GENERALIZED FEKETE MEANS

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1.0. Introduction. In 1954, M. Fekete [3] proposed two sequence-to-sequence summability methods which he called "Taylor-Nörlund" and "Nörlund-Taylor." His methods are extended in this paper by replacing the Taylor method with a more general summability method. Various inclusion properties of the new methods are developed. It is shown that these results hold when certain Hausdorff or quasi-Hausdorff methods are combined with Nörlund methods; in particular, for Cesàro, Euler or Taylor methods. Series-to-sequence analogues of these methods are defined and applied to the problem of summing the geometric series.

2.0. Preliminary definitions. An infinite matrix (f_{nk}) is a T -matrix if and only if $t_n = \sum f_{nk}s_k \rightarrow s$ whenever $s_n \rightarrow s$. An infinite matrix (a_{nk}) is an alpha-matrix if and only if $\sum v_n = s$, where $v_n = \sum a_{nk}u_k$, whenever $\sum u_n = s$. The infinite matrix (g_{nk}) is a gamma-matrix if and only if $t_n = \sum g_{nk}u_k \rightarrow s$ whenever $\sum u_n = s$. Given a T -matrix (f_{nk}) , it has been shown [13], [1, p. 86] that if $g_{nk} = \sum_{j=k}^{\infty} f_{nj}$, if $a_{0k} = g_{0k}$ and if $a_{nk} = g_{nk} - g_{n-1,k}$ when $n \geq 1$, then (g_{nk}) is a gamma-matrix and (a_{nk}) is an alpha-matrix such that $\lim_n \sum_k g_{nk}u_k = \sum_n \sum_k a_{nk}u_k = s$ whenever $\lim_n \sum_k f_{nk}s_k = s$ for bounded $s_n = u_0 + \dots + u_n$, at least. Such triples of T -, gamma- and alpha-matrices define a regular summability method A . When $\lim_n \sum_k f_{nk}s_k = s$, it is convenient to write $A\text{-}\lim s_n = s$.

A particular class of regular methods, the Nörlund methods, is of importance to the subsequent discussion. A sequence $\{p_n\}$ of real numbers is a Nörlund sequence [3] if and only if $p_0 > 0$, $p_n \geq 0$ for $n \geq 1$, and $p_n/P_n \rightarrow 0$, where $P_n = p_0 + \dots + p_n$. Given a Nörlund sequence $\{p_n\}$, let $F_{nk} = p_{n-k}/P_n$ if $n \geq k$, 0 otherwise. Then (F_{nk}) is a T -matrix [4, p. 64] and the summability method defined by it is called the regular Nörlund method (N, p_n) .

The summability method A is said to include the summability method B if $A\text{-}\lim s_n = B\text{-}\lim s_n$ for every sequence $\{s_n\}$ having a convergent B -limit. A totally includes B if $B\text{-}\lim s_n = s$ implies that $A\text{-}\lim s_n = s$ for every s , finite or infinite [12].

3.0. The A -Nörlund method.

3.1. THE DEFINITION.

DEFINITION 3.1.1. Let A be a regular summability method with alpha-matrix (a_{nk}) and T -matrix (f_{nk}) . Let (N, p_n) be a regular Nörlund method such that (N, q_n) is a regular Nörlund method, where $q_n = \sum a_{nk}p_k$. A sequence $\{s_n\}$ is said

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to be summable to s by the A -Nörlund method if and only if $t_n \rightarrow s, (N, q_n)$, where $t_n = \sum f_{nk} s_k$. Write $s_n \rightarrow s, F(A, p_n)$ in this case.

It may happen that the transform of a Nörlund sequence by an alpha-matrix is not a Nörlund sequence. For example, let (a_{nk}) be an alpha-matrix with $a_{uv} < 0$ for some positive integers u and v . Define a Nörlund sequence $\{p_n\}$, where $p_0 = 1, p_v > |a_{u0}/a_{uv}|, p_n = 0$ otherwise. Then $q_u = a_{u0} + a_{uv}p_v < 0$ and $\{q_n\}$ clearly is not a Nörlund sequence. The next two lemmas provide conditions that $\{q_n\}$ be a Nörlund sequence.

LEMMA 3.1.2. *If the alpha-matrix (a_{nk}) of A satisfies $a_{00} > 0, a_{nk} \geq 0$ otherwise; if there exists a constant $M > 0$ such that $a_{nk} \leq M \cdot f_{nk}$, where (f_{nk}) is the T -matrix of A ; if $\{p_n\}$ is a Nörlund sequence such that $q_n = \sum a_{nk} p_k$ exists for every n , then $\{q_n\}$ is a Nörlund sequence.*

Proof. Clearly, $q_n \geq 0$ and $q_0 \geq a_{00}p_0 > 0$. If $q_0 + \dots + q_n = Q_n$, then

$$\begin{aligned} Q_n &= \sum_{k=0}^n \sum_{j=0}^{\infty} a_{kj} p_j = \sum_{j=0}^{\infty} \sum_{k=0}^n a_{kj} p_j \\ &= \sum_{j=0}^{\infty} g_{nj} p_j = \sum_{j=0}^{\infty} \sum_{k=j}^{\infty} f_{nk} p_j = \sum_{k=0}^{\infty} f_{nk} \cdot \sum_{j=0}^k p_j = \sum_{k=0}^{\infty} f_{nk} P_k, \end{aligned}$$

where (g_{nk}) is the gamma-matrix of A , the inversions of order of summation being justified, since all terms are non-negative. Then, $q_n/Q_n = (\sum a_{nk} p_k) / (\sum f_{nk} P_k) = \sum h_{nk} (p_k/P_k)$, where $h_{nk} = (a_{nk} P_k) / (\sum f_{nk} P_k)$. Now, $\sum_k |h_{nk}| = \sum_k h_{nk} \leq (M \cdot \sum f_{nk} P_k) / (\sum f_{nk} P_k) = M$ for every n . Also, $h_{nk} = (a_{nk} P_k) / Q_n \leq (M \cdot f_{nk} P_k) / q_0 \rightarrow 0$ as $n \rightarrow \infty$ for every k . Therefore, by Theorem 4 of [4], $q_n/Q_n \rightarrow 0$ if $p_n/P_n \rightarrow 0$. Hence, $\{q_n\}$ is a Nörlund sequence.

