THE BILINEAR RELATION ON OPEN RIEMANN SURFACES(¹)

BY

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1. **Introduction.** The dependence of harmonic differentials on an open Riemann surface of infinite genus on their periods and indeed the properties of their periods near the ideal boundary seem to be quite difficult problems. Yet upon making the transition from the study of compact surfaces to the study of open surfaces these are natural questions to consider.

On a compact surface, a harmonic differential is uniquely determined by its periods, and any pair ω_1, ω_2 of harmonic differentials satisfy the bilinear relation of Riemann [8],

$$(\omega_1, \omega_2^*) = \sum_{i=1}^{p} \int_{A_i} \omega_1 \int_{B_i} \bar{\omega}_2 - \int_{A_i} \bar{\omega}_2 \int_{B_i} \omega_1,$$

where p is the genus of the surface and $\{A_i, B_i\}$ is a canonical homology basis. These facts so easily derived for compact surfaces are natural starting points for excursions into the colder climate of surfaces of infinite genus. It was the bilinear relation that was first found to be amenable to investigation on (parabolic) surfaces of infinite genus (Ahlfors [2]).

Most recently Kusunoki [5] and Accola [1] have made important contributions to the study of the bilinear relation. Their results have led the present author to some rather general sufficient conditions for the bilinear relation. These are presented in Theorems 1-3 of §4. §5 contains a specialized result concerning parabolic surfaces. This paper concludes with several examples illustrating the theorems.

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2. **Preliminary results.** We briefly review the results we will use; for details, see [3]. Suppose W is an open Riemann surface. The inner product and Dirichlet norm defined as

$$(\omega, \sigma) = \iint_W \omega \bar{\sigma}^*, \quad \| \omega \|^2 = (\omega, \omega)$$

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make the class Γ_h of harmonic differentials into a Hilbert space. Here $\sigma^* = bdx - ady$ denotes the conjugate differential of $\sigma = adx + bdy$, $\bar{\sigma}$ is the complex conjugate of σ.

Let $\{\Omega_n\}$ be an exhaustion of W. That is, Ω_n are (relatively) compact subsurfaces of W bounded by analytic curves with $\Omega_n \subset \Omega_{n+1}$ and $\lim \Omega_n = W$. The class $\Gamma_{h0}(\Omega_n)$ is the class of harmonic differentials in Ω_n that vanish along the boundary, $\partial\Omega_n$, of Ω_n. We can define the closed subspace Γ_{h0} of Γ_h by the requirement that $\omega \in \Gamma_{h0}$ if and only if there exist differentials $\omega_n \in \Gamma_{h0}(\Omega_n)$ with $\lim \| \omega - \omega_n \|_{\Omega_n} = 0$. Thus $\omega \in \Gamma_{h0}$ has zero boundary behavior in the sense that $\omega = 0$ along any smooth piece that can be realized on the ideal boundary. We have the orthogonal decomposition $\Gamma_h = \Gamma_{h0} + \Gamma^*_{he}, \Gamma_{h0} \perp \Gamma^*_{he}$, where Γ_{he} is the class of harmonic exact differentials and Γ^*_{he} is the class of differentials conjugate to those in Γ_{he}.

A 1-chain c on a finite polyhedron P is a relative cycle if ∂c lies on the border of P. The relative cycle c will be called weakly homologous to zero, written $c \approx 0$, if and only if c is the boundary, modulo the border of P, of a 2-chain. We can define weak homology on an infinite polyhedron P by the condition that a (possibly infinite) 1-chain c is weakly homologous to zero ($c \approx 0$) if and only if $c \cap P_n$ is a relative cycle weakly homologous to zero in P_n for all n and any exhaustion $\{P_n\}$ of P. With this definition we can form the weak homology group $\dot{H}(K)$ for the polyhedron K associated with the surface W. The weak singular homology group $\dot{H}(W)$ can be defined analogously, and the method of simplicial approximation determines an isomorphism between $\dot{H}(K)$ and $\dot{H}(W)$; we will always identify these two groups. The group $\dot{H}(K)$ contains the homology group $H(K)$. A canonical homology basis of K is a set $\{A_i, B_i\}$ of cycles that generate $H(K)$ modulo the dividing cycles with $A_i \times A_j = B_i \times B_j = 0$, $A_i \times B_j = \delta_{ij}$. (The symbol $\times$ refers to the intersection number. A dividing cycle is a cycle homologous to a cycle lying outside of any finite subcomplex.) It will be clear in the context whether by "cycle" we refer to the whole homology class (or weak homology class), or to a specific member of the class. In the latter case when we have chosen a rectifiable member of the class we will usually refer to the cycle as a "curve."

If γ is a cycle, $\int_\gamma \omega$ is a bounded linear functional on Γ_h (dependent only on the homology class of γ) and hence there is a unique real differential $\sigma(\gamma) \in \Gamma_{h0}$, called the reproducing differential for γ, such that $\int_\gamma \omega = (\omega, \sigma(\gamma)^*)$ for all $\omega \in \Gamma_h$. In particular, if β is another cycle, $(\sigma(\beta), \sigma(\gamma)^*) = \beta \times \gamma$.

Let Γ be a class of rectifiable "curves" on W. Traditionally, when referring to an extremal length problem, this means that each $\gamma \in \Gamma$ is the sum of countably many rectifiable arcs. A linear density ρ on W is a collection of invariant forms $\rho| dz |$ with $\rho \geqq 0$ and ρ lower semi-continuous. If $\gamma \in \Gamma$ and $\{\Omega_n\}$ is an exhaustion of W, $\int_{\gamma \cap \Omega_n} \rho | dz |$ is a nondecreasing function of n with ∞ admitted as a possible value. Therefore $\int_\gamma \rho | dz | = \lim_{n \to \infty} \int_{\gamma \cap \Omega_n} \rho | dz |$ is independent of the exhaustion chosen and is well defined.

The extremal length $\lambda(\Gamma)$ of Γ is defined as

$$\lambda(\Gamma) = \sup_{\rho} \frac{L(\rho,\gamma)^2}{A(\rho)}, \quad L(\rho,\gamma) = \inf_{\gamma} \int_{\gamma} \rho\,|dz|,$$

where $A(\rho) = \iint_W \rho^2 dx\,dy$ and ρ ranges over all linear densities not identically zero. The extremal length $\lambda(\Gamma)$ is a conformal invariant and depends only on the class Γ and not on the region which contains Γ. The following are the properties of extremal length that we will use.

