LIMITS OF POLYNOMIALS WHOSE ZEROS LIE IN A RADIAL SET

BY

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1. **Introduction.** Let R be an unbounded closed set in the complex plane. The polynomials whose zeros belong to R will be called *R-polynomials*. We say that a sequence of functions is *U-convergent* when it converges everywhere in the plane, and uniformly on every bounded set. The object of study is the class $C(R)$ of all functions not identically zero which can be obtained as U-limits of R-polynomials. It is clear that $C(R)$ consists of entire functions whose zeros belong to R. A set R such that $C(R)$ consists of *all* entire functions not identically zero whose zeros belong to R will be called *regular*. Descriptions of $C(R)$ for a number of special "singular" sets have been known for a long time; cf. Obrechkoff's monograph [6]. For a survey of more recent work, see [3].

Some years ago the second author set the problem of describing the class $C(R)$ for general sets R. He obtained a number of results in terms of the asymptotic directions of R. It is convenient to define the asymptotic directions as the *rays* $\arg z = \theta$ for which R contains a sequence of points $\{z_n\}$ which tends to infinity in such a way that $\arg z_n \to \theta$. The set of the asymptotic directions (rays) $\arg z = \theta$ for R will be denoted by $A(R)$, the set of the corresponding rays $\arg z = j\theta$ by $A^j(R)$ or $A(R^j)$. One can distinguish three kinds of sets R:

(i) *R will be called a set of the first kind* if no set $A^j(R)$, $j = 1, 2, \cdots$, belongs to a half-plane. *Sets R of the first kind are regular* [1].

Suppose now that we have a set R not of the first kind. For such a set, *let k be the smallest positive integer such that $A^k(R)$ belongs to a half-plane.*

(ii) *R will be called a set of the second kind* if the convex hull of $A^k(R)$ is an angle $< \pi$. For a set of the second kind $C(R)$ consists of the entire functions of the form $\exp(cz^k) \cdot g(z)$, where $g(z)$ is an entire function not identically zero of genus $\leqq k - 1$ whose zeros belong to R, and $-\bar{c}$ belongs to the convex hull of $A^k(R)$ [1].

(iii) *R will be called a set of the third kind* if the convex hull of $A^k(R)$ is a half-plane or a line. Some sets of the third kind are regular [1], some are singular; no general results are known. Of the singular sets of the third kind only special sets such as lines, strips and half-planes have been investigated.

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In work of this sort it is convenient to introduce, besides the order and the genus of an entire function $f(z)$, a third number which we shall call its *index*. This is the smallest positive integer q such that $f(z)$ can be written in the form $\exp(cz^q)\cdot g(z)$, where $g(z)$ is entire of genus $\leqq q-1$. The *order* ρ (*index* q) of a set R or the class $C(R)$ is defined as the least upper bound of the orders (indices) of the entire functions in $C(R)$. There exist sets of all orders ρ, $1\leqq\rho\leqq\infty$. It is known that *a set is singular if and only if its order* (*or index*) *is finite* [2]. For a singular set R the zero free functions can be written in the form

$$A\exp(a_1z+\cdots+a_qz^q)\qquad (A\neq 0).$$

Let us denote by K the set of all corresponding points

$$a=(a_1,\cdots,a_q)=(\operatorname{Re}a_1,\operatorname{Im}a_1,\cdots,\operatorname{Re}a_q,\operatorname{Im}a_q)$$

in real $2q$-dimensional space. It is known that *K is a* (*closed*) *convex cone*: if $a\in K$ then $\lambda a\in K$ for all real $\lambda\geqq 0$, and if $a,b\in K$ then $\frac{1}{2}(a+b)\in K$ [2].

2. **Present results.** In this paper we obtain a complete description of the class $C(R)$ for the case where R is any (closed) *radial set*, be it of the first, second or third kind. For a *radial set*, that is, a set such that if $z\in R$ then also $\lambda z\in R$ for all $\lambda\geqq 0$, $A(R)$ coincides with R. *A radial set is regular* if and only if it is of the first kind, that is, *if and only if no power R^j belongs to a half-plane*.

For a singular radial set *we determine the index in terms of the geometry*. The order is equal to the index, and attained by a zero free function. The corresponding cone K is the product of 2-dimensional closed convex cones K_j: $a\in K$ *if and only if* $a_j\in K_j$, $j=1,\cdots,q$. For $j<\frac{1}{2}q$ the set K_j is the plane, for $\frac{1}{2}q\leqq j<q$ the K_j are either the plane, a half-plane, or a line; K_q is always an angle $<\pi$ or a ray. (For certain nonradial sets, such as the half-plane $\operatorname{Re}z\geqq 1$, the cone K cannot be decomposed in this manner [1].)

In order to discuss our results for singular radial sets in more detail we have to describe the geometry of R. Set $R=R_1$, let k_1 be the smallest positive integer such that $R_1^{k_1}$ belongs to a half-plane, and let S_1 be the convex hull of $R_1^{k_1}$. If S_1 is an angle $<\pi$ the set R is of the second kind and $C(R)$ is known. If S_1 is a half-plane $\alpha_1\leqq\arg z\leqq\alpha_1+\pi$ or a line $\arg z=\alpha_1,\ \alpha_1+\pi$ we introduce an auxiliary set R_2. The set R_2 will consist of those rays $\arg z=\theta$ of R_1 for which $k_1\theta\equiv\alpha_1 \pmod{\pi}$. That is, R_2 consists of the rays of R_1 whose k_1th powers fall along the boundary of S_1.

