ON DOMINANT DIMENSIONS OF QF-3 ALGEBRAS

BY

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Introduction. The complete cohomology theory of groups was extended by Nakayama [8] to the case of Frobenius algebras. He [10] also proposed to classify algebras in accordance with the length of a right augmented, two-sided, projective, injective resolution. One of the purposes of this paper is to give a partial extension of the above complete cohomology theory to the case of non-quasi-Frobenius algebras and another is to give an estimation of the length.

Let B be an associative algebra over a commutative field with unit element 1 and M a unital right B -module. If

$$(1) \quad 0 \rightarrow M \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_n$$

is a minimal injective resolution⁽¹⁾ of M , then we shall say that M has dominant dimension $\geq n$, provided that X_k is projective for all $k \leq n$, where it should be noted even the zero module is considered to be projective. The largest such integer⁽²⁾ n is called in this paper the dominant dimension of M and denoted by $\text{domi. dim } {}_B M$. When no such integer exists the dimension is defined to be ∞ . The modules we are chiefly concerned with are B_B which is B considered as a right module over itself and B_{B^e} which is B considered as a right module over $B^e = B \otimes B^0$. In fact $\text{domi. dim } {}_{B^e} B$ is the length considered by Nakayama. In both cases B is a QF-3 algebra in the sense of Thrall [12] if and only if $\text{domi. dim } B \geq 1$ [11]. So our interest is confined to the class of QF-3 algebras.

On the other hand, for QF-3 algebras there exists a result established by Morita [6]. The result is stated as follows: For a given QF-3 algebra B let X be the unique minimal faithful right B -module. Denote by A the B -endomorphism algebra of X as a left operator domain. Then X is fully faithful as a left A -module and the A -endomorphism algebra B' of X is QF-3. Here a left A -module L is said to be fully faithful if every indecomposable projective left A -module as well as every indecomposable injective left A -module is A -isomorphic to a direct summand of L . The above settings about QF-3 algebras are fundamental in this paper and the meanings of notations A , B and B' will be retained throughout.

In §1 the correlation between $\text{domi. dim } {}_B M$ and $\text{domi. dim } {}_{B'} M \otimes_B B'$ is

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(1) The injective resolution (1) is said to be minimal if $\text{Im } \delta_k \supseteq$ the socle of X_{k+1} (cf. Eilenberg [2]).

(2) If X_1 is not projective, we shall say the dominant dimension of M is zero.

discussed and as an application we obtain that if $\text{domi. dim } {}_{B^e} B > 1$, then $B = B'$. The upper bound of dominant dimension of modules is investigated in §2. Especially $\text{domi. dim } {}_{B^e} B \leq \dim A + 1$ holds, provided $0 < \dim A < \infty$.

In [10] Nakayama has conjectured that B is quasi-Frobenius if $\text{domi. dim } {}_{B^e} B = \infty$ and proved it for the case of B being a generalized uni-serial algebra. The author is, however, likely inclined to conjecture that B is quasi-Frobenius if $\text{domi. dim } {}_B B = \infty$. In §3 we shall prove that our conjecture holds for the case of A being generalized uni-serial. Since $\text{domi. dim } {}_{B^e} B \leq \text{domi. dim } {}_B B$, this assures Nakayama's conjecture is true for this case.

In the beginning of §4 we shall introduce as a generalization of Nakayama's automorphism of Frobenius algebra a ring-isomorphism between two subrings of B . Then following Kasch's argument in [4] we have the isomorphisms between (negative dimensional) cohomology groups over B^e -modules and (positive dimensional) homology groups over A^e -modules under a restriction of dimensions. Here it is noted that the restriction is determined by dominant dimension of a B^e -module.

1. Reduction theorems. Let B be a QF-3 algebra and eB the faithful, projective, injective right ideal of B , i.e., e is a suitable idempotent of B and eB is a dominant ideal in the sense of Thrall [12]. Let A be the B -endomorphism algebra of eB considered as a left operator domain of eB . Then $A = eBe$ and from Lemma 17.4 of [6] we know eB is a fully faithful left A -module (for definition cf. Introduction). Denote by B' the A -endomorphism algebra of eB as a right operator domain. Then by Theorem 17.2 of [6], B' is a QF-3 algebra and there exists an idempotent f of B' such that $eB = fB'$, and fB' is a faithful, projective, injective right B' -module. Let Be' be the dual representation module (i.e., $= \text{Hom}_K(eB, K)$). Then by duality it is clear that there exists such an idempotent f' of B' that $B'f' = Be'$. Thus B is obtained as a subalgebra of B' which contains 1 and $B'f'$ and fB' .

We may assume throughout that B is self-basic, that is to say, the basic algebra of B is isomorphic to B itself, for dominant dimensions and homological dimensions are invariant under any category-isomorphism. Now we shall prove

PROPOSITION 1.1. *Let B and B' be QF-3 algebras just introduced above and M a right B -module. Let*

$$(2) \quad 0 \longrightarrow M \xrightarrow{\delta_0} X_1 \xrightarrow{\delta_1} X_2 \xrightarrow{\delta_2} \cdots \longrightarrow X_n$$

be an exact sequence of right B -homomorphisms δ_i , $0 \leq i \leq n$, where X_i is a direct sum of n_i -copies of eB . Then we have an exact sequence

$$(3) \quad M \otimes_B B' \xrightarrow{\Delta_0} Y_1 \xrightarrow{\Delta_1} Y_2 \xrightarrow{\Delta_2} \cdots \longrightarrow Y_n$$

of right B' -homomorphisms Δ_i , where Y_i is a direct sum of n_i -copies of fB' .

Proof. We shall prove that the induced sequence of right B' -homomorphisms

$$(3') \quad M \otimes_B B' \xrightarrow{\Delta_0} X_1 \otimes_B B' \xrightarrow{\Delta_1} X_2 \otimes_B B' \xrightarrow{\Delta_2} \cdots \longrightarrow X_n \otimes_B B'$$

is exact. Put $\delta_i \otimes B' = \Delta_i$ and denote by 1 the unit element of B .

