ON THE HOMOTOPY GROUPS OF THE EXCEPTIONAL LIE GROUPS(1)

BY
P. G. KUMPEL, JR.

1. **Introduction.** Throughout this paper the symbols G_2, F_4, E_6, E_7 denote compact simply connected forms of these exceptional Lie groups. Spin(n) is the universal covering group of SO(n). There are inclusions $G_2 \subset \text{Spin } 8 \subset F_4 \subset E_6$ given by Jacobson [13]. We also use the inclusion $E_6 \subset E_7$ described using representations given by Tits [16] for the Lie algebras of these groups. Q_p denotes the ring of rational numbers whose denominators are powers of the prime p. The following isomorphisms are obtained for $j \geqq 0$:

$$\pi_j(\text{Spin } 8) \otimes Q_3 \approx (\pi_j(G_2) \oplus \pi_j(\text{Spin } 8 / G_2)) \otimes Q_3, \tag{1.1}$$

$$\pi_{j+1}(F_4 / \text{Spin } 8) \otimes Q_3 \approx (\pi_{j+1}(F_4 / G_2) \oplus \pi_j(\text{Spin } 8 / G_2)) \otimes Q_3, \tag{1.2}$$

$$\pi_j(E_6) \otimes Q_2 \approx (\pi_j(F_4) \oplus \pi_j(E_6 / F_4)) \otimes Q_2, \tag{1.3}$$

$$\pi_{j+1}(E_7 / E_6) \otimes Q_2 \approx (\pi_{j+1}(E_7 / F_4) \oplus \pi_j(E_6 / F_4)) \otimes Q_2. \tag{1.4}$$

(1.1) was stated without proof by Harris [8]. The proofs of (1.1), (1.2), (1.3), (1.4) are given in §§3, 4, 5, 6 respectively. §2 contains a discussion of results of Jacobson and Tits as they relate to the inclusions mentioned earlier. Actually, in §5, the proof of the following theorem, which implies (1.3), is given:

THEOREM 1.5. *Let G be a compact, connected, simply connected Lie group, $\sigma: G \to G$ an automorphism of period 2, K the identity component of the fixed point set of σ. Assume the map $H^*(G; R) \to H^*(K; R)$ induced by inclusion is an epimorphism. Then for all i the sequence $0 \to \pi_i(K) \to \pi_i(G) \to \pi_i(G / K) \to 0$ is exact and split when tensored with Q_2. The splitting is given by the map $q: G / K \to G$, $q(gK) = g\sigma(g)^{-1}$, $g \in G$.*

THEOREM 1.5 applies to the following pairs (G, K): (SU($2n+1$), SO($2n+1$)), (SU($2n$), Sp(n)), (Spin ($2n$), Spin ($2n-1$)) and (E_6, F_4). In each of the first three cases, the result was given by Harris [8], and the splitting for (E_6, F_4) was given by Harris provided the primes 2 *and* 3 are neglected. The proof of this theorem is

Received by the editors November 3, 1964.
(1) This research was supported in part by NSF Grant 24026.

based on a suggestion of Harris (see [8]). The idea for using Poincaré duality is from the work of Araki [1]. In [8], Harris obtains a similar result, except that primes p for which G has p-torsion must also be neglected. In [1], Araki obtains the conclusion of 1.5 under the hypothesis that rank $G = \text{rank } K + \text{rank } G/K$. Araki's hypothesis implies that $H^*(G; R) \to H^*(K; R)$ is an epimorphism. It is to be emphasized that the proof of 1.5 does not depend on the classification of symmetric spaces.

In §7, it is shown that by combining (1.4) with similar results of Harris, and using the classification of symmetric spaces, the following theorem is obtained:

THEOREM 1.6. *Let G, K, σ satisfy the hypotheses of Theorem 1.5. In addition, assume G is simple. Then there exists a simple group L, containing G, such that the sequence $0 \to \pi_{j+1}(L/G) \to \pi_{j+1}(L/K) \to \pi_j(G/K) \to 0$ is exact and split when tensored with Q_2. The splitting is given by the Bott suspension map (see [6]) $\pi_j(G/K) \to \pi_{j+1}(L/G)$. Moreover, rank $L - \text{rank } G \leqq 1$ and either L/G or L/M is a symmetric space, where M is locally isomorphic to $G \times S^1$.*

Theorem 1.6 applies to the following triples (L, G, K): $(SO(4n), SU(2n), Sp(n))$, $(Sp(2n+1), SU(2n+1), SO(2n+1))$, Spin $(2n+1)$, $(\text{Spin}(2n), \text{Spin}(2n-1))$ and (E_7, E_6, F_4). The results in the first three cases were obtained by Harris [8].

