LIAPUNOV FUNCTIONS AND L^p SOLUTIONS OF DIFFERENTIAL EQUATIONS

BY

AARON STRAUSS(1)

1. **Introduction.** The following result is known (see [1, p. 148]).

THEOREM 1.1. *If there exists a Liapunov function with a negative definite derivative for the system*

$$(E) \qquad x' = f(t,x) \qquad \left(' = \frac{d}{dt}\right),$$

then the zero solution of (E) *is stable and every solution* $F(t,t_0,x_0)$ *starting with* $|x_0|$ *sufficiently small tends to zero on some sequence of points* $t_n \to +\infty$.

Two questions can now be asked. What further restrictions on the Liapunov function are necessary and sufficient for every such solution to tend to zero as $t \to +\infty$? What further restrictions are necessary and sufficient for every solution to be integrable on the half-line to the right of the initial point t_0? The first question has been answered by many authors [1], [4], [5], [8], [9]. The second question has been asked by Cesari [2, §1.5] and answered by him for second order linear equations [2, §5.6]. Levin and Nohel [6,], [7] also have obtained some results in this direction using Liapunov's second method. The purpose of this paper is to provide a more complete answer to this question. To this end, a new kind of stability (L^p-stability) is introduced and a theory which parallels the theory of asymptotic stability by Liapunov's second method is developed.

2. **Definitions and notation.** The norm of an element y of Euclidean n-space E^n is given by $|y| = \sum_{i=1}^n |y_i|$; the norm of an $n \times n$ matrix $A = (a_{ij})$ is given by $|A| = \sum_{i,j} |a_{ij}|$. Consider the ordinary differential equation

$$(E) \qquad x' = f(t,x) \qquad \left(' = \frac{d}{dt}\right)$$

where x and f belong to E^n, t is a real scalar, and f is defined on the semi-cylinder $D_M = \{(t,x): t \geqq 0, |x| < M\}$, $0 < M \leqq +\infty$. Let $D_M^* = D_M - \{(t, 0): t \geqq 0\}$.

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Assume that $f(t,0) \equiv 0$ for $t \geqq 0$ and that $f(t,x)$ is sufficiently smooth for local existence and uniqueness [3]. (Uniqueness is convenient, not essential, see [4]. It is interesting to note, however, that the mere existence of a Liapunov function for (E) actually implies that the zero function must be a solution and that it is unique in the following sense: if a solution becomes zero at some point, it remains zero thereafter.) For $(t_0, x_0) \in D_M$, let $F(t, t_0, x_0)$ be that solution of (E) for which $F(t_0, t_0, x_0) = x_0$.

DEFINITION 2.1. *$x = 0$ is stable for (E) if for all $\varepsilon > 0$, and all $t_0 \geqq 0$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that if $|x_0| < \delta$, then $F(t, t_0, x_0)$ exists on $[t_0, \infty)$ and satisfies $|F(t, t_0, x_0)| < \varepsilon$ for all $t \geqq t_0$.*

DEFINITION 2.2. *$x = 0$ is asymptotically stable for (E) if it is stable and if for all $t_0 \geqq 0$, there exists a $\delta_0 = \delta_0(t_0) > 0$ such that if $|x_0| < \delta_0$, then $F(t, t_0, x_0) \to 0$ as $t \to \infty$.*

DEFINITION 2.3. *$x = 0$ is L^p-stable for (E) if it is stable and if for all $t_0 \geqq 0$, there exists a $\delta_0 = \delta_0(t_0) > 0$ such that if $|x_0| < \delta_0$, then*

$$\int_{t_0}^{\infty} |F(t, t_0, x_0)|^p dt < \infty. \tag{2.1}$$

DEFINITION 2.4. *$x = 0$ is uniformly L^p-stable if it is L^p-stable and if there exists a number m, $0 < m < M$, such that for every $(t_0, x_0) \in D_m$, there is a neighborhood N of (t_0, x_0) in which the integral* (2.1) *converges uniformly.*

DEFINITION 2.5. *$x = 0$ is (uniformly) L^p-stable in the large if $x = 0$ is stable, the solution $F(t, t_0, x_0)$ exists on $[t_0, \infty)$ for every $(t_0, x_0) \in D_\infty$, and the integral* (2.1) *converges (uniformly in some neighborhood of (t_0, x_0)) for every $(t_0, x_0) \in D_\infty$.*

DEFINITION 2.6. *$V(t,x)$ is a scalar function defined on D_M whose generalized total derivative* [11] *with respect to (E) is given by*

$$\dot{V}_{(E)}(t,x) = \limsup_{h \to 0+} h^{-1}[V(t+h, x+hf(t,x)) - V(t,x)].$$

DEFINITION 2.7. *$V(t,x)$ is positive definite on D_M if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $|x| \geqq \varepsilon$, then $V(t,x) \geqq \delta$ for all $t \geqq 0$. $V(t,x)$ is negative definite if $-V(t,x)$ is positive definite.*

DEFINITION 2.8. *$V(t,x)$ is mildly unbounded on D_∞ if for every $T > 0$, $V(t,x) \to +\infty$ as $|x| \to +\infty$ uniformly on the set $0 \leqq t \leqq T$.*

DEFINITION 2.9. *$V(t,x)$ is locally Lipschitzian on D_M^* if for every $(t_0, x_0) \in D_M^*$, there exists a neighborhood N of (t_0, x_0) and a constant $k = k(t_0, x_0, N)$ such that*

$$|V(t,x_1) - V(t,x_2)| \leqq k|x_1 - x_2|$$

for every (t,x_1) and (t,x_2) in N.

