THE ESSENTIAL SPECTRUM OF ELLIPTIC DIFFERENTIAL OPERATORS IN $L^p(R_n)$

BY

ERIK BALSLEV[1]

Introduction. The main aim of the present paper is to determine the essential spectrum and index of a class of elliptic differential operators in $L^p(R_n)$ (see Definition 1.3). By a well-known theorem of perturbation theory, the essential spectrum and index of an operator A are unchanged under addition of an operator B, which is compact relative to A (see Definition 1.1). §1 contains without proof the definitions and main theorems of this theory.

In §3 we consider the case of an elliptic differential operator P_p with constant coefficients in $L^p(R_n)$ perturbed by lower-order terms. The spectrum of the constant coefficient operator is given in Theorem 3.5. Then we determine a rather large class of operators, which are compact with respect to P_p. Theorem 3.9 contains the main result on the essential spectrum and index of the perturbed constant coefficient operator.

The preliminary work leading to the compactness conditions is done in §2. The graph norm of the elliptic constant coefficient operator is equivalent to the W^p_k-norm (Definition 2.2). Therefore, the problem of finding compactness conditions is essentially reduced to the problem of finding conditions in order that the embedding of $W^p_k(R_n)$ in $L^p(R_n, b)$ with a weight function b be compact. The main result in this direction is stated in Lemmas 2.11 and 2.15.

1. Perturbation of operators in Banach spaces.

1.1. Definition. Let $\mathfrak{B}$ be a Banach space with the norm $\|\cdot\|$, and let A be a closed, densely defined, linear operator in $\mathfrak{B}$. For $x \in D(A)$, the A-norm $\|x\|_A$ is defined by

$$\|x\|_A = \|x\| + \|Ax\|.$$

$D(A)$ provided with the A-norm is a Banach space. Let B be a linear operator with $D(B) \supset D(A)$. Then B is said to be A-defined. If $B|D(A)$ is a bounded operator from $D(A)$ with the A-norm into $\mathfrak{B}$, B is called A-bounded with the A-norm $\|B\|_A$. B is said to be A-ϵ-bounded if there exists, for every $\epsilon > 0$, a number $K(\epsilon) > 0$ such that

$$(*) \qquad \|Bx\| \leqq \epsilon\|Ax\| + K(\epsilon)\|x\| \quad \text{for } x \in D(A).$$

Received by the editors March 6, 1964.

(1) The preparation of this paper was sponsored in part by the National Science Foundation grant G-22982.

If $B|D(A)$ is a compact operator from $D(A)$ with the A-norm into $\mathfrak{B}$, B is called A-compact.

1.2. LEMMA. *If there exists $\epsilon < 1$ such that Definition 1.1, $(*)$ holds, then $A+B$ is closed, and the A-norm and the $(A+B)$-norm are equivalent on $D(A)$. C is A-compact if and only if C is $(A+B)$-compact.*

Furthermore, if A is essentially self-adjoint, and B is symmetric, then $A+B$ is essentially self-adjoint.

1.3. DEFINITION. The Fredholm domain $\Phi(A)$ is the set of complex numbers, λ, such that the null space $\mathfrak{N}(A-\lambda)$ of $A-\lambda$ is of finite dimension $\alpha_\lambda(A)$, and the range $\mathfrak{R}(A-\lambda)$ of $A-\lambda$ is closed and of finite codimension $\beta_\lambda(A)$. The essential spectrum $\sigma_e(A)$ is the complement of $\Phi(A)$. $\Phi_+(A)$ is the set of λ such that $\mathfrak{R}(A-\lambda)$ is closed, $\alpha_\lambda(A) < \infty$ and $\beta_\lambda(A) = \infty$. $\Phi_-(A)$ is the set of λ such that $\mathfrak{R}(A-\lambda)$ is closed, $\alpha_\lambda(A) = \infty$ and $\beta_\lambda(A) < \infty$. $\sigma_s(A)$ is the complement of $\Phi(A) \cup \Phi_+(A) \cup \Phi_-(A)$. For $\lambda \in \Phi(A)$ the index $I_\lambda(A)$ is $\alpha_\lambda(A) - \beta_\lambda(A)$. The spectrum of A is denoted by $\sigma(A)$ and the resolvent set by $\rho(A)$.

1.4. THEOREM. *$\Phi(A)$, $\Phi_+(A)$ and $\Phi_-(A)$ are open sets. In each component of $\Phi(A)$, $I_\lambda(A)$ is constant, and $\alpha_\lambda(A)$ is constant except possibly in a discrete set, where it is larger. In each component of $\Phi_+(A)$ and $\Phi_-(A)$ the same holds for $\alpha_\lambda(A)$ and $\beta_\lambda(A)$, respectively.*

If B is closed and A-compact, $\Phi(A+B) = \Phi(A)$, $\Phi_+(A+B) = \Phi_+(A)$ and $\Phi_-(A+B) = \Phi_-(A)$.

For $\lambda \in \Phi(A)$, $I_\lambda(A+B) = I_\lambda(A)$.

For $\lambda \in \Phi(A)$ $(\Phi_+(A), \Phi_-(A))$ there exists $\epsilon(\lambda) > 0$, such that $\lambda \in \Phi(A+B)$ $(\Phi_+(A+B), \Phi_-(A+B))$ for $\|B\|_{A-\lambda} < \epsilon(\lambda)$.

If A is a self-adjoint operator in a Hilbert space, and B is A-compact, then the resolvent sets of A and $A+B$ are equal except for at most a discrete set of points $S = \{\lambda_i\}$. Any real number λ_i of S is an eigenvalue of A with $\alpha_{\lambda_i}(A) < \infty$ or an eigenvalue of $A+B$ and $(A+B)^*$ with $\alpha_{\lambda_i}(A+B) = \alpha_{\lambda_i}(A+B)^* < \infty$ (possibly both); any nonreal number λ_i of S is an eigenvalue of $A+B$ with $\alpha_{\lambda_i}(A+B) < \infty$, while $\overline{\lambda_i}$ is an eigenvalue of $(A+B)^*$, with $\alpha_{\lambda_i}(A+B)^* = \alpha_{\lambda_i}(A+B)$.

Proof. We refer to [8], [10] and [14].

1.5. DEFINITION. A singular sequence for the operator A is a sequence, $\{\phi_n\} \subset D(A)$, such that

(i) $\|\phi_n\| < K$,

(ii) $\{\phi_n\}$ is noncompact,

(iii) $A\phi_n \to 0$.

1.6. LEMMA. *$\lambda \in \sigma_s(A) \cup \Phi_-(A)$ if and only if there exists a singular sequence for $A-\lambda$.*

Proof. This is proved for Hilbert spaces in [14]. For Banach spaces in general it will be proved in [16].

1.7. Lemma. *Let A be a closed, densely defined operator in a Banach space $\mathfrak{B}$ and let A^* be its adjoint in $\mathfrak{B}^*$. Then A has closed range if and only if A^* has closed range.*

Proof. We refer to [10].

2. **Embedding operators in function spaces.**

2.1. Notations. We use the following notations:

R_n is n-dimensional euclidean space.

$$D_j = i^{-1}\frac{\partial}{\partial x_j}.$$

If $\alpha = (\alpha_1, \cdots, \alpha_n)$ is any n-tuple of non-negative integers, then

$$|\alpha| = \sum_{j=1}^{n} \alpha_j,$$

$$D^\alpha = \prod_{j=1}^{n} D_j^{\alpha_j}.$$

In the following, all functions are complex-valued, measurable functions on R_n. If C is the space of these functions (where two functions are identified if they are equal almost everywhere), then all function spaces considered are subspaces of C, and we shall omit explicit reference to R_n in our notations.

2.2. Definition. Let p be a real number, $1 < p < \infty$, k a non-negative integer. Then W_k^p is the set of functions u, in L^p, for which all derivatives $D^\beta u$ of order $|\beta| \leqq k$ belong to L^p. W_k^p is a Banach space under the W_k^p-norm,

$$\|u\|_{k,p} = \left\{ \sum_{|\beta| \leqq k} \|D^\beta u\|_p^p \right\}^{1/p}.$$

2.3. Definition. Let $b_\beta \in C$. Then $B_{\beta,p}$ is the operator in L^p defined by

$$D(B_{\beta,p}) = \{u \in L^p \mid b_\beta D^\beta u \in L^p\}$$

and

$$B_{\beta,p} u = b_\beta D^\beta u \quad \text{for } u \in D(B_{\beta,p}).$$

We set $B_{0,p} = B_p$.

2.4. Definition. Let T be the embedding operator of W_k^p into L^p, $Tu = u$. Then $B_{\beta,p}$ is said to be W_k^p-bounded if $D(B_{\beta,p}) \supset T(W_k^p)$, and $B_{\beta,p} T$ is a bounded operator from W_k^p into L^p. We set $\|B_{\beta,p}\|_{W_k^p} = \|B_{\beta,p} T\|$.