The results of this paper are contained in the author's doctoral dissertation. The author wishes to thank Professor Bruno Harris of Brown University for suggesting these problems and for his advice and criticism throughout the preparation of the manuscript. These results were announced in the Notices of the American Mathematical Society, abstracts 63T-53, January 1963, and 64T-173, February 1964.

2. **Descriptions of the groups and algebras.** Representations of G_2, F_4 and E_6 are given by Jacobson [13]. Denote by $\mathscr{K}$ the nonassociative algebra of Cayley numbers. This is an 8-dimensional real vector space. It can be described by naming generators $1, i, j, l$ subject to the relations $i^2 = j^2 = l^2 = -1$, $ij = -ji$, $il = -li$, $jl = -lj$. A basis is given by $1, i, j, ij, l, il, jl, (ij)l$. The subalgebra generated by $1, i, j$ is the algebra of quaternions. Every Cayley number is uniquely expressible in the form $a + bl$, a, b quaternions. For quaternion a, $a = a_0 + a_1 i + a_2 j + a_3 ij$, a_i real, define $\bar{a} = a_0 - a_1 i - a_2 j - a_3 ij$, and for $x = a + bl \in \mathscr{K}$ define $\bar{x} = \bar{a} - bl$. The norm of $x \in \mathscr{K}$ is given by $N(x) = x\bar{x} = \bar{x}x$ and is a real number. The associated bilinear form $N(x, y) = (x\bar{y} + y\bar{x})/2$ is nondegenerate and the basis mentioned above is orthonormal. Also

$$N(xy) = N(x)N(y), \quad x, y \in \mathscr{K}. \tag{2.1}$$

Let M_3^8 be the exceptional Jordan algebra of 3 by 3 hermitian matrices with coefficients in $\mathscr{K}$; an element $A \in M_3^8$ is a 3 by 3 Cayley matrix with $\bar{A}^{\text{tr}} = A$. Multiplication in M_3^8 is given by $A \cdot B = (AB + BA)/2$, where juxtaposition denotes usual matrix multiplication. Thus, if $A \in M_3^8$, we may write

$$A = \begin{bmatrix} \alpha_1 & a & b \\ \bar{a} & \alpha_2 & c \\ \bar{b} & \bar{c} & \alpha_3 \end{bmatrix}, \quad \alpha_i \text{ real}, \quad a,b,c \in \mathscr{K}. \tag{2.2}$$

Clearly M_3^8 is a 27-dimensional real vector space.

Define $\varepsilon_1 = \mathrm{diag}\,(1,0,0)$, $\varepsilon_2 = \mathrm{diag}\,(0,1,0)$, $\varepsilon_3 = \mathrm{diag}\,(0,0,1)$, and for $a \in \mathscr{K}$, define a_{ij}, $i \neq j$, $i,j = 1,2,3$, to be the 3 by 3 matrix whose only nonzero entries are a in the (i,j) position and $\bar{a}$ in the (j,i) position. Then (2.2) may be written

$$A = \sum_{i=1}^{3} \alpha_i\varepsilon_i + a_{12} + b_{13} + c_{23}. \tag{2.3}$$

The trace of an element of $\mathscr{K}$ is twice its real part, i.e. $T(a) = a + \bar{a}$, $a \in \mathscr{K}$. The trace and norm of an element $A \in M_3^8$, written as in (2.3), are defined, respectively, by

$$T(A) = \sum_{i=1}^{3} \alpha_i, \tag{2.4}$$

$$N(A) = \alpha_1\alpha_2\alpha_3 + T(a(c\bar{b})) - \alpha_1 N(c) - \alpha_2 N(\bar{b}) - \alpha_3 N(a). \tag{2.5}$$

Multiplication in M_3^8 is characterized by the formulas:

$$\varepsilon_i \cdot a_{ij} = \frac{1}{2}\, a_{ij} = a_{ij} \cdot \varepsilon_j, \tag{2.6}$$

$$a_{ij}^2 = N(a)\,(\varepsilon_i + \varepsilon_j), \tag{2.7}$$

$$2a_{ij} \cdot b_{jk} = (ab)_{ik}, \quad i,j,k \text{ unequal}, \tag{2.8}$$

and by the fact that ε_i are orthogonal idempotents.