1. The extremal length of the class of curves connecting the opposite sides of length b of a rectangle with sides of length a and b is a/b. The extremal length of the class of curves separating the contours of the annulus $r_1 \leqq |z| \leqq r_2$ is $2\pi[\log r_2/r_1]^{-1}$.

2. If Γ_1, Γ_2 are two classes of curves such that every $\gamma_1 \in \Gamma_1$ contains a $\gamma_2 \in \Gamma_2$, then $\lambda(\Gamma_1) \geqq \lambda(\Gamma_2)$.

3. (Hersch [4].) If $\Gamma = \Gamma_1 \cup \Gamma_2$ in the sense that every curve in Γ_1 and Γ_2 is a curve in Γ and conversely if $\gamma \in \Gamma$, then $\gamma \in \Gamma_1$ and/or $\gamma \in \Gamma_2$ then

$$\frac{1}{\lambda(\Gamma)} \leqq \frac{1}{\lambda(\Gamma_1)} + \frac{1}{\lambda(\Gamma_2)}.$$

4. If $\Gamma \supset \Gamma_1 \cup \Gamma_2$ and Γ_1 and Γ_2 are contained in disjoint open sets,

$$\frac{1}{\lambda(\Gamma)} \geqq \frac{1}{\lambda(\Gamma_1)} + \frac{1}{\lambda(\Gamma_2)}.$$

In particular, if $\Gamma = \Gamma_1 \cup \Gamma_2$ from 3,

$$\frac{1}{\lambda(\Gamma)} = \frac{1}{\lambda(\Gamma_1)} + \frac{1}{\lambda(\Gamma_2)}.$$

5. Suppose $\Gamma = \sum_{i=1}^{n} \Gamma_i$ in the sense that every $\gamma \in \Gamma$ is a sum $\gamma = \sum \gamma_i$ of curves $\gamma_i \in \Gamma_i$ and conversely, every such sum is in Γ. Assume the classes Γ_i are contained in disjoint open sets R_i. Then

$$\lambda(\Gamma) \leqq n \max_i \lambda(\Gamma_i).$$

Proof. Set $a_i^2 = A_{R_i}(\rho)$ and assume $A_R(\rho) = \sum A_{R_i}(\rho) = 1$ where $R = \bigcup R_i$. Then $\inf_{\gamma_i \in \Gamma_i} \int_{\gamma_i} \rho |dz| \leqq \lambda(\Gamma_i)^{1/2} a_i$ and $\inf_{\gamma \in \Gamma} \int_{\gamma} \rho |dz|^2 = (\sum \inf_{\gamma_i \in \Gamma_i} \int_{\gamma_i} \rho |dz|)^2 \leqq (\sum \lambda(\Gamma_i)^{1/2} a_i)^2 \leqq n \max_i \lambda(\Gamma_i)$.

If $\omega \in \Gamma_h$, the notation $|\omega|$ is to mean the linear density $|\omega| = |\operatorname{grad} u|\,|dz|$, where $\omega = du$ in the z-parameter neighborhood.

6. If $\lambda(\Gamma) = 0$, then for any $\omega_1, \omega_2 \in \Gamma_h$ there is a sequence of curves $\gamma_n \in \Gamma$ such that

$$\lim_{n\to\infty} \int_{\gamma_n} |\omega_1| = \lim_{n\to\infty} \int_{\gamma_n} |\omega_2| = 0. \blacksquare$$

Proof. Consider the linear density $\rho = |\omega_1| + |\omega_2|$ with ω_1 and ω_2 normalized so that $A(|\omega_i|) = \|\omega_i\|^2 = 1$, $i = 1,2$. By the Schwarz inequality,

$$A(\rho) = \iint (|\omega_1| + |\omega_2|)^2 \leqq (\|\omega_1\| + \|\omega_2\|)^2 = 4.$$

Because

$$\lambda(\Gamma) \geqq \inf_{\gamma\in\Gamma} \frac{1}{A(\rho)} \int_\gamma \rho |dz|^2 \geqq \frac{1}{4} \inf\left(\int_\gamma |\omega_1| + \int_\gamma |\omega_2|\right)^2,$$

the result follows immediately.

7. Suppose E_1, E_2 each consist of a finite number of arcs or full contours on the boundary of a compact bordered Riemann surface. Let u be the harmonic function which is 0 on E_1, 1 on E_2 and whose normal derivative vanishes on the rest of the boundary. Then the extremal length of the class of curves connecting E_1 and E_2 is $\| du \|^{-2}$ while the extremal length of the class of curves separating E_1 and E_2 is $\| du \|^2$.

3. **The bilinear relation.** Suppose W is an arbitrary open Riemann surface. Given any two differentials $\omega_1, \omega_2 \in \Gamma_{h0}$ the bilinear relation is said to be valid on W if there exists an exhaustion $\{\Omega_n\}$ with

$$(\omega_1, \omega_2^*) = \lim_{n\to\infty} \sum_{i=1}^{N(n)} \int_{A_i} \omega_1 \int_{B_i} \bar{\omega}_2 - \int_{B_i} \omega_1 \int_{A_i} \bar{\omega}_2$$

where $\{A_i, B_i\}_{i=1}^{N(n)}$ is a canonical homology basis for Ω_n. If the exhaustions $\{\Omega_n\}$, $\{\Omega_n'\}$ corresponding to any two pairs of differentials may be taken so that Ω_n and Ω_n' have the same basis $\{A_i, B_i\}_{i=1}^{N(n)}$ for all n, then the sequence $\{A_i, B_i\}_{i=1}^{N(n)}$ is independent of the choice of differentials ω_1, ω_2. In this case the *strong form* of the bilinear relation is said to be valid.

The following lemma forms the basis for our investigations.

