LIMIT DISTRIBUTIONS OF THE MINIMAX OF INDEPENDENT IDENTICALLY DISTRIBUTED RANDOM VARIABLES[1]

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Summary. The class of limiting distributions of the normalized minimax (or maximin) of independent identically distributed random variables is obtained and the domains of attraction of the three limiting types are characterized. Asymptotic independence of the minimax and maximin is also demonstrated.

1. **Introduction.** Consider a sequence of independent identically distributed random variables on some probability space with P the probability measure thereupon. These will be double indexed in the fashion

(1.1) $$X_{11}; X_{12}, X_{22}, X_{21}; X_{13}, X_{23}, X_{33}, X_{32}, X_{31}; \cdots$$

and engender a sequence of random matrices of order n

(1.2) $$\chi_n = \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1n} \\ \vdots & \vdots & & \vdots \\ X_{n1} & X_{n2} & \cdots & X_{nn} \end{pmatrix}, \qquad n = 1, 2, \cdots,$$

which, in turn, give rise to two sequences of random variables

(1.3)
$$W_n = \min_i \max_j X_{ij},$$
$$Z_n = \max_j \min_i X_{ij}.$$

Denote by F the common cumulative distribution function (c. d. f.) of the X_{ij}. We seek the classes $\mathscr{L}_1$ and $\mathscr{L}_2$ of limiting distributions of W_n and Z_n (suitably normalized) as well as conditions insuring that the normalized W_n or Z_n based on a stipulated F, actually converge in law to a particular limiting distribution. In view of the relationship $Z_n = -\min_i \max_j (-X_{ji})$, it suffices to deal with W_n. Clearly, $\mathscr{L}_i$, $i = 1, 2$, is independent of the initial indexing procedure.

The problem is redolent of that of the class of limiting distributions of the normalized maximum of independent, indentically distributed variables.

Here, it will be recalled [3] that the class of limiting distributions comprises the three distribution types ($\alpha > 0$):

$$
\begin{aligned}
L_{1\alpha}(x) &= \exp\{-(-x)^{\alpha}\}, & x &\leq 0, \\
&= 1, & x &> 0, \\
L_{2\alpha}(x) &= 0, & x &\leq 0, \\
&= \exp\{-x^{-\alpha}\}, & x &> 0, \\
L_{3}(x) &= \exp\{-e^{-x}\}.
\end{aligned}
$$

(1.4)

In the case of the minimum of independent, identically distributed random variables, the three (nondegenerate) limiting types are given by $L_j^*(x) = 1 - L_j(-x)$, $j = 1, 2, 3$.

It follows readily from (2.0) that W_n, like the *maximum*, converges almost certainly to $+\infty$ or c according as $F(x) < 1$, all x or $F(c - \epsilon) < 1 = F(c + \epsilon)$, all $\epsilon > 0$. On the other hand, the only nondegenerate limiting types possible in the case of W_n are L_j^*, $j = 1, 2, 3$, namely, those corresponding to the *minimum* of independent identically distributed chance variables.

2. Limiting distributions. Let $\{a_n, b_n, n = 1, 2, \cdots\}$ constitute two sequences of constants with $a_n > 0$. Evidently,

$$
(2.0) \qquad P\left\{\frac{1}{a_n}(W_n - b_n) < x\right\} = 1 - [1 - F^n(a_n x + b_n)]^n.
$$

As a first step in the direction adumbrated, we prove (letting $\mathscr{L}(W)$ denote generically the c.d.f. or "law" of a random variable W)

LEMMA 1. *A necessary and sufficient condition that $\mathscr{L}[(W_n - b_n)/a_n]$ converge to a nondegenerate limiting distribution is that there exist an extended real-valued, nondecreasing, right-continuous function $\gamma(x)$ with $\gamma(\infty) = \infty$, $\gamma(-\infty) = -\infty$ such that for x in the (nonempty) set of points at which $\gamma(x)$ is continuous and finite,*

$$
(2.1) \qquad F(a_n x + b_n) = 1 - \frac{\log n - \gamma(x) + o(1)}{n}.
$$

Proof. Suppose that (2.0) converges to a nondegenerate c.d.f. $\Gamma(x)$. Clearly, $F^n(a_n x + b_n) = o(1)$ and for $0 < \Gamma(x) < 1$

$$
n \log[1 - F^n(a_n x + b_n)] = [1 + o(1)] \log[1 - \Gamma(x)],
$$

implying

$$
-nF^n(a_n x + b_n) = [1 + o(1)] \log[1 - \Gamma(x)].
$$

Thus, if

$$
(2.2) \qquad \gamma(x) = \log(-\log[1 - \Gamma(x)]),
$$

we have

$$\log n + n \log F(a_n x + b_n) = \gamma(x) + o(1).$$

This implies $n[\,1 - F(a_n x + b_n)]^2 = o(1)$ and consequently also

$$\log n - n[\,1 - F(a_n x + b_n)] = \gamma(x) + o(1)$$

which is tantamount to (2.1).

Retracing the argument in the reverse direction, (2.1) implies that (2.0) converges to $\Gamma(x) = 1 - \exp\{ - e^{\gamma(x)}\}$ at all continuity points in the domain where γ is finite, and thus at all continuity points.

This leads to

THEOREM 1(^{2}). *The class $\mathscr{L}_1$ of nondegenerate limiting distributions of the normalized minimax W_n of independent identically distributed random variables (as in (1.1)-(1.3)) contains only the types* $(\alpha > 0)$

$$
\begin{aligned}
L_{1\alpha}^*(x) &= 0, && x < 0,\\
&= 1 - \exp\{ - x^\alpha\}, && x \geq 0,\\
(2.3) \qquad L_{2\alpha}^*(x) &= 1 - \exp\{ - (- x)^{-\alpha}\}, && x < 0,\\
&= 1, && x \geq 0,\\
L_3^*(x) &= 1 - \exp\{ - e^x\}.
\end{aligned}
$$

Proof. Suppose that (2.0) converges to a nondegenerate limiting distribution Γ, whence according to the lemma, (2.1) holds.