$B_{\beta,p}$ is W_k^p-ϵ-bounded if $D(B_{\beta,p}) \supseteq T(W_k^p)$, and if there exists, for every $\epsilon > 0$, a constant $K(\epsilon)$ such that

$$\|B_{\beta,p} u\|_p \leqq \epsilon \sum_{|\alpha| = k} \|D^\alpha u\|_p + K(\epsilon)\|u\|_p, \qquad u \in W_k^p.$$

$B_{\beta,p}$ is W_k^p-compact if $D(B_{\beta,p}) \supset T(W_k^p)$ and $B_{\beta,p}T$ is a compact operator from W_k^p into L^p.

2.5. CLASSES OF FUNCTIONS. We set

$$S_{x,r} = \{y \in R_n \mid |x-y| \leqq r\},$$

$$S_x = S_{x,1},$$

$$S' = \{x \in R_n \mid |x| = 1\}.$$

We introduce the following subspaces of C (2.1), $1 < p < \infty$.

(1) $$L_{\text{loc}}^p = \left\{ f \,\middle|\, \int_{S_x} |f(y)|^p dy < K(x) \right\},$$

(2) $$L_{\text{loc}}^\infty = \{ f \mid |f(y)| < K(x) \text{ for a.e. } y \in S_x \},$$

(3) $$M^p = \left\{ f \,\middle|\, \int_{S_x} |f(y)|^p dy < K \text{ for } x \in R_n \right\},$$

(4) $$N^p = \left\{ f \in M^p \,\middle|\, \int_{S_x} |f(y)|^p dy \to 0 \text{ as } |x| \to \infty \right\},$$

(5) $$Q_{l,a}^p = \left\{ f \,\middle|\, \int_{S_x} |f(y)|^p |x-y|^{pl-n+a} dy < K \text{ for } x \in R_n \right\},$$

(6) $$R_{l,a}^p = \left\{ f \in Q_{l,a}^p \,\middle|\, \int_{S_x} |f(y)|^p |x-y|^{pl-n+a} dy \to 0 \text{ as } |x| \to \infty \right\},$$

(7) $$U_{l,a}^p = \left\{ f \,\middle|\, \int_0^1 |f(\rho,\omega)|^p \rho^{pl-1+a} d\rho < K, \int_r^{r+1} |f(\rho,\omega)|^p d\rho < K, \right.$$
$$\left. r \geqq 1, \omega \in S' \right\},$$

(8) $$V_{l,a}^p = \left\{ f \in U_{l,a}^p \,\middle|\, \int_r^{r+1} |f(\rho,\omega)|^p d\rho \to 0 \text{ as } r \to \infty, \text{ uniformly} \right.$$
$$\left. \text{for } \omega \in S' \right\},$$

(9) $$Y_\alpha^p = \left\{ f \,\middle|\, \int_0^1 |f(\rho,\omega)|^\alpha \rho^{n-1} d\rho < K, \int_r^{r+1} |f(\rho,\omega)|^p d\rho < K, \right.$$
$$\left. r \geqq 1, \omega \in S' \right\},$$

(10) $$Z_\alpha^p = \left\{ f \in Y_\alpha^p \,\middle|\, \int_r^{r+1} |f(\rho,\omega)|^p d\rho \to 0 \text{ as } r \to \infty, \text{ uniformly} \right.$$
$$\left. \text{for } \omega \in S' \right\}.$$

(11) The set of almost radial functions

$$\mathrm{AR} = \left\{ f \,\middle|\, \sup_{\omega \in S'} |f(r,\omega)| \leqq C \inf_{\omega \in S'} |f(r,\omega)| \text{ for } 0 < r < \infty \right\}.$$

2.6. Lemma. *The classes of functions defined in* 2.5 *satisfy the following relations:*

(a) $Q^p_{l,a} \subset M^p$; $R^p_{l,a} \subset N^p$; $U^p_{n/p,0} = Y^p_p$; $V^p_{n/p,0} = Z^p_p$.

For $pl - n + a \geqq 0$, $Q^p_{l,a} = M^p$; $R^p_{l,a} = N^p$;

(b) *If* $pl - n < 0$, *then, for every* $c < 0$, *there exists* $a < 0$ *such that* $Q^p_{l,c} \cap N^p \subset R^p_{l,a}$;

(c) $Y^p_p \subset M^p$; $Z^p_p \subset N^p$.

Proof. (a) is obvious, and (c) is easy to prove. For the proof of (b), let $f \in Q^p_{l,c} \cap N^p$. For $1/p_0 + 1/q_0 = 1$ we have, by Hölder's inequality,

$$\begin{aligned} &\int_{S_x} |f(y)|^p |x-y|^{pl-n+a} dy \\ (1) \qquad &\leqq \left\{ \int_{S_x} |f(y)|^p dy \right\}^{1/p_0} \left\{ \int_{S_x} |f(y)|^p |x-y|^{q_0(pl-n+a)} dy \right\}^{1/q_0}. \end{aligned}$$

We choose $q_0 < 1 + c/(pl-n)$. Let

$$a_0 = \frac{c - (q_0 - 1)(pl - n)}{q_0}.$$

Then $a_0 < 0$, and (1) gives, for this choice of q_0 and $a = a_0$:

$$\begin{aligned} &\int_{S_x} |f(y)|^p |x-y|^{pl-n+a_0} dy \\ (2) \qquad &\leqq \left\{ \int_{S_x} |f(y)|^p dy \right\}^{1/p_0} \left\{ \int_{S_x} |f(y)|^p |x-y|^{pl-n+c} dy \right\}^{1/q_0}. \end{aligned}$$

From (2) follows, since $f \in Q^p_{l,c} \cap N^p$,

$$(3) \qquad \int_{S_x} |f(y)|^p |x-y|^{pl-n+a_0} dy \to 0 \quad \text{as } |x| \to \infty,$$

so that $f \in R^p_{l,a_0}$.

2.7. Definition. χ_R is the characteristic function of $S_{0,R}$, i.e.,

$$\chi_R(x) = \begin{cases} 1 & \text{for } |x| \leqq R, \\ 0 & \text{for } |x| > R, \end{cases}$$

$$T_R = \{ x \in R_n \,|\, |x| > R \}.$$

For a real, set

$$D(R,l,a) = \sup_{|x|>R} \int_{S_x} |b(y)|^p |x-y|^{pl-n+a} dy,$$

$$E(R) = \sup_{|x|>R} \int_{S_x} |b(y)|^p dy,$$

$$F(R) = \sup_{\omega \in S'} \sup_{r \geq R} \int_r^{r+1} |b(\rho,\omega)|^p d\rho.$$

$A_{p,R}$ is the operator B_p of 2.3 corresponding to the function $b\chi_R$, and $B_{p,R}$ is the operator corresponding to $b(1-\chi_R)$.

$I_{p,R}$ is B_p corresponding to the function χ_R.

2.8. Lemma. *Criteria for W_k^p-ϵ-boundedness of B_p.*
(a) $k \leqq n/p$. (1) *If there exists $a<0$ such that $D(R, k, a) < \infty$, then $B_{p,R+1}$ is W_k^p-bounded, and*

$$\|B_{p,R+1}\|_{W_k^p}^p \leqq K(a) D(R,k,a),$$

where $K(a)$ is independent of R.

(2) *If there exists $a<0$ such that $b \in Q_{k,a}^p$, then B_p is W_k^p-ϵ-bounded.*

(3) *If B_p is W_k^p-bounded, then $b \in Q_{k,a}^p$ for every $a>0$.*

(b) $k > n/p$. (1) *If $E(R) < \infty$, then $B_{p,R+1}$ is W_k^p-bounded, and*

$$\|B_{p,R+1}\|_{W_k^p}^p \leqq KE(R).$$

(2) *It is necessary in order that B_p be W_k^p-bounded and sufficient in order that B_p be W_k^p-ϵ-bounded, that $b \in M^p$.*

(a2) *is essentially contained in* [5, *Lemma* 16].

Proof. (a) $k \leqq n/p$. Since C_0^∞ is dense in W_k^p, it suffices to consider $u \in C_0^\infty$.

(1) We make use of the following inequality, proved in [5, (5.2), p. 86] (the proof is valid for any k):

For any $r_0 > 0$ and $-1 < a < 0$, there exists $K(a) > 0$ such that, for $0 < r \leqq r_0$, $u \in C_0^\infty$,

$$(1') \quad |u(x)|^p \leqq K(a) \left\{ r^{-a} \int_{S_{x,r}} |x-y|^{pk-n+a} \left(\sum_{|\alpha|=k} |D^\alpha u(y)|^p \right) dy + r^{-pk-a} \int_{S_{x,r}} |x-y|^{pk-n+a} |u(y)|^p dy \right\}.$$

We can obviously assume that $-1 < a < 0$ in the given inequality,

$$D(R,k,a) < \infty.$$

Setting $r = r_0 = 1$, multiplying (1′) by $|b(x)|^p$, integrating with respect to x over T_R, and interchanging the order of integration, we arrive at

$$\begin{aligned}\|B_{p,R+1}u\|_p^p &= \int_{T_{R+1}} |b(x)u(x)|^p dx \\ &\leqq K(a)\int_{T_R}\left\{\int_{S_y}|b(x)|^p|x-y|^{pk-n+a}dx\right\} \\ &\qquad\qquad \cdot\left\{\sum_{|\alpha|=k}|D^\alpha u(y)|^p + |u(y)|^p\right\}dy \\ &\leqq K(a)D(R,k,a)\int_{T_R}\left\{\sum_{|\alpha|=k}|D^\alpha u(y)|^p + |u(y)|^p\right\}dy \\ &\leqq K(a)D(R,k,a)\|u\|^p_{W_k^p}.\end{aligned} \tag{2'}$$

Therefore, $B_{p,R+1}$ is W_k^p-bounded, and $\|B_{p,R+1}\|^p \leqq K(a)D(R,k,a)$.