DEFINITION 2.10. *$V(t,x)$ is a Liapunov function on D_M for (E) if*

(i) $V(t,0) = 0$ *for all* $t \geqq 0$,

(ii) *$V(t,x)$ is continuous on D_M,*

(iii) *$V(t,x)$ is positive definite on D_M,*

(iv) *$V(t,x)$ is locally Lipschitzian on D_M^*,*

(v) *$\dot{V}_{(E)}(t,x) \leqq 0$ on D_M.*

DEFINITION 2.11. *$\mathscr{L}(E)$ is the class of Liapunov functions for (E).*

DEFINITION 2.12. *$\bar{\mathscr{L}}(E)$ is the class of mildly unbounded Liapunov functions for (E).*

Definitions 2.5, 2.8, and 2.12 make sense for the case $M = +\infty$ only. In the other definitions, the value $M = +\infty$ is allowed, but it is tacitly assumed that M is finite in §4. Definitions 2.1, 2.2, 2.6, and 2.7 are standard [1]. Definition 2.9 (and thus 2.10) is slightly more general than usual; in fact, it is by using D_M^* rather than D_M in 2.9 that Theorem 4.5 has a reasonably wide range of application, see Example 4.7. Definitions 2.3, 2.4, 2.5, and 2.8 are apparently new.

3. **Preliminary results.** For future reference, we state three lemmas.

LEMMA 3.1. *Let $V \in \mathscr{L}(E)$ and $(t_0, x_0) \in D_M$. Then*

(a) $\dot{V}_{(E)}(t_0, x_0) = \limsup_{h\to 0^+} h^{-1}[V(t_0+h, F(t_0+h, t_0, x_0)) - V(t_0, x_0)]$.

(b) *If $V(t, F(t, t_0, x_0))$ is a continuously differentiable function of t,*

$$\dot{V}_{(E)}(t_0, x_0) = \frac{d}{dt}V(t, F(t, t_0, x_0)) \text{ evaluated at } t = t_0.$$

(c) *If $V \in C^1$ on D_M, then*

$$\dot{V}_{(E)}(t,x) = \frac{\partial}{\partial t}V(t,x) + \sum_{i=1}^{n} f_i(t,x)\frac{\partial}{\partial x_i}V(t,x).$$

Sketch of the proof. If $x_0 = 0$, then both sides of (a) are zero. If $x_0 \neq 0$, then $(t_0, x_0) \in D_M^*$ and the local Lipschitz property of V on D_M^* gives (a). Also, (b) follows immediately from (a) while (c) follows by adding and subtracting $V(t, F(t+h, t, x))$.

The second lemma is due to Liapunov [8] and is proved everywhere (see [1], [4], [5] for example).

LEMMA 3.2. *If there exists a function $V \in \mathscr{L}(E)$, then $x = 0$ is stable for (E).*

The final lemma is a converse theorem originally due to Persidskiĭ [10] (see [1]) and modified by Krasovskiĭ [5].

LEMMA 3.3. *Let $f(t,x)$ and $f_x(t,x)$ be continuous on D_M (f_x is the Jacobian matrix $(\partial f_i/\partial x_j)$). Let $x = 0$ be stable for (E). Then for any number m, $0 < m < M$, there exists a function $V \in \mathscr{L}(E)$ on D_m.*

4. **L^p-stability.** We now present the main results pertaining to L^p-stability.

THEOREM 4.1. *Let $V \in \mathscr{L}(E)$ be such that $\dot{V}_{(E)}(t,x) \leqq -c|x|^p$ on D_M for some $c > 0$, $p > 0$. Then $x = 0$ is L^p-stable for (E).*

Proof. By Lemma 3.2, $x = 0$ is stable. Let $t_0 \geqq 0$ and choose $\delta_0 > 0$ so that if $|x_0| < \delta_0$, then $(t, F(t, t_0, x_0)) \in D_M$ for $t \geqq t_0$. Define for $t \geqq t_0$

$$\gamma(t) = V(t, F(t, t_0, x_0)) + c \int_{t_0}^{t} |F(s, t_0, x_0)|^p ds. \tag{4.1}$$

Fix $t \in [t_0, \infty)$ and let $x = F(t, t_0, x_0)$. Then

$$\begin{aligned}
&\limsup_{h \to 0^+} h^{-1}[\gamma(t+h) - \gamma(t)] \\
&\leqq \limsup_{h \to 0^+} h^{-1}[V(t+h, F(t+h, t_0, x_0)) - V(t, F(t, t_0, x_0))] + c\,|F(t, t_0, x_0)|^p \\
&= \limsup_{h \to 0^+} h^{-1}[V(t+h, F(t+h, t, x)) - V(t, x)] + c\,|x|^p \\
&= \dot{V}_{(E)}(t, x) + c\,|x|^p \leqq 0.
\end{aligned}$$

Thus $\gamma(t)$ is nonincreasing on $[t_0, \infty)$. Since $\gamma(t_0) = V(t_0, x_0)$, we see that $\gamma(t) \leqq V(t_0, x_0)$ for all $t \geqq t_0$. Hence from (4.1),

$$0 \leqq V(t, F(t, t_0, x_0)) \leqq -c \int_{t_0}^{t} |F(s, t_0, x_0)|^p \, ds + V(t_0, x_0)$$

for all $t \geqq t_0$, so that

$$\int_{t_0}^{\infty} |F(t, t_0, x_0)|^p \, dt \leqq \frac{1}{c} V(t_0, x_0),$$

proving Theorem 4.1.