Define, for any real δ,

$$(2.4) \qquad m = m_n(\delta) = \{\text{greatest integer} \leq n(1 + \delta/\log n)\}.$$

Then, if A denotes the set of points at which $\gamma(x)$ is continuous and finite, for $x \in A$ we have from (2.1),

$$
\begin{aligned}
F(a_m x + b_m) &= 1 - \frac{\log m - \gamma(x)}{m} + o(m^{-1})\\
(2.5) \qquad &= 1 - \frac{\log n + \delta(\log n)^{-1} + o[\,(\log n)^{-1}] - \gamma(x)}{n[\,1 + \delta(\log n)^{-1}]} + o(n^{-1})\\
&= 1 - \frac{\log n - (\gamma(x) + \delta) + o(1)}{n}.
\end{aligned}
$$

Let us call (x, γ) a point of increase if $\gamma(x - \epsilon) < \gamma < \gamma(x + \epsilon)$ for all $\epsilon > 0$. Then at every point of increase (x, γ), the inverse function $h(\gamma)$ is

(^{2})*Added in proof.* Theorem 1 extends to rectangular matrices.

uniquely defined and $h(\gamma) = x$. Further, let $G(x)$ denote an inverse function of $1 - F(x)$ (any monotone version will do in spite of the ambiguity due to intervals where $F(x)$ is constant). It follows from (2.1) that for any sequence of numbers $\epsilon_n \to 0$ and point of increase $(h(\gamma), \gamma)$,

$$(2.6) \qquad \lim_{n \to \infty} \frac{1}{a_n} \left[G\left(\frac{\log n - \gamma + \epsilon_n}{n} \right) - b_n \right] = h(\gamma).$$

Let $m = m_n(\delta)$ be as in (2.4) and note that

$$\frac{\log m - \gamma}{m} = \frac{\log n - (\gamma + \delta) + o(1)}{n}.$$

Consequently, if $(h(\gamma + \delta), \gamma + \delta)$ and $(h(\gamma), \gamma)$ are points of increase, from (2.5) and (2.6)

$$G\left(\frac{\log n - (\gamma + \delta) + o(1)}{n} \right)$$
$$= a_n[h(\gamma + \delta) + o(1)] + b_n = a_m[h(\gamma) + o(1)] + b_m.$$

Applying the preceding to γ_0, γ_1, and γ, we have for all real δ

$$a_n[h(\gamma + \delta) + o(1)] + b_n = a_m[h(\gamma) + o(1)] + b_m,$$
$$(2.7) \qquad a_n[h(\gamma_0 + \delta) + o(1)] + b_n = a_m[h(\gamma_0) + o(1)] + b_m,$$
$$a_n[h(\gamma_1 + \delta) + o(1)] + b_n = a_m[h(\gamma_1) + o(1)] + b_m,$$

whence,

$$(2.8) \qquad \frac{h(\gamma + \delta) - h(\gamma_1 + \delta)}{h(\gamma_1 + \delta) - h(\gamma_0 + \delta)} = \frac{h(\gamma) - h(\gamma_1)}{h(\gamma_1) - h(\gamma_0)}$$

provided all points involved correspond to points of increase of γ and the denominators do not vanish.

The nondegeneracy of the limiting distribution implies that $h(\gamma)$ is not constant for $-\infty < \gamma < \infty$. Now suppose h is constant on some proper subinterval I of $-\infty < \gamma < \infty$, say $h(\gamma) \equiv \beta$ in I while $h(\gamma) \neq \beta$ outside I. Then choosing $\gamma_1 \in I$, $\gamma \notin I$, $\gamma_0 \notin I$ and δ such that $\gamma_0 + \delta \notin I$ but both $\gamma_1 + \delta$ and $\gamma + \delta$ are in I would controvert (2.8). Since at most countably many γ values do not correspond to points of increase, such a choice can be made. Therefore h is strictly monotone.

Next, suppose that $\gamma_0 + i\delta$, $i = 0, 1, \cdots, k^* + 1$ correspond to points of increase. Let $\gamma = \gamma_1 + \delta$, $\gamma_i = \gamma_0 + (i - 1)\delta$ in (2.8) and verify inductively that for all integers k, $1 \leq k \leq k^*$

$$\frac{h(\gamma_0 + (k + 1)\delta) - h(\gamma_0 + k\delta)}{h(\gamma_0 + k\delta) - h(\gamma_0 + (k - 1)\delta)} = \frac{h(\gamma_0 + 2\delta) - h(\gamma_0 + \delta)}{h(\gamma_0 + \delta) - h(\gamma_0)} = \lambda \qquad \text{(say)}.$$

Thus,

$$h(\gamma_0 + (k+1)\delta) - h(\gamma_0 + k\delta) = \lambda^k [\, h(\gamma_0 + \delta) - h(\gamma_0) \,], \qquad 0 \leqq k \leqq k^*,$$

which necessitates

(2.9)
$$h(\gamma_0 + k\delta) = c_1 \lambda^k + d_1, \qquad 0 \leqq k \leqq k^* \text{ for } \lambda \neq 1,$$
$$= c_1 \lambda + d_1, \qquad 0 \leqq k \leqq k^* \text{ for } \lambda = 1.$$

Select an interval (x_1, x_2) such that $-\infty < x_1 < x_2 < \infty$ and $-\infty < \gamma(x_1 +)$ $< \gamma(x_2 -) < \infty$. In the range $\gamma(x_1 -)$ to $\gamma(x_2 +)$ at most a countable set of γ values do not correspond to points of increase. Hence, in arbitrary neighborhoods of $\gamma(x_1 -)$ and $\gamma(x_2 +)$ we can find γ_1^* and γ_2^* such that $\gamma_1^* < \gamma_2^*$ and all of the values $\gamma_1^* + j2^{-k}(\gamma_2^* - \gamma_1^*)$, $j = 0, 1, 2, \cdots, 2^k$, $k = 1$, $2, \cdots$, correspond to points of increase. From (2.9) and monotonicity, for $\gamma(x_1 -) < \gamma < \gamma(x_2 +)$,

(2.10)
$$h(\gamma) = c e^{\gamma/\alpha} + d, \qquad \alpha \neq 0$$

or

(2.10)′
$$h(\gamma) = c\gamma + d.$$

Letting x_1 decrease and x_2 increase so that $\gamma(x_1 -) \to -\infty$ and $\gamma(x_2 +) \to \infty$, it follows that (2.10) or (2.10)′ holds for all γ. Note that $\gamma(x_1 +) = -\infty$ for finite x_1 entails $\alpha > 0$, and $c > 0$ whereas $\gamma(x_2) = +\infty$ for finite x_2 requires $\alpha < 0$, and $c < 0$. Summarizing, we find three possibilities:

(2.11)

(i)
$$h(\gamma) = c\gamma + d, \qquad c > 0,$$
$$\gamma(x) = \frac{x-d}{c}, \qquad -\infty < x < \infty, \quad c > 0,$$

(ii)
$$h(\gamma) = c e^{\gamma/\alpha} + d, \qquad \alpha > 0, \ c > 0,$$
$$\gamma(x) = \alpha \log \frac{x-d}{c}, \qquad x > d, \quad c > 0, \quad \alpha > 0,$$

(iii)
$$h(\gamma) = c e^{-\gamma/\alpha} + d, \qquad \alpha > 0, \quad c < 0,$$
$$\gamma(x) = -\alpha \log \frac{x-d}{c}, \quad x < d, \quad c < 0, \quad \alpha > 0.$$

These correspond via (2.2) to the distribution types $L_3^*(x)$, $L_{1\alpha}^*(x)$, $L_{2\alpha}^*(x)$ respectively of (2.3) and the proof is therefore complete.

