considering this expression for $S_p(z)$ when $z \in N$. In the first part of the proof, we consider only z in N and assume that the substitution of z for a has been made in $S_p(z)$.

For each p , choose $m = m(p)$ so that $m = [(\tau + h - 2\varepsilon)r_p]$, i.e., m is the largest integer not greater than $(\tau + h - 2\varepsilon)r_p$. For each p with $m \geq 1$, integration of $\int_z^u \phi(s) e^{-st} ds$ by parts m times, together with the fact that $\mathcal{F}_{u=\gamma}[\phi^{(k)}(u)] = 0$, yields

$$S_p(z) = \phi(z) + \frac{1}{2\pi i} \int_{\Gamma_p} \frac{e^{(z-\gamma)t}}{f(t)t^m} \mathcal{F}_{u=\gamma} \left[e^{ut} \int_z^u \phi^{(m)}(s) e^{-st} ds \right] dt.$$

For $m \geq 1$, the second term of the right member is split into two terms, $Q_p(z)$ and $T_p(z)$, by replacing the integral from z to u by integrals from z to γ and from γ to u , respectively. Each of these will be shown to approach zero uniformly as $p \rightarrow \infty$.

Let C be the curve formed by the boundary of $P_0 \oplus 3\varepsilon$. Then for p with $m \geq 1$,

$$Q_p(z) = \frac{1}{2\pi i} \int_z^\gamma \phi^{(m)}(s) \int_{\Gamma_p} e^{(z-s)t} t^{-m} dt ds.$$

The inner integral is $(z - s)^{m-1}/(m-1)!$. Writing $\phi^{(m)}(s)$ as a Cauchy integral over $\gamma + C$,

$$Q_p(z) = \frac{m}{2\pi i} \int_{\gamma+C} \phi(\xi) \int_z^\gamma \frac{(z-s)^{m-1}}{(\xi-s)^{m+1}} ds d\xi.$$

Let $M = \max |\phi(\xi)|$ for ξ on $\gamma + C$, and let L be the length of C . For z in N , s in $[z, \gamma]$ and ξ on $\gamma + C$, we have $|z - s| < \varepsilon$ and $|\xi - s| > 2\varepsilon$. Hence $|Q_p(z)| \leq MLm/(\pi \varepsilon 2^{m+2})$. As $p \rightarrow \infty$, $m \rightarrow \infty$ and so $Q_p(z) \rightarrow 0$ uniformly. (In [4, p. 64] a similar term should be $o(1)$ rather than zero.)

We now consider $T_p(z)$. Let D and E be the curves formed by the boundaries of $P_0 \oplus \varepsilon$ and $P_0 \oplus (\tau + h)$. We first rewrite the image of \mathcal{F}_u evaluated at γ that appears in $T_p(z)$. Write \mathcal{F}_u evaluated at γ as an integral over D and $\phi^{(m)}(s)$ as a Cauchy integral over $\gamma + E$. A change in order of integration combined with two translation changes of variables shows that

$$\mathcal{F}_u =_\gamma \left[e^{ut} \int_\gamma^u \phi^{(m)}(s) e^{-st} ds \right] = \frac{1}{2\pi i} \int_E \phi(\zeta + \gamma) \mathcal{F}_{u=0} \left[e^{ut} \int_0^u \frac{m! e^{-st}}{(\zeta - s)^{m+1}} ds \right] d\zeta.$$

By Lemma 3.2,

$$(2) \quad T_p(z) = \frac{1}{(2\pi i)^2} \int_{\Gamma_p} \frac{e^{(z-\gamma)t}}{f(t) t^m} \int_E \phi(\zeta + \gamma) L_\zeta[\zeta^m g(\zeta, t)] d\zeta dt.$$

Next we obtain a bound for $|L_\zeta[\zeta^m g(\zeta, t)]|$ for ζ on E . Consider such a ζ and any fixed t . The conjugate indicator diagram of $z^m g(z, t)$ is a subset of P , and the support function of P is $h(-\theta)$ where $h(\theta)$ is the indicator function of f . If ζ is on E , then ζ is in the complement of the closed convex set $P_0 \oplus (\tau + h - \varepsilon)$. This set has support function $\max \{0, h(-\theta)\} + \tau + h - \varepsilon$. Hence for each ζ there is a $\psi = \psi(\zeta)$ such that $\Re(\zeta e^{-i\psi}) > \max \{0, h(-\psi)\} + \tau + h - \varepsilon$. Let $\Psi = e^{-i\psi}$. In general, if a function v of exponential type has indicator function h_v and Borel transform V , then $V(w/\Psi)$ is given in the half plane $\Re(w) > h_v(-\psi)$ by $\Psi \int_0^\infty v(x\Psi) e^{-wx} dx$. In our case, if ζ is on E and $\psi = \psi(\zeta)$ is chosen as indicated, we have $\Re(\zeta\Psi) > h(-\psi)$ and

$$L_\zeta[\zeta^m g(\zeta, t)] = \Psi \int_0^\infty x^m \Psi^m g(x\Psi, t) e^{-\zeta\Psi x} dx.$$