LEMMA 1. *If Ω is a compact region in W with canonical basis $\{A_i, B_i\}_{i=1}^N$ and $\omega_1, \omega_2 \in \Gamma_h$ such that ω_1 and ω_2 have no period over any component of $\partial\Omega$, then*

$$(\omega_1, \omega_2^*)_\Omega = \sum_{i=1}^{N} \int_{A_i} \omega_1 \int_{B_i} \bar{\omega}_2 - \int_{B_i} \omega_1 \int_{A_i} \bar{\omega}_2 + \int_{\partial\Omega} u\bar{\omega}_2,$$

$$\left| \int_{\partial\Omega} u\bar{\omega}_2 \right| \leqq \int_{\partial\Omega} |\omega_1| \int_{\partial\Omega} |\omega_2|,$$

$$\left| \int_{\partial\Omega} u\bar{\omega}_2 \right| \leqq \sum_i \left| \int_{\gamma_i} u\bar{\omega}_2 \right| \leqq \sum_i \int_{\gamma_i} |\omega_1| \int_{\gamma_i} |\omega_2|,$$

where γ_i are the components of $\partial\Omega$ and $u(p)$ is a function defined on $\partial\Omega$, on γ_i having the form $u(p) = \int_{p_i}^{p} \omega_i$, p_i fixed on γ_i.

Proof. The proof is an application of Green's theorem to the differential

$$T_N\omega_i = \sum_{i=1}^{N} \left(\int_{B_i} \omega_i\right)\sigma(A_i) - \left(\int_{A_i} \omega_i\right)\sigma(B_i).$$

This lemma shows that the bilinear relation is valid on a bordered surface if, for example, $\omega_2 \in \Gamma_{h0}$ and ω_1 has no periods over any border cycle ($\omega_1 \in \Gamma_{hse}$). However in the general case it is difficult to obtain information on the behavior of $\omega_1 \in \Gamma_{hse}$ near the ideal boundary; therefore we assume $\omega_1 \in \Gamma_{h0} \subset \Gamma_{hse}$.

Accola [1] observed that if we are interested in the strong form of the bilinear relation for a sequence $\{A_i, B_i\}_{i=1}^{N(n)}$, then $(\omega_1, T_N\omega_2^*)$, for a fixed ω_2 and N, may be regarded as a bounded linear functional on Γ_{h0}. Consequently the convergence of the sequence of linear functionals $\{(\omega_1, T_N\omega_2^*)\}$ can be treated by Hilbert space methods.

LEMMA 2 (ACCOLA [1]). *Necessary and sufficient that the strong form of the bilinear relation hold with respect to the sequence* $\{A_i, B_i\}_{i=1}^{N(n)}$ *is that there exists* $M < \infty$ *independent of* N *such that* $|(\omega_1, T_N\omega_2^*)| \leqq M\|\omega_1\| \, \|\omega_2\|$ *for all* ω_1, ω_2 *in* Γ_{h0}.

4. **The sufficient conditions.** Assume that W is an open Riemann surface and let $\bar{W}$ be the compactification of W that associates a point with each ideal boundary component. If γ is a curve in W, γ can be completed in $\bar{W}$; denote the completion by $\bar{\gamma}$. Thus if γ is an arc in W with "end points" on the ideal boundary, $\bar{\gamma}$ is an arc or closed curve in $\bar{W}$.

Let $\mathscr{C}$ be the class of curves γ which satisfy the following conditions (see Figure 1).

1. γ has a finite number of components all of which are rectifiable curves.
2. γ is the relative boundary of a subsurface Ω of finite, positive genus, which contains a point p, fixed in advance.
3. Every component $\bar{c}$ of $\bar{\gamma}$ in $\bar{W}$ is weakly homologous to zero when restricted to W.

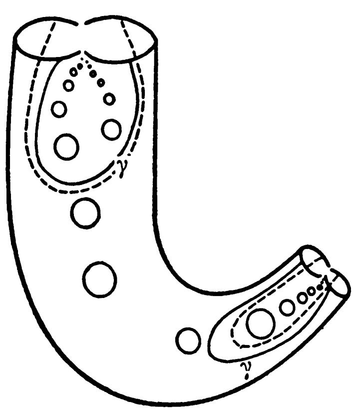

FIGURE 1

4. If $\bar{c}_1$ is any closed curve in $\bar{W}$ which is contained in some $\bar{c}$ of $\bar{\gamma}$, then $\bar{c}_1$ restricted to W is *not* weakly homologous to a sum of cycles in Ω.

THEOREM 1. *If* $\lambda(\mathscr{C}) = 0$, *the bilinear relation is valid.*

Proof. Let ω_1, ω_2 be two differentials in Γ_{h0}. There exists a sequence of curves γ_n in $\mathscr{C}$ such that $\lim \int_{\gamma_n} | \omega_i | = 0$, $i = 1, 2$. Because the curves γ_n cannot shrink to a point and each compact set is of positive $| \omega_i |$ distance from the ideal boundary, it follows that γ_n must tend to the ideal boundary. Denote by Ω_n the possibly, noncompact subsurface determined by γ_n. We may assume all components of $W - \Omega_n$ are of infinite genus.

At most a finite number of the curves γ_n can intersect any given compact set; therefore by condition 2 the subsurfaces Ω_n exhaust W. Hence by choosing a subsequence of γ_n if necessary, we may assume Ω_n has canonical homology basis $\{A_i, B_i\}_{i=1}^{N(n)}$ where $\{A_i, B_i\}$ is a canonical basis of W and $N(n)$ is an increasing function of n.

Fix n and let $\bar{\gamma}_n = \sum_i \bar{\gamma}_{ni}$ be the decomposition of $\bar{\gamma}_n$ into its components $\bar{\gamma}_{ni}$ in $\bar{W}$. Choose a compact subsurface K, with all components of ∂K dividing cycles, so large that the following conditions are satisfied. (A cycle is a dividing cycle if it is homologous to a cycle outside any given compact set.)

(a) $K \cap \Omega_n$ has only one component of positive genus and this has the same canonical homology basis as Ω_n.

(b) The components of ∂K separate the ideal points on $\bar{\gamma}_{ni}$ from those on $\bar{\gamma}_{nj}$, all $i, j, i \neq j$.

(c) Each component of ∂K intersects $\bar{\gamma}_{ni}$ for at most one i. (Hence each component of $W - K$ contains pieces of $\bar{\gamma}_{ni}$ for at most one i.)

Let $\{K_m\}$, $K \subset K_m$, be an exhaustion of W by (relatively) compact subsurfaces with each component of ∂K_m a dividing cycle. We will now show that each component of $\partial(K_m \cap \Omega_n)$ is a dividing cycle for all m, except perhaps for a finite number.