Levin and Nohel [6], [7] obtained a result similar to the above for a particular differential system under the assumption that $V \in C^1$ on D_M.

EXAMPLE 4.2. Consider the first order linear equation

$$x' = \frac{g'(t)}{g(t)} x \tag{4.2}$$

which has the general solution $F(t, t_0, x_0) = g(t)[g(t_0)]^{-1} x_0$. If $g(t)$ is a continuously differentiable, bounded L^1 function on $[0, \infty)$ which does not tend to zero as $t \to \infty$ (see [9]), then $x = 0$ is L^1-stable but not asymptotically stable for (2). On the other hand, if $g(t) = [\log(t+2)]^{-1}$, then the zero solution of (4.2) is asymptotically stable but not L^p-stable for any $p > 0$. Thus asymptotic stability and L^p-stability are different concepts. We do, however, have the following connection.

THEOREM 4.3. *Suppose there is a constant $k > 0$ such that $|f(t,x)| \leqq k$ on D_M. Then if $x = 0$ is L^p-stable for some $p > 0$, it is also asymptotically stable.*

Proof. For $t_0 \geqq 0$, let $\delta_0(t_0)$ be the function described in Definition 2.3. Suppose that for some $t_0 \geqq 0$, there exists an x_0, $|x_0| < \delta_0$, such that $F(t, t_0, x_0) \not\to 0$ as $t \to \infty$. Then there is a number $\varepsilon > 0$ and a sequence $t_n \to \infty$ such that

$|F(t_n, t_0, x_0)| \geqq 2\varepsilon$ for all n. Since $(d/dt)F(t, t_0, x_0)$ is bounded on $t_0 \leqq t < \infty$, there is a number $\lambda > 0$ such that

$$|F(t, t_0, x_0)| \geqq \varepsilon \text{ for } t_n \leqq t \leqq t_n + \lambda,$$

$n = 1, 2, \cdots$, which contradicts (2.1), proving Theorem 4.3.

The converse of Theorem 4.3 does not hold even for autonomous systems because $[\log(t+2)]^{-1}$ is also a solution of the scalar equation

$$x' = \begin{cases} -x^2 \exp(-|x|^{-1}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Now it is natural to ask whether the converse of Theorem 4.1 holds. The following two theorems provide a partial answer. The first deals with the linear case and is completely general. The second deals with the nonlinear case and requires stronger hypotheses.

THEOREM 4.4. *Let $f(t,x) = A(t)x$ in (E) where $A(t)$ is a continuous matrix on $[0, \infty)$. Suppose $x = 0$ is L^p-stable for (E). Then on D_M there exists a function $V \in \mathscr{L}(E)$ such that $\dot{V}_{(E)}(t,x) \leqq -|x|^p$.*

Proof. $F(t, t_0, x_0) = X(t)X^{-1}(t_0)x_0$, where $X(t)$ is that fundamental matrix for $x' = A(t)x$ for which $X(0) = E$, the identity matrix. Since $x = 0$ is stable, there exists a constant $k > 0$ such that $|X(t)| \leqq k$ for all $t \geqq 0$. Define $\lambda(t) = |X(t)|$ and $\phi(t_0, x_0) = |X^{-1}(t_0)x_0|$. Then $|F(t, t_0, x_0)| \leqq \lambda(t)\ \phi\ (t_0, x_0)$, hence $|x_0| \leqq \lambda(t_0)\phi(t_0, x_0)$. For $(t,x) \in D_M$, define

$$V(t,x) = \int_t^\infty [\lambda(s)\phi(t,x)]^p\, ds + \int_0^\infty [\lambda(s)\phi(t,x)]^p\, ds. \tag{4.3}$$

We shall prove the theorem by exhibiting three positive constants k_1, k_2, and k_3 such that if (t,x), (t,x_1), and (t,x_2) are any points in D_M,

$$k_1|x|^p \leqq V(t,x) \leqq k_2|X^{-1}(t)|^p|x|^p, \tag{4.4}$$

$$\dot{V}_{(E)}(t,x) \leqq -|x|^p, \tag{4.5}$$

$$|V(t,x_1) - V(t,x_2)| \leqq k_3(|x_1|^{p-1} + |x_2|^{p-1})|X^{-1}(t)|^p|x_1 - x_2|. \tag{4.6}$$

Note that (4.6) implies that V is locally Lipschitzian on D_M, hence a fortiori, on D_M^*.