A slight modification of the discussion preceding the lemma shows that $a_{00} > 0$ and $a_{nk} \geq 0$ otherwise are necessary for $\{q_n\}$ to be a Nörlund sequence. An example is given after Lemma 3.4.1 below to show that the second condition on the matrices (a_{nk}) and (f_{nk}) is not a necessary condition.

LEMMA 3.1.3. *Let (a_{nk}) be an alpha-matrix and let $\{p_n\}$ be a Nörlund sequence such that $\sum p_n$ converges and $q_n = \sum a_{nk} p_k$ exists for every n . Then, the conditions $a_{00} > 0$ and $a_{nk} \geq 0$ otherwise are necessary and sufficient that $\{q_n\}$ be a Nörlund sequence.*

Proof. The necessity of the conditions is obvious. For sufficiency, one notes that $\sum q_n$ converges and hence that $q_n/Q_n \leq q_n/q_0 \rightarrow 0$.

In particular, Lemma 3.1.3 shows that finite Nörlund sequences, when $p_n \neq 0$ for only finitely many n , are transformed into Nörlund sequences by any alpha-matrix (a_{nk}) with non-negative entries and $a_{00} > 0$.

3.2. PROPERTIES OF $F(A, p_n)$. Since the Nörlund method (N, q_n) is regular, if $t_n \rightarrow s$, then $t_n \rightarrow s, (N, q_n)$. Thus the next theorem is an immediate consequence of the definition of $F(A, p_n)$.

THEOREM 3.2.1. $F(A, p_n)$ is regular and includes A .

In fact, since the matrix of (N, q_n) has non-negative elements, if t_n exists for every n and $t_n \rightarrow +\infty$, it follows that $t_n \rightarrow +\infty$, (N, q_n) [4, Theorem 9], so $F(A, p_n)$ totally includes A .

If A is $(C, 1)$, if $p_0 = 1$, $p_n = 2$ for $n \geq 1$, and if $s_n = (n+1)(-1)^n$ then $\{s_n\}$ is not summable A , but $s_n \rightarrow 0$, $F(A, p_n)$, as a short calculation shows. Hence, in general, $F(A, p_n)$ is not equivalent to the method A .

Since Nörlund methods are consistent [4, p. 65], it follows that if $s_n \rightarrow s$, $F(A, p_n)$ and $s_n \rightarrow s'$, $F(A, r_n)$, then $s = s'$. Thus, for fixed A and various Nörlund sequences, the methods $F(A, p_n)$ are consistent.

However, if A and B are inconsistent methods, if $s_n \rightarrow s$, $F(A, p_n)$ and $s_n \rightarrow s'$, $F(B, p_n)$, it need not follow that $s = s'$ when $\{s_n\}$ is divergent. For example, let A transform $\{s_n\}$ into the sequence $\{t_n\}$, where $t_n = (s_{2n} + s_{2n+2})/2$, and let B transform $\{s_n\}$ into the sequence $\{u_n\}$, where $u_n = (s_{2n+1} + s_{2n+3})/2$. Then, a short calculation shows that the alpha-matrices of A and B satisfy the hypotheses of Lemma 3.1.3. If $\{p_n\}$ is any finite Nörlund sequence, it follows from Theorem 3.2.1 that $(-1)^n \rightarrow 1$, $F(A, p_n)$ while $(-1)^n \rightarrow -1$, $F(B, p_n)$.

DEFINITION 3.2.2. [14, p. 111]. A sequence $\{s_n\}$ is summable to s by the strong-Abel method if and only if $s^*(x) = (1-x) \cdot \sum s_n x^n$ has a positive radius of convergence and defines an analytic function which is regular for $0 \leq x < 1$ and which tends to s as $x \rightarrow 1$ through real values less than 1.

THEOREM 3.2.3. If $s_n \rightarrow s$, $F(A, p_n)$, then $s_n \rightarrow s$ by the product of A and the strong-Abel methods.

Proof. Let $v_0 = t_0$, $v_n = t_n - t_{n-1}$ for $n \geq 1$. Then, if $s_n \rightarrow s$, $F(A, p_n)$, $\sum v_n = s$, (N, q_n) . By Theorem 18 of [4], $\sum v_n = s$ by the strong-Abel method. Since $t^*(x) = v(x) = \sum v_n x^n$, the assertion follows.

If it should happen that $s_n \rightarrow s$, $F(A, p_n)$ and $\sum t_n x^n$ converges for $|x| < 1$, then $s_n \rightarrow s$ by the product of A and the Abel method. That $F(A, p_n)$ is not, in general, equivalent to the product of A and the (strong-)Abel methods can be seen by letting A be $(C, 1)$, $p_0 = 1$, $p_1 = 2$, $p_2 = 1$, $p_n = 0$ for $n \geq 3$, and $s_n = (n+1)(-1)^n$. It can be seen, after some calculations, that $s_n \rightarrow 0$ by the product of the $(C, 1)$ and Abel methods, but $\{s_n\}$ is not summable $F(A, p_n)$.

DEFINITION 3.2.4. An alpha-matrix (a_{nk}) is said to be monotone-preserving with respect to the nondecreasing sequence $\{p_n\}$ if and only if the sequence $\{q_n\} = \{\sum a_{nk} p_k\}$ is nondecreasing.

DEFINITION 3.2.5. An alpha-matrix (a_{nk}) is said to be $(C, 1)$ -preserving if and only if (N, q_n) is equivalent to $(C, 1)$ whenever (N, p_n) is $(C, 1)$ and $q_n = \sum a_{nk} p_k = \sum a_{nk}$.

Clearly, if $\sum_k a_{nk} = L > 0$ for every n , then (a_{nk}) is $(C, 1)$ -preserving, since $q_0 = q_1 = \dots = L$.

THEOREM 3.2.6. *If the alpha-matrix of A is $(C, 1)$ -preserving and monotone-preserving with respect to the nondecreasing Nörlund sequence $\{p_n\}$, then $F(A, p_n)$ includes $F(A, 1)$.*

Proof. Since $\{q_n\}$ is nondecreasing, (N, q_n) includes $(C, 1)$, by Theorem 20 of [4]. If $s_n \rightarrow s$, $F(A, 1)$, then $t_n \rightarrow s$, (N, r_n) , where (N, r_n) is equivalent to $(C, 1)$. Thus, $t_n \rightarrow s$, $(C, 1)$ and therefore $t_n \rightarrow s$, (N, q_n) .

