SOME RESULTS ON STABILITY IN SEMIGROUPS⁽¹⁾

BY

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A semigroup S is called stable if (1) $a, b \in S$ and $Sa \subseteq Sab$ imply that $Sa = Sab$ and (2) $a, b \in S$ and $aS \subseteq baS$ imply that $aS = baS$. This notion was introduced by Koch and Wallace [5], who showed that in a stable semigroup the $\mathcal{D}$ and $\mathcal{J}$ equivalences coincide. Previously, Green [4], to obtain this equality, had imposed certain minimum conditions which were of a somewhat strong nature. Our principal concern here is the connection of stability with the $\mathcal{H}$ -equivalence. In presenting a number of characterizations and consequences of stability, the bicyclic semigroup plays a rather special rôle. As we shall see, the fact that this semigroup cannot be immersed in a stable semigroup has, by itself, a number of consequences.

Frequently, we shall use fairly well-known results in the theory of semigroups which can be found in [1].

The equivalences of Green [4], defined for any semigroup S , are as follows:

$$a\mathcal{R}b \rightleftharpoons a \cup aS = b \cup bS,$$

$$a\mathcal{L}b \rightleftharpoons a \cup Sa = b \cup Sb,$$

$$\mathcal{H} = \mathcal{L} \cap \mathcal{R}, \quad \mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L},$$

$$a\mathcal{J}b \rightleftharpoons a \cup Sa \cup aS \cup SaS = b \cup Sb \cup bS \cup SbS.$$

Since we shall frequently assume the existence of an identity element it should be mentioned that the adjunction of an identity does not disturb the above equivalences⁽²⁾.

For subsets A and B of a semigroup S , we shall write $A \cdot B$ for $\{x \mid Bx \subseteq A\}$. Similarly, $A \cdot B = \{x \mid xB \subseteq A\}$, and $A \cdot B = \{(x, y) \mid xBy \subseteq A\}$. The complement of A in B is $B \setminus A$, the empty set is $\emptyset$, and the notation $A \not\subseteq B$ means $A \cap B \neq \emptyset$. In most instances our notation and terminology follow [1].

PROPOSITION 1. *Let S be a semigroup with identity and let H, L, R, D be*

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(2) While adjoining an identity does not disturb the Green equivalence, there is, nevertheless, conceivably some loss of generality since it is at least conceptually possible to stabilize a semigroup through this adjunction.

$\mathcal{H}$, $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{D}$ classes respectively such that $H = L \cap R \subseteq D$. The following are then equivalent:

- (1) S is stable.
- (2) $sHt \not\propto D$ implies sH is an $\mathcal{H}$ -class in L , Ht is an $\mathcal{H}$ -class in R and sHt is an $\mathcal{H}$ -class in D for all $s, t \in S$ and all H in S .
- (3) $sHt \not\propto H$ implies $sH = H = Ht = sHt$, for all $s, t \in S$ and all H .
- (4) $H \cdot H = H \cdot H \times H \cdot H$ for all H .

Proof. (1) $\rightarrow$ (2). First of all, $sHt \not\propto D$ implies that $sht \in D$ for some $h \in H$, so that $h \not\propto sht$. Hence, there are elements p and q such that $pshtq = h$. Letting $a = ps$ and $b = tq$ we have $h = ahb$. We show next that $hb \not\propto h$. Note that $hbS \subseteq hS = a(hbS)$. Since S is stable, $hbS = hS = ahbS$ and $hb \not\propto h$. Hence, $Hb = Htq$ is an $\mathcal{H}$ -class in R . Since $htq \not\propto h$, we have $Htq = H$. It follows that $Ht \not\propto R$ so by [1], Ht is an $\mathcal{H}$ -class in R . Similarly, sH is an $\mathcal{H}$ -class in L and sHt is an $\mathcal{H}$ -class contained in D .

(2) $\rightarrow$ (3). We immediately have $sHt = H$. Since sH is an $\mathcal{H}$ -class in L and $\mathcal{L}$ is right regular, we have $(sHt)\mathcal{L}(Ht)$. But Ht is already an $\mathcal{H}$ -class in R . Since Ht lies in both L and R we conclude that $H = Ht$. Similarly, $sH = H$ establishing (3).

(3) $\rightarrow$ (4). The statement $H \cdot H \times H \cdot H \subseteq H \cdot H$ is always true. To show that $H \cdot H \subseteq H \cdot H \times H \cdot H$, let $(x, y) \in H \cdot H$ so that $xHy \subseteq H$. By (3), we have $xH = H = xHy = Hy$ and the result follows.