Now (4.4) follows from

$$V(t,x) \leqq 2\int_0^\infty [\lambda(s)\phi(t,x)]^p\, ds \leqq \left\{2\int_0^\infty [\lambda(s)]^p\, ds\right\}|X^{-1}(t)|^p|x|^p,$$

and

$$V(t,x) \geqq \int_0^\infty \left[\lambda(s)\frac{|x|}{\lambda(t)}\right]^p ds \geqq \left\{k^{-p}\int_0^\infty [\lambda(s)]^p\, ds\right\}|x|^p.$$

Note that $\phi(t, x) = |X^{-1}(t)x| = |X(0)X^{-1}(t)x| = |F(0,t,x)|$, hence $\phi(t,F(t,t_0, x_0)) = |F(0,t,F(t,t_0, x_0))| = |F(0,t_0, x_0)| = \phi(t_0, x_0)$. Therefore

$$V(t,F(t,t_0, x_0)) = \int_t^{\infty} [\lambda(s)\phi(t_0, x_0)]^p ds + \int_0^{\infty} [\lambda(s)\phi(t_0, x_0)]^p ds$$

which is a continouuously differentiable function of t.

Thus $(d/dt)V(t,F(t,t_0, x_0)) = -[\lambda(t)\phi(t_0, x_0)]^p$, hence by Lemma 3.1b, $\dot{V}_{(E)}(t_0,x_0) = -[\lambda(t_0)\phi(t_0, x_0)]^p \leqq -|x_0|^p$ for any $(t_0, x_0) \in D_M$, proving (4.5).

Finally, we see that

$$(4.7) \quad |V(t,x_1) - V(t,x_2)| \leqq 2\int_0^{\infty} [\lambda(s)]^p ds \, \big| |X^{-1}(t)x_1|^p - |X^{-1}(t)x_2|^p \big|.$$

Now if r_1 and r_2 are real numbers, then by the mean value theorem,

$$|r_1^p - r_2^p| = |p\xi^{p-1}(r_1 - r_2)| \leq p(r_1^{p-1} + r_2^{p-1})|r_1 - r_2|$$

for some ξ between r_1 and r_2. Hence

$$\begin{aligned} &\big| |X^{-1}(t)x_1|^p - |X^{-1}(t)x_2|^p \big| \\ &\leqq p(|X^{-1}(t)x_1|^{p-1} + |X^{-1}(t)x_2|^{p-1}) \big| |X^{-1}(t)x_1| - |X^{-1}(t)x_2| \big| \\ &\leqq p|X^{-1}(t)|^{p-1}(|x_1|^{p-1} + |x_2|^{p-1})|X^{-1}(t)x_1 - X^{-1}(t)x_2| \\ &\leqq p|X^{-1}(t)^p|(|x_1|^{p-1} + |x_2|^{p-1})|x_1 - x_2|, \end{aligned}$$

which, using (4.7), establishes (4.6), where $k_3 = 2p\int_0^{\infty} [\lambda(s)]^p ds$, and proves Theorem 4.4.

Now if two functions belong to L^p on $[0,\infty)$, so do their sum and difference. Since it is known that if the zero solution of a linear system is stable, all solutions are stable, it follows that if the zero solution is L^p-stable, all solutions are L^p-stable

THEOREM 4.5. *Let $f(t,x)$ and $f_x(t,x)$ be continuous on D_M. Let $x = 0$ be uniformly L^p-stable. Suppose further that for every closed subset Q of D_m^*, there exists a function $\psi_Q(t) \in L^p$ on the interval $[0,\infty)$ such that*

$$(4.8) \qquad |\Phi(t + t_0, t_0, x_0)| \leqq \psi_Q(t),$$

for $(t_0, x_0) \in Q$, $t \geqq 0$, where $\Phi(t, t_0, x_0)$ is the matrix of partial derivatives $(\partial F_i/\partial x_{0j})(t, t_0, x_0)$. Then there exists on D_m a function $V \in \mathscr{L}(E)$ such that $\dot{V}_{(E)}(t,x) = -|x|^p$.

An alternative statement of condition (4.8) is that the zero solution of the first variation $y' = f_x(t,F(t,t_0, x_0))y$ is L^p-stable uniformly with respect to the parameters (t_0, x_0) on closed subsets of D_m^*. In either form, the condition seems too strong and not very natural, yet the following proof relies heavily upon it.

Proof. By Lemma 3.3, there exists on D_m, a function $U \in \mathscr{L}(E)$. In fact, U can be so chosen that U and U_x are continuous on D_m (thus U is locally Lipschitzian in D_m^*) and $U(t, F(t, t_0, x_0)) = U(t_0, x_0)$ for every $(t_0, x_0) \in D_m$.