In general, let k_ν be the smallest integer $>k_{\nu-1}$ $(k_0=0)$ such that $R_\nu^{k_\nu}$ belongs to a half-plane, and let S_ν be the convex hull of $R_\nu^{k_\nu}$. If S_ν is an angle $<\pi$ we go no further. If S_ν is a half-plane $\alpha_\nu \leqq \arg z \leqq \alpha_\nu + \pi$ or a line $\arg z = \alpha_\nu,\ \alpha_\nu + \pi$ we introduce an auxiliary set $R_{\nu+1}$. The set $R_{\nu+1}$ will consist of those rays $\arg z = \theta$ of R_ν for which $k_\nu \theta \equiv \alpha_\nu \pmod{\pi}$. In other words, $R_{\nu+1}$ consists of the rays of R_ν whose k_νth powers fall along the boundary of S_ν.

The above process comes to an end in a finite number of steps since $R_1 \supset R_2 \supset \cdots$, and $R_2^{2k_1}$ is just a ray. Let R_ω be the last set R_μ introduced in the construction, and let k_ω be the smallest integer $> k_{\omega-1}$ such that $R_\omega^{k_\omega}$ belongs to a half-plane. Then the convex hull S_ω of $R_\omega^{k_\omega}$ is an angle $<\pi$ or a ray. Note that since $R_\nu^{2k_1}$ is a ray for any $\nu \geqq 2$, one has

$$k_\omega \leqq 2k_1 \tag{2.1}$$

whenever $\omega \geqq 2$.

THEOREM 2.1. *Let R be a singular radial set. Then the zero free entire functions in $C(R)$ are the functions of the form*

$$A \exp(a_1 z + \cdots + a_N z^N) \qquad (A \neq 0) \tag{2.2}$$

where $N = k_\omega$,

$$a_{k_\nu} \in -\overline{S}_\nu \qquad \text{for } \nu = 1, \cdots, \omega, \tag{2.3}$$

and the other a_j are arbitrary. (Note that by (2.1) all a_j with $j < \frac{1}{2}N$ are arbitrary.)

In (2.3), $-\overline{S}_\nu$ denotes the set of all points $-\bar{z}$ with $z \in S_\nu$.

THEOREM 2.2. *The singular radial set R has index and order equal to $N = k_\omega$. The class $C(R)$ consists of the entire functions which can be represented in the form*

$$\begin{aligned} &Az^m \exp(b_1 z + \cdots + b_N z^N) \\ &\qquad \cdot \prod_p (1 - z/z_p) \exp(z/z_p + \cdots + z^{N-1}/(N-1) z_p^{N-1}) \end{aligned} \tag{2.4}$$

where $A \neq 0$, $z_p \in \underline{R}$, $\sum_p |z_p|^{-N}$ converges, b_j is arbitrary for $j \neq k_\nu$, $\nu = 1, \cdots, \omega$, $b_{k_\omega} \in -\overline{S}_\omega$, and

$$b_{k_\nu} + \frac{1}{k_\nu} \sum_p z_p^{-k_\nu} \in -\overline{S}_\nu, \qquad \nu = 1, \cdots, \omega - 1, \tag{2.5}$$

in the sense that the series

$$\operatorname{Im}(b_{k_\nu} e^{i\alpha_\nu}) + \frac{1}{k_\nu} \sum_p \operatorname{Im}(z_p^{-k_\nu} e^{i\alpha_\nu}) \tag{2.6}$$

converges to a sum

$$\sigma_\nu \begin{cases} \geqq 0 \textit{ if } S_\nu \textit{ is the half-plane } \alpha_\nu \leqq \arg z \leqq \alpha_\nu + \pi, \\ = 0 \textit{ if } S_\nu \textit{ is the line } \arg z = \alpha_\nu, \alpha_\nu + \pi. \end{cases} \tag{2.7}$$

A somewhat less precise form of the above results was obtained by the first author in his (unpublished) Ph.D. thesis [4]. The proofs in the present paper are shorter and quite different. They depend on comparison of $C(R)$ with certain related classes of entire functions, a technique which may be useful also in the study of nonradial sets.

3. **The auxiliary classes $C_h(R)$.** We begin by normalizing the functions in $C(R)$. If $F(z)$ is any entire function not identically zero one can write $F(z) = Az^m f(z)$, with $f(0) = 1$. Furthermore, if $F(z)$ is the U-limit of R-polynomials $F_n(z)$, one can omit suitable factors from the $F_n(z)$ to obtain R-polynomials $f_n(z)$ which are U-convergent to $f(z)$. Thus $f(z) \in C(R)$. We denote the subset of the functions $f(z) \in C(R)$ with $f(0) = 1$ by $C_0(R)$. Every function $f(z) \in C_0(R)$ is the U-limit of normalized R-polynomials

$$f_n(z) = \prod_p (1 - z/z_{np}), \qquad z_{np} \in R. \tag{3.1}$$

More generally, if h is any integer $\geqq 0$ we define $C_h(R)$ as the class of U-limits of finite products of the form

$$f_n(z; h) = \prod_p (1 - z/z_{np}) \exp(z/z_{np} + \cdots + z^h/hz^h_{np}), \qquad z_{np} \in R. \tag{3.2}$$

Note that for $|z| < \min_p |z_{np}|$,

$$f_n(z; h) = \exp(s_{n,h+1} z^{h+1} + s_{n,h+2} z^{h+2} + \cdots) \tag{3.3}$$

where the s_{nj} are given by the formula

$$s_{nj} = -\frac{1}{j} \sum_p z_{np}^{-j}, \qquad j = 1, 2, \cdots. \tag{3.4}$$

It is clear that the product of two functions in $C_h(R)$ is again in $C_h(R)$, and likewise the limit of a U-convergent sequence of functions in $C_h(R)$. If $f(z)$ is the limit of a U-convergent sequence of finite products $f_n(z; h)$ one can write, in a disc about 0 which is free of zeros of $f(z)$,

$$f(z) = \exp(a_{h+1} z^{h+1} + a_{h+2} z^{h+2} + \cdots), \tag{3.5}$$

where

$$a_j = \lim_{n \to \infty} s_{nj}, \qquad j = h + 1, h + 2, \cdots. \tag{3.6}$$

The existence of the limits (3.6) by itself is not sufficient to imply U-convergence of a sequence $\{f_n(z; h)\}$. Something has to be added, as in the following lemma.