(4) $\rightarrow$ (1). Suppose $aS \subseteq saS$. Then $a = sat$, so that $(s, t) \in H_a \cdot H_a$. Since $H_a \cdot H_a \subseteq H_a \cdot H_a \times H_a \cdot H_a$, we have $sH_a \subseteq H_a$, so that $sa \in H_a$. But then, $aS \subseteq saS \subseteq H_aS = aS$ so that $aS = saS$. Similarly, $Sa \subseteq Sat$ implies $Sa = Sat$ and the proof is complete.

COROLLARY 1.1. *With all things as in Proposition 1 we have the following:*

- (1) $SL \cap D = L$ and $RS \cap D = R$.
- (2) $L \cdot L = H \cdot H$ and $R \cdot R = H \cdot H$.
- (3) $SH \cap R = H$ and $HS \cap L = H$.
- (4) If D is regular and $x \in H$ then $xSx \cap D = H$.
- (5) If x, y and xy lie in D then $xy \in R_x \cap L_y$ so that $R_y \cap L_x$ contains an idempotent. Thus, if $D^2 \not\propto D$ then D is regular.
- (6) If e and f are idempotent elements in D such that $ef = fe$ then $ef = fe \in S \setminus D$ and $SefS \subseteq SDS \setminus D$.
- (7) If $x \in D$ and there exists an integer $n = n(x)$ such that $x \in Sx^nS$ then H_x is a group.

Proof. (1) Suppose $ta \in D$ where $a \in L$. Then $ta \not\propto a$ so that $ta \not\propto a$. Hence there exist elements p and q such that $a = ptaq$. From the stability of S we have $pta \not\propto a$. Hence, $Sa = Spta \subseteq Sta \subseteq Sa$, and $Sta = Sa$ which is to say, $ta \not\propto a$. Clearly, $L \subseteq SL \cap D$ so that $L = SL \cap D$. Similarly $RS \cap D = R$.

(2) If $t \in L \cdot L$, then $Lt \subseteq L$ so $at \in La$ for all $a \in L$. Since $at \in R_a S \cap D = R_a$, we see that $at \in La$ so that $t \in H \cdot H$. The other containment is clear since any $\mathcal{H}$ -class contained in L is a left translate of H . In the same way, $R \cdot R = H \cdot H$.

(3) Suppose $ta \in Ra$, where $a \in H$. Then $tR \subseteq R$ so by (2), $tH \subseteq H$, so that $ta \in H$. Clearly $H \subseteq SH \cap R$ so that we have $SH \cap R = H$. Similarly, $HS \cap L = H$.

(4) If D is regular then there exist idempotents e and f such that $xS = eS$ and $Sx = Sf$. Clearly then $xS \cap Sx = xSx$ and the result follows from (1).

(5) From (1) it follows that $xy \in R_x \cap L_y$. The existence of an idempotent in $R_y \cap L_x$ follows from [1].

(6) This is immediate from (1) and the fact that $SDS \setminus D$ is an ideal.

(7) If $x \in Sx^n S$ then x^n lies in D since $SDS \setminus D$ is an ideal. In particular x^2 lies in D and from (1) must lie in H . This implies that H is a group [4].

In his original investigations, Schützenberger, for his definition of the group of an $\mathcal{H}$ -class, dealt with a somewhat restricted type of $\mathcal{D}$ -class. (This restriction was later removed.) A $\mathcal{D}$ -class D is said to be of *elementary type* if for any $a, b \in D$ we have

(i) $Sa = Sb$ and $aS \subseteq bS$ imply $aS = bS$,

(ii) $aS = bS$ and $Sa \subseteq Sb$ imply $Sa = Sb$.

Thus, noting conclusion (3) in Corollary 1.1 we see that if S is a stable semigroup then each $\mathcal{D}$ -class is of elementary type.

The bicyclic semigroup $\mathcal{C}(p, q)$ is the semigroup with identity e generated by two elements p and q subject to $qp = e$. We write $\mathcal{C} = \mathcal{C}(p, q)$.