(2) Proceeding as in (1), choosing $r_0 = 1$, and integrating over R_n, we obtain, for $0 < r \leqq 1$:

$$\begin{aligned}\|B_p u\|_p^p &= \int_{R_n}|B_p u(x)|^p dx \\ &\leqq K(a)r^{-a}\int_{R_n}\left\{\int_{S_{y,r}}|b(x)|^p|x-y|^{pk-n+a}dx\right\}\left\{\sum_{|\alpha|=k}|D^\alpha u(y)|^p\right\}dy \\ &\quad + K(a)r^{-pk-a}\int_{R_n}\left\{\int_{S_{y,r}}|b(x)|^p|x-y|^{pk-n+a}\right\}|u(y)|^p dy \\ &\leqq K(a)\left\{\sup_{y\in R_n}\int_{S_y}|b(x)|^p|x-y|^{pk-n+a}dy\right\} \\ &\qquad \cdot\left\{r^{-a}\int_{R_n}\sum_{|\alpha|=k}|D^\alpha u(y)|^p dy + r^{-pk-a}\int_{R_n}|u(y)|^p dy\right\}.\end{aligned} \tag{3'}$$

Choosing r sufficiently small, we arrive at an inequality of the form

$$\|B_p u\|_p^p < \epsilon^p \sum_{|\alpha|=k}\|D^\alpha u\|_p^p + K(\epsilon)\|u\|_p^p,$$

from which the desired inequality follows, and B_p is W_k^p-ϵ-bounded.

(3) Let $\phi \in C_0^\infty(R_n)$ be a function such that

$$\phi(x) = 1 \quad \text{for } |x| \leqq 1.$$

For $a > 0$, $x \in R_n$, we set

$$f_{a,x}(y) = |x-y|^{k-n/p+a/p}\phi(x-y).$$

It is easy to check that $f_{a,x} \in W_k^p$, and the set

$$F_a = \{f_{a,x} \mid x \in R_n\}$$

is a bounded subset of W^p_k.

Using the boundedness of B_p on the set F_a, we obtain, for $x \in R_n$:

$$\int_{S_x} |b(y)|^p |x-y|^{kp-n+a} dy \leqq \|B_p f_{a,x}\|^p \leqq \|B_p\|^p_{W^p_k} \|f_{a,x}\|^p_{W^p_k} < K,$$

so that $b \in Q^p_{k,a}$.

(b) For $k > n/p$, $M^p = Q^p_{k,n-kp}$ and $E(R) = D(R,k,n-kp)$, hence (b) follows from (a).

2.9. Lemma. *Let $1 < p < r < \infty$, and let g be a function on $S_{0,R}$. We define the operator $G_{p,r}$ by*

$$D(G_{p,r}) = \{ f \in L^r(S_{0,R}) \mid gf \in L^p(S_{0,R}) \}$$

and

$$G_{p,r} f = gf \quad \text{for } f \in D(G_{p,r}).$$

Then $G_{p,r}$ is a bounded operator from $L^r(S_{0,R})$ into $L^p(S_{0,R})$, if

$$\int_{S_{0,R}} |g(x)|^{p/(1-p/r)} dx < \infty.$$

Proof. By Hölder's inequality,

$$\int_{S_{0,R}} |g(x)|^p |f(x)|^p dx \leqq \left\{ \int_{S_{0,R}} |g(x)|^{p/(1-p/r)} dx \right\}^{1-p/r} \left\{ \int_{S_{0,R}} |f(x)|^r dx \right\}^{p/r}$$

hence

$$\|gf\|_{L^p(S_{0,R})} \leqq \left\{ \int_{S_{0,R}} |g(x)|^{p/(1-p/r)} dx \right\}^{1/p-1/r} \|f\|_{L^r(S_{0,R})}$$

2.10. Lemma. *The operator $A_{p,R}$ is W^p_k-compact, if b satisfies one of the following conditions.*

(a) *For $k \leqq n/p$: There exists $a > 0$, such that*

$$\int_{S_{0,R}} |b(x)|^{(n/k)+a} dx < \infty.$$

(b) *For $n/p < k \leqq n/p + 1$: There exists $a > 0$, such that*

$$\int_{S_{0,R}} |b(x)|^{p+a} dx < \infty.$$

(c) *For $k > n/p + 1$:*

$$\int_{S_{0,R}} |b(x)|^p dx < \infty.$$

Proof. For the basic Sobolev-type embedding theorems we refer to [6, Lemma 6] or [7, Vol. II]. From these the proof is as follows:

(a) $k \leqq n/p$. The operator $I_{r,R}$ (2.7) is W_k^p-compact, if there exists $a > 0$ such that

$$\frac{p}{1 - p/r} = n/k + a,$$

and Lemma 2.9 applies.

(b) $n/p < k \leqq n/p + 1$. The operator $I_{r,R}$ is W_k^p-compact for every $r < \infty$. It follows from the assumption on b by Lemma 2.9, that the operator $G_{p,r}$ (2.9) corresponding to the function $b = g$ is bounded for r sufficiently large. Therefore, $A_{p,R}$ is W_k^p-compact.

(c) $k > n/p + 1$. The operator $u \to \chi_R u$ is compact from W_k^p into W_{k-1}^p. Furthermore, $W_{k-1}^p(S_{0,R})$ is contained in $C^0(\bar{S}_{0,R})$, the space of continuous functions on $\bar{S}_{0,R}$ and the embedding of $W_{k-1}^p(S_{0,R})$ in $C^0(\bar{S}_{0,R})$ is a bounded operator. Then the operator $u \to \chi_R u$ is compact from W_k^p into $C^0(\bar{S}_R)$, and $A_{p,R}$ is W_k^p-compact.

2.11. Lemma. *Criteria for W_k^p-compactness of B_p.*

(a) $k \leqq n/p$. *B_p is W_k^p-compact, if*

(i) *For every $R > 0$ there exists $\alpha(R) > 0$ such that*

$$\int_{S_{0,R}} |b(x)|^{(n/k)+\alpha(R)} dx < \infty.$$

(ii) *There exists $a < 0$, such that*

$$b \in R_{k,a}^p.$$

If B_p is W_k^p-compact, then

$$b \in R_{k,a}^p \quad \textit{for every } a > 0.$$

(b) $n/p < k \leqq n/p + 1$. *B_p is W_k^p-compact, if*

(i) *For every $R > 0$ there exists $\alpha(R) > 0$ such that*

$$\int_{S_{0,R}} |b(x)|^{p+\alpha(R)} dx < \infty.$$

(ii) $b \in N^p$.

If B_p is W_k^p-compact, then $b \in N^p$.

(c) $k > n/p + 1$. *B_p is W_k^p-compact if and only if $b \in N^p$.*

Proof. (a) $kp - n \leqq 0$. Suppose that b satisfies (i) and (ii). It follows from (i) by Lemma 2.10 (a), that the operators $A_{p,R}$ are W_k^p-compact for every $R > 0$ (Definition 2.7). By Lemma 2.8 (a2), it follows from (ii) that B_p is W_k^p-bounded, and, by Lemma 2.8 (a1),

$$\|B_p - A_{p,R}\|_{W_k^p} = \|B_{p,R}\|_{W_k^p} \to 0 \quad \text{as } R \to \infty,$$

therefore, B_p is W_k^p-compact.

Suppose, on the other hand, that B_p is W_k^p-compact. By Lemma 2.8 (a3), it follows that $b \in Q_{k,a}^p$ for every $a > 0$. Assume that $b \notin R_{k,a}^p$ for some $a > 0$. Then there exists a sequence of points $x_m \in R_n$, $m = 1, 2, \cdots$, such that

$$|x_m| \to \infty \quad \text{as } m \to \infty$$

and

$$\int_{S_{x_m}} |b(y)|^p |x - y|^{kp-n+a} dy > K > 0.$$

Let $f_{a,x_m} = f_m$ be the functions defined in the proof of Lemma 2.8 (a3). We can assume that the supports of f_k and f_l are disjoint for $k \neq l$. Then $\{f_m\}_{m=1}^{\infty}$ is a bounded sequence in W_k^p,

$$Bf_m(y) \to 0, \text{ as } m \to \infty, \text{ for every } y \in R_n,$$

and

$$\|Bf_m\|_p^p \geqq \int_{S_{x_m}} |b(y)|^p |x - y|^{kp-n+a} dy > K > 0.$$

This contradicts the assumption that B_p was W_k^p-compact; therefore, $b \in R_{k,a}^p$.