LEMMA 3.1 (CF. LINDWART AND PÓLYA [5]; OBRECHKOFF [6]). *Let the* $f_n(z;h)$, $n = 1, 2, \cdots$ *be finite products* (3.2) *such that the associated sequences* $\{s_{nj}\}$ (3.4) *tend to limits* a_j *as* $n \to \infty$, $j = h+1, h+2, \cdots$. *Suppose furthermore that there are an integer* $t > h$ *and a number* M *such that*

$$\sum_p |z_{np}|^{-t} \leqq M, \qquad n = 1, 2, \cdots. \tag{3.7}$$

Then the sequence $\{f_n(z;h)\}$ *is U-convergent to an entire function of index* $\leqq t$.

There are simple geometrical conditions which imply (3.7):

LEMMA 3.2 (CF. LINDWART AND PÓLYA [5]; KOREVAAR [1]). *If the kth powers* z_{np}^k *of the zeros of the* $f_n(z;h)$ *belong to a fixed angle with vertex* 0 *and opening* $< \pi$, *and the numbers* s_{nk}, $n = 1, 2, \cdots$, *form a bounded sequence, then* (3.7) *holds with* $t = k$ *and a suitable constant* M. *If the kth powers* z_{np}^k *all belong to a fixed half-plane* $\alpha \leqq \arg z \leqq \alpha + \pi$, *and the numbers* s_{nk} *and* $s_{n,2k}$, $n = 1, 2, \cdots$, *form bounded sequences, then there are constants* A *and* B *such that*

$$\sum_p |\operatorname{Im}(z_{np}^{-k} e^{i\alpha})| \leqq A, \quad \sum_p |z_{np}|^{-2k} \leqq B, \qquad n = 1, 2, \cdots. \tag{3.8}$$

Here the requirement that the angle ($< \pi$ or $= \pi$) have vertex 0 may be dropped if it is known beforehand that the sequence $\{f_n(z;h)\}$ is U-convergent.

The following lemma helps one determine the zero free functions in $C_h(R)$.

LEMMA 3.3. *Let* $z_n = r_n e^{i\theta_n}$, *where* $r_n \to \infty$ *and* $\theta_n \to \theta$. *Let* $\lambda > 0$, s *a positive integer, and let* ν_n *be the integral part of* $\lambda s r_n^s$. *Then as* $n \to \infty$,

$$\{(1 - z/z_n)\exp(z/z_n + \cdots + z^{s-1}/(s-1)z_n^{s-1})\}^{\nu_n} \to \exp(-\lambda e^{-is\theta} z^s) \tag{3.9}$$

in the sense of U-convergence.

The preceding lemmas may be used to obtain results on the class $C_h(R)$ in terms of asymptotic directions (cf. §1). We omit the proofs because they are very similar to those for the corresponding results on $C(R)$ (cf. Korevaar [1]).

THEOREM 3.4. *Let* R *be a set whose asymptotic directions are such that* $A^{h+1}(R), \cdots, A^{k-1}(R)$ (*where* $k \geqq h+1$) *do not lie in a half-plane. We denote the convex hull of* $A^k(R)$ *by* S.

(i) *Suppose that*

$$\exp(a_{h+1}z^{h+1} + \cdots + a_k z^k + \cdots) \tag{3.10}$$

represents a zero free entire function in $C_h(R)$. *Then* $a_k \in -\bar{S}$, *and if* S *is an angle* $< \pi$ *moreover* $a_j = 0$ *for all* $j > k$. *On the other hand, if* $a_k \in -\bar{S}$ *then*

$$\exp(a_{h+1}z^{h+1} + \cdots + a_k z^k) \tag{3.11}$$

belongs to $C_h(R)$ *for every choice of the complex numbers* $a_{h+1}, \cdots, a_{k-1}$.

(ii) *Suppose that* R^k *belongs to an angle* $< \pi$. *Then* $C_h(R)$ *has index* k, *that is, the maximum of the indices of the functions in* $C_h(R)$ *is* k. *The class* $C_h(R)$ *consists of the entire functions which can be written in the form*

$$\begin{aligned} &\exp(b_{h+1}z^{h+1} + \cdots + b_k z^k) \\ &\quad \cdot \prod_p (1 - z/z_p)\exp(z/z_p + \cdots + z^{k-1}/(k-1)z_p^{k-1}), \end{aligned} \tag{3.12}$$

where $z_p \in R$, $\sum_p |z_p|^{-k}$ *converges, and* $b_k \in -\bar{S}$.