Suppose $\bar{c}_m$ is a component of $\partial(K_m \cap \Omega_n)$ which is not a dividing cycle. Using the notation $c_m = \bar{c}_m \cap K_m$, we see from condition (c) that c_m contains pieces of $\bar{\gamma}_{ni}$ for exactly one i. Furthermore, since $\bar{c}_m$ is a boundary component of $K_m \cap \Omega_n$, $\bar{c}_m$ is homologous to a cycle in Ω_n, necessarily a nondividing cycle. Hence c_m is weakly homologous in K_m to cycles of the canonical homology basis of Ω_n, in fact, to cycles in $K \cap \Omega_n$.

We can regard c_m together with these cycles in $K \cap \Omega_n$ as bounding a connected subregion of $K_m \cap \Omega_n$. But also if $p < m$, $c_m \cap K_p$ and these cycles bound a region (not necessarily connected) in $K_p \cap \Omega_n$. Therefore we can write $c_m \cap K_p = \sum_i c_{pi}$ plus cycles weakly homologous to zero in K_p where $\bar{c}_{pi}$ is a component of $\partial(K_p \cap \Omega_n)$ which is not a dividing cycle. For short, we write $c_m \cap K_p \equiv \sum_i c_{pi}$.

Suppose we can find a c_m for an infinite sequence $\{m\}$. Then there is a subsequence of $\{m\}$ such that the corresponding c_m have the following properties.

(a) c_m is contained in $\bar{\gamma}_{ni}$ for some i and all m.

(b) The decomposition $c_m \cap K \equiv \sum_i c_{0i} = s$ is the same for all m.

Part (b) is possible because there are only a finite number of possible cycles c_{0i} in $K \cap \Omega_n$ $(K_0 = K)$.

The condition (b) implies c_{m+1} contains c_m for all m. For $c_{m+1} \cap K_m$ contains at least one possible c_m, say c_m', and also contains s. On the other hand there is a nondividing component of $\partial(K_m \cap \Omega_n)$ which contains s, namely $\bar{c}_m$, and hence no other nondividing component of $\partial(K_m \cap \Omega_n)$ can contain s or part of s. Consequently $c_m' = c_m$.

Because of (a) and the fact c_m is contained in c_{m+1} for all m, lim c_m exists as a closed curve $\bar{c}$ in $\bar{\gamma}_{ni}$. Furthermore for any m and $j \leqq m$, $c_m \cap K_j$ is weakly homologous in K_j to cycles in $K \cap \Omega_n$ which do not depend on j or m (they are in fact the same cycles s is weakly homologous to in K). Hence $c = \bar{c} \cap W$ is weakly homologous to cycles in $K \cap \Omega_n$. This contradicts our assumptions on the curves $\bar{\gamma}_{in}$ and we conclude that each component of $\partial(K_m \cap \Omega_n)$ is a dividing cycle. (We remark that simple examples show the necessity of condition 4 on $\mathscr{C}$ for the argument just advanced.)

The differentials $\omega_i \in \Gamma_{h0}$, $i = 1,2$, are limits of differentials $\omega_{im} \in \Gamma_{h0}(K_m)$, $\omega_i = \lim \omega_{im}$. It follows from Lemma 1 that

$$\text{(1)}\qquad \begin{aligned} &\left|(\omega_{1m}, \omega_{2m}^*)_{\Omega_n \cap K_m} - \sum_{i=1}^{N(n)} \int_{A_i} \omega_{1m} \int_{B_i} \bar{\omega}_{2m} - \int_{A_i} \bar{\omega}_{2m} \int_{B_i} \omega_{1m}\right| \\ &\qquad \leqq \int_{\gamma_n \cap K_m} |\omega_{1m}| \int_{\gamma_n \cap K_m} |\omega_{2m}|, \end{aligned}$$

since ω_{im} vanishes along ∂K_m and all the components of $\partial(K_m \cap \Omega_n)$ are dividing cycles. Allow $m \to \infty$ to obtain

$$\text{(2)}\qquad \left|(\omega_1, \omega_2^*)_{\Omega_n} - \sum_{i=1}^{N(n)} \int_{A_i} \omega_1 \int_{B_i} \bar{\omega}_2 - \int_{A_i} \bar{\omega}_2 \int_{B_i} \omega_1\right| \leqq \int_{\gamma_n} |\omega_1| \int_{\gamma_n} |\omega_2|.$$

That the bound on the right is valid will be proven below. Assuming (2) then, allow $n \to \infty$ and the truth of the theorem is established.

The only difficulty in proceeding from (1) to (2) is near the "end points" of γ_n. This can be overcome by avoiding the end points by tiny smooth curves t on which $\int_{t \cap K_m} |\omega_{im}| \to \int_t |\omega_i|$ and then allowing t to shrink in such a way that $\int_t |\omega_i| \to 0$. Specifically we can construct an approximation $c_{nk} \in \mathscr{C}$ to γ_n which bounds a subsurface with the same basis as Ω_n and coincides with γ_n except near the end points of γ_n, and for which

$$\text{(3)}\qquad \lim_{m \to \infty} \int_{c_{nk} \cap K_m} |\omega_{im}| = \int_{c_{nk}} |\omega_i|, \quad \lim_{k \to \infty} \int_{c_{nk}} |\omega_i| = \int_{\gamma_n} |\omega_i|.$$

Using relations (3), the above proof shows (1) holds for c_{nk} in place of γ_n; letting $m \to \infty$ and then $k \to \infty$, (2) immediately follows.

To obtain c_{nk} follow the following procedure. For some sufficiently large p, $\Omega_n - \Omega_n \cap K_p$ is the union of rectangular regions R_j with one pair of opposite sides consisting of pieces of γ_n and the third side a piece of ∂K_p. Each region R_j can be mapped on a rectangle (or triangle) R_j' and ω_i is harmonic on and vanishes along the side I corresponding to the ideal boundary, except perhaps at the end points (see [3]). We can modify K_m, $m > p$, so that $\partial K_m \cap R_j'$ is a straight line parallel to I. For all large m, extend ω_{im} to I in R_j' (and beyond) by the reflection principle. Then ω_{im} converges to ω_i on I (except perhaps at the end points). The equalities (3) can now be easily verified.

We will give two variations of Theorem 1 which apply to the strong form of the bilinear relation.