(b) $n/p < k \leqq n/p + 1$. It follows as in (a) by Lemmas 2.10 (b) and 2.8 (b) that B_p is W_k^p-compact, if b satisfies (i) and (ii). Since $N^p = R_{k,a}^p$ for $a > 0$, it follows from (a) that $b \in N^p$, if B_p is W_k^p-compact.

(c) $k > n/p + 1$. We prove as in (b) by Lemmas 2.10 (c) and 2.8 (b) that B_p is W_k^p-compact, if $b \in N^p$. The necessity follows as in (b).

2.12. Remark. As noticed in Lemma 2.6 (b) the function b satisfies condition (ii) of Lemma 2.11 (a), if there exists $c < 0$ such that $b \in Q_{k,c}^p \cap N^p$. This implies a result of Birman [3, Theorem 2.2].

2.13. Lemma. (a) *If* $R \geqq 1$, *and* $F(R) < \infty$ (Definition 2.7), *then* $B_{p,R}$ *is* W_k^p-*bounded, and*

$$\|B_{p,R}\|_{W_k^p}^p \leqq KF(R),$$

where K *is independent of* R.

(b) *Let* $k \leqq n/p$, *and suppose that* b *satisfies the conditions*

(i) *There exist* $\delta > 0$ *and* $a < 0$ *such that*

$$\int_{S_x} |b(y)|^p |x - y|^{kp-n+a} dy < K \quad \text{for } |x| \leqq \delta.$$

(ii) $F(\delta) < \infty$. *Then* B_p *is* W_k^p-ϵ-*bounded.*

(c) *Let* $k > n/p$, *and suppose that* $b \in Y_p^p$; *then* B_p *is* W_k^p-ϵ-*bounded.*

Proof. (a) By a slight adjustment of the proof of Lemma 2.8 (a) for $n = 1$, $k = 1$, we arrive at the inequality

$$\int_R^\infty |b(r)f(r)|^p dr \leqq \left\{ \sup_{R \leqq r < \infty} \int_r^{r+1} |b(\rho)|^p d\rho \right\} \tag{1}$$
$$\cdot \left\{ \epsilon \int_R^\infty |f'(\rho)|^p d\rho + K(\epsilon) \int_R^\infty |f(\rho)|^p d\rho \right\}$$

valid for all $R \geqq 0$, $b \in L^p_{\mathrm{loc}}[0, \infty)$, and $f \in L^p[0, \infty)$, $f' \in L^p[0, \infty)$. A proof of this is also found in [1], Lemma 4.

Let $f \in W^p_k$, $R \geqq 1$. We apply (1) to the function $f(r, \omega) r^{(n-1)/p}$, for fixed $\omega \in S'$, and obtain

$$\int_R^\infty |b(r,\omega)|^p |f(r,\omega)|^p r^{n-1} dr \leqq \left\{ \sup_{R \leqq r < \infty} \int_r^{r+1} |b(\rho,\omega)|^p d\rho \right\} \tag{2}$$
$$\cdot \left\{ \epsilon \int_R^\infty \left| \frac{d}{dr} (f(r,\omega) r^{(n-1)/p}) \right|^p dr + K(\epsilon) \int_R^\infty |f(r,\omega)|^p dr \right\}.$$

Since

$$\frac{d}{dr} (f(r,\omega) r^{(n-1)/p}) = \frac{\partial f(r,\omega)}{\partial r} r^{(n-1)/p} + \frac{(n-1)}{p} \frac{1}{r} f(r,\omega) r^{(n-1)/p},$$

we obtain, from (2),

$$\int_R^\infty |b(r,\omega)|^p |f(r,\omega)|^p r^{n-1} dr \leqq \left\{ \sup_{R \leqq r < \infty} \int_r^{r+1} |b(\rho,\omega)|^p d\rho \right\} \tag{3}$$
$$\cdot \left\{ \epsilon \int_R^\infty \left| \frac{\partial f(r,\omega)}{\partial r} \right|^p r^{n-1} dr + K(\epsilon) \int_R^\infty |f(r,\omega)|^p dr \right\}.$$

Integrating (3) with respect to ω over S' we arrive at

$$\int_{T_R} |b(x)|^p |f(x)|^p dx \leqq \left\{ \sup_{\omega \in S'} \sup_{r \geqq R} \int_r^{r+1} |b(r,\omega)|^p d\rho \right\} \tag{4}$$
$$\cdot \left\{ \epsilon \int_{T_R} \sum_{i=1}^n \left| \frac{\partial f}{\partial x_i} \right|^p dx + K(\epsilon) \int_{T_R} |f(x)|^p dx \right\}.$$

From this (a) immediately follows.

(b) $k \leqq n/p$. Suppose, for simplicity, that $\delta = 1$; the proof in the general case is similar.

Let $B_p = A_{p,1} + B_{p,1}$ (Definition 2.7).

From (4) of the proof of (a) and assumption (ii), it follows that $B_{p,1}$ is W_k^p-ϵ-bounded. From Lemma 2.8 (a2) and assumption (i), it follows that $A_{p,1}$ is W_k^p-ϵ-bounded. Therefore, $B_p = A_{p,1} + B_{p,1}$ is W_k^p-ϵ-bounded.

(c) $k > n/p$. This follows from Lemmas 2.8 (b2) and 2.6 (c).

2.14. Lemma. *For $b \in$ AR we obtain the following criteria for W_k^p-ϵ-boundedness*:

(a) $k \leqq n/p$. *B_p is W_k^p-ϵ-bounded if b satisfies the conditions*:

(i) *There exists $\delta > 0$ such that*

$$\inf_{\omega \in S'} |b(r_1, \omega)| \geqq \inf_{\omega \in S'} |b(r_2, \omega)| \quad \text{for } 0 < r_1 < r_2 \leqq \delta.$$

(ii) *There exists $a < 0$ such that $b \in U_{k,a}^p$.*

If B_p is W_k^p-bounded, then $b \in U_{k,a}^p$ for every $a > 0$.

(b) $k > n/p$. *It is necessary in order that B_p be W_k^p-bounded and sufficient in order that B_p be W_k^p-ϵ-bounded that $b \in Y_p^p$.*

Proof. (a) $k \leqq n/p$. Suppose that b satisfies (i) and (ii). Since b, by assumption (ii), satisfies (ii) of Lemma 2.13 (b), we need only prove that b satisfies (i) of Lemma 2.13 (b) for $\delta = 1$. By assumption (ii),

$$I_a = \int_{S_0} |b(y)|^p |y|^{kp-n+a} dy = \int_{S'} \left\{ \int_0^1 |b(r,\omega)|^p r^{kp-1+a} dr \right\} d\omega < \infty.$$

Set

$$U_x = \{y \in S_x \mid |y| \leqq |x-y|\},$$
$$V_x = \{y \in S_x \mid |x-y| < y\}.$$

Since $U_x \subset S_0$,

$$\int_{U_x} |b(y)|^p |x-y|^{kp-n+a} dy \leqq \int_{U_x} |b(y)|^p |y|^{kp-n+a} dy \leqq I_a. \tag{3}$$

Let $y \in V_x$ and let y' be the point symmetric to y with respect to the hyperplane $\{z \mid |z| = |z-x|\}$, $V'_x = \{y' \mid y \in V_x\}$. Then $|y'| = |y-x| < |y| \leqq 2$, and, since b is of class AR and satisfies (i), we have

$$|b(y)| \leqq \sup_{\omega \in S'} |b(\omega|y|)| \leqq K \inf_{\omega \in S'} |b(\omega|y|)| \leqq K \inf_{\omega \in S'} |b(\omega|y'|)| \leqq K|b(y')|.$$

Therefore,

$$\int_{V_x} |b(y)|^p |x-y|^{kp-n+a} dy \leqq K \int_{V'_x} |b(y')|^p |y'|^{kp-n+a} dy' \leqq K I_a. \tag{4}$$

From (3) and (4) we obtain

$$\int_{S_x} |b(y)|^p |x-y|^{kp-n+a} dy < K \quad \text{for } |x| \leqq 1,$$

so that b satisfies 2.13 (b), (i) for $\delta = 1$. By Lemma 2.13 (b), B_p is W_k^p-ϵ-bounded.

Suppose next that B_p is W_k^p-bounded. It is sufficient to prove our statement for a radial function $b(r)$.

Let

$$\phi \in C_0^\infty(R_1) \text{ and } \phi(t) = 1 \quad \text{for } 0 \leqq t \leqq 1,$$

$$g_\rho(r) = \phi(r-\rho)\rho^{(1-n)/p}, \qquad 0 \leqq r < \infty.$$

Then $\{g_\rho\}_{1 \leqq \rho < \infty}$ is a W_k^p-bounded set.