Next, define on D_m,

$$W(t,x) = \int_t^\infty |F(s,t,x)|^p \, ds = \int_0^\infty |F(s+t,t,x)|^p \, ds. \tag{4.9}$$

Then $W(t,0) = 0$ for $t \geqq 0$ and by the uniformity of the L^p-stability, $W(t,x)$ is continuous on D_m. By uniqueness, $F(s,t,F(t,t_0,x_0)) = F(s,t_0,x_0)$ and therefore $W(t,F(t,t_0,x_0)) = \int_t^\infty |F(s,t_0,x_0)|^p \, ds$, hence

$$\frac{d}{dt} W(t,F(t,t_0,x_0)) = -|F(t,t_0,x_0)|^p. \tag{4.10}$$

To see that $W(t,x)$ is locally Lipschitzian on D_m^*, let $(t_0,x_0) \in D_m^*$ and choose the neighborhood N_1 of (t_0,x_0) given by Definition 2.4 so that N_1 is contained in D_m^*. Let N be an open, convex neighborhood of (t_0,x_0) such that $\bar{N} \subset N_1$, where $\bar{N}$ denotes the topological closure of N. Let

$$k = \sup\left\{\int_0^\infty |F(s+t,t,x)|^{p-1} \psi_{\bar{N}}(s) ds : (t,x) \in \bar{N}\right\}.$$

Let (t,x_1) and (t,x_2) belong to N. By a generalized form of the mean value theorem,

$$F(s+t,t,x_1) - F(s+t,t,x_2) = \Phi(s+t,t,z)(x_1 - x_2)$$

where (t,z) is some point on the line segment joining (t,x_1) to (t,x_2). Since N is convex, $(t,z) \in N$. Hence

$$\begin{aligned}
&\int_0^\infty \big||F(s+t,t,x_1)|^p - |F(s+t,t,x_2)|^p\big| \, ds \\
&\leqq \int_0^\infty p\{|F(s+t,t,x_1)|^{p-1} + |F(s+t,t,x_2)|^{p-1}\} \\
&\qquad \cdot \big||F(s+t,t,x_1)| - |F(s+t,t,x_2)|\big| \, ds \\
&\leqq \int_0^\infty p\{|F(s+t,t,x_1)|^{p-1} + |F(s+t,t,x_2)|^{p-1}\} |\Phi(s+t,t,z)| \, |x_1 - x_2| \, ds.
\end{aligned}$$

It thus follows that

$$|W(t,x_1) - W(t,x_2)| \leqq 2pk|x_1 - x_2|$$

proving that $W(t,x)$ is locally Lipschitzian on D_m^*.

Finally, define on D_m,

$$V(t,x) = U(t,x) + W(t,x) \tag{4.11}$$

and we see that V immediately satisfies (i), (ii) and (iv) of Definition 2.10. Since $V(t,x) \geqq U(t,x)$, V is positive definite. Also

$$V(t, F(t, t_0, x_0)) = U(t_0, x_0) + W(t, F(t, t_0, x_0))$$

so that

$$\frac{d}{dt} V(t, F(t, t_0, x_0)) = -\left|F(t, t_0, x_0)\right|^p,$$

by (4.10). Hence by Lemma 3.1b,

$$\dot{V}_{(E)}(t,x) = -\left|x\right|^p,$$

completing the proof of Theorem 4.5.

An interesting special case occurs when f is bounded, i.e., when there is a constant $k > 0$ such that $\left|f(t,x)\right| \leqq k$ on D_M. For this case, $W(t,x)$ itself is the desired Liapunov function. To prove this, we need only show that $W(t,x)$ is positive definite. Now

$$\left|F(s+t,t,x) - x\right| \leqq \int_t^{t+s} \left|f(u, F(u,t,x))\right| du \leqq ks,$$

hence $0 \leqq s \leqq \left|x\right|(2k)^{-1}$ implies $\left|F(s+t,t,x)\right| \geqq 2^{-1}\left|x\right|$. Therefore

$$W(t,x) \geqq \int_0^{|x|(2k)^{-1}} \left|F(s+t,t,x)\right|^p ds \geqq (4k)^{-1} \left|x\right|^{p+1}.$$

Corollary 4.6. *Suppose the assumptions of Theorem 4.5 hold. If f is independent of t or periodic in t then the corresponding Liapunov function can be chosen independent of t or periodic in t, respectively.*

Proof. Let $m < M$. Then f is bounded on D_m and the function

$$W(t,x) = \int_0^\infty \left|F(s+t,t,x)\right|^p ds$$

satisfies $W \in \mathscr{L}(E)$, $\dot{W}_{(E)}(t,x) = -\left|x\right|^p$. Let ω be the period of f in the periodic case and let ω be arbitrary in the autonomous case. Then

$$F(s+t+\omega, t+\omega, x) = F(s+t,t,x)$$

for all $s \geqq 0$, hence

$$\begin{aligned} W(t+\omega, x) &= \int_0^\infty \left|F(s+t+\omega, t+\omega, x)\right|^p ds \\ &= \int_0^\infty \left|F(s+t,t,x)\right|^p ds \\ &= W(t,x), \end{aligned}$$

proving Corollary 4.6.