Consider right B -homomorphisms θ_i , $i \geq 0: X_i \rightarrow X_i \otimes_B B'$ defined by $\theta_i(x_i) = x_i \otimes 1 \in X_i \otimes_B B'$ for all $x_i \in X_i$, $i \geq 1$ and $\theta_0(x_0) = x_0 \otimes 1 \in M \otimes_B B'$ for all $x_0 \in M$. Then clearly every θ_i is a monomorphism. We shall express X_i , $i \geq 1$, as a direct sum of n_i -copies of eB : $X_i = \bigoplus_{j=1}^{n_i} u_j^i eB$, $u_j^i eB \approx eB$. Then we have $x_i \otimes b' = \sum_{j=1}^{n_i} u_j^i e b_j \otimes b' = \sum_{j=1}^{n_i} u_j^i e \otimes e b_j b'$, where $b' \in B'$, $b \in B$, $j = 1, 2, \dots, n$ and $x_i \otimes b' \in X_i \otimes_B B'$. Since $eB = eB'$, $x_i \otimes b' = \sum_{j=1}^{n_i} u_j^i e b_j b' \otimes 1$ and it follows that θ_i , $i \geq 1$, is an epimorphism and consequently an isomorphism. Thus we obtain the following commutative diagram:

$$(4) \quad \begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{\delta_0} & X_1 & \xrightarrow{\delta_1} & X_2 & \xrightarrow{\delta_2} & \cdots & \longrightarrow & X_n \\ & & \downarrow \theta_0 & & \downarrow \theta_1 & & \downarrow \theta_2 & & & & \downarrow \theta_n \\ & & M \otimes_B B' & \xrightarrow{\Delta_0} & X_1 \otimes_B B' & \xrightarrow{\Delta_1} & X_2 \otimes_B B' & \xrightarrow{\Delta_2} & \cdots & \longrightarrow & X_n \otimes_B B' \end{array}$$

where the upper row is exact. Since θ_i , $i > 0$, is an isomorphism, the exactness of (3') follows from (4).

COROLLARY 1.2. *In Proposition 1.1 if it is assumed that $n > 1$, then M is a B -module which is obtained from a right B' -module by restricting its operator domain B' to B .*

Proof. From (4) we have $M \approx \text{Ker } \delta_1 \approx \text{Ker } \Delta_1$. But since Δ_1 is considered to be a B' -homomorphism, the conclusion follows.

LEMMA 1.3. *In Proposition 1.1 assume*

$$0 \longrightarrow M \otimes_B B' \xrightarrow{\Delta_0} X_1 \otimes_B B'$$

is exact; then M is B -isomorphic to $M \otimes_B B'$.

Proof. By Proposition 1.1 we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{\delta_0} & X_1 & \xrightarrow{\delta_1} & X_2 & \longrightarrow & \cdots \\ & & \downarrow \theta_0 & & \downarrow \theta_1 & & \downarrow \theta_2 & & \\ 0 & \longrightarrow & M \otimes_B B' & \xrightarrow{\Delta_0} & X_1 \otimes_B B' & \xrightarrow{\Delta_1} & X_2 \otimes_B B' & \longrightarrow & \cdots \end{array}$$

As θ_0 is a monomorphism, $\Delta_0 \theta_0$ is a monomorphism. However, $\text{Im } \Delta_0 \theta_0 = \text{Im } \theta_1 \delta_0 = \text{Ker } \Delta_1 = \text{Im } \Delta_0$; hence θ_0 is an isomorphism.

THEOREM 1.4. *Let B and B' be QF-3 algebras as in Proposition 1.1. Assume $\text{domi. dim}_B B > 1$; then $B = B'$.*

Proof. From assumption the following exact sequence is given:

$$0 \rightarrow B \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_n, \quad n \geq 2,$$

where X_i is a direct sum of n_i -copies of eB . Put $\Delta_i = \delta_i \otimes B'$, $i \geq 0$. Then since $B \neq B'$ implies $B \approx B \otimes B'$, by Lemma 1.3 we have only to prove

$$(5) \quad 0 \longrightarrow B \otimes_B B' \xrightarrow{\Delta_0} X \otimes_B B'$$

is exact. Let $\Delta_0(1 \otimes 1) = \sum_i u_i$, where u_i is an element of an indecomposable direct summand of $X_1 \otimes_B B'$. Then $\Delta_0(1 \otimes b') = \sum_i u_i b'$. Suppose $\Delta_0(1 \otimes b') = 0$ for some $b' \neq 0$, $b' \in B'$; then $u_i b' = 0$, hence $u_i b' B' = 0$ for all i . There exists clearly a minimal subideal r of $b' B'$ such that

$$(6) \quad u_i r = 0 \quad \text{for all } i.$$

Since B is self-basic, B' is also self-basic and QF-3. Hence $r \subseteq B'f'$. It follows that $r \subseteq B$, for $B'f' = Be' \subseteq B$. Then (6) contradicts to that δ_0 is a monomorphism. Thus (5) must be exact.

COROLLARY 1.5. *Let B and B' be QF-3 algebras as in Proposition 1.1. If B is properly contained in B' , then*

$$\text{domi. dim}_B B' \geq \text{domi. dim}_B B = 1.$$

REMARK. The inequality of the above relation really holds. Let B , B' and B'' be subalgebras of full matrix ring K_{14} over a commutative field K such that the elements in the following table form K -basis respectively:

$$\begin{aligned} B: e_1 &= c_{11} + c_{22} + c_{33}, & e_2 &= c_{44} + c_{55}, & e_3 &= c_{66} + c_{77}, \\ e_4 &= c_{88} + c_{99} + c_{10,10}, & e_5 &= c_{11,11} + c_{12,12}, \\ e_6 &= c_{13,13} + c_{14,14}, & c_{11,1} &+ c_{12,3}, & c_{13,1} &+ c_{14,2}, & c_{51}, & c_{71}, & c_{10,1}, & c_{94} \\ & & & & & + c_{10,5}, c_{3,4}, & c_{12,4}, & c_{8,6} & + c_{10,7}, & c_{26}, & c_{14,6}, & c_{12,9}, c_{14,8}, & c_{10,11}, & c_{10,13}. \end{aligned}$$

B'' : annexing elements $c_{7,11}$, $c_{5,11}$, $c_{7,13}$, $c_{5,13}$ to the table of B ,

B' : annexing elements c_{39} , c_{28} to the table of B'' . Here $c_{i,j}$ denotes the matrix with 1 in the (i,j) -position and zero elsewhere. Then $\text{domi. dim}_B B = \text{domi. dim}_{B'} B' = 1$ and $\text{domi. dim}_B B' = 3$.

REMARK. The dominant dimension of a QF-3 algebra which is isomorphic to an endomorphism algebra of a fully faithful module is not necessarily larger than 1 (cf. the example of Remark of Theorem 1.9).

Consider the enveloping algebra $B^e = B \otimes_K B^0$ of B , where B^0 is the opposite algebra of B . Then B may be regarded to be a right B^e -module by the following

definition: $a \cdot b \cdot c = b(c \otimes_K a^0)$, for $a, b, c \in B$ and $a^0 \in B^0$. It was shown in [11] that $\text{domi. dim}_{B^*} B \geq 1$ if and only if B is QF-3. In this case B^e is also QF-3 and its unique minimal faithful right B^e -ideal is isomorphic to $eB \otimes_K e'^0 B^0$.