THEOREM 3.2.7. *If the method A is such that the A -transform of a sequence $\{s_n\}$ is summable to s by the Abel method whenever $\{s_n\}$ is summable to s by the Abel method, if the alpha-matrix of A is monotone-preserving with respect to the nondecreasing Nörlund sequence $\{p_n\}$, then $F(A, p_n)$ includes $(C, 1)$ for bounded sequences.*

Proof. Let $s_n \rightarrow s$, $(C, 1)$. Then $s_n \rightarrow s$, (Abel), so $t_n \rightarrow s$, (Abel). Since $\{s_n\}$ is bounded, $\{t_n\}$ is bounded. Hence, by Theorem 92 of [4], $t_n \rightarrow s$, $(C, 1)$. Since $\{q_n\}$ is a nondecreasing Nörlund sequence, (N, q_n) includes $(C, 1)$, and therefore, $t_n \rightarrow s$, (N, q_n) .

The methods (N, p_n) and $F(A, p_n)$, in general, are not comparable. It has already been noted that when A is $(C, 1)$, and $p_0 = 1$, $p_1 = 2$, $p_2 = 1$, $p_n = 0$ for $n \geq 3$, the sequence $\{(n+1)(-1)^n\}$ is not summable $F(A, p_n)$, although $(n+1)(-1)^n \rightarrow 0$, (N, p_n) . On the other hand, for the same A and $\{p_n\}$, if $r_n = 2i^n$, then $\{r_n\}$ is not summable (N, p_n) , although $r_n \rightarrow 0$, $(C, 1) = A$, and hence $r_n \rightarrow 0$, $F(A, p_n)$.

3.3. HAUSDORFF-NÖRLUND METHODS. Let $\{\mu_n\}$ be a sequence of real numbers such that $\mu_0 = 1$, $\Delta^n \mu_0 \rightarrow 0$ and $\Delta^p \mu_n \geq 0$ for all n and p , where $\Delta \mu_n = \mu_n - \mu_{n+1}$, $\Delta^{p+1} \mu_n = \Delta(\Delta^p \mu_n)$. The matrix (f_{nk}) , where $f_{nk} = \binom{n}{k} \Delta^{n-k} \mu_k$ if $n \geq k$, 0 otherwise, is a T -matrix [14, p. 148]. Associated with this T -matrix is the alpha-matrix (a_{nk}) , where $a_{0k} = f_{0k}$, $a_{nk} = (k/n)f_{nk}$ when $n \geq 1$. Let $H(\mu_n)$ denote the regular summability method determined by these matrices.

The $H(\mu_n)$ methods are a subclass of the set of regular Hausdorff methods. This subclass includes the Cesàro, Hölder, and Euler methods [4, Chapter XI], and hence is of some interest. Since the T -matrix for $H(\mu_n)$ has non-negative elements, it is a totally regular matrix [4, Theorem 9] and thus, $\mu_n \neq 0$ for every n [12]. The alpha-matrix of $H(\mu_n)$ transforms Nörlund sequences into Nörlund sequences, for q_n always exists and Lemma 3.1.2 holds with $M = 1$. One can therefore define $H(\mu_n)$ -Nörlund methods for any Nörlund sequence. In order to see how the results of §3.2 hold for these Hausdorff-Nörlund methods, one observes that Pati [8] has shown that if $\{s_n\}$ is summable by the Abel method, the $H(\mu_n)$ -transform of $\{s_n\}$ is also summable by the Abel method to the same sum. Regarding the $(C, 1)$ - and monotone-preserving properties, there are the following results.

LEMMA 3.3.1. *The alpha-matrix of $H(\mu_n)$ is $(C, 1)$ -preserving.*

Proof. $\sum_k a_{0k} = 1$, $\sum_k a_{nk} = \sum_{k=1}^n \binom{n-1}{k-1} \Delta^{n-k} \mu_k = \mu_1$ for $n \geq 1$, and $0 < \mu_1 \leq 1$. If $p_n = 1$ for all n , so (N, p_n) is $(C, 1)$, then $q_0 = 1$ and $q_n = \mu_1$ for $n \geq 1$. Let $q(x) = \sum q_n x^n = 1 + (\mu_1 x)/(1-x)$ and $p(x) = \sum p_n x^n = 1/(1-x)$, so $q(x)/p(x) = 1 - (1 - \mu_1)x = \sum K_n x^n$, and $p(x)/q(x) = 1/[1 - (1 - \mu_1)x] = \sum (1 - \mu_1)^n x^n = \sum L_n x^n$. Since both $\sum |K_n|$ and $\sum |L_n|$ converge, (N, q_n) is equivalent to $(N, p_n) = (C, 1)$ by Theorem 21 of [4].

LEMMA 3.3.2. *The alpha-matrix of $H(\mu_n)$ is monotone-preserving with respect to those nondecreasing sequences $\{p_n\}$ for which $p_0 \leq \mu_1 p_1$.*

Proof. $q_0 - q_1 = p_0 - \mu_1 p_1 \leq 0$. For $n \geq 1$,

$$q_n - q_{n+1} = \sum_{k=1}^n \binom{n-1}{k-1} \Delta^{n-k} \mu_k p_k - \sum_{k=1}^{n+1} \binom{n}{k-1} \Delta^{n-k+1} \mu_k p_k,$$

which, after some simplifications, becomes

$$q_n - q_{n+1} = \sum_{k=1}^n \binom{n-1}{k-1} \Delta^{n-k} \mu_{k+1} (p_k - p_{k+1}).$$

Since $\Delta^p \mu_n \geq 0$, it follows that $q_n - q_{n+1} \leq 0$ if $\{p_n\}$ is nondecreasing. Thus, the sequence $\{q_n\}$ is nondecreasing.