We summarize some properties of $\mathcal{C}(p, q)$. First of all, it has the following form:

$$\begin{array}{cccc} e & q & q^2 & \cdots \\ p & pq & pq^2 & \cdots \\ p^2 & p^2q & p^2q^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

For $x \in \mathcal{C}(p, q)$, L_x consists of those elements in the same column with x , and R_x consists of those elements in the same row with x . Further $E = \{e\} \cup \{p^n q^n : n = 1, 2, 3, \dots\}$, and $e > pq > p^2 q^2 > \dots$. Note that $\mathcal{C}(p, q)$ is a $\mathcal{D}$ -simple inverse semigroup.

(1) If $fSf < eSe \subseteq SfS$, then e is the unit of a bicyclic semigroup contained in J_e (see [1, p. 81]).

(2) If S is a simple semigroup (with or without zero) which contains a nonzero idempotent, then S is completely simple if and only if S does not contain a bicyclic semigroup (Olaf Andersen, see [1, p. 81]).

(3) An element p of a semigroup S lies in a bicyclic semigroup if there is an idempotent e and an element q such that $p, q \in eSe$, $pq = e \neq qp$ [1].

The $\mathcal{L}$ -, $\mathcal{R}$ -, and $\mathcal{H}$ -relations in $\mathcal{C}(p, q)$ are compatible with those in S . In fact we have the following more general proposition.

PROPOSITION 2. *Let a and b be regular elements in a semigroup S , and suppose S is embedded in a semigroup T . Then a and b are $\mathcal{R}$ - ($\mathcal{L}$ -, $\mathcal{H}$ -) related in T if and only if a and b are $\mathcal{R}$ - ($\mathcal{L}$ -, $\mathcal{H}$ -) related in S .*

Proof. Suppose a and b are $\mathcal{R}$ -related in S (notation $aR_S b$). Then $aS = bS \subseteq bT$. So $a \in bT$, $aT \subseteq bT$. Similarly $bT \subset aT$, so $aR_T b$. Dually, $aL_S b$ implies $aL_T b$.

Conversely, suppose $aR_T b$. By regularity there exist idempotents e and f in S such that $eS = aS$ and $fS = bS$. Hence by the first part of the proof, $eT = aT = bT = fT$. Thus $e = fe \in fS$, so $aS = eS \subset fS = bS$, and similarly $bS \subset aS$. Hence $aR_S b$; a dual argument establishes the proposition.

Thus we see that regular elements of S cannot become $\mathcal{R}$ - ($\mathcal{L}$ -, $\mathcal{H}$ -) related in an extension of S unless they already have the relation in S . This does not hold for $\mathcal{D}$, however; in fact, any semigroup can be embedded in a $\mathcal{D}$ -simple semigroup [8]. It may be possible to characterize regularity in terms of the invariance of $\mathcal{R}$ - ($\mathcal{L}$ -, $\mathcal{H}$ -) relations under embeddings.

COROLLARY 2.1. *Let $\mathcal{C}$ be a bicyclic semigroup which is embedded in a semigroup S . Then $\mathcal{C}$ meets an $\mathcal{H}$ -class of S in at most one point. Thus if x is a non-idempotent in $\mathcal{C}$, then no power of x lies in a subgroup of S .*

Proof. Suppose $a, b \in \mathcal{C}$ with $a\mathcal{H}_S b$. By the proposition we have $a\mathcal{H}_{\mathcal{C}} b$, so $a = b$. The remainder of the corollary follows readily.

COROLLARY 2.2. *The bicyclic semigroup cannot be embedded in a stable semigroup.*

Proof. Suppose that $\mathcal{C}(p, q) \subseteq S$. Note that $qp^2 = p \notin R_q \cap L_{p^2}$ so that S is not stable by (5) of Corollary 1.1.

COROLLARY 2.3. *Let S be a 0-simple semigroup. Then S is completely 0-simple if and only if S is stable.*

Proof. If S is completely 0-simple then it follows from the well-known structure theory for such semigroups that S is stable.

On the other hand suppose that S is 0-simple and stable. Let x, y be elements of S such that $xy \neq 0$. (Such elements exist since S is, by definition, not the zero semigroup of order 2.) By (5) of Corollary 1.1 it follows that S contains a nonzero idempotent. (We recall that $\mathcal{D} = \mathcal{J}$ since S is stable.) Now if S were not completely 0-simple it would have to contain a copy of the bicyclic semigroup (see [1, p. 81]). This is ruled out by the stability of S through Corollary 2.2. Thus, S is completely 0-simple.