Similarly as Theorem 1.4 we have

THEOREM 1.6. *If $\text{domi. dim}_{B^*} B > 1$, then $B = B'$.*

Proof. Let $0 \rightarrow B \rightarrow {}^{\gamma_0} Y_1$ be an exact sequence, where Y_1 is isomorphic to a direct sum of n -copies of $eB \otimes e'^0 B^0$. Then by Lemma 1.3 we have only to prove that $0 \rightarrow B \otimes_{B^*} (B')^e \rightarrow {}^{\Delta_0} Y \otimes_{B^*} (B')^e$ is also exact, because if it is proved, $B \approx B \otimes_{B^*} (B')^e$ and consequently $B = B'$. Since any element of $B \otimes_{B^*} (B')^e$ is expressed by $1 \otimes \beta$ for a suitable element β of $(B')^e$, we may take $1 \otimes \beta$ as an element of $\text{Ker } \Delta_0$. Let D be a simple subideal of $\beta \cdot (B')^e$. Then $1 \otimes D \subseteq \text{Ker } \Delta_0$. We shall show $D(e' \otimes e^0) = D$, where $e' \otimes e^0 \in (B')^e = B' \otimes_K (B')^0$. Any simple ideal of B' is monomorphic to $eB' \otimes_K (B'e')^0 = eB \otimes_K (B_e')^0$ and hence monomorphic to $e \cdot 1(N') \otimes_K e'^0 \cdot l(N'^0)$, where N' and $(N')^0$ are the radicals B' and $(B')^0$, and $1(N')$ and $l((N')^0)$ are the left annihilators of N' and $(N')^0$ in B' and $(B')^0$ respectively. Denoting $\overline{B'} = B'/N'$, $\overline{(B')^0} = (B')^0/(N')^0$, we have $e \cdot 1(N') \otimes e'^0 \cdot l((N')^0) \approx \bar{e}' \overline{B'} \otimes_K \bar{e}^0 \overline{(B')^0}$. On the other hand the bottom Loewy constituent of $\bar{e}' \overline{B'} \otimes_K \bar{e}^0 \overline{(B')^0}$ is isomorphic to the top Loewy constituent of $\bar{e}' \overline{B'} \otimes_K \bar{e}^0 \overline{(B')^0}$, since $\overline{B'} \otimes_K \overline{(B')^0}$ is almost symmetric. Thus we obtain $D(e' \otimes e^0) = D$. Now let γ be an element of D and express $\gamma = \sum_v b_v \otimes c_v^0 \in B' \otimes_K (B')^0$, $b_v \in B'$ and $c_v^0 \in (B')^0$. Then $\gamma = \gamma(e' \otimes e^0) = \sum_v b_v e' \otimes c_v^0 e^0$. Here we notice $b_v e' \in Be'$ and $c_v^0 e^0 \in B^0 e^0$ because $B'e' = Be'$ and $(B')^0 e^0 = B^0 e^0$. Then it follows that

$$\begin{aligned} \Delta_0(1 \otimes \gamma) &= \Delta_0 \left(\left(\sum_v b_v e' \otimes c_v^0 e^0 \right) \otimes (1 \otimes 1^0) \right) \\ &= \delta_0 \left(\sum_v b_v e' \otimes c_v^0 e^0 \right) \otimes (1 \otimes 1^0) \neq 0, \end{aligned}$$

where $1 \otimes 1^0$ is the unit of $(B')^e$. But this contradicts $1 \otimes \gamma \in \text{Ker } \Delta_0$. This completes the proof.

COROLLARY 1.7. *Let B and B' be QF-3 algebras as in Proposition 1.1. If B is a proper subalgebra of B' , then $\text{domi. dim}_{B^*} B = 1$.*

REMARK. Consider the algebra A being fixed. Then so far as we estimate the upper bound of dominant dimensions of B 's over QF-algebras B^e connected with A , we have only to consider such QF-3 algebras B' that B' consists of all A -endomorphisms of fully faithful A -modules.

Let A be an algebra, M_0 a fully faithful left A -module and M a left A -module. Denote by B the left A -endomorphism algebra of $M_0 \oplus M$ considered as a right operator domain and similarly denote by B_0 the left A -endomorphism algebra of M_0 as a right operator domain. Let N be a right B -module and

$$(7) \quad 0 \longrightarrow N \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_n$$

an exact sequence with X_k , $1 \leq k \leq n$, which is a projective, injective right B -module. Without loss of generality we may assume that X_k is isomorphic to a direct sum of n_k -copies of eB , where $M_0 \oplus M \approx eB$ and e is an idempotent of B . The projection $f: M_0 \oplus M \rightarrow M_0$ may be considered as an idempotent of B and it holds $B_0 = fBf$ and $M_0 = eBf$. Then multiplying (7) with f on the right hand we obtain

$$(8) \quad 0 \longrightarrow Nf \longrightarrow X_1f \longrightarrow X_2f \longrightarrow \cdots \longrightarrow X_nf,$$

where X_kf is a direct sum of n_k -copies of eBf .

Now we shall divide two cases.

Case 1. We shall take a right B -module B as N ; then in place of (7) and (8) we obtain

$$(7') \quad 0 \longrightarrow B \xrightarrow{\delta_0} X_1 \xrightarrow{\delta_1} X_2 \xrightarrow{\delta_2} \cdots \xrightarrow{\delta_{n-1}} X_n,$$

and

$$(8') \quad 0 \longrightarrow Bf \longrightarrow X_1f \longrightarrow X_2f \longrightarrow \cdots \longrightarrow X_nf.$$

eBf is right fBf -projective, injective and $Bf \approx fBf \oplus (1-f)Bf$. Hence we have

$$0 \longrightarrow B_0 \longrightarrow X'_1 \longrightarrow X'_2 \longrightarrow \cdots \longrightarrow X'_n,$$

where X_1 is the injective hull of $\delta_0(B_0)$ and X'_k is the injective hull of $\delta_{k-1}(X'_{k-1})$. Here we notice that if $X'_j = 0$ for some j , then B_0 is projective, injective B_0 -module for X_j is B_0 -projective, and this implies B_0 is quasi-Frobenius.

Case 2. In this case we shall take B^e -module B as N ; then we have to take in place of the ring B , B_0 , projective, injective right B -modules X_i and the idempotent f , $B^e = B \otimes_K B^0$, $B_0^e = fBf \otimes_K (fBf)^0$, projective, injective right B^e -modules Y_i and an idempotent $(f \otimes_K f^0)$ of B^e . Then in place of (8) we obtain an exact sequence

$$(8'') \quad 0 \longrightarrow B(f \otimes f^0) \longrightarrow Y_1(f \otimes f^0) \longrightarrow \cdots \longrightarrow Y_n(f \otimes f^0)$$

of B^e -homomorphisms, where $B(f \otimes f^0) = fBf = B_0$. Here we notice that if (8'') is split, B_0 is B_0^e -projective (injective) and B_0 is separable. Thus we have proved

THEOREM 1.8. *Let A be an algebra, M_0 a fully faithful left A -module and M a left A -module. Denote by B the left A -endomorphism algebra of $M_0 \oplus M$ considered as a right operator domain and similarly denote by B_0 the left A -endomorphism algebra of M_0 as a right operator domain. If B_0 is not quasi-Frobenius, then*

$$\text{domi. dim } {}_B B \leq \text{domi. dim } {}_{B_0} B_0.$$

THEOREM 1.9. *Let B and B_0 be QF-3 algebras as in Theorem 1.8. If B_0 is not separable, then $\text{domi. dim } {}_B B \leq \text{domi. dim } {}_{B_0} B_0$.*

REMARK. The inequality of Theorem 1.8 really holds for the following example. Let A be a subalgebra of the full matrix K_4 over a commutative field K such that the elements

$$e_1 = c_{11}, \quad e_2 = c_{22} + c_{33} + c_{44}, \quad c_{21}, \quad c_{31}$$

form a K -basis. Take $M_0 \approx Ae_1 \oplus Ae_2 \oplus (e_2A)^* \oplus (e_1A)^*$ and $M \approx Ae_1/Ac_{21}$ in Theorem 1.8. Then we have $\text{domi. dim } {}_{B_0} B_0 = 2$ and $\text{domi. dim } {}_B B = 1$.