Jacobson [13, III] has shown that the group of norm-preserving linear transformation of M_3^8 is a compact form of E_6, and [13, II] that the group of all automorphisms of the Jordan algebra M_3^8 is the compact F_4. In the same series of papers, it is shown that

$$\{t \in F_4 \colon t(\varepsilon_i) = \varepsilon_i,\ i = 1,2,3\} \approx \mathrm{Spin}\ 8, \tag{2.9}$$

$$\{t \in F_4 \colon t(1_{ij}) = 1_{ij},\ i \neq j,\ i,j = 1,2,3, \quad t(\varepsilon_i) = \varepsilon_i, i = 1,2,3\} \approx G_2. \tag{2.10}$$

Moreover, if ε is a primitive idempotent, that is, an idempotent which is not the sum of two orthogonal idempotents (equivalently, $T(\varepsilon) = 1$), then

$$\{t \in F_4 \colon t(\varepsilon) = \varepsilon\} \approx \mathrm{Spin}\ 9. \tag{2.11}$$

We now proceed to describe the compact Lie algebra $\mathscr{E}_7$ and its subalgebras $\mathscr{E}_6$ and $\mathscr{F}_4$ (compact forms) using the work of J. Tits [16].

Let $\mathscr{Y}$ be the complex simple 3-dimensional Lie algebra. $\mathscr{Y}$ has basis $\{e,f,h\}$

over the complex numbers, C, with multiplication given by $[eh] = 2e$, $[fh] = -2f$, $[ef] = h$. The compact real form of $\mathscr{Y}$, henceforth denoted by Y, has real basis $\{ih,\ e+f,\ i(e-f)\}$, $i = (-1)^{1/2}$. If $(\ ,\ )$ denotes the Killing form of $\mathscr{Y}$, one computes that $(h,h) = 8$, $(e,f) = -4$, $(e,h) = (f,h) = (e,e) = (f,f) = 0$. Following [16], the compact Lie algebra $\mathscr{E}_7$ is

$$\mathscr{E}_7 = \mathscr{D}(M_3^8) \oplus (Y \otimes M_3^8) \tag{2.12}$$

where $\mathscr{D}(M_3^8)$ is the Lie algebra of derivations of M_3^8, $\mathscr{D}(M_3^8) = \mathscr{F}_4$, and where the multiplication in $\mathscr{E}_7$ is given by

$$[d, y \otimes a] = y \otimes d(a), \tag{2.13}$$

$$[y \otimes a, y' \otimes a'] = \frac{1}{2}\,(y,y')\,\langle a,a'\rangle + [yy'] \otimes aa', \tag{2.14}$$

for $y, y' \in Y$, $d \in \mathscr{D}(M_3^8)$, $a, a' \in M_3^8$. The symbol $\langle a,a'\rangle$ denotes the inner derivation of M_3^8 defined by

$$\langle a,a'\rangle\,(a'') = a(a'a'') - a'(aa''), \quad a'' \in M_3^8. \tag{2.15}$$

We have noted that $\mathscr{D}(M_3^8) = \mathscr{F}_4$ (see paragraph preceding (2.9)). Tits has shown that the subalgebra

$$\mathscr{D}(M_3^8) \oplus \frac{ih}{2} \otimes M_3^8(0) \tag{2.16}$$

is isomorphic to the compact $\mathscr{E}_6$. Here $M_3^8(0)$ denotes the set of elements of trace zero in M_3^8. Observe that $\frac{1}{2}\,(ih/2,\ ih/2) = -1$, so the multiplication described in (2.14) applied to an element of (2.16) is

$$\left[\frac{ih}{2} \otimes a,\ \frac{ih}{2} \otimes a'\right] = -\,\langle a,a'\rangle. \tag{2.17}$$