Using Lemma 3.1 together with each of the lower bounds for $\Re(\zeta\Psi)$, there is a $K = K(\varepsilon)$ such that

$$(3) \quad |L_\zeta[\zeta^m g(\zeta, t)]| \leq (K + |f(t)|) m! / (\tau + h - 2\varepsilon)^{m+1}.$$

Choose p_0 sufficiently large so that if $p > p_0$, then: $m \geq 1, \tau + h - 2\varepsilon - [(\tau + h - 2\varepsilon)r_p] / r_p < \varepsilon, \lambda_p < \exp(\varepsilon r_p), R_p < (\rho + 1) r_p, |f(t)| > \exp[-(\tau + \varepsilon)r_p]$ when t is on Γ_p , and $m! / m^m < (8\pi m)^{1/2} e^{-m}$. Let $M' = \max |\phi(\zeta + \gamma)|$ for

ζ on E and L' be the length of E . Then if $z \in N$ and $p > p_0$, it follows from (2) and (3) that

$$|T_p(z)| \leq LM'(K + 1)(\tau + h - 2\epsilon)^{-1/2} r_p^{1/2} \exp(-\epsilon r_p).$$

Therefore, as $p \rightarrow \infty$, $T_p(z) \rightarrow 0$ uniformly in N . Hence $S_p(z) \rightarrow \phi(z)$ uniformly in N .

To complete the proof of the theorem it suffices to show that if $\gamma' \in R$ and $\gamma' \neq \gamma$, then the series relative to (f, Σ, γ) converges uniformly to ϕ in some neighborhood of γ' . By what has been shown, the exponential series of ϕ relative to (f, Σ, γ') converges uniformly to ϕ in a neighborhood of γ' . Hence it will suffice to show that the series are the same. This will be done by showing that $\Delta = \mathfrak{F}_{khy}[\phi] - \mathfrak{F}_{khy'}[\phi] = 0$. In the definitions of $\mathfrak{F}_{khy}[\phi]$ and $\mathfrak{F}_{khy'}[\phi]$ choose a as γ and γ' , respectively. Let Λ be a curve in R from γ to γ' . Choose $\delta > 0$ sufficiently small so that $z + P_0 \oplus \delta \subset R + P_0$ for all z on Λ , and let c be the curve formed by the boundary of $P_0 \oplus \delta$. If z is on Λ and w is on $P_0 \oplus \delta$, then $[z, z + w]$ is in $R + P_0$ since the endpoints are in the convex set $z + P_0 \oplus \delta$. Let $g(r)$ be a parametrization of Λ where $g(r_0) = \gamma$ and $g(r'_0) = \gamma'$ and $r_0 < r'_0$. For each w on c , let L_w and L'_w be the directed segments $[\gamma, \gamma + w]$ and $[\gamma', \gamma' + w]$, and let K_w be the curve with parametrization $g(r) + w$, $r_0 \leq r \leq r'_0$. Now $\Lambda + L'_w - K_w - L_w$ is homotopic to a point curve in $R + P_0$. Since $\phi \in A(R + P_0)$, the integrals of $\phi(s)e^{-st}$ with respect to s over $L_w - L'_w$ and $\Lambda - K_w$ are equal. Writing \mathcal{F}_u as an integral over c , Δ may be expressed as

$$\frac{1}{(2\pi i)^2} \int_{c_k} \frac{(t - \zeta_k)^h}{f(t)} \int_c F(w)e^{wt} \int_{\Lambda - K_w} \phi(s)e^{-st} ds dw dt.$$

The part of this expression arising from the integration over Λ is zero by Cauchy's theorem; while the part due to integrating over $-K_w$ may be written, after a translation of variables, as

$$- \frac{1}{2\pi i} \int_{c_k} \frac{(t - \zeta_k)^h}{f(t)} \int_{\Lambda} e^{-st} \mathcal{F}_{u=s}[\phi(u)] ds dt.$$

Since $\mathcal{F}_{u=s}[\phi(u)] = 0$ for all s on Λ , $\Delta = 0$. This completes the proof of the theorem.