A fully faithful left A -module M_0 is said to be minimal if each indecomposable summand of M_0 is either projective or injective and not isomorphic to others.

Let M_0 be a minimal fully faithful left A -module. Then the endomorphism algebra of M_0 is quasi-Frobenius if and only if A is a quasi-Frobenius algebra. Thus, to estimate the upper bound of dominant dimensions of modules B_B and B_{B^e} for a fixed algebra A we must divide the class of QF-3 algebras which are not quasi-Frobenius into two subclasses according as whether

- (i) A is a quasi-Frobenius algebra, or
- (ii) A is not a quasi-Frobenius algebra,

and by Theorems 1.8 and 1.9 we have only to consider QF-3 algebras B in each subclass such that

- (i) B is the endomorphism algebra of $M_0 \oplus M$, where M_0 is a minimal faithful left A -module and M is an indecomposable, not projective, left A -module,
- (ii) B is the endomorphism algebra of M_0 , where M_0 is a minimal fully faithful left A -module.

2. Dominant dimension and injective dimension. Let M be a right B -module. Suppose the dominant dimension of M is infinite, that is to say, we have a minimal injective resolution of M : $0 \rightarrow M \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_n \rightarrow \cdots$, where every X_k is a projective right B -module. If the injective dimension of M is finite, i.e., $X_{n-1} = 0$ for some n , then we obtain $X_{n-1} \approx \text{Im } \delta_{n-1} \oplus \text{Ker } \delta_{n-1}$ and obtain successively $X_{k-1} \approx \text{Im } \delta_{k-1} \oplus \text{Ker } \delta_{k-1}$ for X_k is projective, and consequently M is projective and injective. Thus we have

PROPOSITION 2.1. *Let M be a right B -module. Assume $\text{domi. dim } {}_B M = \infty$ and $\text{inj. dim } {}_B M < \infty$. Then M is projective and injective.*

COROLLARY 2.2. *If M is not a projective injective right B -module, then from condition $\text{inj. dim } {}_B M < \infty$ it follows that $\text{domi. dim } {}_B M < \infty$.*

Since self injective algebras are quasi-Frobenius, we obtain

COROLLARY 2.3. *Let B be a QF-3 algebra which is not quasi-Frobenius. If $\text{inj. dim } {}_B B < \infty$, then $\text{domi. dim } {}_B B < \infty$.*

REMARK. As an example of a QF-3 algebra which is not quasi-Frobenius and whose injective dimension $= \infty$ we can show a subalgebra B of the full matrix ring K_{10} over a commutative field K such that elements

$$\begin{aligned} e_1 &= c_{11} + c_{22} + c_{33}, & e_2 &= c_{44} + c_{55} + c_{66}, & e_3 &= c_{77} + c_{88} + c_{99}, \\ e_4 &= c_{10,10}, c_{61}, c_{81} + c_{92}, c_{10,1}, c_{24} + c_{35}, c_{94}, c_{27}, c_{57} + c_{68}, c_{6,10} \end{aligned}$$

form a K -basis.

COROLLARY 2.4. *If a QF-3 algebra B is not separable, then the condition $0 < \text{inj. dim}_{B^e} B < \infty$ implies $\text{domi. dim}_{B^e} B < \infty$.*

Let B be a QF-3 algebra and eB a minimal faithful right ideal, where e is an idempotent of B . There exists an idempotent e' of B such that $Be' \approx \text{Hom}_K(eB, K)$. Then $\text{Hom}_K(eBe, K)$ is left eBe -isomorphic to eBe' . Since eBe is a projective, faithful right eBe -module, eBe' is an injective, faithful left eBe -module and similarly eBe' is an injective, faithful right $e'Be'$ -module. It is well known that by the contravariant functor $\text{Hom}(_, eBe')$ the duality holds between the category of finitely generated left eBe -modules and one of finitely generated right $e'Be'$ -modules.

Now let β be a right B -homomorphism: $eB \rightarrow eB$. Then multiplying with the idempotent e' on the right-hand side we have right $e'Be'$ -homomorphism $\alpha: eBe' \rightarrow eBe'$. Conversely for a right $e'Be'$ -homomorphism $\gamma: eBe' \rightarrow eBe'$ we have a right B -homomorphism $\text{Hom}(Be', \gamma): \text{Hom}_{e'Be'}(Be', eBe') \rightarrow \text{Hom}_{e'Be'}(Be', eBe')$ as follows: $(\text{Hom}(Be', \gamma)(\phi))(x) = \gamma(\phi(x))$ for $\phi \in \text{Hom}(Be', eBe')$, $x \in Be'$.

Further it holds $\text{Hom}_{e'Be'}(Be', eBe') \approx \text{Hom}_{e'Be'}(Be', \text{Hom}_K(e'Be', K)) \approx \text{Hom}_K(Be' \otimes_{e'Be'} e'Be', K) \approx \text{Hom}_K(Be', K) \approx eB$.

Now we shall prove

PROPOSITION 2.5. *Let α and β be just introduced homomorphisms. Then there exists an isomorphism $\sigma: \text{Hom}_{e'Be'}(Be', eBe') \rightarrow eB$ such that the following diagram is commutative:*

$$\begin{array}{ccc} \text{Hom}_{e'Be'}(Be', eBe') & \xrightarrow{\text{Hom}(Be', \alpha)} & \text{Hom}_{e'Be'}(Be', eBe') \\ \downarrow \sigma & & \downarrow \sigma \\ eB & \xrightarrow{\beta} & eB. \end{array}$$

Proof. To begin with we identify eB with $\text{Hom}_{e'Be'}(Be', eBe')$. Then by the assumption the following diagram is commutative:

$$\begin{array}{ccc} \theta \in \text{Hom}_{e'Be'}(Be', eBe') & \longrightarrow & \beta(\theta) \in \text{Hom}_{e'Be'}(Be', eBe') \\ \rho \downarrow & & \rho \downarrow \\ \theta \circ e' \in \text{Hom}_{e'Be'}(e'Be', eBe') & \longrightarrow & \beta(\theta \circ e') \in \text{Hom}_{e'Be'}(e'Be', eBe'), \end{array}$$

where

$$\begin{aligned}(\rho\theta)(e'xe') &= (\theta \circ e')(xe') \\ &= \theta(e'xe') \text{ for } xe' \in Be' \text{ and } \theta \in \text{Hom}(Be', eBe').\end{aligned}$$