Let $\{A_i, B_i\}$ be a canonical basis of W and let $N(n)$ be an increasing function of n. Define the class $\mathscr{C}_n$ of curves as follows:

$$\mathscr{C}_n = \{\gamma \in \mathscr{C} \mid \text{ if } \gamma = \partial\Omega \text{ then } \Omega \text{ has canonical basis } \{A_i, B_i\}_{i=1}^{N(n)}\}.$$

The following theorem is proven exactly as Theorem 1, making use of Lemma 2.

THEOREM 2. *If $\lambda(\mathscr{C}_n) \leqq M < \infty$ for all m, the strong form of the bilinear relation is valid.*

The question of whether the hypothesis of Theorem 2 implies that of Theorem 1 will be discussed in a forthcoming paper. However, we observe here that if the classes $\mathscr{C}_n$ can be chosen to be contained in mutually disjoint open sets, then by the Hersch inequality, the hypothesis of Theorem 1 will be satisfied.

Suppose $\mathscr{C}_n^0 \subset \mathscr{C}_n$ is a class of curves such that $\mathscr{C}_n^0 = \sum_{i=1}^{m(n)} \mathscr{C}_{ni}^0$ in the sense that every $\gamma_n \in \mathscr{C}_n^0$ can be written $\gamma_n = \sum_{i=1}^{m(n)} \gamma_{ni}$ with $\gamma_{ni} \in \mathscr{C}_{ni}^0$ and $\bar{\gamma}_{ni} \cap W \sim 0$. Assume furthermore that the classes $\mathscr{C}_{ni}^0$ are contained in open regions R_{ni} with $R_{ni} \cap R_{nj} = \varnothing$, $i \neq j$. If $R_n = \sum_{i=1}^{m(n)} R_{ni}$ then for a suitable choice of $\gamma_n \in \mathscr{C}_n^0$ (see §2.6),

$$\sum_{i=1}^{m(n)} \int_{\gamma_{ni}} |\omega_1| \int_{\gamma_{ni}} |\omega_2| \leqq 2 \sum_{i=1}^{m(n)} \lambda(\mathscr{C}_{ni}^0) \|\omega_1\|_{R_{ni}} \|\omega_2\|_{R_{ni}}$$
$$\leqq 2 \max_i \lambda(\mathscr{C}_{ni}^0) \|\omega_1\|_{R_n} \|\omega_2\|_{R_n}$$

by the Schwarz inequality. Again using Lemma 2 the following theorem is immediate.

THEOREM 3. *If $\lambda(\mathscr{C}_{ni}^0) \leqq M < \infty$ for all n and i, the strong form of the bilinear relation is valid.*

If in addition to the above assumptions, $R_{ni} \cap R_{mj} = \varnothing$ for $m \neq n$, then

$$\sum_{n=1}^{\infty} \left[m(n) \max_i \lambda(\mathscr{C}_{ni}^0) \right]^{-1} \leqq \sum_{n=1}^{\infty} \frac{1}{\lambda(\mathscr{C}_n^0)} \leqq \frac{1}{\lambda(\mathscr{C})}$$

so that if $m(n) = O(n)$ it can be asserted that the hypothesis of Theorem 3 implies that of Theorem 1. It does not necessarily imply that of Theorem 2 either (see example 3 of §6).

Theorem 3 includes the sufficient condition of Accola [1].

With the help of an estimate derived in the proof of Theorem 1, a general approximation theorem for the strong form of the bilinear relation can be proved. This theorem may be regarded as a generalization of Theorem 2.

THEOREM 2*. *If* $\lambda(\mathscr{C}_n) \geqq \delta > 0$ *for all* n, *then*

$$\lim_{n\to\infty} \frac{1}{\lambda(\mathscr{C}_n)} \left[(\omega_1, \omega_2^*) - \sum_{i=1}^{N(n)} \int_{A_i} \omega_1 \int_{B_i} \bar{\omega}_2 - \int_{A_i} \bar{\omega}_2 \int_{B_i} \omega_1\right] = 0$$

for all $\omega_1, \omega_2 \in \Gamma_{h0}$.

Proof. Given $\omega_2 \in \Gamma_{h0}$ define the Γ_{h0} differential $T_N\omega_2$ as

$$T_N\omega_2 = \sum_{i=1}^{N(n)} \left(\int_{B_i} \bar{\omega}_2\right)\sigma(A_i) - \left(\int_{A_i} \bar{\omega}_2\right)\sigma(B_i).$$

It follows from what we have shown above ((2) and §2.6) that for any $\omega_1 \in \Gamma_{h0}$,

$$|(\omega_1, \omega_2^*)_{\Omega_n} - (\omega_1, T_N\omega_2^*)| \leqq 2\lambda(\mathscr{C}_n)\|\omega_1\| \, \|\omega_2\|$$

or

$$|(\omega_1, T_N\omega_2^*)| \leqq \|\omega_1\| \, \|\omega_2\|(1 + 2\lambda(\mathscr{C}_n)).$$

Let τ_N denote the projection of $T_N\omega_2^*$ into Γ_{h0}. Then $(\omega_1, T_N\omega_2^*) = (\omega_1, \tau_N)$. Setting $\omega_1 = \tau_N$ we obtain

$$\|\tau_N\| \leqq \|\omega_2\|(1 + 2\lambda(\mathscr{C}_n)). \tag{4}$$

Consider the sequence of differentials $\rho_N = (\omega_2^* - \tau_N)\lambda(\mathscr{C}_n)^{-1}$. Using (4) we find that there exists an $M < \infty$ such that

$$\|\rho_N\| \leqq (\|\omega_2\| + \|\tau_N\|)\lambda(\mathscr{C}_n)^{-1} \leqq M < \infty, \text{ all } n.$$

Hence foı at least some subsequence of $\{n\}$, $\lim_{n\to\infty}\rho_{N(n)} = \rho \in \Gamma_{h0}$. Furthermore, for any cycle C,

$$(\sigma(C), \rho) = \lim_{n\to\infty} (\sigma(C), \omega_2^* - T_N\omega_2^*)\lambda(\mathscr{C}_n)^{-1} = 0.$$

Therefore $\rho \in \Gamma_{h0}^{\perp} = \Gamma_{he}^*$ and $(\omega_1, \rho) = 0$ for any $\omega_1 \in \Gamma_{h0}$. In other words,

$$\lim_{n\to\infty} \lambda(\mathscr{C}_n)^{-1}[(\omega_1, \omega_2^*) - (\omega_1, T_N\omega_2^*)] = \lim_{n\to\infty} (\omega_1, \rho_N) = 0.$$

This holds for some subsequence of $\{n\}$ but actually it is independent of the subsequence chosen. Theorem 2* is now established.