THEOREM 3.2. *If Σ is (τ, S) associated with f , γ is in region R , and $\phi \in K \cap A(R + P_S \oplus \tau)$, then the exponential series of ϕ relative to (f, Σ, γ) converges to ϕ in $A(R + S)$.*

Proof. Upon noting that compact convergence of the series to ϕ in every $\beta + R$ where β is in S implies compact convergence to ϕ in $S + R$, it suffices to prove the theorem when S is a single point β . This case will follow from the last theorem by a translation.

Let $g(z) = e^{-\beta z}f(z)$. Then the conjugate indicator diagram P' of g is $P - \beta$, and $P'_0 = P_\beta - \beta$. Let the Borel transform of g be $G(z) = F(z + \beta)$. Since Σ is (τ, β) associated with f , it is $(\tau, 0)$ associated with g . Let $R' = R + \beta$. Let the operator $\mathcal{G} : A(R' + P') \rightarrow A(R')$ and the functionals $\mathfrak{G}_{kh\gamma}$ on $A(R' + P')$ be defined with g as \mathcal{F} and $\mathfrak{F}_{kh\gamma}$ were defined with f .

$\phi \in A(R + P_\beta \oplus \tau)$ implies that $\phi \in A(R' + P'_0 \oplus \tau)$. It is easy to verify that $\mathcal{G}_{u=z+\beta}[\phi(u)] = \mathcal{F}_{u=z}[\phi(u)]$. Hence if $z' = z + \beta \in R'$ and $\phi \in K$, then $\mathcal{G}[\phi(z')] = \mathcal{G}_{u=z'}[\phi(u)] = \mathcal{G}_{u=z+\beta}[\phi(u)] = \mathcal{F}_{u=z}[\phi(u)] = 0$. By the preceding theorem the exponential series of ϕ relative to $(g, \Sigma, \gamma + \beta)$ converges to ϕ in $A(R')$. But this is the same as the series relative to (f, Σ, γ) as can be verified by showing that $\mathfrak{G}_{kh\gamma+\beta}[\phi] = \mathfrak{F}_{kh\gamma}[\phi]$.

COROLLARY 3.1. *If Σ is (τ, P) associated with f , γ is in region R , and $\phi \in K \cap A(R + P \oplus \tau)$, then the exponential series of ϕ relative to (f, Σ, γ) converges to ϕ in $A(R + P)$.*

In particular this corollary states that $K \cap A(R + P \oplus \tau) \subset B(R + P)$ if such (τ, P) associated contours exist. If $(0, P)$ associated contours exist, then $K = B(R + P)$.

4. Specific expansion theorems. In this section the existence of sequences of contours that are (τ, S) associated with a general f will be established for certain choices of τ and S . These results will then be used together with the theorems of the preceding section to give expansion theorems.

If f is of exponential type σ , it follows from a well-known theorem (see [1 p. 43] and [10, p. 277]) that there exists a sequence of circular contours with center at the origin that is $(\sigma, 0)$ associated with f . The theorem asserts that for each $\varepsilon > 0$ there are circles of arbitrarily large radius on which $|f(z)| > \exp\{-(\sigma + \varepsilon)|z|\}$. $(\sigma, 0)$ associated contours may be constructed by choosing Γ_p for each p as a circle on which $|f(z)| > \exp\{-(\sigma + 1/p)|z|\}$ and such that the radii $r_p \rightarrow \infty$ while $r_p < r_{p+1}$. This is a special case of the next theorem.

DEFINITION 4.1. If P is the conjugate indicator diagram of f and $\delta \geq 0$, S_δ is the set $\bigcap_{\beta \in P} \{z \mid |z - \beta| \leq \delta\}$. That is, S is the set of all points z having the property that the closed disk with center at z and radius δ contains P .

DEFINITION 4.2. If $d(s, p)$ is the distance from s to p , and set $S \neq \emptyset$, then $\sigma(S) = \sup d(s, p)$ over $(s, p) \in S \times P$.

THEOREM 4.1. *If $\delta \geq 0$, $S_\delta \neq \emptyset$, and \tilde{S}_δ is the convex hull of S_δ , then there exists a sequence of circular contours with center at the origin that are $(\delta, \tilde{S}_\delta)$ associated with f .*

Proof. In view of Theorem 2.2, it will suffice to construct (δ, S_δ) associated contours. Suppose $\alpha \in S$. Then $f(z)\exp(-\alpha z)$ has conjugate indicator diagram

$P - \alpha$ and is of type $\sigma_\alpha \leq \delta$ since $P - \alpha$ is contained in the closed disk of radius δ with center at the origin.