In this case $\text{Hom}(Be', eBe')$ is defined to be a left eBe -module by $(b * \theta)(xe') = b(\theta(xe'))$, for $b \in eBe$ and $xe' \in Be'$ and $\text{Hom}(Be', eBe')$ is fully faithful as a left eBe -module. Hence the right B -endomorphism algebra of $\text{Hom}(Be', eBe')$ is eBe and for a right B -homomorphism β there exists an element b_β of eBe such that $\beta(\theta) = b_\beta * \theta$. Thus we have

$$\begin{aligned}(\beta(\theta) \circ e')(xe') &= \beta(\theta \circ e')(xe') \\ &= \beta(\theta(e'xe')) \\ &= (b_\beta * \theta)(e'xe') \\ &= b_\beta(\theta(e'xe')).\end{aligned}$$

On the other hand, by a correspondence $\tau : \theta \circ e' \leftrightarrow (\theta \circ e')(e')$, $\text{Hom}_{e'Be'}(e'Be', eBe') \approx eBe'$. Hence β induces $e'Be'$ -homomorphism $\alpha' : eBe' \rightarrow eBe'$ such that $\alpha'((\theta \circ e')(e')) = b_\beta(\theta \circ e')(e')$.

Then it is sufficient to show that α' induces β .

Let $\eta \in \text{Hom}_{e'Be'}(Be', eBe')$; then $\eta(xe') \in eBe'$. There exists $\theta \in \text{Hom}_{e'Be'}(Be', eBe')$ such that $\eta(xe') = (\theta \circ e')(e')$, because we have only to put $\theta \circ e' = \tau^{-1}(\eta(xe'))$. Then we have

$$\begin{aligned}(\text{Hom}(Be', \alpha')(\eta))(xe') &= \alpha'\eta(xe') \\ &= \alpha'((\theta \circ e')(e')) \\ &= b_\beta((\theta \circ e')(e')) \\ &= b_\beta(\eta(xe')) \\ &= (b_\beta * \eta)(xe') \\ &= \beta(\eta)(xe'), \text{ for all } xe' \in Be'.\end{aligned}$$

This completes the proof.

COROLLARY 2.6. *Let B be a QF-3 algebra. Let M_i , $i = 1, 2$, be a submodule of X_i respectively and X_i free right B -modules. Let ϕ be a right B -homomorphism: $M_1 \rightarrow M_2$ and ψ a right $e'Be'$ -homomorphism: $M_1e' \rightarrow M_2e'$ defined by $\psi(xe') = \phi(x) \cdot e'$ for $x \in M_1$. Then there exists a natural isomorphism σ such that the following diagram is commutative:*

$$\begin{array}{ccc} M_1 & \xrightarrow{\phi} & M_2 \\ \sigma \downarrow & & \downarrow \sigma \\ \text{Hom}_{e'Be'}(e', M_1e') & \xrightarrow{\text{Hom}(Be', \psi)} & \text{Hom}_{e'Be'}(Be', M_2e'). \end{array}$$

Proof. Since B is a QF-3 algebra, for a suitable integer n , X_i can be embedded in a direct sum X of n -copies of eB and we have a right B -homomorphism $\Phi: X \rightarrow X$ such that the following diagram is commutative:

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ M_1 & \xrightarrow{\phi} & M_2 \\ \downarrow & & \downarrow \\ X_1 & \xrightarrow{\Phi} & X_2. \end{array}$$

Then from Proposition 2.5 the conclusion follows.

Let

$$(9) \quad 0 \longrightarrow M \xrightarrow{\beta_0} X_1 \xrightarrow{\beta_1} X_2 \xrightarrow{\beta_2} \cdots \longrightarrow X_n$$

be an exact sequence of right B -homomorphisms β_p , where X_p , $1 \leq p \leq n$, are projective, injective right B -modules. By multiplying X_p with e' on the right-hand side we have an exact sequence of right $e'Be'$ -homomorphisms:

$$(10) \quad 0 \longrightarrow Me' \xrightarrow{\alpha_0} X_1e' \xrightarrow{\alpha_1} X_2e' \xrightarrow{\alpha_2} \cdots \longrightarrow X_ne',$$

with X_pe' being injective right $e'Be'$ -modules. Then from Proposition 2.5 and Corollary 2.6 there exist natural isomorphisms σ_i , $i = 0, 1, 2, \dots, n$:

$$(11) \quad \begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{\beta_0} & X_1 & \xrightarrow{\beta_1} & \cdots \longrightarrow X_n \\ \sigma_0 \downarrow & & \downarrow & & \downarrow & & \downarrow \sigma_n \\ 0 \rightarrow \text{Hom}(Be', Me') & \xrightarrow{\text{Hom}(Be', \alpha_0)} & \text{Hom}(Be', X_1e') & \xrightarrow{\text{Hom}(Be', \alpha_1)} & \cdots \rightarrow & \text{Hom}(Be', X_ne'). \end{array}$$

If (9) is minimal in the sense that $\text{Im } \beta_i \supseteq$ the socle of X_{i+1} , then (10) is minimal in the sense of that $\text{Im } \alpha_i \supseteq$ the socle of $X_{i+1}e'$. We have now proved

PROPOSITION 2.7. *Let B be a QF-3 algebra with Be' and eB as unique minimal faithful left and right B -modules respectively. Let M be a right B -module. Then from the assumption $0 < \text{inj. r. dim}_{e'Be'} Me' < \infty$, it follows that*

$$\text{domi. dim}_B M \leq \text{inj. r. dim}_{e'Be'} Me' + 1.$$

THEOREM 2.8. *Let B be a QF-3 algebra as in Proposition 2.7. Then*

$$\text{domi. dim}_B B \leq \text{inj. r. dim}_{e'Be'} Be' + 1 = \text{proj. l. dim}_{eBe} eB + 1,$$

provided $0 < \text{inj. r. dim}_{e'Be'} Be' < \infty$.

COROLLARY 2.9. *Let A be an algebra which is not quasi-Frobenius. If the projective dimension of a fully faithful injective left A -module is finite, then the dominant dimension of QF-3 algebra B which is connected with A is also finite.*

THEOREM 2.10. *Let B be a QF-3 algebra as in Proposition 2.7. From the assumption $0 < \dim eBe$ ($= \text{proj. dim}_{eBe} eBe$) $< \infty$ it follows that $\text{domi. dim}_{B^e} B \leq \dim eBe + 1$.*

Proof. Let us consider again B to be a right B^e -module. Then in place of e' , $e'Be'$ we can take $e' \otimes_K e^0$ and $e'Be' \otimes_K e^0 B^0 e^0$. Hence from the assumption $0 < \text{inj. r. dim}_{e'Be' \otimes_{e^0 B^0 e^0} eBe'} < \infty$ it follows that $\text{domi. dim}_{B^e} B < \text{inj. r. dim}_{e'Be' \otimes_{e^0 B^0 e^0} eBe'} + 1$. On the other hand, eBe' is left eBe -isomorphic to $\text{Hom}_K(eBe, K)$ and right $e'Be'$ -isomorphic to $\text{Hom}_K(e'Be', K)$. Hence $\text{Hom}_K(eBe', K)$ is right eBe -isomorphic to eBe and is left $e'Be'$ -isomorphic to $e'Be'$. Thus $\text{inj. r. dim}_{e'Be' \otimes_{e^0 B^0 e^0} eBe'} = \text{proj. l. dim}_{eBe \otimes_{e^0 B^0 e^0} eBe} = \dim eBe$. And we have proved $\text{domi. dim}_{B^e} B \leq \dim eBe + 1$.