DEFINITION. A c.d.f. F will be said to belong to the minimax domain of attraction of a c.d.f.[3] L^* (in symbols, $F \in \mathscr{D}_{mM}(L^*)$) if there exist sequences b_n and $a_n > 0$ for which $\lim_{n \to \infty} (1 - [\, 1 - F^n(a_n x + b_n) \,]^n) = L^*(x)$. In similar fashion, we may say that F belongs to the "maximum" domain of

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[3] Of course, if $F \in \mathscr{D}_{mM}(L^*(x))$, then $F \in \mathscr{D}_{mM}(L^*(ax+b))$ for every $a > 0$ and real b so it is the distribution "type" that is pertinent.

attraction of L (written $F \in \mathscr{D}_M(L)$), F belongs to the "minimum" domain of attraction of L^* (written $F \in \mathscr{D}_m(L^*)$) or F belongs to the maximin domain of attraction of L (symbolically, $F \in \mathscr{D}_{Mm}(L)$) according as $\lim_{n\to\infty} F^n(\tilde{a}_n x + \tilde{b}_n) = L(x)$, $\lim_{n\to\infty} (1 - [1 - F(\bar{a}_n x + \bar{b}_n)]^n) = L^*(x)$ or $\lim_{n\to\infty} (1 - [1 - F(a'_n x + b'_n)]^n)^n = L(x)$.

Next, for any c.d.f. F, define

$$(2.12) \qquad a_n^* = G\left(\frac{\log n - 1}{n}\right) - G\left(\frac{\log n}{n}\right), \qquad n \geq 1,$$

where G is again an inverse function of $1 - F$. Then $a_{m_n(\delta)}^*$ is implicitly defined by (2.12) and (2.4) and

$$a_{m_n(\delta)}^* = G\left(\frac{\log n - 1 - \delta + \bar{\epsilon}'_n}{n}\right) - G\left(\frac{\log n - \delta + \bar{\epsilon}_n}{n}\right)$$

where $\bar{\epsilon}_n$, $\bar{\epsilon}'_n$ are both $o(1)$ and depend on δ.

As will be seen in the ensuing theorems, the domain to which F is attracted depends on the limit of $a_{m_n(\delta)}^*/a_n^*$ or more precisely upon that of the related quantity $r_n(1, \delta)$ (see (2.13)). Define

$$(2.13) \qquad \begin{aligned} r_n(\beta, \delta) &= r_n(\beta, \delta; \epsilon_n, \epsilon'_n) \\ &= \frac{1}{a_n^*}\left[G\left(\frac{\log n - \delta - \beta + \epsilon'_n}{n}\right) - G\left(\frac{\log n - \delta + \epsilon_n}{n}\right)\right] \end{aligned}$$

where ϵ'_n, ϵ_n are sequences of real numbers approaching zero.

THEOREM 2. *In order that* $F \in \mathscr{D}_{mM}(L_3^*)$ *with* $a_n = a_n^*$, *it is necessary that*

$$(2.14) \qquad \lim_{n\to\infty} r_n(\beta, \delta) = \beta \text{ for all real } \beta, \delta \text{ and } \epsilon_n = o(1) = \epsilon'_n.$$

Conversely, if $\lim r_n(\beta, \delta)$ *exists*[4] *for all* β, δ *and* $\epsilon'_n = o(1) = \epsilon_n$ *and if for all real* δ *and* $\epsilon_n = o(1) = \epsilon'_n$,

$$(2.14)' \qquad \lim_{n\to\infty} r_n(1, \delta) = 1,$$

then $F \in \mathscr{D}_{mM}(L_3^*)$ *with* $a_n = a_n^*$ *and* $b_n = G(\log n/n)$.

Necessity. (2.14) is a simple consequence of (2.6) and (2.10)'.

Sufficiency. Set $a_n = a_n^*$. It follows directly from (2.14)' that

$$\lim_{n\to\infty} \frac{1}{a_n}\left[G\left(\frac{\log n - \delta - k + \epsilon'_n}{n}\right) - G\left(\frac{\log n - \delta + \epsilon_n}{n}\right)\right] = k$$

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[4] It is not supposed that this limit is independent of ϵ_n and ϵ'_n. Also, an example can be adduced to show that (2.14)' by itself is insufficient.

for all $\epsilon_n = o(1) = \epsilon_n'$, all real δ and positive integers k, hence for all negative integers k, whence

$$\lim_{n \to \infty} \frac{1}{a_n} \left[G\left(\frac{\log n - \delta + \epsilon_n'}{n} \right) - G\left(\frac{\log n - \delta + \epsilon_n}{n} \right) \right]$$

$$(2.15) \qquad = \lim_{n \to \infty} \frac{1}{a_n} \left[G\left(\frac{\log n - \delta + \epsilon_n'}{n} \right) - G\left(\frac{\log n - \delta - k}{n} \right) \right.$$

$$\left. + G\left(\frac{\log n - \delta - k}{n} \right) - G\left(\frac{\log n - \delta + \epsilon_n}{n} \right) \right] = 0$$

for all real δ and all ϵ_n, ϵ_n' which are $o(1)$. This shows further that $\lim r_n(\beta, \delta; \epsilon_n, \epsilon_n')$ is independent of ϵ_n and ϵ_n'.

Consider next the subsequence $r_{n'}(\beta, \delta; \epsilon_{n'}, \epsilon_{n'}')$ of $r_n(\beta, \delta; \epsilon_n, \epsilon_n')$ where $n' = m_n(\delta')$. This subsequence may be delineated as

$$\frac{1}{a_{m_n(\delta')}} \left[G\left(\frac{\log n - (\delta + \delta') - \beta + \epsilon_n^{*'}}{n} \right) - G\left(\frac{\log n - (\delta + \delta') + \epsilon_n^{*}}{n} \right) \right]$$

where $\epsilon_n^{*'}$ and ϵ_n^{*} are again $o(1)$. In view of $(2.14)'$ and (2.15), this subsequence has the same limit as $r_n(\beta, \delta + \delta'; \epsilon_n, \epsilon_n')$. Thus, $\lim r_n(\beta, \delta)$ is independent of δ (as well as ϵ_n, ϵ_n'). Consequently, for any positive integer k and all real δ

$$k \lim_{n \to \infty} \frac{1}{a_n} \left[G\left(\frac{\log n - \delta - 1/k}{n} \right) - G\left(\frac{\log n - \delta}{n} \right) \right]$$

$$= \sum_{i=0}^{k-1} \lim_{n \to \infty} \frac{1}{a_n} \left[G\left(\frac{\log n - \delta - (i+1)/k}{n} \right) - G\left(\frac{\log n - \delta - i/k}{n} \right) \right]$$

$$= 1,$$

which readily implies for all integers j, k, real δ and $\epsilon_n = o(1) = \epsilon_n'$, that

$$(2.16) \quad \lim_{n \to \infty} \frac{1}{a_n} \left[G\left(\frac{\log n - \delta - j/k + \epsilon_n'}{n} \right) - G\left(\frac{\log n - \delta + \epsilon_n}{n} \right) \right] = j/k.$$