First, we assert that for each $\varepsilon > 0$, there is an $r_0 = r_0(\varepsilon)$ such that if $|z| > r_0$, then $|f(z)\exp(-\alpha z)| < \exp\{(\delta + \varepsilon)|z|\}$ for all $\alpha \in S_\delta$. Suppose not; then for some $\varepsilon > 0$ and each natural n there are z_n and α_n with $|z_n| > n$ and $\alpha_n \in S_\delta$ for which $|f(z_n)\exp(-\alpha_n z_n)| \geq \exp\{(\delta + \varepsilon)|z_n|\}$. The sequence $\{\alpha_n\}$ has a convergent subsequence, which we suppose to be $\{\alpha_n\}$, converging to some β in S_δ since S_δ is nonempty and compact. Choose N sufficiently large so that if $n > N$, then $|\alpha_n - \beta| < \varepsilon/2$. Combining inequalities, $|f(z_n)\exp(-\beta z_n)| \geq \exp\{(\delta + \varepsilon/2)|z_n|\}$. Since $|z_n|$ may be taken arbitrarily large, this contradicts the fact that $f(z)\exp(-\beta z)$ is of type $\sigma_\beta \leq \delta$. This proves the assertion.

Next, it will be shown that for each $\varepsilon > 0$, there are circles of arbitrarily large radius on which $|f(z)\exp(-\alpha z)| > \exp\{-(\delta + \varepsilon)|z|\}$ for all $\alpha \in S_\delta$. We may assume $f(0) = 1$. Let $\{\zeta_n\}$ be the zeros of f with $|\zeta_n| = \rho_n$ and $\rho_n \leq \rho_{n+1}$ and let $\phi(z) = \prod_{n=1}^\infty (1 + z/\rho_n^2)$. Titchmarsh [10, p. 277] shows that $\limsup_{r \rightarrow \infty} r^{-1} \log |\phi(-r^2)| \geq 0$. Hence there is a sequence of $r_n \rightarrow \infty$, for which $\log |\phi(-r_n^2)| \geq -\varepsilon r_n/2$. Since $|f(z)f(-z)| \geq |\phi(-r^2)|$ where $r = |z|$, $|f(z)f(-z)| \geq \exp(-\varepsilon r_n/2)$ for $|z| = r_n$. By our assertion, there is an $r_0 = r_0(\varepsilon)$ such that if $|z| > r_0$, then $|f(z)\exp(-\alpha z)| < \exp\{(\delta + \varepsilon/2)|z|\}$. These two inequalities show that for $|z| = r_n > r_0$, $|f(z)\exp(-\alpha z)| > \exp\{-(\delta + \varepsilon)r_n\}$.

By choosing Γ_p as a circle on which $|f(z)\exp(-\alpha z)| > \exp\{-(\delta + 1/p)|z|\}$ and such that the radii $r_p \rightarrow \infty$ while $r_p < r_{p+1}$, the lemma is established.

COROLLARY 4.1. *If S is a bounded, nonempty set with convex hull \tilde{S} , then there exists a sequence of circular contours with center at the origin that are $(\sigma(S), \tilde{S})$ associated with f .*

Proof. The theorem says there exists a sequence of $(\sigma(S), \tilde{S}_{\sigma(S)})$ associated contours. These are also $(\sigma(S), S)$ associated since $S \subset \tilde{S}_{\sigma(S)}$. The desired conclusion follows from Theorem 2.2.

COROLLARY 4.2. *If δ is the length of a longest diameter of P , then there exists a sequence of circular contours with center at the origin that are (δ, P) associated with f .*

Proof. This follows from the fact that $P \subset S_\delta$.

These facts on associated contours together with the results of §3 give the following theorems, of which the second is the most general. Each of these theorems could be stated in a slightly stronger way by noting that each solution is expansible in an exponential series relative to f in which the terms involving the $\exp(\zeta_k z)$ appear in an order of non-decreasing $|\zeta_k|$. This has not been done for the sake of simplicity.

THEOREM 4.2. $K \cap A(R + P_0 \oplus \sigma) \subset B(R)$ if f is of exponential type σ .

Proof. This follows from Theorem 3.1 and the remarks at the beginning of this section.

THEOREM 4.3. *If $\delta \geq 0, S \subset \tilde{S}_\delta$, and $S \neq \emptyset$, then $K \cap A(R + P_S \oplus \delta) \subset B(R + S)$.*

Proof. This follows from Theorem 3.2 and Theorem 4.1 since $(\delta, \tilde{S}_\delta)$ association implies (δ, S) association.