3. In case A is generalized uni-serial. Throughout this section we assume A is a generalized uni-serial algebra and B is an endomorphism algebra of a fully faithful A -module. We shall prove that if B is not quasi-Frobenius, the dominant dimension of B , considered as a right B -module, is finite. Since $\text{domi. dim}_{B^e} B \leq \text{domi. dim}_B B$, this implies Nakayama's conjecture holds for the same situation.

Let B be a QF-3 algebra and eB and Be' unique minimal faithful right ideal and unique minimal faithful left ideal, respectively.

Let

$$(12) \quad \cdots \longrightarrow X_{i-1} \xrightarrow{\delta_{i-1}} X_i \xrightarrow{\delta_i} X_{i+1} \xrightarrow{\delta_{i+1}} \cdots$$

be an exact sequence of right $e'Be'$ -homomorphisms with X_i being isomorphic to a direct sum of n_i -copies of eBe' . As was shown in §2 we have a sequence of right B -homomorphisms

$$(13) \quad \begin{array}{ccccccc} \Delta_{i-1} = \text{Hom}(Be', \delta_{i-1}) & & \Delta_i = \text{Hom}(Be', \delta_i) & & & & \\ \cdots \rightarrow \text{Hom}(Be', X_{i-1}) & \longrightarrow & \text{Hom}(Be', X_i) & \longrightarrow & \text{Hom}(Be', X_{i+1}) & \rightarrow & \cdots \end{array}$$

This sequence, however, is not always exact. For the sake of (13) being exact what condition does (12) satisfy?

Let ϕ be an element of $\text{Ker } \Delta_i$, i.e., $\delta_i \phi$ is the zero mapping. On the other hand, let ψ be any element of $\text{Im } \Delta_{i-1}$. Then there exists a homomorphism $\Psi: Be' \rightarrow X_{i-1}$ such that $\psi = \delta_{i-1} \Psi$. Since it holds that $\text{Im } \delta_{i-1} = \text{Ker } \delta_i$ and $\text{Im } \Delta_{i-1} \subseteq \text{Ker } \Delta_i$, we have

PROPOSITION 3.1. *In order that $\text{Ker } \Delta_i = \text{Im } \Delta_{i-1}$, it is necessary that δ_{i-1} satisfies the following condition: For a homomorphism $\phi: Be' \rightarrow \text{Im } \delta_{i-1}$, there always exists a homomorphism $\psi: Be' \rightarrow X_{i-1}$ such that the following diagram is commutative:*

$$(14) \quad \begin{array}{ccc} & & X_{i-1} \\ & \nearrow \psi & \downarrow \delta_{i-1} \\ Be' & \xrightarrow{\phi} & \text{Im } \delta_{i-1}. \end{array}$$

REMARK. Since $Be' \approx (1 - e')Be' \oplus e'Be'$ and $e'Be'$ is $e'Be'$ -projective, (14) can be replaced by the following diagram:

$$\begin{array}{ccc} & & X_{i-1} \\ & \nearrow \psi & \downarrow \delta_{i-1} \\ (1 - e')Be' & \xrightarrow{\phi} & \text{Im } \delta_{i-1}. \end{array}$$

Hereafter we assume $eBe (= A)$ is a generalized uni-serial algebra. And we assume the following sequence of right B -homomorphisms:

$$0 \longrightarrow B \xrightarrow{\Delta_0} Z_1 \xrightarrow{\Delta_1} \cdots \longrightarrow Z_n \xrightarrow{\Delta_n} \cdots$$

is infinite and exact with Z_i being isomorphic to n_i -copies of eB . As is shown in §2, we have an exact sequence of $e'Be'$ -homomorphisms

$$0 \longrightarrow Be' \xrightarrow{\delta_0} Z_1e' \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_n} Z_ne' \longrightarrow \cdots.$$

Then from Proposition 2.5 there exists a natural equivalence σ :

$$(15) \quad \begin{array}{ccccccc} 0 & \longrightarrow & B & \xrightarrow{\Delta_0} & Z_1 & \xrightarrow{\Delta_1} & \cdots \longrightarrow Z_n \xrightarrow{\Delta_n} \cdots \\ & & \sigma \downarrow & & \sigma \downarrow & & \sigma \downarrow \\ & & \text{Hom}(Be', \delta_0) & & \text{Hom}(Be', \delta_1) & & \text{Hom}(Be', \delta_n) \\ 0 & \longrightarrow & \text{Hom}(Be', Be') & \longrightarrow & \text{Hom}(Be', Z_1e') & \longrightarrow & \cdots \longrightarrow \text{Hom}(Be', Z_ne') \longrightarrow \cdots. \end{array}$$

Let us impose further an assumption that B is not quasi-Frobenius. Then there exists at least an indecomposable summand S of Be' which is not injective as a right $e'Be'$ -module and for S we have an exact sequence of homomorphisms

$$(16) \quad 0 \longrightarrow S \xrightarrow{\tau_0} Y_1e' \xrightarrow{\tau_1} \cdots \longrightarrow Y_ne' \xrightarrow{\tau_n} \cdots,$$

where τ_i are minimal and Y_ie' is nonzero, indecomposable summand of Z_ie' , for S is not injective and $e'Be'$ is generalized uni-serial. By (15),

$$(17) \quad 0 \longrightarrow \text{Hom}(Be', S) \longrightarrow \text{Hom}(Be', Y_1e') \longrightarrow \cdots$$

must be exact. But this is a contradiction.

To show the reason why this is a contradiction, we shall divide our discussion.

(i) $e'Be'$ is quasi-Frobenius.

In this case $e' = e$, $(1 - e')Be' = (1 - e)Be$ and from Theorem 1.8 we may assume that $(1 - e)Be$ is indecomposable and $S = (1 - e)Be$. Since the sequence (16) is infinite, there exist two integers i and $i + k$ such that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & Y_i e & \longrightarrow & Y_{i+k} e & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \text{Ker } \tau_i & \longrightarrow & \text{Ker } \tau_{i+k} & \longrightarrow & 0 \\ & & \uparrow & & & & \\ & & 0 & & 0 & & \end{array}$$

is commutative, where all row and column sequences of homomorphisms are exact.