Applying the W_k^p-boundedness of B_p on this set, we obtain

$$\int_\rho^{\rho+1} |b(r)|^p (r/\rho)^{n-1} dr \leqq \|B_p g_\rho\|_p^p < K \qquad \text{for } 1 \leqq \rho < \infty.$$

From this it follows easily that

$$\int_\rho^{\rho+1} |b(r)|^p dr < K \quad \text{for } 1 \leqq \rho < \infty.$$

Since the function f, defined for $0 < r < \infty$,

$$f(r) = r^{k-(n/p)+(a/p)}\phi(r),$$

belongs to W_k^p for every $a > 0$, we conclude that

$$\int_0^1 |b(r)|^p r^{kp-1+a} dr < \infty \quad \text{for every } a > 0,$$

so that $b \in U_{k,a}^p$ for every $a > 0$.

(b) $k > n/p$. By Lemma 2.13 (c) the operator B_p is W_k^p-ϵ-bounded if $b \in Y_p^p$. Suppose next that B_p is W_k^p-bounded. We can assume that b is a radial function $b(r)$. Then, by Lemma 2.8 (b),

$$\int_0^1 |b(r)|^p r^{n-1} dr < \infty.$$

This, together with (a), shows that $b \in Y_p^p$.

2.15. Lemma. *This is a generalization of a result by Birman* (cf. [3]). *The method of the proof is the same as far as the condition at* ∞ *goes.*

(a) $k \leqq n/p$. B_p *is* W_k^p*-compact if*:

(i) *There exists* $a > 0$ *such that* $b \in Z_{(n/k)+a}^p$.

(ii) *For every* $R > 1$ *there exists* $\alpha(R) > 0$ *and* $K(R)$ *such that*

$$\int_1^R |b(r,\omega)|^{(n/k)+\alpha(R)}\,dr < K(R) \quad \text{for } \omega \in S'.$$

If B_p is W_k^p-compact and $b \in \mathrm{AR}$, then

$$b \in V_{k,a}^p \quad \text{for every } a > 0.$$

(b) $n/p < k \leqq (n/p) + 1$. *B is W_k^p-compact if*:
(i) *There exists $a > 0$ such that $b \in Z_{p+a}^p$.*
(ii) *For every $R > 1$ there exists $\alpha(R) > 0$ and $K(R)$ such that*

$$\int_1^R |b(r,\omega)|^{p+\alpha(R)}\,dr < K(R) \quad \text{for } \omega \in S'.$$

If B_p is W_k^p-compact and $b \in \mathrm{AR}$, then $b \in Z_p^p$.
(c) $k > (n/p) + 1$. *B_p is W_k^p-compact if $b \in Z_p^p$.*
If B_p is W_k^p-compact and $b \in \mathrm{AR}$, then

$$b \in Z_p^p.$$

Proof. (1) Sufficiency. In each case it follows, from the corresponding part of Lemma 2.10, that the operators $A_{p,R}$ are W_k^p-compact for $0 < R < \infty$. From Lemma 2.13, it follows that B_p is W_k^p-bounded, and that

$$\|B_{p,R}\|_{W_k^p} \to 0 \quad \text{as } R \to \infty.$$

Therefore B_p is W_k^p-compact.

(2) Necessity when $b \in \mathrm{AR}$. In view of Lemma 2.14 we only have to prove that

$$\int_r^{r+1} |b(\rho,\omega)|^p\,d\rho \to 0 \quad \text{as } r \to \infty \text{ uniformly for } \omega \in S'.$$

This is proved indirectly using the functions g_ρ of the proof of Lemma 2.14 (a), proceeding as in the proof of the necessity part of Lemma 2.11 (a).

2.16. Remark. We obtain sufficient conditions for W_k^p-ϵ-boundedness and W_k^p-compactness of the operators $B_{\beta,p}$ by substitution of $k - |\beta|$ for k in the conditions of Lemmas 2.8, 2.11, 2.13, 2.14 and 2.15. We need only notice that the mapping $f \to D^\beta f$ defines a bounded operator from W_k^p into $W_{k-|\beta|}^p$, and the same mapping defines a W_k^p-ϵ-bounded operator from W_k^p into L^p. The first is obvious, the second is well known (cf. [5]).

3. **Differential operators.**

3.1. Definition. Let $P(\xi_1, \cdots, \xi_n)$ be a polynomial of degree $k \geqq 1$ in n real variables $\xi_1, \cdots, \xi_n, n \geqq 1$,

$$P(\xi) = P(\xi_1, \cdots, \xi_n) = \sum_{|\alpha| \leqq k} a_\alpha \xi^\alpha.$$

$P(D_1, \cdots, D_n)$ is the formal differential operator with constant coefficients obtained by substitution of $D_1, \cdots, D_n$ for $\xi_1, \cdots, \xi_n$ on $P(\xi)$.

$P(D_1, \cdots, D_n)$ is elliptic if there exists $\alpha > 0$ such that the principal part $Q(\xi)$ of $P(\xi)$ satisfies the inequality

$$|Q(\xi)| \geqq \alpha|\xi|^k, \qquad \xi \in R_n.$$

The formal adjoint operator of $P(D_1, \cdots, D_n)$, denoted by $P'(D_1, \cdots, D_n)$, is the operator corresponding to the polynomial

$$P'(\xi) = P'(\xi_1, \cdots, \xi_n) = \sum_{|\alpha| \leqq k} (-1)^{|\alpha|} a_\alpha \xi^\alpha.$$

The set of points $\xi \in R_n$ such that $P(\xi) = 0$ is denoted by $\mathfrak{Z}(P)$. The set of values assumed by $P(\xi)$ for $\xi \in R_n$ is denoted by $\mathfrak{R}(P)$, and its complement in the complex plane by $C\,\mathfrak{R}(P)$.

3.2. Definition. Let $1 \leqq p \leqq \infty$. The maximal differential operator P_p in L^p associated with $P(D_1, \cdots, D_n)$ is defined by

$$D(P_p) = \{f \in L^p \mid P(D_1, \cdots, D_n)f \in L^p\}$$

and

$$P_p f = P(D_1, \cdots, D_n)f \quad \text{for } f \in D(P_p).$$

Here $P(D_1, \cdots, D_n)f$ is taken in the sense of distributions.

The minimal operator P_{p0} is the closure of the restriction of P_p to C_0^∞.

Any closed extension of P_{p0} is denoted by P_{pc}.

For $1 \leqq p < \infty$ the adjoint operator P_{pc}^* of P_{pc} is the usual Banach-space adjoint in L^q, where $1/p + 1/q = 1$ for $p > 1$ and $q = \infty$ for $p = 1$, i.e., $D(P_{pc}^*)$ consists of those elements $f \in L^q$ for which there exists a $g \in L^q$ such that

$$(*) \qquad \int_{R_n} f \cdot P_{pc}\phi \, dx = \int_{R_n} g \cdot \phi \, dx, \qquad \phi \in D(P_{pc}),$$

and $P_{pc}^* f = g$ for $f \in D(P_{pc}^*)$.

The adjoint $P_{\infty c}^*$ in L^1 of $P_{\infty c}$ is the operator defined for those elements f in L^1 for which there exists a g in L^1 such that $(*)$ holds, for all $\phi \in D(P_{\infty c})$, by $P_{\infty c}^* f = g$. $P_{\infty c}^*$ is uniquely defined because $D(P_{\infty c})$ is total on L^1 (cf. [12]).

3.3. Lemma.

(i) $P_{p0}^* = P_q'$ *for* $1 \leqq p \leqq \infty$,

(ii) $P_p^* = P_{q0}'$ *for* $1 < p \leqq \infty$,

(iii) $P_1^* \supseteq P_{\infty 0}'$.

Proof. By the definition of distribution derivatives, $(*)$ holds for all $\phi \in C_0^\infty$ and $f \in D(P_q')$, $g = P_q' f$, $1 \leqq p \leqq \infty$. Thus, P_q' is the adjoint in L^q of the restriction of P_p to C_0^∞. Therefore, P_{p0} exists, $P'_q = P^*_{p0}$, and, for $1 < p \leqq \infty$, $P'_{q0} = P^*_p$ (cf. [12]). It is clear that $P_1^* \supseteq P'_{\infty 0}$, but, in general, equality does not hold (cf. [12]).

3.4. Lemma. *For $p = 2$ and, when $P(D_1, \cdots, D_n)$ is elliptic, for $1 < p < \infty$, the minimal and the maximal operators coincide*: $P_p = P_{p0}$.

Proof. We refer to [6, Theorem 1.11].

3.5. Theorem. *For $1 \leqq p < \infty$, let P_{pc} be any closed extension of P_{p0}, and let $P_{\infty c}$ be a closed extension of P_1^** (Definition 3.2).