5. **Parabolic surfaces.** Assume W is a parabolic surface and let K be a parametric disk. Consider the class Γ of curves

$\Gamma = \{\gamma | \gamma \subset W - K, \gamma \sim \partial K, \gamma \text{ is the union of a finite number of analytic Jordan curves}\}$.

The extremal length $\lambda(\Gamma) = 0$; in fact this condition is necessary and sufficient that W be parabolic (Ohtsuka [6]).

In spite of the "smallness" of the ideal boundary of W, the only sufficient conditions known for the bilinear relation are those given by Theorems 1–3. (These theorems contain the sufficient conditions given by Kusunoki [5].) In this section we present a result which is valid in some cases in which Theorems 1–3 do not apply. We will show a weaker form of the bilinear relation is valid if W satisfies some additional geometric conditions.

It is no restriction to assume every curve $\gamma \in \Gamma$ is the relative boundary of a compact region Ω. Writing γ as the sum of its components $\gamma = \sum_{i=1}^{m} \gamma_i$, define the harmonic functions $\omega_i, \hat{\omega}_i$ on $\Omega - K$ by the boundary conditions

$$\omega_i = \begin{cases} 1 \text{ on } \gamma_i, \\ 0 \text{ on } \partial K, \end{cases} \qquad \hat{\omega}_i = \begin{cases} 1 \text{ on } \gamma_i, \\ 0 \text{ on } \partial K, \\ 0 \text{ on } \gamma_j, j \neq i. \end{cases}$$

$$\frac{\partial \omega_i}{\partial n} = 0 \text{ on } \gamma_j,\ j \neq i,$$

For $\gamma = \partial\Omega$ in Γ define

$$M_k(\Omega) = \sum_{i=1}^{m} \frac{\| d\hat{\omega}_i \|}{\| d\omega_i \|}, \quad \gamma = \sum_{i=1}^{m} \gamma_i.$$

Note that $M_k(\Omega)$ depends on γ and k, the radius of K. The terms $\| d\hat{\omega}_i \|^2$, $\| d\omega_i \|^2$ are respectively the extremal length of the class of curves separating γ_i from γ_j and $\partial K, j \neq i$, and the extremal length of the class of curves separating γ_i from K. Hence $\| d\hat{\omega}_i \|^2 > \| d\omega_i \|^2$. Therefore, if $\lim \sup_{\partial\Omega \in \Gamma} M_k(\Omega) < \infty$, then m must be uniformly bounded and W can have only a finite number of boundary components. In fact it can be shown that if $\{\gamma' = \partial\Omega'\}$ is the class of curves in $\Gamma \cap \mathscr{C}$, then $\lim_{\Omega' \to W} M_k(\Omega') = n$, the number of boundary components.

THEOREM 4. *Suppose there exists a class $\Gamma_1 \subset \Gamma$ for which $\lambda(\Gamma_1) = 0$ and $\lim \sup_{\partial\Omega \in \Gamma_1} M_k(\Omega) \leqq M_k < \infty$. Then given any $\omega_1, \omega_2 \in \Gamma_h$ there exists an exhaustion $\{\Omega_n\}$ with basis $\{A_i, B_i\}_{i=1}^{N(n)}$ and numbers $a_{in}(k)$, $b_{in}(k)$ such that*

$$(\omega_1, \omega_2^*) = \lim_{k \to 0} \lim_{n \to \infty} \sum_{i=1}^{N(n)} [a_i - a_{in}(k)] \int_{B_i} \bar{\omega}_2 - [b_i - b_{in}(k)] \int_{A_i} \bar{\omega}_2,$$

where $a_i = \int_{A_i} \omega_1$, $b_i = \int_{B_i} \omega_1$.

Proof. Choose a sequence $\gamma_n \in \Gamma_1$ such that $\lim \int_{\gamma_n} |\omega_i| = 0$, $i = 1, 2$. The sequence $\{\gamma_n\}$ can intersect any compact set for only a finite number of n and hence $\{\gamma_n\}$ can be chosen so that $\partial\Omega_n = \gamma_n - \partial K$ with $\{\Omega_n \cup K\}$ an exhaustion of W. Choose a canonical basis $\{A_i, B_i\}$ of W in such a way that $\{A_i, B_i\}_{i=1}^{N(n)}$ is a basis for Ω_n.

Writing γ_n as the sum of its components, $\gamma_n = \sum_{i=1}^{m(n)}\gamma_{in}$ define the harmonic functions ω_{in}, $\hat{\omega}_{in}$ as above. These functions satisfy the relations

$$\| d\omega_{in} \|^2 = \int_{\gamma_{in}} d\omega_{in}^* = \int_{\partial K} d\omega_{in}^*,$$

$$(\omega, d\hat{\omega}_{in}^*) = \int_{\gamma_i} \omega, \text{ any } \omega \in \Gamma_h.$$

Consequently the differential $\tau_n \in \Gamma_h(\Omega_n)$, defined as

$$\tau_n = \left[\omega_1 - \sum_{i=1}^{N(n)} a_i\sigma_n(B_i) - b_i\sigma_n(A_i)\right] - \left[\sum_{j=1}^{m} c_{jn} d\omega_{jn}^* - \sum_{i=1}^{N(n)} a_{in}\sigma_n(B_i) - b_{in}\sigma_n(A_i)\right],$$

where σ_n is the reproducing differential in Ω_n and c_{jn}, a_{in}, b_{in} are suitably defined, is exact in Ω_n. Furthermore, we find

$$| c_{in} | \leqq \frac{\| \omega_1 \| \| d\hat{\omega}_{in} \|}{\| d\omega_{in} \|^2}$$

and

$$| a_{in} | \leqq \sum_{j=1}^{m} \frac{\| \omega_1 \| \| d\hat{\omega}_{jn} \|}{\| d\omega_{jn} \|^2} \| \sigma_n(A_i) \| \| d\omega_{jn} \| \leqq M_k(\Omega) \| \sigma_n(A_i) \| \| \omega_1 \|.$$

A similar expression holds for $| b_{in} |$. Here the norms are taken over the relevant subsurfaces of W.