Since all $Y_1 e, Y_2 e, \dots, Y_{i+k} e$ are projective, $S \approx \text{Im } \tau_{i+p}$ for some p , $1 \leq p \leq k$, because (16) is minimal. Then for the following diagram

$$\begin{array}{ccccccc} & & & & Y_{i+p} e & & \\ & & & & \downarrow \tau_{i+p} & & \\ 0 & \longrightarrow & (1 - e)Be & \xrightarrow{\phi} & \text{Im } \tau_{i+p} \approx S & \approx & (1 - e)Be \longrightarrow 0, \end{array}$$

there is no homomorphism $\psi: (1 - e)Be \rightarrow Y_{i+p} e$ such that $\tau_{i+p} \psi = \phi$, because otherwise $(\phi^{-1} \delta_{i+p}) \psi$ is an isomorphism: $(1 - e)Be \rightarrow (1 - e)Be$ and $(1 - e)Be$ is isomorphic to a direct summand of $Y_{i+p} e$. Thus S is injective. But this contradicts the selection of S . It follows from Proposition 3.1 that (17) is not exact. We arrive at the above stated contradiction.

(ii) $e'Be'$ is not quasi-Frobenius.

In this case we may assume by Theorem 1.8 that every indecomposable injective right $e'Be'$ -module which is not projective is isomorphic to a direct summand of $(1 - e')Be'$. We can select S among them.

(a) If we have in (16) the following commutative diagrams for all j , $1 \leq j \leq i$,

$$\begin{array}{ccccccc} 0 & \longrightarrow & Y_j e' & \longrightarrow & Y_{j+k} e' & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \text{Ker } \tau_j & \longrightarrow & \text{Ker } \tau_{j+k} & \longrightarrow & 0 \end{array}$$

in which rows are exact and column mappings are injections, then as in (i) we have a diagram

$$\begin{array}{ccccccc} & & & & Y_{i+p} e' & & \\ & & & & \downarrow \tau_{i+p} & & \\ 0 & \longrightarrow & S & \xrightarrow{\phi} & \text{Im } \tau_{i+p} & \longrightarrow & 0, \end{array}$$

in which there exists no homomorphism $\psi : S \rightarrow Y_{i+p}e'$ such that $\tau_{i+p}\psi = \phi$, for otherwise $S \approx Y_{i+p}e'$ and $\text{Im } \tau_{i+p-1} = 0$, which contradicts (11). So we arrive again at the above contradiction.

(b) Now, the case remains where we have the next diagram:

$$\begin{array}{ccccc}
 & & Y_{i+p}e' & & \\
 & & \downarrow \tau_{i+p} & & \\
 Y_je' & \xrightarrow{\tau} & \text{Im } \tau_{i+p} \approx \text{Im } \tau_j & \longrightarrow & 0 \\
 & & \downarrow & & \\
 & & 0 & &
 \end{array}$$

where Y_je' , $1 \leq j < i$, is injective but not projective and not isomorphic to Y_{i+p} . Hence Y_je' is considered as a direct summand of $(1 - e')Be'$. Since $e'Be'$ is generalized uni-serial, we may assume there exists an epimorphism $\theta : Y_{i+p}e' \rightarrow Y_je'$ such that $\tau_j\theta = \tau_{i+p}$. Then we know that there exists no homomorphism $\psi : Y_je' \rightarrow Y_{i+p}$ such that $\tau_{i+p}\psi = \tau_j$, for otherwise $\tau_j\theta\psi(x) = \tau_j(x)$ holds for all $x \in Y_je'$; then $Y_je' = \theta\psi(Y_je') + \text{Ker } \tau_j$. But Y_je' is uni-serial, and hence $\theta\psi$ must be an isomorphism. It follows that Y_je' is isomorphic to a direct summand of $Y_{i+p}e'$, and consequently $Y_je' \approx Y_{i+p}e'$. But this is a contradiction. By Proposition 3.1, (17) is not exact. Thus in this case too we arrive at the first pointed contradiction.

THEOREM 3.2. *Let A be a generalized uni-serial algebra and B a QF-3 algebra connected with A . If B is not quasi-Frobenius, then the dominant dimension of B is finite.*

REMARK. The class of QF-3 algebras, each of them obtained as the endomorphism algebra of a fully faithful module over a generalized uni-serial algebra, contains properly the class of generalized uni-serial algebras. In fact, the example at the remark of Theorem 1.8. is an endomorphism algebra of a fully faithful module over a generalized uni-serial algebra, but not generalized uni-serial.

4. Isomorphisms between cohomology groups and homology groups. To begin with we shall intend to generalize the notion "Nakayama's automorphism of Frobenius algebra." Let B be a QF-3 algebra with eB with the unique minimal faithful right B -ideal. Then the unique minimal faithful left B -ideal Be' is B -isomorphic to $\text{Hom}_K(eB, K)$ and by an identification of Be' with $\text{Hom}_K(eB, K)$, eB and Be' form an orthogonal pair (inner product) with respect to K : for any element $x \in eB$ and any element $y \in Be'$ there corresponds an element (x, y) of K , and $(xb, y) = (x, by)$ holds for any element $b \in B$.

Then for an element ρ of eBe , $(\rho x, y) = (\rho, xy)$ holds, where $xy \in eBe'$. In this pairing it is immediate that eBe and eBe' also form an orthogonal pair, and

since K is contained in the center of B this makes it possible for us to identify every element ρ of eBe with an element of $\text{Hom}_K(eBe', K)$, that is to say, for any elements x, y there exists an element ρ' of $e'Be'$ such that

$$(\rho, xy) = (xy, \rho').$$

Since in the right pairing eBe' and $e'Be'$ form an orthogonal pair, the correspondence: $\rho \rightarrow \rho'$ is unique and gives ring-isomorphism $\sigma: eBe \rightarrow e'Be'$. When B is a Frobenius algebra, σ is Nakayama's automorphism. It is known that σ is determined uniquely up to inner automorphisms of $e'Be'$.

Let $l_1, l_2, \dots, l_k$ be a left K -basis of eB and $r_1, r_2, \dots, r_k$ its dual right K -basis of Be' ; that is to say, in the above pairing $(l_i, r_k)_k = \delta_{ik}$ holds. Then, if $\rho l_i = \sum_{j=1}^k \lambda_{ij} l_j$ for $\rho \in eBe$, we have $r_j \rho' = \sum_{i=1}^k r_i \lambda_{ij}$.

Next we shall define the notion "Spur" in the sense of Kasch. Let M be a left B -module and f an element of $\text{Hom}_K(M, K)$. The Spur f is defined to be the following mapping: $M \rightarrow Be'$ such that

$$\text{Spur } f: M \ni x \rightarrow \sum_{i=1}^k r_i f(l_i x) \in Be'.$$

It is easy to show that $\text{Spur } f \in \text{Hom}_B(M, Be')$, and

$$\text{Spur } (f\rho)(x) = (\text{Spur } f)(x)\rho' \text{ for } \rho \in eBe \text{ and } \rho' \in e'Be'.$$

Now, similarly as Kasch we can prove several propositions for preparation.