Then the following holds:

(i) *If $\mathfrak{Z}(P) \neq \emptyset$, the range, $\mathfrak{R}(P_{pc})$, of P_{pc} is not closed.*

(ii) *If $P(D_1, \cdots, D_n)$ is elliptic and $\mathfrak{Z}(P) = \emptyset$, then P_{pc} has a bounded inverse on L^p which can be represented as convolution with a function in L^1.*

(iii) *If $P(D_1, \cdots, D_n)$ is elliptic,*

$$\sigma(P_{pc}) = \sigma_s(P_{pc}) = \sigma_e(P_{pc}) = \mathfrak{R}(P),$$
$$\rho(P_{pc}) = \Phi(P_{pc}) = \mathbf{C}\,\mathfrak{R}(P).$$

Proof. Let $\mathfrak{F}$ be the Fourier transformation applied to tempered distributions and let $*$ denote convolution.

(i) Suppose that $\mathfrak{Z}(P) \neq \emptyset$ and let $1 \leqq p \leqq 2$; then $\mathfrak{F}$ maps L^p into L^q, $1/p + 1/q = 1$.

It is then clear that there is no solution in L^p of the homogeneous equation

$$P(D_1, \cdots, D_n)f = 0,$$

so that P_p has an inverse P_p^{-1} (cf. [13]). We shall show that P_p has nonclosed range and, consequently, that P_p^{-1} is unbounded.

Let f be a function in L^p such that $\mathfrak{F}f$ has compact support S disjoint from $\mathfrak{Z}(P)$. We prove that $f \in \mathfrak{R}(P_p)$. Let $\psi(\xi)$ be a function in C_0^∞ such that

$$\psi(\xi) = \begin{cases} 1 & \text{for } \xi \in S, \\ 0 & \text{for } \xi \text{ near } \mathfrak{Z}(P). \end{cases}$$

Set

$$\chi(\xi) = \begin{cases} \dfrac{\psi(\xi)}{P(\xi)} & \text{for } \xi \notin \mathfrak{Z}(P), \\ 0 & \text{for } \xi \in \mathfrak{Z}(P). \end{cases}$$

Then

$$\mathfrak{F}^{-1}(\mathfrak{F}f(\xi) \cdot \chi(\xi)) = f * \mathfrak{F}^{-1}\chi.$$

Since $\chi \in C_0^\infty$, it is clear that $\mathfrak{F}^{-1}\chi \in L^1$.

Hence,

$$y = f * \mathfrak{F}^{-1}\chi \in L^p,$$

and

$$P(D_1, \cdots, D_n)y = f,$$

so that $f \in \mathfrak{R}(P_p)$.

Let ξ_0 be a point of R_n such that the closed ball with center ξ_0 and radius r_0 has exactly one point, ξ_1, in common with $\mathfrak{Z}(P)$, and $|\xi_0 - \xi_1| = r_0$.

Define the function $f_k(r) \in C^k[0, \infty)$ such that

$$f_k(r) = \begin{cases} 1 & \text{for } 0 \leqq r \leqq r_0/3, \\ (r_0 - r)^{k+1} & \text{for } 2r_0/3 \leqq r \leqq r_0, \\ 0 & \text{for } r_0 < r. \end{cases}$$

Let $g_k \in C^k(R_n)$ be defined by

$$g_k(\xi) = f_k(|\xi - \xi_0|).$$

It is clear that

$$\frac{\partial^k g_k}{\partial \xi_i^k} \in C_0^0 \quad \text{for } 1 \leqq i \leqq n,$$

and, consequently, that there exists k_0 such that

$$\mathfrak{F}^{-1} g_{k_0} \in L^p.$$

Define the functions a_s for non-negative integers s by

$$a_s(\xi) = \begin{cases} [P(\xi)]^{-s} g_{k_0}(\xi) & \text{for } |\xi - \xi_0| < r_0, \\ 0 & \text{for } |\xi - \xi_0| \geqq r_0. \end{cases}$$

Then $\mathfrak{F}^{-1}a_0 = \mathfrak{F}^{-1}g_{k_0} \in L^p$, while, for s sufficiently large, $a_s \notin L^q$, and, hence, $\mathfrak{F}^{-1}a_s \notin L^p$. Let $s_0 \geqq 0$ be the largest integer such that

$$\mathfrak{F}^{-1}a_{s_0} \in L^p. \tag{1}$$

Then

$$\mathfrak{F}^{-1}a_{s_0+1} \notin L^p,$$

and

$$P(D_1, \cdots, D_n)\, \mathfrak{F}^{-1}a_{s_0+1} = \mathfrak{F}^{-1}a_{s_0}.$$

Therefore,

$$\mathfrak{F}^{-1}a_{s_0} \notin \mathfrak{R}(P_p).$$

We choose a sequence $\xi_m \to \xi_0$ such that each of the closed balls with center ξ_m and radius r_0 does not intersect $\mathfrak{Z}(P)$.

Let

$$a_{s_0,m} = a_{s_0}(\xi + \xi_0 - \xi_m).$$

Then

$$\mathfrak{F}^{-1} a_{s_0,m}(x) = \exp(i(\xi_m - \xi_0) \cdot x)\, \mathfrak{F}^{-1} a_{s_0}(x),$$

and it is clear that

$$\mathfrak{F}^{-1} a_{s_0,m} \underset{m \to \infty}{\longrightarrow} \mathfrak{F}^{-1} a_{s_0} \quad \text{in } L^p. \tag{2}$$

Since each function $a_{s_0,m}(\xi)$ has compact support disjoint from $\mathfrak{Z}(P)$,

$$\mathfrak{F}^{-1} a_{s_0,m} \in \mathfrak{R}(P_p). \tag{3}$$

From (1), (2) and (3), it follows that $\mathfrak{R}(P_p)$ is not closed.

Suppose next that $2 \leqq q \leqq \infty$ and that $P(\xi) = 0$. Then $P'(-\xi) = 0$, and, consequently, P'_p has nonclosed range for $1 \leqq p \leqq 2$. It follows, by Lemma 1.7, that $(P'_p)^*$ has nonclosed range. Hence, by Lemma 3.3, P_{q0} has nonclosed range for $2 \leqq q < \infty$, and $(P'_1)^*$ has nonclosed range. This implies that P_{qc} has nonclosed range for every operator P_{qc} with

$$P_{q0} \subseteq P_{qc} \subseteq P_q \quad \text{for } 2 \leqq q < \infty$$

and

$$(P'_1)^* \subseteq P_{\infty c} \subseteq P_\infty.$$

In particular, all the operators P_q have nonclosed range for $2 \leqq q \leqq \infty$. By Lemmas 1.7 and 3.3, the operators P_{p0} and, consequently, all the operators P_{pc} have nonclosed range for $1 \leqq p \leqq 2$. This concludes the proof of the first part of the theorem.

(ii) Suppose, next, that $P(D_1, \cdots, D_n)$ is elliptic, and that $\mathfrak{Z}(P) = \emptyset$. For $n > 2$, the first assumption implies that k is even. For $n = 2$, it is easy to see that both assumptions together imply that $k \geqq 2$. For the simple case $n = k = 1$ and for a different treatment of the case $n = 1$ in general, we refer to [1]. Therefore, we can assume that $k \geqq 2$.

It is clear that the homogeneous equation

$$P(D_1, \cdots, D_n)y = 0$$

has no nontrivial tempered solution. Therefore P_p, has an inverse, for $1 \leqq p \leqq \infty$, given by

$$P_p^{-1} f = \left\{ \mathfrak{F}^{-1} \frac{1}{P(\xi)} \right\} * f.$$

We shall show that $\mathfrak{F}^{-1} 1/P(\xi) \in L^1$, from which it follows that P_p^{-1} is an everywhere defined, bounded operator in L^p, for $1 \leqq p \leqq \infty$. The proof extends an argument used in [6] in the proof of Lemma 8.

A simple calculation shows that, in view of the ellipticity, there exists $c > 0$ such that

$$\left| \frac{\partial^r}{\partial \xi_i^r} \frac{1}{P(\xi)} \right| < c|\xi|^{-k-r} \quad \text{for } |\xi| \geqq 1,$$

while

$$\left| \frac{\partial^r}{\partial \xi_i^r} \frac{1}{P(\xi)} \right| < c \quad \text{for } |\xi| < 1.$$

It follows that

$$\frac{\partial^r}{\partial \xi_i^r} \frac{1}{P(\xi)} \in L^1 \quad \text{for } r > n - k.$$

Therefore, for $r > n - k$,

$$|x|^r \left| \mathfrak{F}^{-1} \left(\frac{1}{P(\xi)} \right) (x) \right| < K \quad \text{for } x \in R_n. \tag{1}$$

From (1), for $r = n - k + 1$, follows

$$\int_{S_0} \left| \mathfrak{F}^{-1} \left(\frac{1}{P(\xi)} \right) \right| dx < \infty. \tag{2}$$

Using (1), for $r > n$, we obtain

$$\int_{R_n - S_0} \left| \mathfrak{F}^{-1} \left(\frac{1}{P(\xi)} \right) (x) \right| dx < \infty \tag{3}$$

By (2) and (3), $\mathfrak{F}^{-1}(1/P(\xi)) \in L^1$.

(iii) The statement concerning the spectrum of an elliptic operator P_p in L^p, $1 \leqq p \leqq \infty$, follows immediately by application of (i) and (ii) to the operators $P_p - \lambda$.