EXAMPLE 4.7. Consider the scalar equation

$$x' = -\frac{1}{2}x^3 \tag{4.12}$$

for which

$$F(t, t_0, x_0) = x_0[x_0^2(t - t_0) + 1]^{-1/2}$$

and

$$\frac{\partial}{\partial x_0}F(t, t_0, x_0) = [x_0^2(t - t_0) + 1]^{-3/2}.$$

Then $x = 0$ is stable for (4.12). Fix M, $0 < M < +\infty$. Then

$$|F(t + t_0, t_0, x_0)| = (t + |x_0|^{-2})^{-1/2} \leqq (t + M^{-2})^{-1/2}$$

for $t \geqq 0$, $(t_0, x_0) \in D_M$. Furthermore, if $|x_0| \geqq \eta > 0$,

$$\left|\frac{\partial}{\partial x_0}F(t + t_0, t_0, x_0)\right| = [|x_0|^2 t + 1]^{-3/2} \leqq [\eta^2 t + 1]^{-3/2}.$$

Thus all the hypotheses of Theorem 4.5 are satisfied for $p > 2$. Hence, for $p > 2$

$$V(t,x) = \int_t^\infty [((s - t) + |x|^{-2})^{-1/2}]^p ds = 2(p - 2)^{-1}|x|^{p-2}.$$

Thus $V \in C^1$ on D_M^*, in fact, by Lemma 3.1c,

$$\dot{V}_{(E)}(t,x) = [2(\operatorname{sgn} x)|x|^{p-3}][-2^{-1}(\operatorname{sgn} x)|x|^3] = -|x|^p$$

for $x \neq 0$ as we would expect. If $x = 0$, $\dot{V}_{(E)}(t,0) = 0$ by definition. If $p \geqq 3$, $V(t,x)$ is locally Lipschitzian on D_M. But if $2 < p < 3$, then $V(t,x)$ is actually not Lipschitzian in any neighborhood of $x = 0$. Note also that

$$\frac{\partial}{\partial x_0}F(t, t_0, 0) \equiv 1 \notin L^p$$

on $[0, \infty)$. This is the significance of requiring the local Lipschitz condition on D_M^* only (in Definition 2.9) and not on D_M.

Consider the systems

$$(L) \qquad x' = A(t)x,$$

$$(PL) \qquad x' = A(t)x + g(t,x),$$

where $A(t)$ is continuous on $[0, \infty)$, g is sufficiently smooth for local existence and uniqueness, and $g(t,0) \equiv 0$ for $t \geqq 0$.

THEOREM 4.8. *Suppose $x = 0$ is L^p-stable for (L) for some $p > 0$. Let*

$$|X^{-1}(t)|^p |g(t,x)||x|^{-1} \to 0 \quad \text{as } |x| \to 0, \tag{4.13}$$

uniformly on the set $0 \leqq t < \infty$, *where* $X(t)$ *is a fundamental matrix for* (L). *Then* $x = 0$ *is* L^p *-stable for* (PL).

Proof. Choose $V(t,x)$ satisfying (4.4), (4.5), and (4.6) of Theorem 4.4. Then

$$\begin{aligned}
\dot{V}_{(PL)}(t,x) &= \limsup_{h \to 0^+} h^{-1}\{V(t+h, x+h[A(t)x+g(t,x)]) - V(t,x)\} \\
&\leqq \limsup_{h \to 0^+} h^{-1}\{V(t+h, x+h[A(t)x+g(t,x)]) - V(t+h, x+hA(t)x)\} \\
&\quad + \limsup_{h \to 0^+} h^{-1}\{V(t+h, x+hA(t)x) - V(t,x)\} \\
&\leqq \limsup_{h \to 0^+} h^{-1}\{k_3(|x+h[A(t)x+g(t,x)]|^{p-1} + |x+hA(t)x|^{p-1}) \\
&\qquad \cdot |X^{-1}(t+h)|^p |hg(t,x)|\} + \dot{V}_{(L)}(t,x) \\
&\leqq 2k_3 |x|^{p-1} |X^{-1}(t)|^p |g(t,x)| - |x|^p \\
&= -|x|^p(1 - 2k_3|X^{-1}(t)|^p |g(t,x)| |x|^{-1}).
\end{aligned}$$

Hence $V \in \mathscr{L}(PL)$ such that $\dot{V}_{(PL)}(t,x) \leqq -2^{-1}|x|^p$ on D_{M_1}, where $0 < M_1 \leqq M$, and M_1 is such that $|x| < M_1$ implies $|X^{-1}(t)|^p |g(t,x)| |x|^{-1} \leqq (4k_3)^{-1}$. An application of Theorem 4.1 now establishes Theorem 4.8.

EXAMPLE 4.9. Consider on $[1, \infty)$, the systems

$$\begin{aligned} x' &= -(3t)^{-1}x, \\ y' &= -(3t)^{-1}y, \end{aligned} \tag{4.14}$$

$$\begin{aligned} x' &= -(3t)^{-1}x, \\ y' &= -(3t)^{-1}y + t^{-\alpha}x^{\beta}. \end{aligned} \tag{4.15}$$

A fundamental matrix for (4.14) is

$$\begin{pmatrix} t^{-1/3} & 0 \\ 0 & t^{-1/3} \end{pmatrix}$$

that is, (4.14) has the solutions $x(t) = c_1 t^{-1/3}$, $y(t) = c_2 t^{-1/3}$ and hence the zero solution of (4.14) is L^p-stable for $p > 3$. Let $\alpha < 1$ and choose β such that $1 < \beta < 4 - 3\alpha$. Let $\gamma = 3\alpha + \beta - 4$. Then there is a solution of (4.15) given by

$$\bar{x}(t) = c_1 t^{-1/3}, \; \bar{y}(t) = t^{-1/3}(c_2 - 3c_1^{\beta}\gamma^{-1}t^{-\gamma/3}).$$