3.4. QUASI-HAUSDORFF-NÖRLUND METHODS. Let $G(u)$ be a nondecreasing function on $[0, 1]$ such that $G(0) = G(0+) = 0$, $G(1) = 1$ and $\int_0^1 (1/u) dG$ exists. Let $\mu_n = \int_0^1 u^n dG$. The matrix (f_{nk}) , where $f_{nk} = \binom{k}{n} \Delta^{k-n} \mu_{n+1}$ if $k \geq n$, 0 otherwise, is a T -matrix [9], [10]. Associated with this T -matrix is the alpha-matrix (a_{nk}) , where $a_{nk} = \binom{k}{n} \Delta^{k-n} \mu_n$ if $k \geq n$, 0 otherwise. Let $H^*(\mu_n)$ denote the regular summability method determined by these matrices.

These $H^*(\mu_n)$ methods are a subclass of the set of regular quasi-Hausdorff methods. This subclass includes the Taylor method. Since the alpha-matrix of $H^*(\mu_n)$ method has $a_{00} = 1$ and $a_{nk} \geq 0$ and has bounded row-sums, it follows from Lemma 3.1.3 that $H^*(\mu_n)$ -Nörlund methods may be defined for any Nörlund sequence for which $\sum p_n$ converges. The next lemma deals with general Nörlund sequences.

LEMMA 3.4.1. *If there exists a $b > 0$ such that $G(u) = 0$ for $0 \leq u \leq b$, then the alpha-matrix of $H^*(\mu_n)$ transforms Nörlund sequences into Nörlund sequences.*

Proof. Let $p(x) = \sum p_n x^n$, where $\{p_n\}$ is a Nörlund sequence. This series converges for $|x| < 1$ [4, p. 65]. Hence, for every n ,

$$\int_0^1 \left(\frac{u^n}{n!} \right) \cdot p^{(n)}(1-u) dG = \sum_{k=n}^{\infty} \binom{k}{n} p_k \cdot \int_0^1 u^n (1-u)^{k-n} dG = q_n$$

exists, the interchange of summation and integration being justified, since the integrands are non-negative and continuous on $[b, 1]$. Furthermore, for $k \geq n$,

$$\begin{aligned} f_{nk} - b \cdot a_{nk} &= \binom{k}{n} \int_0^1 u^{n+1} (1-u)^{k-n} dG - \binom{k}{n} \int_0^1 b \cdot u^n (1-u)^{k-n} dG \\ &= \binom{k}{n} \int_b^1 u^n (1-u)^{k-n} (u-b) dG \geq 0. \end{aligned}$$

Thus, $a_{nk} \leq (1/b) \cdot f_{nk}$ for all n and k , and the hypotheses of Lemma 3.1.2 are satisfied.

In order to see that the hypothesis of Lemma 3.4.1 is not a necessary condition, let $H^*(\mu_n)$ be generated by $G(u) = u^2$, $0 \leq u \leq 1$. Then $\mu_n = 2/(n+2)$. A short calculation shows that the alpha-matrix (a_{nk}) of $H^*(\mu_n)$ is given by $a_{nk} = 2(n+1)/(k+1)(k+2)$ for $k \geq n$, 0 otherwise; and the T -matrix (f_{nk}) of this method is given by $f_{nk} = 2(n+1)(n+2)/(k+1)(k+2)(k+3)$ for $k \geq n$, 0 otherwise. If $\{p_n\}$ is any sequence of non-negative terms with $p_0 > 0$ such that $q_n = \sum a_{nk} p_k$ exists for every n , the proof of Lemma 3.1.2 shows that $Q_n = q_0 + \dots + q_n = \sum f_{nk} P_k$, where $P_n = p_0 + \dots + p_n$. It is easily seen for this method that $0 \leq q_n = Q_n - Q_{n-1} = 2Q_n/(n+2) - 2P_{n-1}/(n+2) < 2Q_n/(n+2)$, so $q_n/Q_n \rightarrow 0$ and $\{q_n\}$ is a Nörlund sequence. Furthermore, since $(n+2)a_{nk} = (k+3)f_{nk}$ for $k \geq n$, there is no $M > 0$ such that $a_{nk} \leq M \cdot f_{nk}$. Hence, this example also shows that the second hypothesis of Lemma 3.1.2 is not a necessary condition.

The results of §3.2 hold for some quasi-Hausdorff-Nörlund methods. Ramanujan [10] has shown that for bounded $\{s_n\}$, if $\{s_n\}$ is summable Abel, then the $H^*(\mu_n)$ -transform of $\{s_n\}$ is also summable Abel to the same sum. Also, it is easy to see that the alpha-matrix of $H^*(\mu_n)$ is $(C, 1)$ -preserving, since

$$\sum_{k=0}^{\infty} a_{nk} = \sum_{k=n}^{\infty} \binom{k}{n} \int_0^1 u^n (1-u)^{k-n} dG = \int_0^1 (1/u) dG \text{ for every } n.$$

LEMMA 3.4.2. *The alpha-matrix of $H^*(\mu_n)$ is monotone-preserving with respect to those nondecreasing sequences $\{p_n\}$ such that q_n exists for every n and*

$$\lim_k \binom{k+1}{n+1} \Delta^{k-n} \mu_n p_k = 0$$

for every n .

Proof. For any $n \geq 0$,

$$\begin{aligned} q_n - q_{n+1} &= \lim_m \left[\sum_{k=n}^{m+1} \binom{k}{n} \Delta^{k-n} \mu_n p_k - \sum_{k=n+1}^{m+1} \binom{k}{n+1} \Delta^{k-n-1} \mu_{n+1} p_k \right] \\ &= \lim_m \left[\sum_1 - \sum_2 \right]. \end{aligned}$$

In $\sum_2$ one can let $k = j + 1$ and then replace $\Delta^{j-n} \mu_{n+1}$ by $\Delta^{j-n} \mu_n - \Delta^{j-n+1} \mu_n$. Then $\sum_2$ can be written as $\sum_3 - \sum_4$. Letting $j + 1 = k$ in $\sum_4$, one finds that $\sum_1$ and $\sum_4$ can be combined. When the resulting expressions are simplified, one has

$$q_n - q_{n+1} = \lim_m \left[\sum_{k=n}^m \binom{k+1}{n+1} \Delta^{k-n} \mu_n(p_k - p_{k+1}) + \binom{m+2}{n+1} \Delta^{m+1-n} \mu_n p_{m+1} \right].$$

Hence,

$$q_n - q_{n+1} = \sum_{k=n}^{\infty} \binom{k+1}{n+1} \Delta^{k-n} \mu_n(p_k - p_{k+1}) \leq 0$$

since $\Delta^p \mu_n \geq 0$.