Next, let $r_i'(r_i'')$ be a monotone sequence of rationals converging from below (above) to the real number β. Then, from (2.16)

$$r_i' \leqq \lim_{n \to \infty} \frac{1}{a_n} \left[G\left(\frac{\log n - \beta - \delta + \epsilon_n'}{n} \right) - G\left(\frac{\log n - \delta + \epsilon_n}{n} \right) \right] \leqq r_i''$$

and consequently $\lim_{n \to \infty} r_n(\beta, \delta) = \beta$ for all real β, δ and sequences ϵ_n, ϵ_n' tending to zero. In particular, setting $b_n = G(\log n / n)$, for all real x

$$\lim_{n \to \infty} \frac{1}{a_n} \left[G\left(\frac{\log n - x + \epsilon_n}{n} \right) - b_n \right] = x.$$

Hence, for any $\epsilon > 0$ and $n > N_\epsilon$,

$$a_n(x - \epsilon) + b_n \leqq G\left(\frac{\log n - x + \epsilon_n}{n}\right) \leqq a_n(x + \epsilon) + b_n$$

or equivalently,

$$F[a_n(x - \epsilon) + b_n] \leqq 1 - \frac{\log n - x + \epsilon_n}{n} \leqq F[a_n(x + \epsilon) + b_n].$$

Thus, for $n > N_\epsilon^*$

$$1 - \frac{\log n - x + 2\epsilon + \epsilon_n}{n} \leqq F(a_n x + b_n) \leqq 1 - \frac{\log n - x - 2\epsilon + \epsilon_n}{n}$$

which is tantamount to (2.1) for $\gamma(x) = x$ and therefore proves sufficiency.

THEOREM 3. *In order that* $F \in \mathscr{D}_{mM}(L_{1\alpha}^*)$ *with* $a_n = a_n^*(e^{1/\alpha} - 1)^{-1}$, *it is necessary that*

$$(2.17) \qquad \lim_{n \to \infty} r_n(\beta, \delta) = e^{\delta/\alpha}(e^{\beta/\alpha} - 1)(e^{1/\alpha} - 1)^{-1}$$

for all real β, δ *and* $\epsilon_n' = o(1) = \epsilon_n$. *Conversely, if* $\lim r_n(\beta, \delta)$ *exists*[4] *for all real* β, δ *and* $\epsilon_n = o(1) = \epsilon_n'$ *and if for some* $\alpha > 0$ *and all real* δ *and* $\epsilon_n' = o(1) = \epsilon_n$,

$$(2.17)' \qquad \lim_{n \to \infty} r_n(1, \delta) = e^{\delta/\alpha},$$

then $F \in \mathscr{D}_{mM}(L_{1\alpha}^*)$ *with* $a_n = a_n^*(e^{1/\alpha} - 1)^{-1}$, $b_n = G(\log n/n) - a_n$.

THEOREM 4. *In order that* $F \in \mathscr{D}_{mM}(L_{2\alpha}^*)$ *with* $a_n = a_n^*(1 - e^{-1/\alpha})^{-1}$, *it is necessary that*
$$(2.18) \qquad \lim r_n(\beta, \delta) = e^{-\delta/\alpha}(e^{-\beta/\alpha} - 1)(e^{-1/\alpha} - 1)^{-1}$$
for all real δ, β *and* $\epsilon_n = o(1) = \epsilon_n'$. *Conversely, if* $\lim r_n(\beta, \delta)$ *exists*[4] *for all real* δ, β *and* $\epsilon_n' = o(1) = \epsilon_n$ *and if for some* $\alpha > 0$ *and all real* δ *and* $\epsilon_n = o(1) = \epsilon_n'$,

$$(2.18)' \qquad \lim_{n \to \infty} r_n(1, \delta) = e^{-\delta/\alpha},$$

then $F \in \mathscr{D}(L_{2\alpha}^*)$ *with* $a_n = a_n^*(1 - e^{-1/\alpha})^{-1}$, $b_n = G(\log n/n) + a_n$.

Proofs. The necessity of (2.17) or (2.18) follows directly from (2.6) and (2.11).

The proof of sufficiency for Theorem 4 requires only minor emmendations from that of Theorem 3 and therefore only the latter will be given. Set $a_n = a_n^*(e^{1/\alpha} - 1)^{-1}$. Then, as in the proof of Theorem 2, a simple consequence of (2.17)' is

$$(2.19) \lim_{n \to \infty} \frac{1}{a_n}\left[G\left(\frac{\log n - \delta - k + \epsilon_n'}{n}\right) - G\left(\frac{\log n - \delta + \epsilon_n}{n}\right)\right] = e^{k/\alpha}(e^{k/\alpha} - 1)$$

for all real δ, *all* integers k and all $\epsilon_n = o(1) = \epsilon_n'$. Again, the case $k = 0$ implies that $\lim r_n(\beta, \delta; \epsilon_n, \epsilon_n')$ is independent of ϵ_n and ϵ_n'.

Exactly as in Theorem 2, $\lim e^{-\delta/\alpha} r_n(\beta, \delta)$ is independent of δ. Consequently, using also (2.19)

$$e^{\delta/\alpha}(e^{1/\alpha} - 1)$$

$$= \lim_{n \to \infty} \sum_{i=0}^{k-1} \frac{1}{a_n} \left[G\left(\frac{\log n - \delta - (i+1)/k}{n} \right) - G\left(\frac{\log n - \delta - i/k}{n} \right) \right]$$

$$= \sum_{i=0}^{k-1} \exp\left[\frac{\delta + i/k}{\alpha} \right]$$

$$\cdot \lim_{n \to \infty} \frac{e^{-\delta/\alpha}}{a_n} \left[G\left(\frac{\log n - \delta - 1/k}{n} \right) - G\left(\frac{\log n - \delta}{n} \right) \right],$$

implying for all positive integers k, real δ and $\epsilon_n = o(1) = \epsilon_n'$ that

$$\lim_{n \to \infty} \frac{1}{a_n} \left[G\left(\frac{\log n - \delta - 1/k + \epsilon_n'}{n} \right) - G\left(\frac{\log n - \delta + \epsilon_n}{n} \right) \right] = e^{\delta/\alpha}(e^{1/k\alpha} - 1),$$

which, in turn, readily yields

$$(2.20) \quad \lim_{n \to \infty} \frac{1}{a_n} \left[G\left(\frac{\log n - \delta - r + \epsilon_n'}{n} \right) - G\left(\frac{\log n - \delta + \epsilon_n}{n} \right) \right] = e^{\delta/\alpha}(e^{r/\alpha} - 1)$$

for all rational r, real δ and $\epsilon_n = o(1) = \epsilon_n'$. This is extended to real r exactly as in Theorem 2. From this extended version of (2.20), for x real, all $\epsilon > 0$ and $n > N_\epsilon$,

$$e^{x/\alpha} - 1 - \epsilon < \frac{1}{a_n} \left[G\left(\frac{\log n - x + \epsilon_n}{n} \right) - G\left(\frac{\log n}{n} \right) \right] < e^{x/\alpha} - 1 + \epsilon.$$