In the same way one proves the following

COROLLARY 2.4. *A semisimple semigroup is completely semisimple if and only if it is stable.*

COROLLARY 2.5. *If S is a stable semigroup which has a 0-simple ideal K then K is completely 0-simple.*

It should be noted that our Corollary 2.4 is, in point of fact, equivalent to Theorem 2.3 of Munn [7]. The minimal conditions m_i^* and m_r^* are together, for a semigroup with identity, equivalent to stability. This last statement follows immediately from (1) of Corollary 1.1.

We note that it may be possible to embed a nonstable semigroup in a stable semigroup, in fact a group.

EXAMPLE (CROISOT [3]). Let S be a countable cancellative semigroup without idempotents, and let L be the free cancellative semigroup generated by the elements a_{ij} and b_{ij} , where i and j are positive integers. Let $P = S * L$ be the free product of S and L . Denoting the elements of S by $s_1, s_2, \dots$, the relations $s_i = a_{ij}s_jb_{ij}$ are imposed. Let S be so chosen that some element, say s_1 , is not a product in S . Then, in P , it is easy to see that both $s_1 \cdot s_1$ and $s_1 \cdot s_1$ are empty while $s_1 \cdot s_1$ contains (a_{11}, b_{11}) . Moreover, since $\{s_1\}$ is itself an $\mathcal{H}$ -class it is immediate that P is not stable. It also follows that the elements of S form a single $\mathcal{J}$ -class. Moreover, it is easy to take S so that each $\mathcal{D}$ -class is degenerate.

Another example illustrating this point can be found in [1, p. 51], due to Olaf Andersen.

PROPOSITION 3. *Let S be a semigroup; the following are equivalent:*

- (1) $\mathcal{C}$ is not embeddable in S .
- (2) $SeS / (SeS \setminus J_e)$ is completely 0-simple for each nonminimal idempotent $e \in S$.
- (3) $eSe \setminus H_e$ is an ideal in eSe for each nonminimal idempotent $e \in S$.

Proof. (1) $\rightarrow$ (2). Let $I = SeS \setminus J_e$ and suppose that $I \neq \emptyset$. Then SeS / I is 0-simple. Since $\mathcal{C}$ is not embeddable in S it is not embeddable in SeS / I . Thus, SeS / I is completely 0-simple.

(2) $\rightarrow$ (3). Again letting $I = SeS \setminus J_e$, we suppose that SeS / I is completely 0-simple. Which is to say, $J_e \cup \{I\}$ is a completely 0-simple semigroup with $\{I\}$ as zero element. Now from the known structure of completely 0-simple semigroups, we know that H_e , the maximal subgroup of $J_e \cup \{I\}$, is precisely those elements in $J_e \cup \{I\}$ for which e is a two-sided identity. Which is to say, $eSe \cap J_e = H_e$. Thus, $eSe \setminus H_e$ lies in I which completes the argument.

(3) $\rightarrow$ (1). Since $\mathcal{C}$ fails to satisfy (3) it cannot be embedded in S . The proof is complete.

We may summarize some of these results in the following

COROLLARY 3.1. *If S satisfies any of the following then S does not contain a*

copy of $\mathcal{C}(p, q)$, so that $SeS/I(e)$ is completely 0-simple for each nonminimal idempotent e .

(1) S is stable.

(2) S is periodic (i.e., each element generates a finite sub-semigroup).

(3) Some power of every element lies in a subgroup.

(4) For each $x \in S$ there exist integers $n = n(x)$, $m = m(x)$, $n < m$ such that $x^n \in x^m S$.

(1) and (2) are rather clear. To see (3) and (4) we have only to call upon Corollary 2.1.

We note that Corollary 3.1 contains certain results of Croisot [2] and Munn [7].

We should note the following: If S is periodic then S is stable. To see this, suppose $Sa \subseteq Sab$. Then $Sa \subseteq Sab^n$ for each n . Thus for some idempotent e , a power of b , we have $Sa \subseteq Sab \subseteq Sab^2 \subseteq \dots \subseteq Sae$. It is then immediate that $Sa = Sae$ so that $Sa = Sab$.

Let A be a subset of a semigroup S having an identity. The congruence of Teissier [9] is defined as follows:

$x \equiv y(\mathcal{C}_A) \Leftrightarrow$ either $x = y$ or there exist elements $x_1, x_2, x_3, \dots, x_n, y_1, y_2, \dots, y_n$ of S such that $x \in x_1 A y_1 \cap x_2 A y_2 \cap \dots \cap x_n A y_n \ni y$.