Incidentally, the same kind of calculation, using μ_{n+1} in place of μ_n , shows that the T -matrix of $H^*(\mu_n)$ preserves monotonicity of suitable sequences.

The Taylor method $T(a)$ of Fekete's original formulation is the quasi-Hausdorff method $H^*(a^n)$, where $0 < a < 1$. This method is generated by the function G , where $G(u) = 0$ for $0 \leq u \leq a$, $G(u) = 1$ for $a < u \leq 1$. This observation, together with Lemma 3.4.1, shows that the alpha-matrix of any Taylor method transforms Nörlund sequences into Nörlund sequences. In fact, the alpha-matrix of $T(a)$ transforms nondecreasing Nörlund sequences into nondecreasing Nörlund sequences. To see this, let $\{p_n\}$ be such a sequence and let $p(x) = \sum p_n x^n$. Then $D^{n+1}[x \cdot p(x)]/(n+1)! = \sum_{k=n}^{\infty} \binom{k+1}{n+1} p_k x^{k-n}$ converges for $|x| < 1$, where $D = d/dx$. Then, for any a , $0 < a < 1$, it follows that $\lim_k a^n \binom{k+1}{n+1} p_k (1-a)^{k-n} = 0$, and, by Lemma 3.4.2, the transform of $\{p_n\}$ by the alpha-matrix of $T(a)$ is nondecreasing.

4.0. The Nörlund- A method.

4.1. THE DEFINITION.

DEFINITION 4.1.1. Let A be a regular summability method which has a T -matrix (f_{nk}) with no zero rows and such that $f_{nk} \geq 0$ for all n and k . Let (N, p_n) be a regular Nörlund method such that $Q_n = \sum f_{nk} P_k$ exists for each n , where $P_n = p_0 + \dots + p_n$. A sequence $\{s_n\}$ is said to be summable to s by the Nörlund- A method if and only if $T_n/Q_n \rightarrow s$, where $T_n = \sum f_{nk} S_k$ and $S_n = \sum_{k=0}^n p_{n-k} s_k$. Write $S_n \rightarrow s, G(A, p_n)$, in this case.

LEMMA 4.1.2. For any Nörlund sequence $\{p_n\}$ there exists an $r = r(p_n, A) > 0$ such that for all n , $Q_n \geq r$.

Proof. For all n , $P_n \geq p_0$. Since $f_{nk} \geq 0$, it follows from Theorem 9 in [4] that $\liminf Q_n \geq \liminf P_n \geq p_0$. Hence there exists an $N > 0$ such that $Q_n \geq p_0/2$ for all $n \geq N$. Furthermore, $Q_n > 0$ for all n , since (f_{nk}) has no zero rows. Set $r = \min(Q_0, Q_1, \dots, Q_{N-1}, p_0/2)$. Then for all n , $Q_n \geq r > 0$.

4.2. PROPERTIES OF $G(A, p_n)$.

THEOREM 4.2.1. $G(A, p_n)$ is regular and includes (N, p_n) .

Proof. $T_n/Q_n = (\sum f_{nk} S_k) / (\sum f_{nk} P_k) = \sum d_{nk} (S_k/P_k)$, where $d_{nk} = f_{nk} P_k / Q_n$. Then, $d_{nk} \leq f_{nk} P_k / r \rightarrow 0$ as $n \rightarrow \infty$ for each k . Also, $\sum_k |d_{nk}| = \sum_k d_{nk} = 1$ for every n . This shows that (d_{nk}) is a T -matrix. Therefore, if $s_n \rightarrow s, (N, p_n)$, then $S_k/P_k \rightarrow s$. Hence, $T_n/Q_n \rightarrow s$. Since (N, p_n) is regular, $G(A, p_n)$ is also regular.

When A is $(C, 1)$, $p_0 = p_2 = 1$, $p_n = 0$ otherwise, and $s_n = (-1)^n$, it is easily seen that $s_n \rightarrow 0$, $G(A, p_n)$, but $\{s_n\}$ is not summable (N, p_n) . Thus, in general, $G(A, p_n)$ is not equivalent to (N, p_n) .

DEFINITION 4.2.2 [6]. A summability method A is said to be translatable to the left if and only if $A\text{-}\lim r_n = A\text{-}\lim s_n$ whenever $A\text{-}\lim s_n$ exists and $r_n = s_{n-1}$, $r_0 = 0$.

THEOREM 4.2.3. *If the method A is translatable to the left and if $\{p_n\}$ is a finite Nörlund sequence, then $G(A, p_n)$ includes A .*

Proof. There exists an $N > 0$ such that $P_n = P_N > 0$ for all $n \geq N$. Thus, $Q_n \rightarrow P_N$. Also, if $A\text{-}\lim s_n = s$, then $T_n = p_0 \cdot \sum_k f_{nk} s_k + p_1 \cdot \sum_k f_{nk} s_{k-1} + \cdots + p_N \cdot \sum_k f_{nk} s_{k-N} \rightarrow P_N s$. Therefore, $T_n/Q_n \rightarrow s$.

When $\{p_n\}$ has only finitely many nonzero terms and A is translatable to the left, A is not, in general, equivalent to $G(A, p_n)$. For example, when A is $(C, 1)$, $p_0 = 1$, $p_1 = 2$, $p_2 = 1$ and $p_n = 0$ for $n \geq 3$, the sequence $\{s_n\}$, with $s_n = (n+1)(-1)^n$, is not summable A , but $s_n \rightarrow 0$, (N, p_n) and hence $s_n \rightarrow 0$, $G(A, p_n)$.