(iii) *Suppose that* R^k *belongs to the half-plane* $\alpha \leqq \arg z \leqq \alpha + \pi$, *but not to an angle* $< \pi$. *Then* $C_h(R)$ *has index* q *with* $k < q \leqq 2k$. *All functions in* $C_h(R)$ *can be written in the form*

$$\begin{aligned} &\exp(b_{h+1}z^{h+1} + \cdots + b_k z^k + \cdots + b_q z^q) \\ &\quad \cdot \prod_p (1 - z/z_p)\exp(z/z_p + \cdots + z^{q-1}/(q-1)z_p^{q-1}), \end{aligned} \tag{3.13}$$

where $z_p \in R$, $\sum_p |z_p|^{-q}$ *converges, and*

$$b_k + \frac{1}{k}\sum_p z_p^{-k} \in -\bar{S}, \tag{3.14}$$

in the sense that the series

$$\operatorname{Im}(b_k e^{i\alpha}) + \frac{1}{k}\sum_p \operatorname{Im}(z_p^{-k} e^{i\alpha}) \tag{3.15}$$

converges to a sum

$$\sigma \begin{cases} \geqq 0 & \text{if } S \text{ is the half-plane } \alpha \leqq \arg z \leqq \alpha + \pi, \\ = 0 & \text{if } S \text{ is the line } \arg z = \alpha, \alpha + \pi. \end{cases} \tag{3.16}$$

4. **Initial results on $C(R)$ for radial sets R.** Regular radial sets will be disposed of in a few lines. With every singular radial set we associate the sets R_ν and S_ν and the integers k_ν defined in §2. It will be shown that $C(R)$ contains the classes $C_{k_\mu}(R_{\mu+1})$, $\mu = 1, \cdots, \omega - 1$, and this result is used to prove one direction of Theorems 2.1 and 2.2.

LEMMA 4.1. *A radial set* R *is regular if and only if no power* R^j, $j = 1, 2, \cdots$, *belongs to a half-plane.*

Proof. (i) Suppose that no power R^j belongs to a half-plane. Then the same is true for the sets $A^j(R)$ $(= R^j)$, hence by §1 (i) the set R is regular. (That all zero free entire functions belong to $C(R)$ could also be derived from Theorem 3.4.)

(ii) Suppose that R^k belongs to a half-plane. Since R^k is a radial set the half-plane will have the form $\alpha \leqq \arg z \leqq \alpha + \pi$. It now follows from

Theorem 3.4 that all functions in $C(R)$ are of order $\leqq 2k$, hence R cannot be regular.

LEMMA 4.2. *For singular radial sets R one has the following inclusion relations*:

$$C(R) \supset C_0(R_1) \supset C_{k_1}(R_2) \supset \cdots \supset C_{k_{\omega-1}}(R_\omega). \tag{4.1}$$

Proof. It is clear from §3 that $C_0(R_1) \subset C(R_1) = C(R)$. Now let $h = k_{\mu-1}$ and $k = k_\mu$, where $1 \leqq \mu \leqq \omega - 1$. We will show that

$$C_h(R_\mu) \supset C_k(R_{\mu+1}).$$

Let $f(z) \in C_k(R_{\mu+1})$. Then $f(z)$ is the limit of a U-convergent sequence of finite products

$$f_n(z;k) = \prod_p (1 - z/z_{np})\exp(-s_{n1}z - \cdots - s_{nk}z^k), \qquad n = 1, 2, \cdots, \tag{4.2}$$

where the z_{np} belong to $R_{\mu+1}$ and the s_{nj} are given by (3.4). Clearly

$$f_n(z;k) = \exp(-s_{n,h+1}z^{h+1} - \cdots - s_{nk}z^k)\, f_n(z;h). \tag{4.3}$$

By its definition the point $-\bar{s}_{nk}$ belongs to the convex hull of $R^k_{\mu+1}$. However, by the definition of $R_{\mu+1}$ the set $R^k_{\mu+1}$ is a line through 0, the boundary of $\tilde{S}_\mu$. It follows that the point $\bar{s}_{nk}$ also belongs to S_μ, hence $-s_{nk} \in -\bar{S}_\mu$. Applying the first part of Theorem 3.4 to R_μ one concludes that

$$\exp(-s_{n,h+1}z^{h+1} - \cdots - s_{nk}z^k)$$

belongs to $C_h(R_\mu)$. Since $z_{np} \in R_{\mu+1} \subset R_\mu$ it is clear that the function $f_n(z;h)$ belongs to $C_h(R_\mu)$. Thus the products $f_n(z;k)$ (4.3) belong to $C_h(R_\mu)$, and likewise their U-limit $f(z)$.

THEOREM 4.3. *Suppose that R is a singular radial set.*

(i) *Let $a_1, \cdots, a_N$, where $N = k_\omega$, be any N-tuple of complex numbers such that*

$$a_{k_\nu} \in -\bar{S}_\nu, \qquad \nu = 1, \cdots, \omega. \tag{4.4}$$

Then the zero free entire function

$$\exp(a_1 z + \cdots + a_N z^N) \tag{4.5}$$

belongs to $C(R)$.

(ii) *More generally, let $f(z)$ be any entire function which can be written in the form*

$$\exp(b_1 z + \cdots + b_N z^N) \prod_p (1 - z/z_p)\exp(z/z_p + \cdots + z^{N-1}/(N-1)z_p^{N-1}) \tag{4.6}$$

where $N = k_\omega$, $z_p \in R$, $\sum_p |z_p|^{-N}$ converges, $b_N \in -\bar{S}_\omega$, and

$$b_{k_\nu} + \frac{1}{k_\nu} \sum_p z_p^{-k_\nu} \in -\bar{S}_\nu, \qquad \nu = 1, \cdots, \omega - 1, \tag{4.7}$$

in the sense explained in the statement of Theorem 2.2. *Then* $f(z) \in C(R)$.

Proof. (i) Theorem 3.4 shows that for $\nu = 1, \cdots, \omega$ the function

$$\exp(a_{k_{\nu-1}+1} z^{k_{\nu-1}+1} + \cdots + a_{k_\nu} z^{k_\nu}) \tag{4.8}$$

will belong to the corresponding class $C_{k_{\nu-1}}(R_\nu)$. However, by Lemma 4.2, the classes $C_{k_{\nu-1}}(R_\nu)$ are part of $C(R)$, hence the functions (4.8) all belong to $C(R)$. Forming their product one concludes that the function (4.5) belongs to $C(R)$.