Setting $b_n = G(\log n / n) - a_n$, for all $\epsilon > 0$ and $n > N_\epsilon$,

$$a_n(e^{x/\alpha} - \epsilon) + b_n < G\left(\frac{\log n - x + \epsilon_n}{n} \right) < a_n(e^{x/\alpha} + \epsilon) + b_n,$$

which is tantamount to (2.1) with $\gamma(x) = \alpha \log x$, $\alpha > 0$, $x > 0$, and completes the proof.

In contradistinction to the case of the minimum of independent identically distributed random variables, virtually all the standard distributions (including the Cauchy) $\in \mathscr{D}_{mM}(L_3^*)$ if indeed there is a nondegenerate limit distribution. Notwithstanding, $\mathscr{D}_{mM}(L_{1\alpha}^*)$ and $\mathscr{D}_{mM}(L_{2\alpha}^*)$ are not empty as is seen from the following

EXAMPLE. Let the inverse of $1 - F$ be defined by

$$G(v) = c_1 e^{\rho u^2/2} + c_2, \qquad u = -\log v + \left(1 - \frac{1}{\log v}\right) \cdot \log\left(-\log v\right)$$

where $c_1 + c_2 > 0, \rho \neq 0$ and v is small[5]. If $v = (\log n - \gamma)/n$, then $u = \log n + (\gamma + o(1))/\log n$, whence

$$G\left[\frac{\log n - \gamma}{n}\right] = c_1 \exp\left[\rho(\log n)^2/2 + \rho\gamma + o(1)\right] + c_2$$

$$= a_n(x + o(1)) + b_n,$$

where $a_n = |c_1| \exp\left[\rho(\log n)^2/2\right]$, $b_n = c_2$, $x = (\text{sgn}\,\rho) \cdot \exp(\rho\gamma)$ and $c_1 < 0$, $c_2 > 0$ for $\rho < 0$ while $c_1 > 0$ if $\rho > 0$. Hence for sufficiently large n,

$$1 - F(a_n x + b_n) = \frac{\log n - \gamma + o(1)}{n}.$$

Thus the limiting distribution of the normalized minimax is of the type $L_{1\alpha}^*$ or $L_{2\alpha}^*$ depending on the sign of ρ.

Next, we note a link between $\mathscr{D}_M(L_3)$ and $\mathscr{D}_{mM}(L_3^*)$. If $F \in \mathscr{D}_M(L_3)$, then as in Lemma 1, for all real x [3],

$$(2.23) \qquad\qquad F(a_n' x + b_n') = 1 - \frac{1}{n} e^{-x} + o(n^{-1}).$$

Suppose that (2.23) continues to hold for $x_n = x/\log n - \log\log n$, i.e.,

$$
(2.24) \qquad
\begin{aligned}
&F\left(a_n'\left(\frac{x}{\log n} - \log\log n\right) + b_n'\right) \\
&\qquad = 1 - \frac{1}{n}\exp\left\{-\left(\frac{x}{\log n} - \log\log n\right)\right\} + o(n^{-1}).
\end{aligned}
$$

Then, setting

$$a_n = a_n'(\log n)^{-1}, \quad b_n = b_n' - a_n' \log\log n,$$

$$(2.25) \qquad F(a_n x + b_n) = 1 - \frac{\log n}{n}\left[1 - \frac{x}{\log n} + o\left(\frac{1}{\log n}\right)\right] + o(n^{-1}),$$

$$= 1 - \frac{\log n}{n} + \frac{x}{n} + o(n^{-1}),$$

so that $F \in \mathscr{D}_{mM}(L_3^*)$ according to (2.1) and (2.11).

On the other hand, (2.24) may be contravened and (2.25) fail to be the correct normalization even though the domains of attraction are unchanged. For example if $F(x) = 1 - \exp\{-x/(1-x)\}$, $0 < x < 1$, then, as noted

---

[5]The nature of $G(v)$ for v bounded away from zero is unimportant from the standpoint of which limit distribution F is attracted to. Thus (as may be necessary for monotonicity), G may be modified for v bounded sufficiently far from zero.

in [3], $F \in \mathscr{D}_M(L_3)$ with $a_n' = (\log n)^{-2}$, $b_n' = (1 + \log n)^{-1} \log n$. However, $F \in \mathscr{D}_{mM}(L_3^*)$ with

$$a_n = (1 + \log n - \log \log n)^{-2} (\log n)^{-1},$$

$$b_n = (1 + \log n - \log \log n)^{-1} (\log n - \log \log n).$$

3. **Asymptotic independence of minimax X_{ij} and maximin X_{ij}.** The joint distribution of W_n and Z_n is easy to obtain in principle but difficult to analyze. Let $u < v$ and $A_{st}^{(n)}$ denote the event that all rows of χ_n in (1.2) have an element greater than v and all columns have an element less than or equal to u while exactly s of the n^2 elements are no greater than u and exactly t of the elements are greater than v. Then

$$P\{Z_n \leqq u, W_n > v\} = \sum_{s=n}^{n^2-n} \sum_{t=n}^{n^2-s} P\{A_{st}^{(n)}\}$$

(3.1)

$$= \sum_{s=n}^{n^2-n} \sum_{t=n}^{n^2-s} a_{st}^{(n)} F^s(u) [1 - F(v)]^t [F(v) - F(u)]^{n^2-s-t}$$

where $a_{st}^{(n)}$ is the number of ways of arranging s white balls, t red balls and $n^2 - s - t$ green balls in an $n \times n$ square array in which each column has at least one white ball and each row at least one red ball.

Equation (3.1) does not seem amenable to establishing the asymptotic independence of W_n and Z_n. On the other hand it hints at the feasibility of an "order statistics" approach. For this reason we shall first concern ourselves with $\min_i \max_j Y_{ij}$ and $\max_j \min_i Y_{ij}$ where the Y_{ij} are obtained by assigning the integers 1 to n^2 at random among the n^2 positions of the $n \times n$ matrix. The study of this minimax and maximin will be facilitated by the use of a construction which simultaneously yields the random variables X_{ij}, Y_{ij} and related random variables, whose distributions are linked to those occurring in the classical occupancy problem [2].

The construction is rather long and will be divided into several stages. Two of these stages will then be reexamined in more detail.