THEOREM 4.2.4. *Let (f_{nk}) and (F_{nk}) be the T -matrices of the methods A and B , both matrices having no zero rows and no negative elements. If there exists a matrix (e_{nk}) with $e_{nk} \geq 0$ for all n and k , such that $(F_{nk}) = (e_{nk})(f_{nk})$ and $\lim_n e_{nk} = 0$ for every k , then $G(B, p_n)$ totally includes $G(A, p_n)$ for all sequences $\{s_n\}$ for which the product $(e_{nk})(f_{kj})(S_j)$ is associative.*

Proof. Let $T_n = \sum f_{nk} S_k$, $Q_n = \sum f_{nk} P_k$, $T_n^* = \sum F_{nk} S_k$ and $Q_n^* = \sum F_{nk} P_k$, all of which are assumed to exist for every n . Now, $T_n^* = \sum_j F_{nj} S_j = \sum_k e_{nk} (\sum_j f_{kj} S_j) = \sum_k e_{nk} T_k$. Similarly, $Q_n^* = \sum_k e_{nk} Q_k$, the inversions being justified since all terms are non-negative. Then $T_n^*/Q_n^* = \sum_k d_{nk} (T_k/Q_k)$, where $d_{nk} = (e_{nk} Q_k) / (\sum_k e_{nk} Q_k)$. Now, $d_{nk} \leq e_{nk} Q_k / r \rightarrow 0$ as $n \rightarrow \infty$ for each k , r being that of Lemma 4.1.2 determined by $\{p_n\}$ and B . Also, $\sum_k |d_{nk}| = \sum_k d_{nk} = 1$ for each n . Hence, (d_{nk}) is a T -matrix with non-negative elements. Thus, if $T_n/Q_n \rightarrow s$, then $T_n^*/Q_n^* \rightarrow s$ for s finite or infinite, and the assertion is proved.

Clearly, if $\{s_n\}$ is a sequence for which both $\sum f_{nk} |S_k|$ and $\sum F_{nk} |S_k|$ converge, then the product $(e_{nk})(f_{kj})(S_j)$ is associative. Also, if $\{s_n\}$ is such that $\{S_n\}$ lies in an angle less than π , in the sense of [1, p. 124], then the proof of 5.7 III on p. 124 of [1] shows that this product again is associative. Another simple case of associativity arises when both matrices (e_{nk}) and (f_{nk}) are row-finite. In the latter case, $G(B, p_n)$ totally includes $G(A, p_n)$ for all sequences $\{s_n\}$.

If the matrices (f_{nk}) and (F_{nk}) are lower-triangular, that is, if they have only zeros above the main diagonal, and if $f_{nn} \neq 0$ for all n , then there exists a two-sided reciprocal of (f_{nk}) which is a lower-triangular matrix [1, pp. 19-22]. If $(f_{nk})^{-1}$ denotes this reciprocal, the product matrix $(F_{nk})(f_{nk})^{-1}$ suffices for (e_{nk}) in the above theorem, provided that this product matrix has all of its elements non-negative and has column-limits zero.

Similarly, if (f_{nk}) and (F_{nk}) are upper-triangular, that is, if they have only zeros below the main diagonal, and if $f_{nn} \neq 0$ for all n , then there exists a left-hand reciprocal of (f_{nk}) which is also an upper-triangular matrix [1, pp. 6, 19]. The product matrix $(F_{nk})(f_{nk})_L^{-1}$, where $(f_{nk})_L^{-1}$ denotes this reciprocal, exists and would suffice for (e_{nk}) of Theorem 4.2.4 provided that this product matrix has all its elements non-negative.

4.3. NÖRLUND-HAUSDORFF METHODS. The $H(\mu_n)$ methods, defined in §3.3 above, satisfy the hypotheses for A of Definition 4.1.1. The following considerations show how the results of §4.2 hold for $G(H(\mu_n), p_n)$.

The (C, k) and Euler methods, which are $H(\mu_n)$ methods, are translatable to the left [2, p. 419], [14, p. 131]. Therefore, $G(H(\mu_n), p_n)$ includes $H(\mu_n)$ for finite Nörlund sequences in these cases. The same result holds for all $H(\mu_n)$ methods if only bounded sequences are considered, for Parameswaran [7] has shown that any regular Hausdorff method is translatable to the left for such sequences.

The T -matrix (f_{nk}) of $H(\mu_n)$ has a two-sided reciprocal of the form (c_{nk}) , where $c_{nk} = \binom{n}{k} \Delta^{n-k} (1/\mu_k)$ when $n \geq k$, 0 otherwise [4, p. 262]. If (F_{nk}) is the T -matrix of another method, $H(\mu_n^*)$, then $(F_{nk})(c_{nk})$ exists and equals (e_{nk}) , where $e_{nk} = \binom{n}{k} \Delta^{n-k} (\mu_k^*/\mu_k)$ when $n \geq k$, 0 otherwise. The hypotheses of Theorem 4.2.4 would be satisfied when $\Delta^p(\mu_n^*/\mu_n) \geq 0$ for $n \geq 0$, $p \geq 0$ and $\lim_n \Delta^n(\mu_0^*/\mu_0) = 0$ [4, pp. 252–255].

4.4. NÖRLUND-QUASI-HAUSDORFF METHODS. If $\mu_n \neq 0$ for all n , the $H^*(\mu_n)$ methods of §3.4 satisfy the hypotheses for A in Definition 4.1.1. The results of §4.2 hold for some of these methods.

Any regular quasi-Hausdorff method is translatable to the left for bounded sequences [11]. Therefore, when $\mu_n \neq 0$ for all n , $G(H^*(\mu_n), p_n)$ includes $H^*(\mu_n)$ for finite Nörlund sequences $\{p_n\}$ and bounded $\{s_n\}$.

The T -matrix (f_{nk}) of $H^*(\mu_n)$, when $\mu_n \neq 0$ for all n , has a two-sided reciprocal of the form (d_{nk}) , where $d_{nk} = \binom{k}{n} \Delta^{k-n} (1/\mu_{n+1})$ when $k \geq n$, 0 otherwise. If (F_{nk}) is the T -matrix of another method $H^*(\mu_n^*)$, then $(F_{nk})(d_{nk})$ exists and equals (e_{nk}) , where $e_{nk} = \binom{k}{n} \Delta^{k-n} (\mu_{n+1}^*/\mu_{n+1})$ when $k \geq n$, 0 otherwise. Thus, the matrices of these methods would satisfy the hypotheses of Theorem 4.2.4 whenever $\Delta^p(\mu_{n+1}^*/\mu_{n+1}) \geq 0$ for all n and p .