THEOREM 4.4. *If S is a bounded, nonempty set, then $K \cap A(R + P_S \oplus \sigma(S)) \subset B(R + S)$.*

Proof. This follows from Theorem 3.2 and Corollary 4.1.

THEOREM 4.5. *If δ is the length of a longest diameter of P , then $K \cap A(R + P \oplus \delta) \subset B(R + P)$.*

Proof. This follows from Theorem 3.2 and Corollary 4.2.

5. Coefficient uniqueness and some consequences.

THEOREM 5.1. *If R is a region containing γ and $\phi \in B(R + P)$, then in any expansion of ϕ in a series relative to f , the sum of the coefficients of $z^h \exp(\zeta_k z)$ is $\mathfrak{F}_{kh\gamma}[\phi] / h!$.*

Proof. Suppose $\phi = \sum_{i=1}^\infty u_i$ where

$$u_i(z) = \sum_{j=1}^{n_i} \left(\sum_{q=0}^{m_j} c_{jq} z^q \right) e^{\zeta_j z} .$$

Then, since $\mathfrak{F}_{kh\gamma}$ is a continuous linear functional on $A(R + P)$, it follows from Theorem 2.1 that $\mathfrak{F}_{kh\gamma}[\phi] = \sum_{i=1}^\infty \mathfrak{F}_{kh\gamma}[u_i] = h! \sum_{i=1}^\infty c_{khi}$.

Application of this theorem together with earlier expansion theorems will now be made for the purpose of characterizing functions that are mean periodic in a region with respect to two functions of exponential type. Generalizations to more than two functions are obvious. In this direction, let f_1 and f_2 be two functions of exponential type. Subscripts will be used in an obvious way on the notations developed for f in the preceding sections. In particular, P_1 and P_2 will denote the conjugate indicator diagrams of f_1 and f_2 in this section.

THEOREM 5.2. *Let sequences of contours Σ_1 and Σ_2 be (τ, P_1) and (τ, P_2) associated with f_1 and f_2 , respectively. Let R be a region with $\gamma_1 \in R + P_2$ and $\gamma_2 \in R + P_1$. Let ϕ in $A(R + P_1 + P_2 \oplus \tau)$ be mean periodic with respect to f_1 and f_2 in $(R + P_1) \cup (R + P_2)$. Then the expansions of ϕ relative to $(f_1, \Sigma_1, \gamma_1)$ and $(f_2, \Sigma_2, \gamma_2)$ in $R + P_1 + P_2$ are identical and contain only terms arising from the common zeros of f_1 and f_2 , multiplicity of zeros taken into account.*

Proof. It follows from Corollary 3.1 that the two series in question converge to ϕ in $A(R + P_1 + P_2)$. Let $f = f_1 f_2$. Then since $P_1 + P_2 \supset P, R + P_1 + P_2 \supset R + P$ and $\phi \in B(R + P)$. By the preceding theorem if $\gamma \in R$, then the coefficient of

$z^h \exp(\zeta_k z)$ in each series must be $\mathfrak{F}_{kh\gamma}[\phi]/h!$. If ζ_k is a zero of f_1 of order greater than h but not a zero of f_2 of order greater than h , then the coefficient of $z^h \exp(\zeta_k z)$ in the series relative to $(f_1, \Sigma_1, \gamma_1)$ must be zero since the coefficient of this term is zero in the expansion relative to $(f_2, \Sigma_2, \gamma_2)$. Similar remarks apply interchanging the roles of f_1 and f_2 .

COROLLARY 5.1. *Let δ be the larger of the lengths of the longest diameters of P_1 and P_2 . Let R be a region. Let ϕ in $A(R + P_1 + P_2 \oplus \delta)$ be mean periodic with respect to f_1 and f_2 in $(R + P_1) \cup (R + P_2)$. Then ϕ is representable in $A(R + P_1 + P_2)$ as an exponential series relative to the g.c.d. of f_1 and f_2 in which the terms containing the $\exp(\zeta_k z)$ appear in an order of nondecreasing $|\zeta_k|$.*

Proof. This follows from the theorem and Corollary 4.2.

In particular, if f_1 and f_2 have only a finite number of common zeros, then any function which is mean periodic with respect to f_1 and f_2 is an exponential polynomial (with polynomial coefficients) in its region of analyticity. In any case, a function of exponential type that is mean periodic with respect to f_1 and f_2 must be an exponential polynomial, the exponent coefficients being the common zeros of f_1 and f_2 in the conjugate indicator diagram of ϕ . In fact, every vertex of the polygonal conjugate indicator diagram of ϕ is a zero of the g.c.d. of f_1 and f_2 .

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