A simple calculation shows that $\bar{y}(t) \notin L^q$ on $[1, \infty)$ for $3 < q < 3(\gamma+1)^{-1}$, hence it is not true that the zero solution of (4.15) is L^p-stable for $p > 3$. Therefore condition (4.13) in Theorem 4.8 cannot be weakened to either of the following two conditions:

$$|X^{-1}(t)|^p|g(t,x)||x|^{-1} \to 0 \text{ as } |x| \to 0 \text{ for each fixed } t \geqq 0,$$

$$|X^{-1}(t)|^r \ |g(t,x)||x|^{-1} \to 0 \text{ as } |x| \to 0, \text{ uniformly on the set } 0 \leqq t < \infty,$$

where

$$r < \inf\left\{p : \int_0^\infty |X(t)|^p \, dt < \infty\right\}.$$

Finally, we remark that since two fundamental matrices, $X_1(t)$ and $X_2(t)$, for (L) are related by $X_1(t) = X_2(t)C$, where C is a constant, nonsingular matrix, the validity of (4.13) is independent of the choice of the fundamental matrix for (L).

5. **L^p-stability in the large.** If f is defined in all of D_∞, then it is reasonable to ask whether the integral (2.1) exists for all initial values $(t_0, x_0) \in D_\infty$, that is, whether $x = 0$ is L^p-stable in the large. In this section we establish necessary and sufficient conditions for this to occur.

The first theorem, which has significance elsewhere, is basic to the development of this section.

THEOREM 5.1. *Let $f(t,x)$ be continuous and locally Lipschitzian on D_∞. Then the solution $F(t,t_0,x_0)$ of* (E) *can be continued to $[t_0,\infty)$ for every $(t_0,x_0) \in D_\infty$ if and only if there exists a nonnegative, mildly unbounded scalar function $V(t,x)$ defined on D_∞ satisfying* (i), (ii), (iv) *and* (v) *of Definition* (2.10). *Furthermore, this function $V(t,x)$ is positive definite if an only if the zero solution of* (E) *is stable.*

Proof. Assume there is such a function and suppose that for some $(t_0,x_0) \in D_\infty$, $F(t,t_0,x_0)$ cannot be continued to $[t_0,\infty)$. Then there is a number $T > t_0$ such that $|F(t,t_0,x_0)| \to \infty$ as $t \to T^-$. Since V is mildly unbounded we may choose τ so close to T that $V(s, F(\tau,t_0,x_0)) > V(t_0,x_0)$ uniformly on the set $0 \leqq s \leqq T$ and so that $t_0 < \tau < T$. In particular then, $V(\tau, F(\tau,t_0,x_0)) > V(t_0,x_0)$, a contradiction because

$$V(t, F(t,t_0,x_0)) \leqq V(t_0,x_0)$$

for all $t_0 \leqq t < T$ by the nonpositive nature of $\dot{V}_{(E)}$.

Conversely, suppose that the solution $F(t,t_0,x_0)$ exists on $[t_0,\infty)$ for every $(t_0,x_0) \in D_\infty$. For each integer m, let $\phi_m(x)$ be a continuously differentiable function of x such that

$$\phi_m(x) = \begin{cases} 1 & \text{if } |x| \leqq m, \\ 0 & \text{if } |x| \geqq m+1. \end{cases}$$

Then consider the equation

$$(E_m) \qquad x' = \phi_m(x) f(t,x)$$

whose solution we denote by $F_m(t, t_0, x_0)$. Then $F_m(t, t_0, x_0)$ exists on all of $[0, \infty)$ for every $(t_0, x_0) \in D_\infty$, because if $|x_0| \geqq m + 1$, then $F_m(t, t_0, x_0) \equiv x_0$. Define, for $(t, x) \in D_\infty$,

$$V_m(t,x) = |F_m(0,t,x)|.$$

Then $V_m(t, 0) = 0$ for $t \geqq 0$, $V_m(t, x)$ is continuous and locally Lipschitzian on D_∞. If $(t, x) \in D_m$, let $h > 0$ be so small that $(t + h, F(t + h, t, x)) \in D_m$. In D_m, (E_m) is the same as (E), hence

$$\begin{aligned} V_m(t+h, F(t+h,t,x)) &= V_m(t+h, F_m(t+h,t,x)) \\ &= |F_m(0, t+h, F_m(t+h,t,x))| \\ &= |F_m(0,t,x)| \\ &= V_m(t,x). \end{aligned}$$

Thus $\dot{V}_{m(E)}(t, x) \equiv 0$ on the set D_m (but not necessarily on D_∞). Since $\phi_m = \phi_n$ on $D_m \cap D_n$, $V_m = V_n$ on $D_m \cap D_n$. Define on D_∞

$$V(t,x) = \lim_{m \to \infty} V_m(t,x).$$

Then it immediately follows that $V(t, 0) = 0$ for $t \geqq 0$, $V(t, x)$ is continuous and locally Lipschitzian on D_∞, and $\dot{V}_{(E)}(t, x) \equiv 0$ on D_∞, because in any bounded subset N of D_∞, $V(t, x) = V_m(t, x)$ on N for n sufficiently large.