(ii) Let us assume for definiteness that the product in (4.6) is infinite. Then $f(z)$ is the U-limit of the finite products

$$\phi_n(z) = \exp(c_1 z + \cdots + c_N z^N) \prod_{p \leq n} (1 - z/z_p), \tag{4.9}$$

where

$$c_j = c_j(n) = b_j + \frac{1}{j} \sum_{p \leq n} z_p^{-j}, \qquad j = 1, \cdots, N-1; \; c_N = b_N. \tag{4.10}$$

Because of (4.7) the number c_{k_ν} will either belong to $-\bar{S}_\nu$ or be very close to it. Indeed, set

$$d_{k_\nu} = c_{k_\nu} + ie^{-i\alpha_\nu} \frac{1}{k_\nu} \sum_{p > n} \operatorname{Im}(z_p^{-k_\nu} e^{i\alpha_\nu}), \qquad \nu = 1, \cdots, \omega - 1 \tag{4.11}$$

(compare the statement of Theorem 2.2). Then

$$d_{k_\nu} - c_{k_\nu} \to 0 \quad \text{as} \quad n \to \infty, \tag{4.12}$$

and

$$d_{k_\nu} e^{i\alpha_\nu} = \rho_\nu + i\sigma_\nu$$

where ρ_ν and σ_ν are real and σ_ν is the sum of the series (2.6). It thus follows from (2.7) that

$$d_{k_\nu} \in -\bar{S}_\nu, \qquad \nu = 1, \cdots, \omega - 1. \tag{4.13}$$

Setting $d_j = c_j$ for $j \neq k_\nu$, $\nu = 1, \cdots, \omega - 1$ we will have $d_j - c_j \to 0$ as $n \to \infty$ for $j = 1, \cdots, N$, hence $f(z)$ is also the U-limit of the modified products

$$\psi_n(z) = \exp(d_1 z + \cdots + d_N z^N) \prod_{p \leq n} (1 - z/z_p). \tag{4.14}$$

Using (4.13) and the fact that $d_N = b_N \in -\bar{S}_\omega$ we obtain from part (i) of the theorem that

$$\exp(d_1 z + \cdots + d_N z^N)$$

belongs to $C(R)$. It follows that the functions $\psi_n(z)$, and hence their U-limit $f(z)$, belong to $C(R)$.

5. **First reduction theorem. Proof of Theorem 2.1.** In this section we complete our investigation of the zero free functions in $C(R)$ for radial sets R. The principal tool is a kind of converse to Lemma 4.2. It shows that for every zero free entire function in $C(R)$ certain related functions belong to the classes $C_{k_\mu}(R_{\mu+1})$.

THEOREM 5.1 (FIRST REDUCTION THEOREM). *Let R be a singular radial set. Set $h = k_{\mu-1}$, $k = k_\mu$, where $1 \leqq \mu \leqq \omega - 1$, and suppose that*

$$f(z) = \exp(a_{h+1} z^{h+1} + \cdots + a_k z^k + \cdots) \tag{5.1}$$

is a zero free entire function in $C_h(R_\mu)$. Then

$$g(z) = \exp(a_{k+1} z^{k+1} + a_{k+2} z^{k+2} + \cdots) \tag{5.2}$$

is a zero free entire function in $C_k(R_{\mu+1})$.

Proof. Write $f(z)$ as the limit of a U-convergent sequence of finite products $f_n(z;h)$ as in (3.2), with the z_{np} in R_μ. Since $f(z)$ is free of zeros,

$$r_n = \min_p |z_{np}| \to \infty \quad \text{as} \quad n \to \infty. \tag{5.3}$$

The s_{nj} (3.4) will tend to a_j as $n \to \infty$, $j = h+1, h+2, \cdots$ (3.6).

We next use the geometry of the set R_μ. Writing $\alpha_\mu = \alpha$ it is known that R_μ^k belongs to the half-plane $\alpha \leqq \arg z \leqq \alpha + \pi$ (and not to an angle $< \pi$). Setting $z_{np} = r_{np}\exp(i\theta_{np})$ one observes that

$$\sin(k\theta_{np} - \alpha) \geqq 0. \tag{5.4}$$

By Lemma 3.2 there are constants A and B such that

$$\sum_p r_{np}^{-k} \sin(k\theta_{np} - \alpha) \leqq A, \qquad \sum_p r_{np}^{-2k} \leqq B, \qquad n = 1, 2, \cdots. \tag{5.5}$$

We now rotate all zeros z_{np} of $f_n(z;h)$ to the (or a) *nearest ray of $R_{\mu+1}$.* Their new positions w_{np} are given by

$$w_{np} = r_{np}\exp\{i(\alpha + s\pi)/k\}, \tag{5.6}$$

where the integer $s = s_{np}$ is chosen so as to minimize

$$|\theta_{np} - (\alpha + s\pi)/k|,$$

subject to the condition that $\arg z = (\alpha + s\pi)/k$ be a ray of $R_{\mu+1}$. (When there are two possibilities for w_{np} one makes an arbitrary choice.) In terms of the w_{np} define

$$t_{nj} = -\frac{1}{j}\sum_p w_{np}^{-j}, \tag{5.7}$$

and

$$\begin{aligned} g_n(z;k) &= \prod_p (1 - z/w_{np})\exp(z/w_{np} + \cdots + z^k/kw_{np}^k) \\ &= \exp(t_{n,k+1}z^{k+1} + t_{n,k+2}z^{k+2} + \cdots), \qquad n = 1, 2, \cdots. \end{aligned} \tag{5.8}$$

Clearly $g_n(z;k) \in C_k(R_{\mu+1})$. *It will be shown that the* $g_n(z;k)$ *are U-convergent to* $g(z)$, *and thus* $g(z) \in C_k(R_{\mu+1})$.