3.6. **Lemma.** *Let P_p be the operator defined in* 3.2, *for $1 < p < \infty$, associated with an elliptic differential operator $P(D_1, \cdots, D_n)$. Let B_p be the maximal operator in L^p associated with the differential expression*

$$\mathfrak{B}y = \sum_{|\beta| < k} b_\beta(x) D^\beta y.$$

Suppose that the functions b_β satisfy the following conditions.

(a) *For $|\beta| < k - n/p$, $b_\beta \in M_p$ (in particular, if $b_\beta \in Y_p^p$).*

(b) *For* $|\beta| \geqq k - n/p$, $b_\beta \in Q^p_{k-|\beta|,a}$ *for some* $a < 0$, *or* b_β *satisfies the conditions* (i) *and* (ii) *of* 2.13 (b) *with* $k - |\beta|$ *substituted for* k (*in particular, if* $b_\beta \in \mathrm{AR}$ *and satisfies the conditions* (i) *and* (ii) *of* 2.14 (a) *with* $k - |\beta|$ *substituted for* k).

Then B_p *is* P_p-ϵ-*bounded.*

Proof. It is well known (cf. [5, 2.3] and [6]), that $D(P_p) \subset W^p_k$, and, for $f \in D(P_p)$,

$$\|f\|_{W^p_k} < K\{\|f\|_p + \|P_p f\|_p\}. \tag{1}$$

From (1), Lemmas 2.8 (a2) and (b2), 2.13 (b) and (c), 2.14 (a) and Remark 2.16, the lemma follows.

3.7. THEOREM. *Let* P_p *and* B_p *be the operators of Lemma* 3.6 *and let* C_p *be the maximal operator in* L^p *associated with the differential expression*

$$\mathfrak{C} y = \sum_{|\gamma| < k} c_\gamma(x) D^\gamma y.$$

Suppose that the functions c_γ *satisfy the following conditions.*

(a) *For* $|\gamma| < k - (n/p) - 1$, $c_\gamma \in N^p$ (*in particular, if* $c_\gamma \in Z^p_p$).

(b) *For* $k - (n/p) - 1 \leqq |\gamma| < k - (n/p)$,

c_γ *satisfies the conditions* (i) *and* (ii) *of* 2.11 (b)

or

c_γ *satisfies the conditions* (i) *and* (ii) *of* 2.15 (b).

(c) *For* $|\gamma| \geqq k - (n/p)$,

c_γ *satisfies the conditions* (i) *and* (ii) *of* 2.11 (a)

or

c_γ *satisfies the conditions* (i) *and* (ii) *of* 2.15 (a).

Then C_p *is* $(P_p + B_p)$-*compact.*

Proof. By Lemmas 1.2 and 3.6, $P_p + B_p$ is closed, and, for $f \in D(P_p + B_p)$,

$$\|f\|_{W^p_k} \leqq K\{\|f\|_p + \|\{P_p + B_p\}f\|_p\}. \tag{1}$$

Then the theorem follows from Lemmas 2.11 and 2.15 and Remark 2.16.

3.8. REMARK. The necessary conditions of Lemmas 2.8 and 2.14 in order that $B_{0,p}$ be W^p_k-bounded are also necessary in order that $B_{0,p}$ be P_p-bounded. The necessary conditions of Lemmas 2.11 and 2.15 in order that $B_{0,p}$ be W^p_k-compact are also necessary in order that $B_{0,p}$ be $(P_p + B_p)$-compact when B_p satisfies the conditions of Lemma 3.6.

Proof. The first statement follows immediately from Lemmas 2.8 and 2.14 and the inequality

(1) $$\|f\|_{P_p} \leqq K\|f\|_{W_k^p} \quad \text{for } f \in D(P_p).$$

The second half follows from Lemmas 2.11 and 2.15 together with (1) when $B_p = 0$; the general case follows from this by Lemmas 1.2 and 3.6.

3.9. THEOREM. *Let P_p and C_p be defined as in Theorem* 3.7. *Then*

$$\sigma_s(P_p + C_p) = \Re(P),$$

$$\Phi(P_p + C_p) = C\,\Re(P),$$

$$I_\lambda(P_p + C_p) = 0 \quad \text{for } \lambda \in \Phi(P_p + C_p).$$

If the coefficients of $P(\xi)$ are real, so that P_2 is self-adjoint, the additional conclusion of Theorem 1.4 *holds.*

Proof. This is an immediate application of Theorems 1.4, 3.5 and 3.7, for $B_p = 0$.

3.10. REMARK. The perturbation in Theorem 3.7 includes, in particular, a lower-order operator whose coefficients are bounded functions converging to 0 at ∞. This case is treated in [5, Theorem 28]. It also includes bounded functions in L^p, first considered in [15] for $P = \Delta$, $p = 2$, $n = 3$. An investigation of the case $p = 2$ is found in [3] (cf. 2.12 and 2.15).

Certain singularities are allowed at finite points. We illustrate this in the case where the unperturbed operator is $-\Delta$ in L^2, and the perturbing operator has powers of r as coefficients. The operator

$$-\Delta + \sum_{i=1}^{n} c_i r^{-1+\epsilon_i} \frac{\partial}{\partial x_i} + c_0 r^{-2+\epsilon_0}, \qquad \epsilon_i > 0, \quad i = 0, 1, \cdots, n,$$

satisfies the conditions of 3.9 for $n \geqq 4$. For $n = 3$, the function $r^{-2+\epsilon_0}$ is to be replaced by $r^{-3/2+\epsilon_0}$.

3.11. REMARK. The results of 3.6–3.8 remain valid if the top-order coefficients of P_p are replaced by uniformly continuous, bounded functions $a_\alpha(x)$, $|\alpha| = k$, provided that the unperturbed operator is uniformly elliptic. This follows immediately from the fact that the inequalities 3.6(1), 3.7(1) and 3.8(1) still hold (cf. [5] and [6]).

Concerning a perturbation of P_p by top-order terms we can at least obtain the following result:

Let $a_\alpha(x)$ be continuous functions on R_n converging to 0 at ∞, $|\alpha| = k$. Suppose that the differential operator,

$$P(D_1, \cdots, D_n) + \sum_{|\alpha| = k} a_\alpha(x) D^\alpha,$$

is elliptic.

Let A_p be the corresponding maximal operator in L^p. Then

$$\Phi(A_p + C_p) \cup \Phi_+(A_p + C_p) = \Phi(A_p) \cup \Phi_+(A_p) = C\,\Re(P).$$

Proof. By Theorems 1.4, 3.5 and 3.7 and Remark 3.11 it suffices to prove that $\Phi(A_p) \cup \Phi_+(A_p) = \Phi(P_p) \cup \Phi_+(P_p) = \Phi(P_p)$. Let ω_R be a function in C^∞ such that

$$\omega_R(x) = 0 \quad \text{for } |x| \leqq R+1,$$
$$\omega_R(x) = 1 \quad \text{for } |x| \geqq R+2.$$

Then it is easy to show that, if $\{\phi_n\}$ is a singular sequence for P_p (or A_p), then $\{\omega\phi_n\}$ is also a singular sequence for P_p (respectively, A_p). Let $P_{p,R}$ and $A_{p,R}$ denote the minimal operators in $L^p(R_n - S_{0,R})$ associated with the same differential expressions. We can consider $P_{p,R}$ and $A_{p,R}$ as restrictions of P_p and A_p.

Suppose that $\lambda \in \Phi(P_p) \cup \Phi_+(P_p)$. Then there exists $\epsilon(\lambda)$ such that

$$\lambda \in \Phi(P_p + B) \cup \Phi_+(P_p + B) \quad \text{for } \|B\|_{P_p - \lambda} < \epsilon(\lambda).$$

It is clear that then also $\lambda \in \Phi(P_p + B)_R \cup \Phi_+((P_p + B)_R)$, since $(P_p + B)_R$ is a closed restriction of $P_p + B$.

Choose R so large that $\|D_{p,R}\|_{P_p - \lambda} < \epsilon(\lambda)$, where D_p corresponds to the differential expression

$$\sum_{|\alpha| = k} a_\alpha(x) D^\alpha.$$

Then $\lambda \in \Phi(A_{p,R}) \cup \Phi_+(A_{p,R})$, and, by Lemma 1.6, $A_{p,R} - \lambda$ does not have a singular sequence. Then $A_p - \lambda$ does not have a singular sequence $\{\phi_n\}$, because then $\{\omega\phi_n\}$ would be a singular sequence for $A_{p,R} - \lambda$. By Lemma 1.6, $\lambda \in \Phi(A_p) \cup \Phi_+(A_p)$.

Reversing the argument, we show that $\Phi(A_p) \cup \Phi_+(A_p) \subset \Phi(P_p) \cup \Phi_+(P_p)$ and the proof is complete.