STAGE 1. Let the integer 1 be assigned to one of the n^2 positions (i, j), $1 \leqq i \leqq n$, $1 \leqq j \leqq n$, at random. Select one of the remaining $n^2 - 1$ positions at random to hold the integer 2. Continue in this fashion until each column has at least one element and let S be the number of integers thus required. The first S integers will be called "*little*."

STAGE 2. Assign the integer n^2 to one of the remaining $n^2 - S$ positions at random. Then select one of the remaining $n^2 - S - 1$ positions to hold $n^2 - 1$. Continue until each row has at least one of the elements n^2, $n^2 - 1$, $\cdots$ or until all $n^2 - S$ remaining integers are exhausted. Let T be the number of integers thus required. These integers will be called "*big*."

STAGE 3. Assign the remaining $n^2 - S - T$ positions (if any) at random to the remaining integers. Define Y_{ij} to be the integer in position (i,j) and note that

$$\max_j \min_i Y_{ij} = S,$$

$$\min_i \max_j Y_{ij} = n^2 - T + 1 \quad \text{if} \quad S + T < n^2.$$

Define T^* by

$$\min_i \max_j Y_{ij} = n^2 - T^* + 1$$

and observe that $T^* = T$ if $S + T < n^2$ and $T^* \geq T$ in general.

STAGE 4. If U is uniformly distributed on $(0,1)$, $x = F^{-1}(U)$, where F^{-1} is an inverse of F, has the c.d.f. F. If F^{-1} is not unique select it to be compatible with G, i.e., $F^{-1}(y) = G(1 - y)$. Now select n^2 independent observations on U and let $X_i = F^{-1}(U_i)$ where the U_i is the ith ordered observations. Thus $U_1 \leq U_2 \leq \cdots \leq U_{n^2}$ and $X_1 \leq X_2 \leq \cdots \leq X_{n^2}$. Let $X_{ij} = X_{Y_{ij}}$. Then

$$\max_j \min_i X_{ij} = X_S = F^{-1}(U_S),$$

$$\min_i \max_j X_{ij} = X_{n^2 - T^* + 1} = G(1 - U_{n^2 - T^* + 1}).$$

The random variables X_{ij} so constructed are independent, indentically distributed with common c.d.f. F.

We now elaborate on Stage 1. First note that if exactly i columns are occupied when r integers have been placed, the probability that $r + 1$ will be in a new column is $n(n - i)/(n^2 - r) \geq (n - i)/n$. Let U_{is} be independent and uniformly distributed on $(0,1)$, $1 \leq i \leq n$, $s \geq 1$.

We shall select the positions of the "little" numbers via the random variables U_{is}. Concomitantly, the U_{is} will be used for two related occupancy problems. We proceed as follows.

Divide $(0,1)$ into n^2 equal intervals corresponding to the n^2 positions. Let the interval into which U_{11} falls determine the position (i,j) of the integer 1. Then, redivide $(0,1)$ into $n^2 - 1$ equal intervals, the first $n(n - 1)$ of which correspond to positions in the unoccupied columns and the remainder to the other vacant positions. Let the value of U_{21} determine the position of the integer 2. However, if it falls in an unoccupied column, i.e., $U_{21} \leq p_{21} = n(n - 1)/(n^2 - 1)$, use U_{31} to determine the position of the integer 3; if 2 lands in an already occupied column, i.e., $U_{21} > p_{21}$, use U_{22} to locate the position of 3. Continue inductively. If U_{is} has assigned the integer r to an unoccupied column, divide $(0,1)$ into $n^2 - r$ equal intervals, the first $n(n - i)$ of which correspond to the positions in the unoccupied columns while the others correspond to the remaining vacant positions. In this case, $U_{i+1,1}$ is

used to determine the position of the integer $r + 1$. On the other hand, if U_{is} has consigned r to an already occupied column, again divide $(0, 1)$ into $n^2 - r$ equal intervals but now let the first $n(n - i + 1)$ of these correspond to the positions in the vacant columns while the others correspond to remaining unoccupied positions. In this case, let $U_{i,s+1}$ determine the position of $r + 1$.

Then $S = S_1 + S_2 + \cdots + S_n$ where $S_1 + S_2 + \cdots + S_i$ is the integer whose placement first leads to the occupation of i columns. If we have observed r integers occupying i columns prior to the observations of $U_{i+1,s}$, it will lead to an $(i + 1)$st column if $U_{i+1,s} \leq p_{i+1,s} = n(n - i)/(n^2 - r)$. Note that the $p_{i+1,s}$ are random, $p_{i+1,s} \geq (n - i)/n$ and that S_i is the smallest subscript s for which $U_{is} \leq p_{is}$.

In the classical occupancy problems, successive observations fall into one of n cells independently and with equal probability. When i cells have been occupied, the probability that the next observation will occupy an $(i + 1)$st cell is $p_{i+1}^{(1)} = (n - i)/n$. In such an experiment let $S^{(1)} = S_1^{(1)} + S_2^{(1)} + \cdots + S_n^{(1)}$ where $S_1^{(1)} + \cdots + S_i^{(1)}$ is the number of observations taken when i cells first become occupied. The *same* uniformly distributed variables U_{is}, $1 \leq i \leq n$, $s \geq 1$ are used to construct $S_i^{(1)}$ as follows. Let $S_i^{(1)}$ be the smallest subscript for which $U_{is} \leq p_i^{(1)} = (n - i + 1)/n$. We also define $S_i^{(2)}$ as the smallest subscript s for which $U_{is} \leq p_i^{(2)} = (n - i + 1)/(n - 3 \log n)$ and $S^{(2)} = S_1^{(2)} + S_2^{(2)} + \cdots + S_n^{(2)}$.

The second stage may be elaborated similarly using uniformly and independently distributed random variables V_{it} where $T = T_1 + T_2 + \cdots + T_n$, T_i is the smallest subscript t for which $V_{it} \leq q_{it} = m_{it}/(n^2 - r_{it})$, where m_{it} is the number of unoccupied cells in the $(n - i + 1)$ rows which are still unoccupied by "big" numbers and r_{it} is the total number of positions occupied when V_{it} is observed. Let $q_i^{(1)} = p_i^{(1)}$, $q_i^{(2)} = p_i^{(2)}$, $q_i^{(3)} = (n - i + 1)(n - 9 \log n)/n^2$, and let $T_i^{(j)}$ be the smallest subscript t for which $V_{it} \leq q_i^{(j)}$ and $T^{(j)} = \sum_{i=1}^{n} T_i^{(j)}$. Note that $q_i^{(3)} < q_i^{(1)} < q_i^{(2)}$. If $S + T < 3n \log n$, $p_i^{(1)} \leq p_{is} \leq p_i^{(2)}$ and $q_{it} \leq q_i^{(2)}$. If no row has more than 9 $\log n$ "little" elements, $q_i^{(3)} \leq q_{it}$.