Now if the functions $u_i(p)$, $0 \leqq i \leqq m(n)$, are defined on $\gamma_{in} (1 \leqq i \leqq m(n))$ and ∂K $(i = 0)$ by $u_i(p) = \int_{p_i}^{p} \omega_1 - \sum_{j=1}^{m} c_{jn} d\omega_{jn}^*$, p_i fixed on γ_{in}, p_0 fixed on ∂K, then

$$(\tau_n, \omega_2^*) = \sum_{i=1}^{m} \int_{\gamma_{in}} u_i\bar{\omega}_2 - \int_{\partial K} u_0\bar{\omega}_2$$

and we find

$$| (\tau_n, \omega_2^*) | \leqq 2 \int_{\gamma_n} | \omega_1 | \int_{\gamma_n} | \omega_2 | + \int_{\partial K} | \omega_1 | \int_{\partial K} | \omega_2 | + \int_{\partial K} | \omega_2 | \int_{\gamma_n} | \omega_1 |.$$

On the other hand since W is parabolic, $\Gamma_h = \Gamma_{h0} = \Gamma_{h0}^*$ and ω_2 can be approximated $\omega_2 = \lim \omega_{2n}$ by differentials $\omega_{2n} \in \Gamma_{h0}^*(\Omega_n \cup K)$. In fact $\omega_2 = \omega_{2n} + dv_n$ in $\Omega_n \cup K$ where $dv_n \in \Gamma_{he}(\Omega_n \cup K)$ and hence $\int_{A_i}\omega_{2n} = \int_{A_i}\omega_2$, $\int_{B_i}\omega_{2n} = \int_{B_i}\omega_2$ for $i = N(n)$. We find

$$| (\tau_n, dv_n^*) | \leqq (M_k(\Omega) + 1) \| \omega_1 \| \| dv_n \|.$$

Putting our estimates together in the right-hand side of the inequality

$$|(\tau_n, \omega_{2n}^*)| \leqq |(\tau_n, \omega_2^*)| + |(\tau_n, dv_n^*)|,$$

we easily obtain the desired result.

This theorem has interest because we can give an example of a surface which satisfies the conditions of Theorem 4 but not of Theorems 1–3. The example is in the next section.

6. **Examples.** We will illustrate each of our sufficient conditions by an example.

1. Consider the half disk $|z| < 1$, $y > 0$, with a sequence of slits s_i, symmetric about the y-axis, approaching an interval $(-a, a)$ on the base. Let $A(R)$ be the class of curves $\{\gamma_r\}$ where γ_r is the pair of semi-circles $|z \pm a| = r$ with $R < r < 1$ and r small enough so that these semi-circles do not intersect. Then $\lim_{R\to 0}\lambda(A(R)) = 0$.

Form a two sheeted surface by taking a replica of this slit region and either (1) identify the corresponding edges of the slits, or (2) cross identify across the slits and identify the sheets along $|z| = 1$. With method (2), each curve γ_r of $A(R)$ together with its partner on the second sheet form an admissible curve of $\mathscr{C}$. With method (1), we must use curves γ_r' contained in $\{\gamma_r\}$ which connect the base to a slit s_i without intersecting any other slit. In either case we see $\lambda(\mathscr{C}) = 0$. (If $a = 0$, simple modifications must be made in this argument. Without the identification along $|z| = 1$ in (2), or if this is done in (1), the hypothesis of Theorem 1 may not be satisfied.)

2. To obtain a surface for which Theorem 2 is applicable, we consider the half disk as above with horizontal slits s_i tending towards an interval (a,b) on the base. If z_i, z_i', are the endpoints of s_i, we assume that there is an angle $\alpha > 0$ such that $\alpha < \arg(z_i - a) < \pi - \alpha$, $\alpha < \arg(z_i' - b) < \pi - \alpha$ for all i. Furthermore, if w_i is the distance between s_{i-1} and s_i and d_i is the distance from s_i to the base, we assume $w_i/d_i \geqq (\sin\alpha)^{-1}$ for all i. A two sheeted surface may be constructed by either method (1) or (2) above. The class $\mathscr{C}_n$ can be taken to include circular arcs lying in four suitable half-annuli (or two, if $b = a$) with outer radii $d_n + w_n$ and inner radii $d_n(\sin\alpha)^{-1}$. It follows that $\lambda(\mathscr{C}_n)$ is bounded for all n.

These surfaces also satisfy the hypothesis of Theorem 1.

3. Consider a series of half-annuli in $|z| < 1$, $y > 0$, the radii of the inner contours being one half that of the outer contours. The first generation consists of one half-annulus, outer radius 1. The second generation is two nonintersecting half-annuli, outer radii $1/4$, lying between the first half-annulus and $y = 0$. The ith generation is 2^{i-1} half-annuli lying in the regions bounded by the half-annuli of the $(i-1)$th generation and $y = 0$. The ith generation of slits s_i is any finite number of arcs in the region bounded by the annuli of the $(i-1)$th generation and the ith generation. Construct a two sheeted surface with respect to the slits by method (2) above. It is clear that the hypothesis of Theorem 3 is satisfied.

4. The final example is of a small class of parabolic surfaces which do not satisfy the conditions of Theorems 1–3 but on which Theorem 4 is applicable. Consider the region R shown in Figure 2 which is obtained from a rectangle by cutting out an infinite number of smaller rectangles R_i in such a way that $\lim_{i\to\infty} a_i / b_i = \lim_{i\to\infty} a'_i / b_i = 0$.

A Riemann surface W is constructed from R and a duplicate of R by identifying all the corresponding edges except the sides I and β. The edges β are to be regarded as the boundary of a parametric disk on W. It follows from §5 that W is parabolic.