From the results of [9], it is known that in order for A to be a class of some congruence it is necessary and sufficient that for any c and d in S one has $cAd \cap A \rightarrow cAd \subseteq A$. As we have seen, any $\mathcal{H}$ -class in a stable semigroup has this property. Thus we have the following

PROPOSITION 4. *Let H be an $\mathcal{H}$ -class of a stable semigroup S . Then there exists a congruence on S in which H is a single class.*

Suppose $I = S \setminus D$, where D is a $\mathcal{D}$ -class in S , a stable semigroup with identity. If H is an $\mathcal{H}$ -class contained in D we have the following diagram, the maps being canonical.

A more precise connection between $\mathcal{C}_H$ and $\mathcal{H}$ is given by the following

PROPOSITION 5. *Let S be a semigroup with identity and let H and H' be $\mathcal{H}$ -classes of S . If H and H' lie in the same $\mathcal{D}$ -class then $\mathcal{C}_H = \mathcal{C}_{H'}$. If either H or H' is nondegenerate and S is stable then $\mathcal{C}_H = \mathcal{C}_{H'}$ implies H and H' lie in the same $\mathcal{D}$ -class hence both are nondegenerate. Thus if S is stable and H is nondegenerate*

$$H \equiv H'(\mathcal{D}) \leftrightarrow \mathcal{E}_H = \mathcal{E}_{H'}.$$

Proof. Suppose then that $p \equiv q(\mathcal{E}_H)$. If $p = q$ then of course $p \equiv q(\mathcal{E}_{H'})$. If $p \neq q$ then there are elements $x_1, \dots, x_n, y_1, \dots, y_n$ such that

$$p \in x_1 H y_1 \times x_2 H y_2 \times \dots \times x_n H y_n \ni q.$$

Since H and H' lie in the same $\mathcal{D}$ -class, there exist elements r and s such that $H = rH's$. Thus we have

$$p \in x_1 r H' s y_1 \times x_2 r H' s y_2 \times \dots \times x_n r H' s y_n \ni q$$

which is to say, $p \equiv q(\mathcal{E}_{H'})$. In the same way $p \equiv q(\mathcal{E}_{H'})$ implies $p \equiv q(\mathcal{E}_H)$.

Now suppose that $\mathcal{E}_H = \mathcal{E}_{H'}$, that S is stable and that say, H contains at least two distinct elements a and b . Since $a \neq b$, it follows from the very definition of $\mathcal{E}_H$ that $H \subseteq SH'S$. If H' were degenerate we would immediately conclude that $a = b$. Thus, H' is nondegenerate and so just as we had $H \subseteq SH'S$, we have $H' \subseteq SHS$. Thus, we see that H and H' lie in the same $\mathcal{L}$ -class. Since S is stable, H and H' lie in the same $\mathcal{D}$ -class and the proof is complete.

To see that nondegeneracy is needed in the second half of the above⁽³⁾, we note that if S is the semigroup of order 3 having elements e, f, z with z as zero, $e^2 = e, f^2 = f, ef = fe = 0$, i.e., the resorbing semigroup of order 3, then $H = \{e\}$ and $H' = \{f\}$ are distinct $\mathcal{D}$ -classes but $\mathcal{E}_H = \mathcal{E}_{H'}$.

In studying the Teissier congruence the "double orbits" xHy arise. Thus it might have been reasonable to think that in a stable, compact, or perhaps finite semigroup, $sHy \times aHb$ implies equality. However, the following example shows this is not so. Let S be the full transformation semigroup on 4 elements $\mathcal{T}_4$ (see [1]). Following the description in [1] we note that if H is the $\mathcal{H}$ -class

$$(1341) \quad (3143)$$

$$(3413) \quad (4314)$$

$$(4134) \quad (1431)$$

then $(1313)H(1111) \times (1313)H(1242)$. Both contain the element (1111) . However, (3333) is contained in the first but not the second.

For the notion of inverse hull we follow, as usual, [1]. This inverse semigroup is defined for any right cancellative semigroup S having no idempotent $\neq 1$.

PROPOSITION 6. *The only semigroups having stable inverse hulls are left groups. If S is a semigroup having the property that each of its sub-semigroups has a stable inverse hull then S is a torsion group.*

⁽³⁾ Our thanks to the referee for this remark.