In outline, the derivation of the asymptotic distribution of (S, T) consists of showing that $S + T = 2n \log n + O_p(n)$ and that with probability approaching one no row contains more than 9 $\log n$ "little" elements. Thus with large probability S is sandwiched between $S^{(1)}$ and $S^{(2)}$ which are relatively close together. Similarly T and $T^{(1)}$ are sandwiched between $T^{(2)}$ and $T^{(3)}$. But $S^{(1)}$ and $T^{(1)}$ are independent and their asymptotic distributions are related to the solution of the occupancy problem.

We commence the derivation by presenting some immediate consequences of the construction.

(i) If $p_i^{(1)} \leq p_{is} \leq p_i^{(2)}$, $S_i^{(1)} \geq S_i \geq S_i^{(2)}$. If $q_i^{(3)} \leq q_{it} \leq q_i^{(2)}$, $T_i^{(3)} \geq T_i \geq T_i^{(2)}$. Since $q_i^{(3)} < q_i^{(1)} < q_i^{(2)}$, $T_i^{(3)} \geq T_i^{(1)} \geq T_i^{(2)}$.

(ii) The $T_i^{(j)}$ and $S_i^{(j)}$ have geometric distributions. Each $S_i^{(j)}$ is independent of each $T_{i'}^{(j')}$. Furthermore $E(S^{(1)}) = \sum_{i=1}^{n-1} n/(n-i) = n[\log n + O(1)]$ and $\sigma^2_{S(1)} \sim n^2$.

(iii) $S_i^{(1)} = S_i^{(2)} + \Delta_i$, where $\Delta_i \geqq 0$, $P\{\Delta_i = 0\} = p_i^{(1)}/p_i^{(2)}$, and

$$P\{\Delta_i = r\} = \frac{p_i^{(2)} - p_i^{(1)}}{p_i^{(2)}} (1 - p_i^{(1)})^{r-1} p_i^{(1)}, \qquad r \geqq 1.$$

(iv) $T_i^{(3)} = T_i^{(2)} + \Xi_i$, where $\Xi_i \geqq 0$, $P\{\Xi_i = 0\} = q_i^{(3)}/q_i^{(2)}$, and

$$P\{\Xi_i = r\} = \frac{q_i^{(2)} - q_i^{(3)}}{q_i^{(2)}} (1 - q_i^{(3)})^{r-1} q_i^{(3)}, \qquad r \geqq 1.$$

LEMMA 2.

$$\Delta = \sum_{i=1}^{n} \Delta_i = O_p(\log n)^2,$$

$$\Xi = \sum_{i=1}^{n} \Xi_i = O_p(\log n)^2.$$

Proof.

$$E(\Delta_i) = \frac{p_i^{(2)} - p_i^{(1)}}{p_i^{(2)}} \sum_{r=1}^{\infty} r p_i^{(1)} [1 - p_i^{(1)}]^{r-1} = \frac{p_i^{(2)} - p_i^{(1)}}{p_i^{(2)} p_i^{(1)}},$$

$$E(\Delta) = \sum_{i=1}^{n} \frac{3 \log n}{n - i + 1} = O(\log n)^2.$$

Since $\Delta \geqq 0$, it follows that $\Delta = O_p(\log n)^2$. Similarly we find

$$E(\Xi_i) = \frac{q_i^{(2)} - q_i^{(3)}}{q_i^{(2)} q_i^{(3)}},$$

$$E(\Xi) = \frac{12n \log n - 27(\log n)^2}{n - 9 \log n} \sum_{i=1}^{n} \frac{1}{n - i + 1} = O(\log n)^2$$

and

$$\Xi = O_p(\log n)^2.$$

LEMMA 3.

$$\mathscr{L}\left(\frac{S^{(1)} - n \log n}{n}, \frac{T^{(1)} - n \log n}{n} \right) \to L_3 \times L_3.$$

Proof. According to known results on the classical occupancy problem [2], the probability of occupying all n cells with $n[\log n + x]$ observations approaches $\exp(-e^{-x})$. Hence

$$P\{S^{(1)} \leqq n[\log n + x]\} \to L_3(x).$$

Since $S^{(1)}$ and $T^{(1)}$ are independent and identically distributed, the lemma follows.

To prepare for the next step we digress with a minor technical lemma.

LEMMA 4. *If X has the binomial distribution $\mathscr{B}([am \log m], m^{-1})$, $a \geqq 1$, $m \geqq 1$ then $P\{X \geqq 4a \log m\} = o(m^{-2a})$ as $m \to \infty$.*

Proof. Since [1], $P\{X \geqq b\} \leqq E[e^{t(X-b)}]$ for $t > 0$,

$$P\{X \geqq 4a \log m\} \leqq E\{\exp[X - 4a \log m]\}$$

$$= \left[\frac{e}{m} + \left(1 - \frac{1}{m}\right)\right]^{[am \log m]} e^{-4a \log m}$$

$$\leqq \exp\left\{am \log m \left[\log\left(1 + \frac{e-1}{m}\right) - \frac{4}{m}\right]\right\} = o(m^{-2a}).$$

LEMMA 5. *With probability approaching one, no row has more than $9 \log n$ "little" elements.*

Proof. Consider the first $M = 2n \log n$ positions selected at random from the $n \times n$ matrix. After all of these are selected, the probability that the next will be in the first row is no greater than $n/(n^2 - 2n \log n)$. Hence, in the course of assigning the first M positions, the number of occupied cells in the first row is less than a variable with the binomial distribution $\mathscr{B}(M, m^{-1})$ where $m = n - 2 \log n$. Applying Lemma 4 with $2 < a < 9/4$, the probability that the number of occupied positions in the first row exceeds $9 \log n$ is $o(n^{-4})$. Let A_M, (A_S) be the event that *some* row contains more than $9 \log n$ of the first M, (S) integers assigned. Then $P(A_S) \leqq P(A_M) + P(S > M)$. But $P(A_M) = o(n^{-3})$. From Lemma 3, $S^{(1)} = n \log n + O_p(n)$. Since $p_i^{(1)} \leqq p_{is}$, $S \leqq S^{(1)}$ and hence $P(S > M) = o(1)$. Thus $P(A_S) \to 0$ which is the desired result.