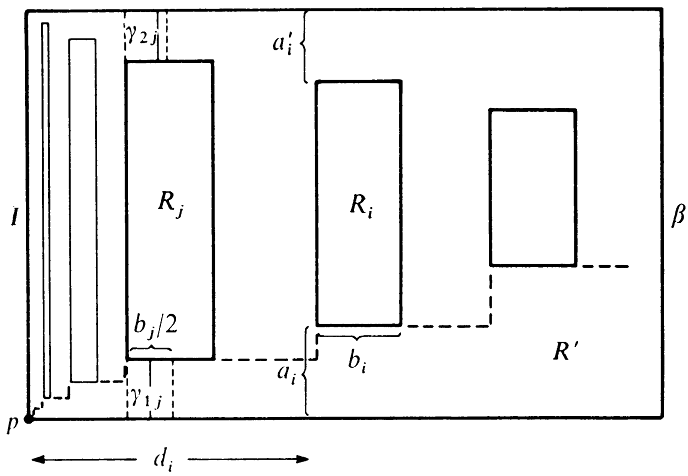

FIGURE 2

Assume furthermore that the numbers b_i are chosen in such a way that $\sum_{i=1}^{\infty} b_i / d_i = \infty$. Then W cannot satisfy the hypothesis of Theorem 1. To prove this it is enough to consider the classes C_1 and C_2 of arcs γ in R,

$$C_1 = \{\gamma \mid \gamma \text{ separates some } R_i \text{ from all } R_{i+j},\ 1 \leqq m \leqq j < \infty, \text{ for some } m\},$$

$$C_2 = \{\gamma \mid \gamma \text{ connects } R_i \text{ with } P \text{ for some } i\}.$$

Obviously $\lambda(C_1) \geqq \min\ l_i / \max\ d_i > 0$ where l_i is the height of R_i. But also $\lambda(C_2) = \infty$ as the following reasoning shows. Consider the subregion R' of R constructed within the dotted lines as shown in Figure 2 and the linear density $\rho = 1/|z|$ in R' where P is the point $r = 0$. Since $a_i / b_i \to 0$, $A(\rho) < \infty$. On the other hand if γ is any arc in C_2,

$$\int_{\gamma \cap R'} \rho |dz| \geqq \sum_{j=i}^{\infty} \log \left(1 + \frac{b_j}{d_j}\right) = \infty, \text{ for some } i.$$

It follows that $\lambda(C_2 \cap R') = \infty$ and hence $\lambda(C_2) = \infty$. Therefore $\lambda(C_1 \cup C_2) > 0$ by Hersch's inequality. With a little more argument, we easily see that indeed W does not satisfy the hypothesis of Theorem 2 or 3 either.

Finally it will be shown that for a suitable choice of a'_i, W satisfies the hypothesis of Theorem 4. The class of admissible exhaustions $\{\Omega_i\}$ of W are those for which $\Omega_i \cap R$ is bounded by vertical straight lines $\gamma_{1i}, \gamma_{2i}, \beta$ of which γ_{1i}, γ_{2i} lie in the two rectangles, height a_i, a_i', width $b_i/2$, bounded by the dotted lines shown in Figure 2. Using the notation of §5, a_i' is to be chosen so small that $\| d\hat{\omega}_{2i} \| / \| d\hat{\omega}_{1i} \| \leqq 1/2$ first for all admissible regions Ω_i for fixed i, and then for every i. This is possible because $\| d\hat{\omega}_{2i} \| \to 0$ as $a_i \to 0$, whereas $\| d\hat{\omega}_{1i} \| \geqq k > 0$ for all small a_i'.

The ratio $(\| d\hat{\omega}_1 \| + \| d\hat{\omega}_2 \|)/\| d\omega_p \|$, $p = 1,2$, will now be shown to be uniformly bounded for all admissible regions Ω. The class of curves C which separate γ_1 from $\beta(\partial\Omega = \gamma_1 + \gamma_2 - \beta)$ can be written as the union $C = C_1 \cup C_2$ of the class C_1 of curves which separate both γ_1 and γ_2 from β and the class C_2 consisting of those curves of C which meet γ_2. According to Hersch's inequality

$$\frac{1}{\| d\omega_1 \|^2} \leqq \frac{1}{\| d\hat{\omega}_1 + d\hat{\omega}_2 \|^2} + \frac{1}{\lambda(C_2)}$$

since $\hat{\omega}_1 + \hat{\omega}_2$ is the harmonic function which is 0 on β and 1 on $\gamma_1 + \gamma_2$.

By assumption then,

$$\frac{\| d\hat{\omega}_1 \| + \| d\hat{\omega}_2 \|}{\| d\omega_1 \|} \leqq \frac{\| d\hat{\omega}_1 \| + \| d\hat{\omega}_2 \|}{\| d\hat{\omega}_1 \| - \| d\hat{\omega}_2 \|} + \frac{\| d\hat{\omega}_1 \| + \| d\hat{\omega}_2 \|}{\lambda(C_2)^{1/2}} \leqq 3 + \frac{3\| d\hat{\omega}_1 \|}{2\lambda(C_2)^{1/2}}.$$

In addition $\lambda(C_2) \geqq 2b_i/2a_i$. Since $\| d\hat{\omega}_1 \| \to 0$ as $\Omega \to W$, the above ratio is indeed uniformly bounded. Similarly $(\| d\hat{\omega}_1 \| + \| d\hat{\omega}_2 \|)/\| d\omega_2 \|$ is uniformly bounded and therefore W fulfills the requirements of Theorem 4.

7. **Conclusion.** We first note that a *medium form* of the bilinear relation, stronger than the bilinear relation defined in §3, but weaker than the strong form, is usually valid when the hypothesis of Theorem 1 is satisfied (see, for instance, example 1).

The general situation in which our method is applicable is essentially the following. Suppose D is a subclass of the class of curves satisfying conditions 1,2,3 of §4. Let $\Gamma \subset \Gamma_{h0}$ be the class of differentials which have no period over any component of $\partial\Omega$ (assuming $\partial\Omega$ can be realized) for all $(\partial\Omega) \cap W = \gamma$ contained in D. Then under suitable assumptions on $\lambda(D)$ the bilinear relation is valid for all pairs of differentials in Γ. Theorems 1-3 are, we feel, the most important realizations of this idea, namely for $\Gamma = \Gamma_{h0}$.

By obtaining additional knowledge of the differentials in Γ_{h0} it is possible to relax the topological restrictions on $\mathscr{C}$. For example, Pfluger [7] has shown the bilinear relation valid on a small class of symmetric transcendental hyperelliptic (parabolic) surfaces which are not all included under Theorem 1. In this case the additional knowledge comes from the symmetry of the surface.

Kobori and Sainouchi [9] have recently used a somewhat different metric condition than extremal length to give sufficient conditions for the bilinear relation similar to those in [1] and [5].

It remains to learn the limitations of the methods used here. That is, how close are our theorems to necessary conditions? No surface is known on which the bilinear relation does not hold. No necessary conditions, given in terms of the geometry of the surface, are known for the bilinear relation.

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