We will need an estimate for $t_{nj} - s_{nj}$. Since for real x,

$$|e^{ix} - 1| \leqq |x|,$$

one has the initial inequality

$$\begin{aligned} |t_{nj} - s_{nj}| &= \frac{1}{j}\left| \sum_p r_{np}^{-j}[\exp\{-ij(\alpha + s_{np}\pi)/k\} - \exp(-ij\theta_{np})] \right| \\ &\leqq \sum_p r_{np}^{-j}\frac{1}{j}|\exp[ij\{\theta_{np} - (\alpha + s_{np}\pi)/k\}] - 1| \\ &\leqq \sum_p r_{np}^{-j}|\theta_{np} - (\alpha + s_{np}\pi)/k|. \end{aligned} \tag{5.9}$$

It will next be shown that *there is a constant* C (depending only on the set R_μ) *such that for all* n *and* p

$$|\theta_{np} - (\alpha + s_{np}\pi)/k| \leqq C\sin(k\theta_{np} - \alpha). \tag{5.10}$$

Note that the complement of R_μ is an open radial set. Let us look at the rays of the special form $\arg z = (\alpha + s\pi)/k$ which belong to this complement. There will be a positive number $\delta < \pi/2k$ such that all rays within an angle δ of a special ray likewise belong to the complement of R_μ. Now consider the rays $\arg z = \theta$ of R_μ such that

$$\min_s |\theta - (\alpha + s\pi)/k| < \delta.$$

By the definition of δ the minimizing rays $\arg z = (\alpha + s\pi)/k$ must belong to R_μ and hence to $R_{\mu+1}$. For the rays $\arg z = \theta$ considered,

$$\min_s |k\theta - (\alpha + s\pi)| < k\delta < \tfrac{1}{2}\pi,$$

hence if $\arg z = (\alpha + s_\theta\pi)/k$ is the ray of $R_{\mu+1}$ nearest $\arg z = \theta$,

$$|k\theta - (\alpha + s_\theta \pi)| < \tfrac{1}{2}\pi \sin(k\theta - \alpha).$$

For all other rays $\arg z = \theta$ in R_μ one has

$$k\delta \leqq \min_s |k\theta - (\alpha + s\pi)| \leqq \tfrac{1}{2}\pi,$$

hence $\sin(k\theta - \alpha) \geqq \sin k\delta$. It follows that for those rays $\arg z = \theta$ (again denoting by $\arg z = (\alpha + s_\theta \pi)/k$ the nearest ray of $R_{\mu+1}$),

$$|k\theta - (\alpha + s_\theta \pi)| \leqq k\pi \leqq \frac{k\pi}{\sin k\delta} \sin(k\theta - \alpha).$$

This completes the proof of (5.10).

Combining (5.9) with (5.10), and using (5.5) and (5.3), one obtains the estimate

$$\begin{aligned} |t_{nj} - s_{nj}| &\leqq C \sum_p r_{np}^{-j} \sin(k\theta_{np} - \alpha) \\ &\leqq C \sum_p r_{np}^{k-j} r_{np}^{-k} \sin(k\theta_{np} - \alpha) \leqq A C r_n^{k-j},\ j \geqq k,\ n = 1, 2, \cdots. \end{aligned} \tag{5.11}$$

Since the s_{nj} tend to a_j as $n \to \infty$ it immediately follows that

$$t_{nj} \to a_j \quad \text{as } n \to \infty,\ j = k+1, k+2, \cdots. \tag{5.12}$$

The second inequality (5.5) and (5.12) enable us to apply Lemma 3.1 to the functions $g_n(z; k)$. The conclusion is that the $g_n(z; k)$ are U-convergent. The limit function $G(z)$ will have the form $\exp(b_{k+1} z^{k+1} + b_{k+2} z^{k+2} + \cdots)$. Here $b_j = \lim t_{nj}$, hence by (5.12) $b_j = a_j$ and thus $G(z) = g(z)$. It follows that $g(z) \in C_k(R_{\mu+1})$.

We now have all the ingredients for the

Proof of Theorem 2.1. Let R be a singular radial set. Suppose that

$$\exp(a_1 z + a_2 z^2 + \cdots) \tag{5.13}$$

represents a zero free entire function in $C(R)$. Repeated application of Theorem 5.1 then shows that for $\nu = 1, \cdots, \omega$,

$$\exp(a_{k_{\nu-1}+1} z^{k_{\nu-1}+1} + \cdots + a_{k_\nu} z^{k_\nu} + \cdots) \tag{5.14}$$

represents a zero free entire function in $C_{k_{\nu-1}}(R_\nu)$. Thus by Theorem 3.4

$$a_{k_\nu} \in -\overline{S}_\nu, \qquad \nu = 1, \cdots, \omega. \tag{5.15}$$

For $\nu = \omega$ the set S_ν is an angle $< \pi$, hence Theorem 3.4 shows in addition that $a_j = 0$ for all $j > k_\omega$.

It follows that all zero free entire functions in $C(R)$ have indeed the form given by (2.2) and (2.3). The converse was obtained earlier as part (i) of Theorem 4.3.