First of all, if a semigroup S has an inverse hull then it is, at any rate, right cancellative. We need only show that S is left simple, which is to say, $Sx = S$ for each $x \in S$. Now let Sa be a proper subset of S . Consider the extended regular representation of S , $a \rightarrow \rho_a$, where ρ_a is the inner right translation defined by a . Now in the inverse hull, if i is the identical mapping of S^1 and γ is the inverse of ρ_a then

$$\gamma\rho_a = i \quad \text{but} \quad \rho_a\gamma \neq i.$$

But then, γ and ρ_a generate a copy of $\mathcal{C}(p, q)$. This is impossible by Corollary 2.2. Thus, $Sa = S$ for each $a \in S$.

Now for the second statement in the proposition let x be any element of S . Since the inverse hull of an infinite cyclic semigroup is $\mathcal{C}(p, q)$, the semigroup generated by x is finite and contains an idempotent e . The element e is then a right identity and each x has an inverse with respect to e . Thus S is a torsion group.

SOME REMARKS. In the following, let a, x, y be arbitrary elements of a semigroup S and let L and R be $\mathcal{L}$ - and $\mathcal{R}$ -classes contained in the $\mathcal{J}$ -class J . Consider the following statements:

- (1) S is stable.
- (2) $a = xay$ implies $xa\mathcal{R}a$ and $ay\mathcal{L}a$.
- (3) $a = xay$ implies $xa\mathcal{H}a$ and $ay\mathcal{H}a$.
- (4) $a = xay$ implies $xa\mathcal{L}a$ and $ay\mathcal{R}a$.
- (5) $SL \cap J = L$ and $RS \cap J = R$.
- (6) $x, y, xy \in J$ imply $xy \in R_x \cap L_y$.
- (7) S contains no copy of $\mathcal{C}(p, q)$.

We have already seen in the proofs that $(1) \leftrightarrow (2) \leftrightarrow (3) \rightarrow (4) \leftrightarrow (5) \rightarrow (6) \rightarrow (7)$.

Condition (4) could be considered to be a weakened form of stability; in this form also the conclusion $\mathcal{D} = \mathcal{J}$ holds. To see this, suppose $a\mathcal{J}b$ so that $S^1aS^1 = S^1bS^1$. Thus, $a = xby$ and $b = cad$ for appropriate x, y, b , and d . We then have

$$a = x(cad)y = (xc)a(dy).$$

Using (4) we have $xca\mathcal{L}a$ which in turn implies $ca\mathcal{L}a$. In the same way, $ad\mathcal{R}a$. Since $\mathcal{L}$ is right regular $cad\mathcal{L}ad$ and since $ad\mathcal{R}a$ we have by definition $cad\mathcal{D}a$.

Needless to say, the absence of $\mathcal{C}(p, q)$ in S does not imply its stability. One can however make the following observation.

If (7) holds and each $\mathcal{J}$ -class contains an idempotent then S is stable.

Letting $I(e)$ denote $SeS \setminus J_e$ we note that $SeS / I(e)$ is 0-simple and, since it does not contain a copy of $\mathcal{C}(p, q)$, it is completely 0-simple. Thus, we see, in particular, that if a is an element such that J_a contains an idempotent e but contains no bicyclic semigroup then $J_a = D_a$ and a is regular.

Now suppose a is a regular element so that there exist idempotents e and f in D_a such that $e\mathcal{R}a$ and $f\mathcal{L}a$. We shall show that (2) holds. To this end let $a = xay$.

Then $a = xay = x(ea)y = xe(ay)$. Dually $a = xay = x(af)y = (xa)(fy)$. Note that xe, ay, xa, fy lie in J_a . Since xe, ay , and $xey = a$ lie in J_a we see that $xey\mathcal{L}ay$ so that $a\mathcal{L}ay$. Dually, $a\mathcal{R}xa$ so that (2) holds which implies stability.

If L and R are $\mathcal{L}$ - and $\mathcal{R}$ -classes contained in a $\mathcal{D}$ -class D and $L \cap R$ contains an idempotent then $LR = D$. There need not be an idempotent present in $L \cap R$ to have this equality. However, we note the following: *If S is stable and D is a $\mathcal{D}$ -class containing $\mathcal{L}$ - and $\mathcal{R}$ -classes L and R then $LR = D$ if and only if $L \cap R$ contains an idempotent. Thus if D is a semigroup it is completely simple.*

Finally we note that if S is stable cancellative and simple then S is a group. If S is stable and cancellative without idempotents then each $\mathcal{J}$ -class is trivial.

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