If $R_a\colon M_3^8 \to M_3^8$ denotes right multiplication by $a \in M_3^8$, then $\langle a,a'\rangle = -[R_a R_{a'}]$, for

$$\begin{aligned} -[R_a, R_a']\,(a'') &= -(a'' R_a R_{a'} - a'' R_{a'} R_a) \\ &= -((a''a)a' - (a''a')a) \\ &= a(a'a'') - a'(aa'') = \langle a,a'\rangle(a''). \end{aligned}$$

Jacobson [12, p.145] has described the Lie algebra $\mathscr{E}_6$ by

$$\mathscr{E}_6 = \mathscr{D}(M_3^8) \oplus \{R_a : a \in M_3^8(0)\}. \tag{2.18}$$

The representations (2.16) and (2.18) are isomorphic under the identification of $ih/2 \otimes a$ with R_a, because of (2.17) and the remarks following it.

Of course, $\mathscr{D}(M_3^8) \oplus (ih/2 \otimes M_3^8)$ is the direct sum of $\mathscr{E}_6$ with a 1-dimensional subalgebra.

Later, in §6, we will see that these inclusions $\mathscr{F}_4 \subset \mathscr{E}_6 \subset \mathscr{E}_7$ give rise to inclusions $F_4 \subset E_6 \subset E_7$ of the simply connected Lie groups.

3. **The fibration** $G_2 \to \mathrm{Spin}\ 8 \to \mathrm{Spin}\ 8/G_2$. In this section we prove (1.1). Recall, from §2, that F_4 is the group of all automorphisms of M_3^8, and that

$$\begin{aligned} \mathrm{Spin}\ 8 &= \{\tau \in F_4 : \tau(\varepsilon_i) = \varepsilon_i,\ i = 1,2,3\}, \\ G_2 &= \{\tau \in \mathrm{Spin}\ 8 : \tau(1_{ij}) = 1_{ij},\ i \neq j,\ i,j = 1,2,3\}. \end{aligned}$$

Let $\eta : M_3^8 \to M_3^8$ be defined by $\eta(\varepsilon_i) = \varepsilon_{i+1}$, $\eta(a_{ij}) = a_{i+1,j+1}$, $i \neq j$, $i,j = 1,2,3$ (addition of subscripts modulo 3), and extend linearly. Define $\sigma : \mathrm{Spin}\,8 \to \mathrm{Spin}\ 8$ by $\sigma(\tau) = \eta\tau\eta^{-1}$, $\tau \in \mathrm{Spin}\ 8$. Then σ is an automorphism of Spin 8, and since $\eta^3 =$ identity of M_3^8, σ has order 3. Moreover, the fixed set of σ is exactly the subgroup G_2.

Define a map $q : \mathrm{Spin}\ 8/G_2 \to \mathrm{Spin}\ 8$ by $q(\tau G_2) = \tau\sigma(\tau)^{-1}$, $\tau \in \mathrm{Spin}\ 8$. Let $p : \mathrm{Spin}\ 8 \to \mathrm{Spin}\ 8/G_2$ be the natural projection and $j : G_2 \to \mathrm{Spin}\ 8$ the inclusion.

THEOREM 3.1. *The exact homotopy sequence of the fiber space*

$$G_2 \to \mathrm{Spin}\ 8 \to \mathrm{Spin}\ 8/G_2$$

is split when tensored by Q_3. The splitting is given by the map

$$q : \mathrm{Spin}\ 8/G_2 \to \mathrm{Spin}\ 8$$

and we obtain $\mathscr{C}_3$-isomorphisms

$$\pi_j(\mathrm{Spin}\ 8) \approx \pi_j(G_2) \oplus \pi_j(\mathrm{Spin}\ 8/G_2), \quad j \geqq 0.$$

Proof. We compute the composition

$$p^*q^* : \pi_j(\mathrm{Spin}\ 8/G_2) \to \pi_j(\mathrm{Spin}\ 8/G_2).$$

To do this, we use the fact that there is a homeomorphism $\mathrm{Spin}\ 8/G_2 \to S^7 \times S^7$. Here we consider S^7 as the unit 7-sphere of Cayley numbers of norm 1. The homeomorphism is defined by sending τG_2, $\tau \in \mathrm{Spin}\ 8$ into the pair (u,v), where u,v are Cayley numbers determined by $\tau(1_{12}) = u_{12}$, $\tau(1_{13}) = v_{13}$. See Jacobson [13, II, p. 93].