6. **Second reduction theorem. Proof of Theorem 2.2.** Let R be a singular radial set, and let q be the index of $C(R)$. Since R^{k_1} belongs to a half-plane, $q \leqq 2k_1$ (Theorem 3.4). On the other hand, by Theorem 4.3, $q \geqq N = k_\omega$. By Lemma 4.2 the classes $C_{k_\mu}(R_{\mu+1})$ all have index $\leqq q$.

We will prove another reduction theorem, more general than the one in §5, which shows that for every entire function in $C(R)$ certain related functions belong to the classes $C_{k_\mu}(R_{\mu+1})$. We start by describing the construction of these functions.

CONSTRUCTION 6.1. Let $h = k_{\mu-1}$, $k = k_\mu$, where $1 \leqq \mu \leqq \omega - 1$, and suppose that $f(z) \in C_h(R_\mu)$. Then $f(z)$ can be written in the form

$$\begin{aligned} f(z) &= \exp(c_{h+1}z^{h+1} + \cdots + c_q z^q) \\ &\quad \cdot \prod_p (1 - z/z_p)\exp(z/z_p + \cdots + z^{q-1}/(q-1)z_p^{q-1}), \end{aligned} \tag{6.1}$$

where q is the index of $C(R)$, $z_p \in R_\mu$ and $\sum_p |z_p|^{-q}$ converges. One can say a little more. The set R_μ^k belongs to the half-plane $\alpha \leqq \arg z \leqq \alpha + \pi$ where $\alpha = \alpha_\mu$. Thus by Theorem 3.4, writing $z_p = \exp(i\theta_p)$, the series

$$\sum_p r_p^{-k} \sin(k\theta_p - \alpha) = \sum_p - \operatorname{Im}(z_p^{-k} e^{i\alpha}) \tag{6.2}$$

of non-negative terms converges.

We now rotate all zeros z_p *of* $f(z)$ *to the* (or a) *nearest ray of* $R_{\mu+1}$. Their new positions w_p are given by

$$w_p = r_p \exp\{ i(\alpha + s_p\pi)/k \}, \tag{6.3}$$

where the integer s_p is chosen so as to minimize $|\theta_p - (\alpha + s_p\pi)/k|$, subject to the condition that the ray $\arg z = (\alpha + s_p\pi)/k$ belong to $R_{\mu+1}$. For some points z_p there may be two candidates for w_p. In this case we let the choice of w_p depend on a given sequence $\{ f_n(z;h) \}$ which is U-convergent to $f(z)$, as explained in the proof of Theorem 6.2 below.

In terms of the numbers w_p define

$$d_j = c_j - \frac{1}{j}\sum_p (w_p^{-j} - z_p^{-j}), \qquad k < j < q; \; d_q = c_q. \tag{6.4}$$

The convergence of the series (6.2) and the argument used in the proof of Theorem 5.1 show that the series in (6.4) are absolutely convergent (cf. (5.9)-(5.11)).

Finally set

$$\begin{aligned} g(z) &= \exp(d_{k+1}z^{k+1} + \cdots + d_q z^q) \\ &\quad \cdot \prod_p (1 - z/w_p)\exp(z/w_p + \cdots + z^{q-1}/(q-1)w_p^{q-1}); \end{aligned} \tag{6.5}$$

it is clear that $\sum_p |w_p|^{-q}$ converges.

THEOREM 6.2 (SECOND REDUCTION THEOREM). *Let $f(z) \in C_h(R_\mu)$, and let $g(z)$ be the function associated with $f(z)$ by Construction* 6.1. *Then $g(z) \in C_k(R_{\mu+1})$. If $f(z)$ has its index equal to q then so does $g(z)$.*

Proof. As in the proof of Theorem 5.1 we write $f(z)$ as the limit of a U-convergent sequence of functions $f_n(z;h)$ of the form (3.2), with the $z_{np} \in R_\mu$. For each fixed n the z_{np} will be numbered in such a way that $z_{np} \to z_p$ as $n \to \infty$, $p = 1, 2, \cdots$. Writing $z_{np} = r_{np}\exp(i\theta_{np})$ one again has (5.4) and (5.5). For the numbers s_{nj} defined by (3.4) one will have, as $n \to \infty$,

$$s_{nj} \to \begin{cases} c_j & \text{for } h < j < q, \\ c_q - \dfrac{1}{q}\sum_p z_p^{-q} & \text{for } j = q, \\ -\dfrac{1}{j}\sum_p z_p^{-j} & \text{for } j > q. \end{cases} \tag{6.6}$$

Next introduce numbers w_{np} as in the proof of Theorem 5.1. For every p for which there was only one candidate w_p in Construction 6.1 one will have $w_{np} \to w_p$ as $n \to \infty$. However, for a p for which there were two candidates w_p' and w_p'' it is known only that every w_{np} (with n large) will be close to either w_p' or w_p''. One can resolve this difficulty as follows. Pick a subsequence of n's such that the corresponding sequence $\{w_{n1}\}$ converges, and denote the limit (which is either w_1, or w_1', or w_1'') by w_1. Next pick a subsequence of the subsequence such that the corresponding sequence $\{w_{n2}\}$ converges, etc. For the diagonal sequence of n's, $w_{np} \to w_p$ for every p. The corresponding sequence of functions $f_n(z;h)$ will again be denoted by $\{f_n(z;h)\}$.