3.12. Lemma. *Let T_2 be a differential operator of order $2s$ in L^2 which satisfies the conditions*:

(i) *T_2 is a closed operator with $D(T_2) \subset W_s^2 \cap W_{2s,\mathrm{loc}}^2$,*

(ii) $\|u\|_{W_s^2} \leqq K(\|u\|_2 + \|T_2 u\|_2)$ *for $u \in D(T_2)$,*

(iii) *for every $R > 0$, $u \in D(T_2)$,*

$$\|u\|_{W_{2s}^2(S_{0,R})} \leqq K_R(\|u\|_{L^2(R_n)} + \|T_2 u\|_{L^2(R_n)}).$$

Let B_2 be the maximal operator in L^2 associated with the differential expression

$$\mathfrak{B}y = \sum_{|\beta| < 2s} b_\beta(x) D^\beta y.$$

Suppose that the coefficients b_β satisfy the following conditions for some $R > 0$:

(a) *The functions $b_\beta \chi_R$ satisfy the conditions of Lemma 3.6 for $k = 2s$, $p = 2$, $0 \leqq |\beta| < 2s$,*

(b) *the functions* $b_\beta(1-\chi_R)$ *satisfy the conditions of Lemma* 3.6 *for* $k = s$, $p = 2$, $0 \leqq |\beta| \leqq s-1$,

(c) $b_\beta \chi_R = 0$ *for* $s \leqq |\beta| \leqq 2s-1$.

Then B_2 *is* T_2-ϵ-*bounded.*

Proof. Let $B_2 = A_{2,R} + B_{2,R}$, as in Definition 2.7.

Then $A_{2,R}$ is T_2-ϵ-bounded by (iii), (a) and Lemmas 2.8 (a2) and (b2), 2.13 (b) and (c), 2.14 (a) and Remark 2.16 for $k = 2s$, $p = 2$.

$B_{2,R}$ is T_2-ϵ-bounded by (ii), (b) and the same lemmas and remark for $k = s$, $p = 2$.

Therefore, B_2 is T_2-ϵ-bounded.

3.13. Theorem. *Let* T_2 *and* B_2 *be defined as in* 3.12. *Let* C_2 *be the maximal operator in* L^2 *associated with the differential expression*

$$\mathfrak{C} y = \sum_{|\gamma|<2s} c_\gamma(x) D^\gamma y.$$

Suppose that the coefficients c_γ *satisfy the following conditions*:

(a) *The conditions of Theorem* 3.7 *for* $k = 2s$, $p = 2$,

(b) $\int_{S_x} |c_\gamma(y)|^2 dy \to 0$ *as* $|x| \to \infty$ *for* $0 \leqq |\gamma| < s - n/2$ (*in particular, if* $\int_r^{r+1} |c_\gamma(\rho,\omega)|^2 d\rho \to 0$ *as* $r \to \infty$ *uniformly for* $\omega \in S'$),

(c) $\int_{S_x} |c_\gamma(y)|^2 |x-y|^{2(s-|\gamma|)-n+a} dy \to 0$ *as* $|x| \to \infty$ *for some* $a < 0$, *or* $\int_r^{r+1} |c_\gamma(\rho,\omega)|^2 d\rho \to 0$ *as* $r \to \infty$ *uniformly for* $\omega \in S'$, $s - n/2 \leqq |\gamma| < s$,

(d) ess $\sup_{|x|>R} |c_\gamma(x)| \to 0$ *as* $R \to \infty$, $|\gamma| = s$,

(e) $c_\gamma(x) = 0$ *for* $|x| > K$ *for some* K, $s+1 \leqq |\gamma| < 2s$.

Then C_2 *is* $T_2 + B_2$-*compact.*

Proof. By Lemmas 1.2 and 3.12 it suffices to prove that C_2 is T_2-compact. Let $C_2 = A_{2,R} + C_{2,R}$ as in Definition 2.7.

From (a) and 3.12 (iii) it follows, by Lemmas 2.11 and 2.15, Remark 2.16 and Theorem 1.4, that the operators $A_{2,R}$ are T_2-compact for $0 < R < \infty$. From (a)–(e) and 3.12 (ii) it follows, by Lemmas 2.8 and 2.13 that the operator C_2 is T_2-bounded, and that

$$\|C_{2,R}\|_{T_2} \to 0 \quad \text{as } R \to \infty.$$

Therefore, C_2 is T_2-compact.

3.14. Remark. The conditions (i)–(iii) of 3.12 on the operator T_2 have been established in the following papers, mainly in connection with conditions for essential self-adjointness: [5, Theorem 24], [9, Lemmas 3 and 5], [11] and, for $n = 1$, [4].

The conditions are either directly verified or can be established under obvious additional assumptions.

3.15. Remark. The perturbation results can be generalized to the case where R_n is replaced by any open subset of R_n with a sufficiently smooth boundary, and the differential operators are defined by regular boundary conditions. A discussion of this more general situation is found in [2].

The results can also be extended to the case $p = 1$ with certain modifications.

ACKNOWLEDGEMENT. The author wishes to express his thanks to Professor T. Kato for stimulating conversations and references in connection with the present work. He is also indebted to Dr. Figà-Talamanca for a discussion concerning the proof of Theorem 3.5.

Added in proof. There exists $a < 0$, such that for $R \geqq \delta > 0$, $k \geqq 1$

$$D(R,k,a) \leqq K \cdot F(R) \qquad \text{(Definition 2.7)}.$$

Hence Lemma 2.13 (a) follows from 2.8 (a1). Also 2.13 (b), (i) and (ii) imply that there exists $a < 0$, such that $b \in Q^p_{k,a}$, hence 2.13 (b) follows from 2.8 (a2). Likewise 2.15 (a), (i) and (ii) imply 2.11 (a), (i) and (ii), and 2.15 (b), (i) and (ii) imply 2.11(b), (i) and (ii), hence the sufficiency part of Lemma 2.15 follows from the sufficiency part of Lemma 2.11.

The results of the present work have for $p = 2$ been improved recently by P. A. Rejto [18] and the author [17]. Among other things, it is shown that B_p is W^p_k-compact if there exists $a < 0$, such that $b \in R^p_{k,a}$, so that assumption (ai) in Lemma 2.11 is superfluous. Accordingly, Theorems 3.7, 3.9 and 3.13 hold without the corresponding assumption. It is likely that the same holds for general p.

REFERENCES

1. E. Balslev and T. W. Gamelin, *The essential spectrum of a class of ordinary differential operators,* Pacific J. Math. (3) **14** (1964), 755-776.

2. E. Balslev, *Perturbation of differential operators,* Ph.D. Thesis, Univ. of California, Berkeley, Calif., 1963.

3. M. Š. Birman, *On the spectrum of singular boundary value problems,* Mat. Sb. (N. S.) **97** (1961), 125-174.

4. I. Brinck, *Self-adjointness and spectra of Sturm-Liouville operators,* Math. Scand. **7** (1959), 219-239.

5. F. E. Browder, *On the spectral theory of elliptic differential operators.* I, Math. Ann. **142** (1961), 22-130.

6. ———, *Functional analysis and partial differential equations.* II, Math. Ann. **145** (1962), 81-226.

7. N. Dunford and J. T. Schwartz, *Linear operators.* I, II, Interscience, New York, 1958, 1963.

8. I. C. Gohberg and M. G. Kreĭn, *The basic propositions on defect numbers, root numbers and indices of linear operators,* Uspehi Mat. Nauk **12** (1957), 43-118; English transl., Amer. Math. Soc. Transl. (2) **13** (1960), 185-264.

9. T. Ikebe and T. Kato, *Uniqueness of the self-adjoint extension of singular elliptic differential operators,* Arch. Rational Mech. Anal. **9** (1962), 77-92.

10. T. Kato, *Perturbation theory for nullity, deficiency and other quantities of linear operators,* J. Analyse Math. **6** (1958), 261-322.

11. N. Nilsson, *Essential self-adjointness and the spectral resolution of Hamiltonian operators*, Kungl. Fysiogr. Sällsk. Lund Förhandlingar (1) **29** (1959), 1-19.

12. G. C. Rota, *Extension theory of differential operators*. I, Comm. Pure Appl. Math. **11** (1958), 23-65.

13. L. Schwartz, *Théorie des distributions*. II, Hermann, Paris, 1959.

14. F. Wolf, *On the essential spectrum of partial differential boundary problems*, Comm. Pure Appl. Math. **12** (1959), 211-228.

15. F. R. D. Agudo and F. Wolf, *Propriétés spectrales de l'opérateur* $\Delta + \sum_i a_i(\partial/\partial x_i) + b$ *à coefficients complexes*, Atti Acad. Naz. Lincei Rend Cl. Sci. Fis. Mat. Nat. **25** (1958), 273-275.

16. E. Balslev and C. F. Schubert, *Essential spectrum and singular sequences*, Pacific J. Math. (to appear).

17. E. Balslev, *The essential spectrum of self-adjoint elliptic differential operators in* $L^2(R_n)$, Math. Scand. (to appear).

18. P. A. Rejto, *On the essential spectrum of the hydrogen energy and related operators*, Pacific J. Math. (to appear).

University of California,
Berkeley, California

University of California,
Los Angeles, California