THEOREM 5. *If the first n^2 integers are assigned at random to the positions of the $n \times n$ matrix yielding entries Y_{ij}, $S = \max_j \min_i Y_{ij}$, and $n^2 - T^* + 1 = \min_i \max_j Y_{ij}$, then $S = S^{(1)} + O_p(\log n)^2$, $T^* = T^{(1)} + O_p(\log n)^2$ where $S^{(1)}$ and $T^{(1)}$ are independent. Furthermore*

$$\lim_{n \to \infty} \mathscr{L}\left(\frac{S - n \log n}{n}, \frac{T^* - n \log n}{n}\right) = L_3 \times L_3.$$

Proof. With probability approaching one, $S^{(1)} < (3/2)n \log n$ in which case $p_i^{(1)} \leqq p_{is} \leqq p_i^{(2)}$. Applying Lemma 5, we have with probability approaching one, $q_i^{(3)} \leqq q_{it}$, $T \leqq T^{(3)} \leqq (3/2)n \log n$ and $q_i^{(3)} \leqq q_{it} \leqq q_i^{(2)}$. Thus with probability approaching one, $S^{(1)} \geqq S \geqq S^{(2)}$, $T^{(3)} \geqq T \geqq T^{(2)}$, and $T^{(3)} \geqq T^{(1)} \geqq T^{(2)}$.

It follows from Lemma 2 that $S = S^{(1)} + O_p(\log n)^2$, $T = T^{(1)} + O_p(\log n)^2$, and $T^* = T$ in probability (i.e., with probability approaching one). Applying Lemma 3 completes the proof of Theorem 5.

To study the asymptotic behavior of (W_n, Z_n), we require a lemma which is essentially a corollary of §2. Here we assume that $F \in \mathscr{D}_{mM}(L_j^*)$ for some $j, b_n, a_n > 0$ and $F \in \mathscr{D}_{Mm}(L_k)$ for some choice of $k, b_n', a_n' > 0$. Let $h(\gamma)$ and $\bar{h}(\gamma)$ be the corresponding functions.

LEMMA 6. *If* $\gamma_n = O_p(1)$ *as* $n \to \infty$, *and* $F \in \mathscr{D}_{mM}(L_j^*) \cap \mathscr{D}_{Mm}(L_k)$,

(3.2)
$$\frac{1}{a_n}\left\{G\left[\frac{\log n - \gamma_n}{n}\right] - b_n\right\} = h(\gamma_n) + o_p(1),$$

$$\frac{1}{a_n'}\left\{F^{-1}\left[\frac{\log n + \gamma_n}{n}\right] - b_n'\right\} = \bar{h}(\gamma_n) + o_p(1).$$

Proof. Equation (2.6) states the first result for fixed $\gamma_n = \gamma$. Since $h(\gamma)$ is continuous and monotone, any bounded γ interval I can be subdivided into a finite number of subintervals over each of which $h(\gamma)$ increases by less than ϵ. There is an N such that for $n > N$

(3.3)
$$\left|\frac{1}{a_n}\left[G\left(\frac{\log n - \gamma}{n}\right) - b_n\right] - h(\gamma)\right| < \epsilon$$

for the γ which are end points of these intervals. Since G and h are monotone (3.3) holds with ϵ replaced by 2ϵ for all γ on I and $n > N$. It follows that for every deterministic sequence $\gamma_n = O(1)$, the first line of (3.2) holds with $o_p(1)$ replaced by $o(1)$. In view of the properties of the "in probability" calculus, [1], [4], (3.2) is thus valid when $\gamma_n = O_p(1)$ is random. The second part of (3.2) follows in a similar fashion from the analogue of equation (2.6).

LEMMA 7.

$$W_n = G\left(\frac{T^* + o_p(n)}{n^2}\right), \qquad Z_n = F^{-1}\left(\frac{S + o_p(n)}{n^2}\right).$$

Proof. Given $S = s = n \log n + zn$, the Sth order statistic U_S from the sample of n^2 observations on U (see Stage 4) has mean s/n^2 and variance $s(n^2 - s + 1)/(n^2 + 1)^2(n^2 + 2)$. Applying the Chebyshev inequality and the fact that $S = n \log n + O_p(n)$ it follows that

$$U_S - \frac{S}{n^2} = O_p\left[\frac{(n \log n)^{1/2}}{n^2}\right] = o_p(n^{-1}).$$

Hence, recalling the equations of Stage 4,

$$Z_n = F^{-1}\left(\frac{S + o_p(n)}{n^2}\right).$$

The statement about W_n follows similarly.

THEOREM 6. *If*

$$\mathscr{L}\left(\frac{W_n - b_n}{a_n}\right) \to L^* \ and \ \mathscr{L}\left(\frac{Z_n - b_n'}{a_n'}\right) \to L,$$

then

$$\mathscr{L}\left(\frac{W_n - b_n}{a_n}, \ \frac{Z_n - b_n'}{a_n'}\right) \to L^* \times L.$$

Proof. First we note that the existence of the limits L^* and L implies the existence of corresponding $h(\gamma)$ and $\bar{h}(\gamma)$ and the relevance of Lemma 6. According to Lemma 7 and the fact that $T^* - T^{(1)} = O_p(\log n)^2$,

$$W_n = G\left[\frac{T^* + o_p(n)}{n^2}\right] = G\left[\frac{T^{(1)} + o_p(n)}{n^2}\right].$$

Define γ_n by setting $(\log n - \gamma_n)/n$ equal to $[T^{(1)} + o_p(n)]/n^2$ in the argument of G. Then $\gamma_n = O_p(1)$ and by Lemma 6

$$\frac{W_n - b_n}{a_n} = h(\gamma_n) + o_p(1) = h\left[\frac{n \log n - T^{(1)}}{n} + o_p(1)\right] + o_p(1).$$

Since h is uniformly continuous on bounded intervals and $(n \log n - T^{(1)})/n = O_p(1)$,

$$h\left[\frac{n \log n - T^{(1)}}{n} + o_p(1)\right] = h\left[\frac{n \log n - T^{(1)}}{n}\right] + o_p(1).$$

Hence

(3.4)
$$\frac{W_n - b_n}{a_n} = h\left(\frac{n \log n - T^{(1)}}{n}\right) + o_p(1)$$

and

$$\mathscr{L}\left\{h\left(\frac{n \log n - T^{(1)}}{n}\right)\right\} \to L^*.$$

Similarly

(3.5)
$$\frac{Z_n - b_n'}{a_n'} = \bar{h}\left(\frac{S^{(1)} - n \log n}{n}\right) + o_p(1)$$

and

$$\mathscr{L}\left\{\bar{h}\left(\frac{S^{(1)} - n\log n}{n}\right)\right\} \to L.$$

But $S^{(1)}$ and $T^{(1)}$ are independent. Hence

$$\mathscr{L}\left\{h\left(\frac{n\log n - T^{(1)}}{n}\right), \bar{h}\left(\frac{S^{(1)} - n\log n}{n}\right)\right\} \to L^* \times L,$$

$$\left(\frac{W_n - b_n}{a_n}, \frac{Z_n - b_n'}{a_n'}\right) = \left[h\left(\frac{n\log n - T^{(1)}}{n}\right), \bar{h}\left(\frac{S^{(1)} - n\log n}{n}\right)\right] + o_p(1)$$

and Theorem 6 follows.

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