One finally defines numbers t_{nj} and a sequence $\{g_n(z;k)\}$ of functions in $C_k(R_{\mu+1})$ as in the proof of Theorem 5.1. *It will be shown that the $g_n(z;k)$* (5.8) *are U-convergent to $g(z)$* (6.5), *and thus $g(z) \in C_k(R_{\mu+1})$.*

We first prove that for $j > k$

$$t_{nj} - s_{nj} = -\frac{1}{j}\sum_p (w_{np}^{-j} - z_{np}^{-j}) \to -\frac{1}{j}\sum_p (w_p^{-j} - z_p^{-j}) \tag{6.7}$$

as $n \to \infty$. Taking $r \neq$ all r_p it is clear that

$$\sum_{r_{np} < r} (w_{np}^{-j} - z_{np}^{-j}) \to \sum_{r_p < r} (w_p^{-j} - z_p^{-j})$$

as $n \to \infty$. Furthermore, by the argument used in the proof of Theorem 5.1 (cf. (5.11)),

$$\left| \frac{1}{j}\sum_{r_{np} \geqq r} (w_{np}^{-j} - z_{np}^{-j}) \right| \leqq ACr^{k-j}, \qquad n = 1, 2, \cdots.$$

The same inequality holds with w_{np}, z_{np} and r_{np} replaced by w_p, z_p and r_p. Taking r large one obtains (6.7).

Combining (6.6) and (6.7) one finds that as $n \to \infty$,

$$t_{nj} \to \begin{cases} d_j & \text{for } k < j < q, \\ d_q - \frac{1}{q}\sum_p w_p^{-q} & \text{for } j = q, \\ -\frac{1}{j}\sum_p w_p^{-j} & \text{for } j > q, \end{cases} \tag{6.8}$$

where the d_j are given by (6.4). By the second inequality (5.5)

$$\sum_p |w_{np}|^{-2k} \leqq B, \qquad n = 1, 2, \cdots. \tag{6.9}$$

Thus by Lemma 3.1 our functions $g_n(z;k)$ are U-convergent. For $|z| < \min r_p$ the limit function $G(z)$ can be written in the form

$$\exp(e_{k+1}z^{k+1} + e_{k+2}z^{k+2} + \cdots).$$

Here $e_j = \lim t_{nj}$, hence by (6.8) and (6.5), $G(z) = g(z)$. It follows that $g(z) \in C_k(R_{\mu+1})$.

Finally, suppose that $f(z)$ has its index equal to q, the index of $C(R)$. If $q > 1$ then either $c_q \neq 0$ or the series $\sum_p r_p^{-q+1}$ diverges (or both). Hence in that case $g(z)$ also has its index equal to q. If $q = 1$ there is nothing to prove.

We can now give the

Proof of Theorem 2.2. Let R be a singular radial set, and let q be the index of the class $C(R)$. Repeated application of Theorem 6.2 shows that the classes $C_{k_\mu}(R_{\mu+1})$ also have index q. However, $R_\omega^{k_\omega}$ is contained in an angle $< \pi$, hence by part (ii) of Theorem 3.4 the class $C_{k_{\omega-1}}(R_\omega)$ has index k_ω. *It follows that* $q = k_\omega = N$.

Suppose now that $f(z)$ is any function in $C_0(R)$. Then $f(z)$ can be written in the form

$$\exp(b_1 z + \cdots + b_N z^N) \prod_p (1 - z/z_p)\exp(z/z_p + \cdots + z^{N-1}/(N-1)z_p^{N-1}) \tag{6.10}$$

where $z_p \in R$ and $\sum_p |z_p|^{-N}$ converges. Repeated application of Theorem 6.2 shows that $C_{k_{\nu-1}}(R_\nu)$ contains an entire function of the form

$$\begin{aligned} &\exp(b_{k_{\nu-1}}^{(\nu)} z^{k_{\nu-1}+1} + \cdots + b_N^{(\nu)} z^N) \\ &\cdot \prod_p (1 - z/z_p^{(\nu)})\exp(z/z_p^{(\nu)} + \cdots + z^{N-1}/(N-1)[z_p^{(\nu)}]^{N-1}), \end{aligned} \tag{6.11}$$

where $|z_p^{(\nu)}| = |z_p|$, and

$$b_j^{(\nu)} = b_j^{(\nu-1)} - \frac{1}{j}\sum_p \{[z_p^{(\nu)}]^{-j} - [z_p^{(\nu-1)}]^{-j}\} = \cdots$$

$$= b_j - \frac{1}{j}\sum_p \{[z_p^{(\nu)}]^{-j} - z_p^{-j}\}, \qquad k_{\nu-1} < j < N, \tag{6.12}$$

$$b_N^{(\nu)} = b_N.$$

For $\nu = 1, \cdots, \omega - 1$ one can apply part (iii) of Theorem 3.4 to the function in (6.11). Thus one finds that the series

$$\operatorname{Im}(b_{k_\nu}^{(\nu)} e^{i\alpha_\nu}) + \frac{1}{k_\nu}\sum_p \operatorname{Im}([z_p^{(\nu)}]^{-k_\nu} e^{i\alpha_\nu}) \tag{6.13}$$

converges to a sum σ_ν which is $\geqq 0$ if S_ν is the half-plane $\alpha_\nu \leqq \arg z \leqq \alpha_\nu + \pi$, and $= 0$ if S_ν is the line $\arg z = \alpha_\nu$, $\alpha_\nu + \pi$. Combining (6.13) and (6.12) (with $j = k_\nu$) one concludes that the series

$$\operatorname{Im}(b_{k_\nu} e^{i\alpha_\nu}) + \frac{1}{k_\nu}\sum_p \operatorname{Im}(z_p^{-k_\nu} e^{i\alpha_\nu}) \tag{6.14}$$

also converges to the sum σ_ν.

For $\nu = \omega$ one uses part (ii) of Theorem 3.4 to show that $b_N \in -\bar{S}_\omega$.

It follows that all functions in $C(R)$ have indeed the form given in Theorem 2.2. The converse was obtained earlier as part (ii) of Theorem 4.3.

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