Suppose that $V(t, x)$ is not mildly unbounded. Then for some $T > 0$, it is not true that for every $M > 0$, there exists $Q > 0$ such that $V(t, x) \geqq M$ for all $|x| = Q$ and $t \in [0, T]$. Hence there exists $M > 0$, and sequences $\{x_n\}$, $\{t_n\}$, $|x_n| = n - 1$, $t_n \in [0, T]$ such that $V(t_n, x_n) < M$. Thus, since $(t_n, x_n) \in D_n$,

$$V(t_n, x_n) = V_n(t_n, x_n) = |F_n(0, t_n, x_n)| < M$$

for every n. Now by continuous dependence on initial values and global existence of solutions of (E), there exists $N = N(T, M) > 0$ such that $|F(t, 0, x)| < N$ for all $|x| < M$, $t \in [0, T]$. Thus, if $n > N$, $F(t, 0, x) \equiv F_n(t, 0, x)$ for all $|x| < M$, $t \in [0, T]$. Let $n \geqq N + 1$. Then

$$F(t, 0, F_n(0, t_n, x_n)) \equiv F_n(t, 0, F_n(0, t_n, x_n))$$

and therefore $|F_n(t, 0, F_n(0, t_n, x_n))| < N$ for all $t \in [0, T]$. But at $t = t_n$, this inequality states that $|x_n| < N$, a contradiction because $|x_n| = n - 1 \geqq N$. Hence $V(t, x)$ is mildly unbounded.

To prove the last assertion observe that if the function $V(t, x)$ defined above is positive definite, then the zero solution of (E) is stable by Lemma 3.2. Conversely, let $x = 0$ be stable for (E). Let $\varepsilon > 0$ and choose $\delta > 0$ such that $|x| < \delta$ implies $|F(t, 0, x)| < \varepsilon$ for $t \geqq 0$. Let $m > \varepsilon$. Then clearly $x = 0$ is stable for (E_m); also,

$V(t,x) = |F_m(0,t,x)|$ on D_m. Let $|x| \geqq \varepsilon$. Suppose there is some $t_0 \geqq 0$ such that $|F_m(0,t_0,x)| < \delta$. Then

$$(5.1) \qquad |F_m(t,0,F_m(0,t_0,x))| = |F(t,0,F_m(0,t_0,x))| < \varepsilon$$

for all $t \geqq 0$, by stability. But $t = t_0$, (5.1) becomes $|x| < \varepsilon$, a contradiction. Hence $|x| \geqq \varepsilon$ implies $V(t,x) \geqq \delta$ for all $t \geqq 0$, thus $V(t,x)$ is positive definite and the proof is complete.

If $f_x(t,x)$ is continuous, then we can choose V so that $V_x(t,x)$ is continuous, merely by using the square of the Euclidean norm in the construction in place of the absolute value norm.

Corresponding to Theorem 4.1, we have

THEOREM 5.2. *Let $V \in \overline{\mathscr{L}}(E)$ such that $\dot V_{(E)}(t,x) \leqq -c|x|^p$ on D_∞ for some $c > 0$, $p > 0$. Then $x = 0$ is L^p-stable in the large.*

The proof follows that of Theorem 4.1, because, since V is mildly unbounded, $(t, F(t,t_0,x_0)) \in D_\infty$ for all $t \geqq t_0$ and every $(t_0,x_0) \in D_\infty$.

For linear systems, L^p-stability implies L^p-stability in the large. Furthermore, (4.4) implies that the Liapunov function constructed by (4.3) is mildly unbounded. Hence Theorem 4.4 is already a converse theorem for L^p-stability in the large.

The analogue of Theorem 4.5 is

THEOREM 5.3. *Let $f(t,x)$ and $f_x(t,x)$ be continuous on D_∞. Let $x = 0$ be uniformly L^p-stable in the large. Suppose further that for every closed subset Q of D^*_∞, there exists a function $\psi_Q(t) \in L^p$ on the interval $[0,\infty)$ such that*

$$|\Phi(t+t_0,t_0,x_0)| \leqq \psi_Q(t) \quad \textit{for } (t_0,x_0) \in Q, \qquad t \geqq 0.$$

Then there exists on D_∞ a function $V \in \overline{\mathscr{L}}(E)$ such that $\dot V_{(E)}(t,x) = -|x|^p$.

Proof. Construct U by Theorem 5.1. Because $x = 0$ is stable, U is actually positive definite. Hence $U \in \overline{\mathscr{L}}(E)$ and we form the desired function V by $V(t,x) = W(t,x) + U(t,x)$ as before, completing the proof.

Three final remarks are in order. First, a careful examination of the proof of Theorem 4.8 shows that this theorem does not have an "L^p-stable in the large" analogue. Second, Theorem 5.1 can be used to generalize the standard "asymptotic stability in the large" theorems; that is, the condition $V(t,x) \to \infty$ as $|x| \to \infty$ uniformly in t on the set $0 \leqq t < \infty$ (V is radially unbounded) may be replaced by the weaker condition that V is mildly unbounded, see [1], [4], [5]. Finally, many of these results can be extended to the case where (E) is replaced by a functional differential equation of the form

$$(F) \qquad x'(t) = f(t, x(t+\theta)), \qquad -h \leqq \theta \leqq 0,$$

where $h > 0$, see [5, Chapter 6].

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UNIVERSITY OF WISCONSIN,
MADISON, WISCONSIN