PROPOSITION 4.1. *Let epe be an element of $\text{Hom}_K(Be, K)$ defined by $epe(x) = (epe, x)$ for $x \in Be'$. Then the correspondence:*

$$epe \rightarrow \text{Spur } epe \in \text{Hom}_B(Be', Be')$$

gives the ring-isomorphism σ .

We shall omit the proof (cf. Kasch [4, Hilfssatz 1]).

Let X be a direct sum of m -copies of Be' : $X = \bigoplus_{j=1}^m Be'x_j$ and $\sum_{j=1}^m \gamma_j x_j$, $\gamma_j \in Be'$ be an element of X . Then we have

PROPOSITION 4.2. *Let X^* be a direct sum of m -copies of eBe : $X^* = \bigoplus_{j=1}^m \varepsilon_j eBe$, $\varepsilon_j eBe \approx eBe$. And by $\sum_{j=1}^m \varepsilon_j \rho_j$, $\rho_j \in eBe$, we shall denote an element of $\text{Hom}_K(X, K)$ defined by*

$$\left(\sum_k \varepsilon_k \rho_k \right) \left(\sum_j \gamma_j x_j \right) = \sum_k (\rho_k, \gamma_k).$$

Then the mapping $X^ \rightarrow \text{Hom}_B(X, Be')$ defined by*

$$X^* \ni f \rightarrow \text{Spur } f \in \text{Hom}_B(X, Be')$$

gives a semi-linear isomorphism connected with σ .

be a commutative diagram, where the row is exact with projective, injective left B -modules for $l \leq m, k+1 \leq n$, and σ and τ are epimorphism and monomorphism respectively. Without loss of generality $X_i, -n < i \leq m$, may be assumed to be isomorphic to a direct sum of n_i -copies of Be' . Multiplying X_i with e on the left-hand side we obtain an exact sequence

$$(19) \quad \begin{array}{ccccccccccc} \rightarrow eX_l & \rightarrow & \cdots & \rightarrow & eX_1 & \rightarrow & eX_0 & \rightarrow & eX_{-1} & \rightarrow & eX_{-2} & \rightarrow & \cdots & \rightarrow & eX_{-(k+1)} & \rightarrow \\ & & & & & & \searrow & & \nearrow & & & & & & & \\ & & & & & & eM & & & & & & & & & \\ & & & & \nearrow & & & & \searrow & & & & & & & \\ & & & & 0 & & & & & & 0 & & & & & \end{array}$$

of eBe -homomorphisms with eX_i injective eBe -modules. Taking the dual of M and $X_i, i = -n, \dots, m$, we obtain an exact sequence

$$(20) \quad \begin{array}{ccccccccccc} \rightarrow (eX)_{-(k+1)}^* & \rightarrow & \cdots & \rightarrow & (eX_{-2})^* & \rightarrow & (eX_{-1})^* & \rightarrow & (eX_0)^* & \rightarrow & (eX_1)^* & \rightarrow & \cdots & \rightarrow & (eX_l)^* & \rightarrow \\ & & & & & & \searrow & & \nearrow & & & & & & & \\ & & & & & & (eM)^* & & & & & & & & & \\ & & & & \nearrow & & & & \searrow & & & & & & & \\ & & & & 0 & & & & & & 0 & & & & & \end{array}$$

of right eBe -homomorphisms with $(eX_i)^*$ projective right eBe -modules. Then from Theorem 4.3 it follows that

$$X_{i+1}^* \otimes_{eBe} e'N^\theta \approx \text{Hom}_B(X_{-(i+1)}, N).$$

However, since we can take (18) and (20) as parts of complete projective resolutions of ${}_B M$ and $(eM)_{eBe}^*$ respectively, we obtain

$$\text{Tor}_i^{eBe}((eM)^*, e'N^\theta) \approx \text{Ext}_B^{-(i+1)}(M, N).$$

This completes the proof.

Now, if we take ${}_B B$ as M , then for a left B^e -module N we obtain

$$H_k^{eBe \otimes_K (e'Be')^0}((eBe')^*, e'Ne^\theta) \approx H_{B^e}^{-(k+1)}(B, N).$$

Let η be a ring-isomorphism: $eBe \otimes_K (e'Be')^0 \rightarrow eBe \otimes_K (eBe)^0$ defined by $\eta(\rho \otimes \rho'^0) = \rho \otimes (\rho'^{\sigma^{-1}})^0$, where σ is Nakayama's ring-isomorphism. Then using $\eta, (eBe)^*$ and $e'N^\theta$ can be considered as left eBe^e -module and it is easy to show that

$$(eBe')^* \overset{eBe^e}{\approx} eBe$$

and $(\rho_1 \otimes \rho_2)x^\theta = \rho_1^\sigma x \rho_2^\sigma$ for $x \in N, x^\theta \in e'Ne^\theta, \rho_1, \rho_2 \in eBe$.

THEOREM 4.5. *Let B be a QF-3 algebra with eB and Be' as unique minimal faithful right and left ideals respectively and N a two sided B -module. If $\text{domi. dim } {}_B B = n$, then*

$$H_k^{eBe}(eBe, e'Ne^0) \approx H_B^{-(k+1)}(B, N), \quad 0 < k < n,$$

where the above groups denote the homology group of eBe with coefficients in N^0 and the cohomology group of B with coefficients in N respectively.

REFERENCES

1. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, N.J., 1956.
2. S. Eilenberg, *Homological dimension and syzygies*, Ann. of Math. (2) **64** (1956), 328–336.
3. J. P. Jans, *Projective injective modules*, Pacific J. Math. **9** (1959), 1103–1108.
4. F. Kasch, *Dualitätseigenschaften von Frobenius-Erweiterungen*, Math. Z. **77** (1961), 219–227.
5. Y. Kawada, *A generalization of Morita's theorem concerning generalized uni-serial algebras*, Proc. Japan. Acad. **34** (1958), 404–406.
6. K. Morita, *Duality for modules and its applications to the theory of rings with minimum condition*, Sci. Rep. Tokyo. Kyoiku Daigaku **6** (1958), no. 150, 83–142.
7. ———, *Category-isomorphisms and endomorphisms rings of modules*, Trans. Amer. Math. Soc. **103** (1962), 451–469.
8. T. Nakayama, *On the complete cohomology of Frobenius algebras*, Osaka Math. J. **9** (1957), 165–187.
9. ———, *Note on complete cohomology of a quasi-Frobenius algebra*, Nagoya Math. J. **13** (1958), 115–121.
10. ———, *On algebras with complete homology*, Abh. Math. Sem. Univ. Hamburg **22** (1958), 300–307.
11. H. Tachikawa, *A characterization of QF-3 algebras*, Proc. Amer. Math. Soc. **13** (1962), 701–703.
12. R. M. Thrall, *Some generalizations of quasi-Frobenius algebras*, Trans. Amer. Math. Soc. **64** (1948), 173–183.
13. D. W. Wall, *Algebras with unique minimal faithful representations*, Duke Math. J. **25** (1958), 321–329.

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