5.0. Comparison of the methods $F(A, p_n)$ and $G(A, p_n)$. The methods $F(A, p_n)$ and $G(A, p_n)$, in general, are not comparable, for let A be the method which transforms the sequence $\{s_n\}$ into the sequence $\{t_n\}$, where $t_n = (2s_n + s_{n+1})/3$. A short calculation shows that the alpha-matrix of A is (a_{nk}) , where $a_{00} = 1$, $a_{01} = 1/3$, $a_{nn} = 2/3$ and $a_{n, n+1} = 1/3$ for $n \geq 1$, $a_{nk} = 0$ otherwise. The matrices for A satisfy Lemma 3.1.2 with $M = 2$, and also satisfy Definition 4.1.1. Let (N, p_n) be defined by $p_0 = 1$, $p_1 = 3$, $p_n = 0$ for $n \geq 2$. Then $q_0 = q_1 = 2$, $q_n = 0$ for $n \geq 2$. If $s_n = 3(-1)^{n+1}$, one can easily see that $t_n = (-1)^{n+1}$ and that $t_n \rightarrow 0$, (N, q_n) . However, $T_n/Q_n = (-1)^n/2$ and $\{s_n\}$ is not summable $G(A, p_n)$.

On the other hand, if $s_n^* = (-3)^{n+1}$, then $T_n^* = 0$ for $n \geq 1$, so $s_n^* \rightarrow 0$, $G(A, p_n)$, while $(q_0 t_n^* + q_1 t_{n-1}^*)/Q_n = -(-3)^{n-1}$ and $\{s_n^*\}$ is not summable $F(A, p_n)$.

However, the two methods coincide when A is the Euler method (E, r) , defined as the Hausdorff method $H(\mu_n)$, where $\mu_n = (1+r)^{-n}$ for positive r [4, p.248].

THEOREM 5.0.1. $F((E, r), p_n) \equiv G((E, r), p_n)$.

Proof. It suffices to show that $\sum q_{n-k} t_k \equiv T_n$ for all n . Now, $q_0 = p_0$ and, for $n \geq 1$,

$$q_n = (1+r)^{-n} \cdot \sum_{j=1}^n \binom{n-1}{j-1} r^{n-j} p_j.$$

Also,

$$t_n = (1+r)^{-n} \cdot \sum_{m=0}^n \binom{n}{m} r^{n-m} s_m \text{ for all } n.$$

After combining terms, letting $i = m + j$ and simplifying, one sees that

$$\begin{aligned} \sum_{k=0}^n q_{n-k} t_k &= (1+r)^{-n} \cdot \sum_{j=0}^n \sum_{i=j}^n \binom{n}{i} r^{n-i} p_j s_{i-j} \\ &= (1+r)^{-n} \cdot \sum_{i=0}^n \binom{n}{i} r^{n-i} S_i = T_n. \end{aligned}$$

A related result for Taylor-Nörlund and Nörlund-Taylor methods is the following theorem.

THEOREM 5.0.2. For sequences $\{s_n\}$ which have $T(a)$ -transforms, $F(T(a), p_n) \equiv G(T(a), p_n)$.

Proof. As in the previous theorem, it suffices to show that $\sum q_{n-k} t_k \equiv T_n$ for all n , assuming that t_n exists for each n . Now, the series defining q_j converge absolutely for each j , and hence, by Mertens' theorem [5, p. 321], $q_j t_{n-j} = \sum_k c_k^{(j)}$, the Cauchy product of these series. Then,

$$\sum_{j=0}^n q_j t_{n-j} = \sum_{j=0}^n \sum_{k=0}^{\infty} c_k^{(j)} = \sum_{k=0}^{\infty} \sum_{j=0}^n c_k^{(j)}.$$

Now,

$$\sum_{j=0}^n c_k^{(j)} = a^{n+1} (1-a)^k \cdot \sum_{j=0}^n \sum_{r=0}^k \binom{r+j}{j} \binom{k+n-j-r}{n-j} p_{r+j} s_{k+n-j-r}.$$

When the rectangular array (j, r) is summed along the diagonals $j+r=i$, $i=0, \dots, n+k$, and the resulting expressions are simplified and combined, one sees that

$$\sum_{j=0}^n q_j t_{n-j} = \sum_{k=0}^{\infty} \binom{n+k}{n} a^{n+1} (1-a)^k S_{k+n} = T_n.$$

It follows immediately from this theorem that $G(T(a), p_n)$ includes $F(T(A), p_n)$. However, when $s_n = (a-1)^{-n}$, $\{s_n\}$ fails to have a $T(a)$ -transform, so $\{s_n\}$ is not

summable $F(T(a), P_n)$ for any Nörlund sequence, although $s_n \rightarrow 0$, $G(T(a), p_n)$ when $p_0 = 1$, $p_1 = (1 - a)^{-1}$, $p_n = 0$ for $n \geq 2$.

6.0. Series-to-sequence analogues. The methods $F(A, p_n)$ and $G(A, p_n)$, which were developed above, are applicable to the sequence of partial sums of a series. In this section, analogous methods are defined which operate directly on the terms of the series.

6.1. THE A -NÖRLUND METHOD.

DEFINITION 6.1.1. Let A be a regular summability method with alpha-matrix (a_{nk}) and gamma-matrix (g_{nk}) . Let (N, p_n) be a regular Nörlund method such that (N, q_n) is a regular Nörlund method, where $q_n = \sum a_{nk} p_k$. A series $\sum u_k$ is said to be summable to s by the A -Nörlund method if and only if $t'_n \rightarrow s$, (N, q_n) , where $t'_n = \sum g_{nk} u_k$. Write $\sum u_k = s$, $F'(A, p_n)$ in this case.

If (a_{nk}) is the alpha-matrix of A and $v_n = \sum a_{nk} u_k$, Vermes [13] has shown that $v_0 + \dots + v_n = t'_n$. Therefore, an equivalent form of Definition 6.1.1 is: $\sum u_k = s$, $F'(A, p_n)$, if and only if $\sum v_n = s$, (N, q_n) .