Since σ leaves G_2 pointwise fixed, it induces a map

$$\sigma : \mathrm{Spin}\ 8/G_2 \to \mathrm{Spin}\ 8/G_2.$$

We identify $\mathrm{Spin}\ 8/G_2$ with $S^7 \times S^7$ as above and compute σ. In fact, if $\tau \in \mathrm{Spin}\ 8$ is such that τG_2 corresponds to the pair (u,v) in $S^7 \times S^7$, then

$$\tau(1_{23}) = \tau(2(1_{21} \cdot 1_{13})) = 2\tau(1_{12}) \cdot \tau(1_{13}) = 2u_{12}v_{13} = (\bar{u}v)_{23}.$$

Hence

$$\begin{aligned}\sigma(\tau)(1_{12}) &= \eta\tau\eta^{-1}(1_{12})\\ &= \eta\tau(1_{31})\\ &= \eta\tau(1_{13})\\ &= \eta(v_{13}) = v_{21} = \bar{v}_{12},\end{aligned}$$

and by similar calculation

$$\sigma(\tau)(1_{13}) = (\bar{v}u)_{13}.$$

Thus if G_2 corresponds to $(u,v) \in S^7 \times S^7$, $\sigma(\tau)G_2$ corresponds to the pair $(\bar{v}, \bar{v}u) \in S^7 \times S^7$, and $\sigma: S^7 \times S^7 \to S^7 \times S^7$ given by

$$\sigma(u,v) = (\bar{v}, \bar{v}u).$$

Since $\bar{v}$ is the multiplicative inverse of v, if we identify $\pi_j(S^7 \times S^7)$ with $\pi_j(S^7) \oplus \pi_j(S^7)$, it follows that the induced map $\sigma_*: \pi_j(S^7 \times S^7) \to \pi_j(S^7 \times S^7)$ is given by

$$\sigma_*((\alpha,\beta)) = (-\beta, -\beta + \alpha), \quad \alpha, \beta \in \pi_j(S^7). \tag{3.2}$$

We now compute $p_* q_*$. From the definition, if $\tau \in$ Spin 8,

$$pq(\tau G_2) = \tau\sigma(\tau)^{-1} G_2.$$

Let $T = pq$. To simplify the notation, let $K =$ Spin 8, and $H = G_2$. We have a multiplication

$$\mu: K/H \times K/H \to K/H$$

defined by $\mu(k_1 H, k_2 H) = k_1 \sigma(k_1)^{-1} k_2 H$. One easily verifies that

(a) $\mu(\varepsilon H, kH) = kH$, ε the identity of K, $k \in K$,

(b) $\mu(kH, \sigma(kH)) = kH$,

(c) $T(kH) = \mu(kH, \varepsilon H)$.

Identifying $\pi_j(K/H \times K/H)$ with $\pi_j(K/H) \oplus \pi_j(K/H)$ we obtain

(a)′ $\mu_*(0, a) = a$,

(b)′ $\mu_*(a, \sigma_*(a)) = a$,

(c)′ $T_*(a) = \mu_*(a, 0)$, $a \in \pi_j(K/H)$.

Therefore,

$$\begin{aligned}T^*(a) &= \mu_*(a,0) = \mu_*(a, \sigma_*(a) - \sigma_*(a))\\ &= \mu_*((a, \sigma_*(a)) - (0, \sigma_*(a))\\ &= \mu_*(a, \sigma_*(a)) - \mu_*(0, \sigma_*(a))\\ &= a - \sigma_*(a), \ a \in \pi_j(K/H).\end{aligned}$$

Therefore if $\alpha, \beta \in \pi_j(S^7)$, we have

$$
\begin{aligned}
T_*((\alpha,\beta)) &= (\alpha,\beta) - \sigma_*(\alpha,\beta) \\
&= (\alpha,\beta) - (-\beta, -\beta+\alpha) = (\alpha+\beta,\ 2\beta-\alpha).
\end{aligned}
$$