Definition 6.1.1 is more general than Definition 3.1.1, since there exist regular summability methods to which correspond alpha- and gamma-matrices, but not T -matrices [1, p. 87]. However, if a method has a T -matrix, then there always exist alpha- and gamma-matrices for that method. The explicit form of these matrices is given in §2.0 above. The relation between $F(A, p_n)$ and $F'(A, p_n)$, when A has a T -matrix, is investigated in the next theorem.

THEOREM 6.1.2. Let A have a T -matrix (f_{nk}) and a gamma-matrix (g_{nk}) . Let $s_n = u_0 + \dots + u_n$ be such that $\sum f_{nk} s_k$, $\sum g_{nk} u_k$ and $\lim_k g_{nk} s_k = w_n$ exist for each n . If $w_n \rightarrow 0$, (N, q_n) , then $s_n \rightarrow s$, $F(A, p_n)$, if and only if $\sum u_k = s$, $F'(A, p_n)$.

Proof. Let $s_{-1} = 0$. As on p. 86 of [1],

$$\begin{aligned} \sum_{k=0}^{\infty} f_{nk} s_k &= \lim_m \left[\sum_{k=0}^m g_{nk} s_k - \sum_{k=0}^m g_{n, k+1} s_k \right] \\ &= \lim_m \left[\sum_{k=0}^{m+1} g_{nk} (s_k - s_{k-1}) - g_{n, m+1} s_{m+1} \right]. \end{aligned}$$

This shows that, in the notation of 3.1.1 and 6.1.1 above, $t_n + w_n = t'_n$ for every n , and the assertion of the theorem follows.

If $\{s_n\}$ is bounded, then $\lim_k g_{nk} s_k = 0$ for every n , since $\lim_k g_{nk} = 0$ when A has a T -matrix [1, p. 86]. Hence, for series with bounded partial sums, $t_n = t'_n$, and the methods $F(A, p_n)$ and $F'(A, p_n)$ are identical. Another case in which these methods are identical is when the gamma-matrix of A has only finitely many nonzero terms in each row, for then, $\lim_k g_{nk} s_k = 0$ for every sequence $\{s_n\}$.

6.2. THE NÖRLUND- A METHOD. A natural series-to-sequence analogue of Definition 4.1.1 would use the series-to-sequence form for (N, p_n) as follows: A series $\sum u_k$ is summable to s by the Nörlund- A method if and only if $T_n/Q_n \rightarrow s$,

where $T_n = \sum f_{nk} S_k$, $S_n = P_n u_0 + \cdots + P_0 u_n$, $P_n = p_0 + \cdots + p_n$ and $Q_n = \sum f_{nk} P_k$. However, since the expressions $P_n u_0 + \cdots + P_0 u_n$ and $p_n s_0 + \cdots + p_0 s_n$ are identical, this series-to-sequence analogue of $G(A, p_n)$ is identical to applying $G(A, p_n)$ to the sequence of partial sums of $\sum u_k$.

6.3. THE GEOMETRIC SERIES. The regions of the complex plane in which the geometric series is summable (E, r) and $T(a)$ are well known [4, p. 178], [13]. For both methods, a series-to-series transformation changes the geometric series into a series of the form $\sum b_n X^n$, where $X = X(z, r)$ or $X = X(z, a)$. The regions of summability result from imposing the conditions that X exist and $|X| < 1$. These regions can be extended by the methods defined above.

THEOREM 6.3.1. *If $\{p_n\}$ is a nondecreasing Nörlund sequence with $p_0(r+1) \leq p_1$, then $\sum z^k = 1/(1-z)$, $F'(E, r, p_n)$ for all z satisfying $|r+z| \leq r+1$, $z \neq +1$.*

Proof. By Lemma 3.3.1 and Theorem 3.2.6, it suffices to consider the case $(N, q_n) = (C, 1)$. Upon transforming the geometric series by the alpha-matrix of (E, r) , one finds that $v_0 = 1$, and, when $n \geq 1$,

$$v_n = (1+r)^{-n} \cdot \sum_{j=1}^n \binom{n-1}{j-1} r^{n-j} z^j = \frac{c_n z}{1+r},$$

where $c_n = [(r+z)/(1+r)]^{n-1}$. Now, for all z such that $|r+z| \leq r+1$, $z \neq +1$, $\sum_{k=1}^{\infty} c_k = [1 - (r+z)/(r+1)]^{-1}$, $(C, 1)$ [5, p. 481]. By Theorem 47 of [4], it follows that $(C, 1) - \sum v_k = 1 + [z/(1+r)] \cdot (C, 1) - \sum c_k = 1/(1-z)$ for these z 's.

THEOREM 6.3.2. *If $\{p_n\}$ is a nondecreasing Nörlund sequence, then $\sum z^k = 1/(1-z)$, $F'(T(a), p_n)$ for all z such that $z \neq +1$, $|z| < 1/(1-a)$, $|az| \leq |1 - (1-a)z|$.*

Proof. It follows from the discussion at the end of §3.4 above that, as in the previous theorem, it suffices to consider the case $(N, q_n) = (C, 1)$. Using the alpha-matrix for $T(a)$, one gets for all n , $v_n = a^n \cdot \sum_{k=n}^{\infty} \binom{k}{n} (1-a)^{k-n} z^k = (az)^n / [1 - (1-a)z]^{n+1}$ if and only if $(1-a)|z| < 1$. Then, $\sum v_k = [1 - (1-a)z]^{-1} \cdot \sum [az/(1-z+az)]^k = 1/(1-z)$, $(C, 1)$, if and only if $|az| \leq |1 - (1-a)z|$, $z \neq +1$ [5, p. 481].

If one transforms $s_n = 1 + z + \cdots + z^n$ by the T -matrix of $T(a)$, one finds that $t_n = 1/(1-z) - az \cdot v_n/(1-z)$ if and only if $(1-a)|z| < 1$. Since $\sum v_n$ is $(C, 1)$ -summable whenever $z \neq +1$, $|az| \leq |1 - (1-a)z|$, then $v_n \rightarrow 0$, $(C, 1)$ [5, p. 485]. Hence for these z 's, $t_n \rightarrow 1/(1-z)$, $(C, 1)$, so the assertion of Theorem 6.3.2 holds for $F(T(a), p_n)$ and $G(T(a), p_n)$ as well.

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