To complete the proof of Theorem 3.1, tensor the exact homotopy sequence of the fiber space $G_2 \to \text{Spin } 8 \to \text{Spin } 8/G_2$ with Q_3. Exactness is preserved, since Q_3 has no torsion. Since division by 3 is now permissible, we may define the map

$$S: \pi_i(S^7 \times S^7) \otimes Q_3 \to \pi_i(S^7 \times S^7) \otimes Q_3$$

by $S(\gamma,\delta) = ((2\gamma-\delta)/3,\ (\gamma+\delta)/3)$, $\gamma,\ \delta \in \pi_i(S_7)$. Then

$$T_*S(\alpha,\beta) = T_*((2\gamma-\delta)/3, (\gamma+\delta)/3) = (\gamma,\delta),$$

and

$$ST_*(\alpha,\beta) = S(\alpha+\beta, 2\beta-\alpha) = ((2\alpha+2\beta-2\beta+\alpha)/2,\ 3\beta/3) = (\alpha,\beta).$$

Hence T_* is an automorphism after tensoring by Q_3. Thus the sequence splits, and 3.1 is proved.

4. **The fibration** Spin $8/G_2 \to F_4/G_2 \to F_4/\text{Spin } 8$. In this section it is shown that the exact homotopy sequence of the fiber space Spin $8/G_2 \to F_4/G_2 \to F_4/\text{Spin } 8$ is split when tensored by Q_3, thereby proving (1.2). We retain the notation of the preceding sections, and we continue to describe these groups in terms of M_3^8.

We begin by considering the set of all 3 by 3 matrices with Cayley coefficients, together with addition and ordinary matrix multiplication. This is a nonassociative algebra $\mathscr{A}$ over the real numbers. If A is a 3 by 3 real nonsingular matrix, the mapping $\mathscr{A} \to \mathscr{A}$ defined by $B \to ABA^{-1}$, $B \in \mathscr{A}$, is an automorphism of the algebra $\mathscr{A}$. The product ABA^{-1} is unambiguous since A has real coefficients. If τ is an automorphism of the Cayley numbers, τ leaves the subalgebra of real numbers pointwise fixed. Hence if $B = (b_{ij}) \in \mathscr{A}$, the map $B \to \tau(B) = (\tau(b_{ij}))$ is an automorphism of $\mathscr{A}$ and if A has real coefficients, $\tau(ABA^{-1}) = A\tau(B)A^{-1}$. Also note that if $A \in O(3)$, $B \in M_3^8$ then $ABA^{-1} \in M_3^8$, and the map $B \to ABA^{-1}$ is an automorphism of the Jordan algebra M_3^8.

Let $\tau \in G_2$. Then there is an automorphism f of the Cayley numbers such that

$$\left(\sum_{i=1}^{3} \alpha_i e_i + a_{12} + b_{23} + c_{13}\right) = \sum_{i=1}^{3} \alpha_i e_i + f(a)_{12} + f(b)_{23} + f(c)_{13}$$

(see Jacobson [13, II]). Thus, if $B \in M_3^8$, $B = (b_{ij})$, then $\tau(B) = (f(b_{ij}))$, so τ is of the form described above. Hence we may state the

Lemma 4.1. *If $A \in O(3)$, $B \in M_3^8$, then $ABA^{-1} \in M_3^8$ and $\tau(ABA^{-1}) = A\tau(B)A^{-1}$.*

PROPOSITION 4.2. *There is a* 1-*parameter subgroup* ψ *of* F_4 *such that*
(a) $\psi(1)=\eta$, *i.e.*, $\psi(1)x\psi(1)^{-1}=\sigma(x)$, $x\in \text{Spin } 8$,
(b) $\psi(t)\tau=\tau\psi(t)$, $\tau\in G_2$, $t\in R$.

Proof. First note that η is given by $\eta(B)=ABA^{-1}$, $B\in M_3^8$ where

$$A=\begin{bmatrix} 0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\in \mathrm{SO}(3).$$

There is a 1-parameter subgroup $\phi(t)$ of SO(3) such that $\phi(1)=A$. (Every point of a compact connected Lie group lies on a 1-parameter subgroup.) Define $\psi(t)\colon M_3^8\to M_3^8$ by $\psi(t)(B)=\phi(t)B\phi(t)^{-1}$. From the preceding remarks, ψ is a subgroup of F_4. Since $\psi(1)(B)=ABA^{-1}$, $\psi(1)=n$ and (a) is satisfied. If $\tau\in G_2$, then for all $B\in M_3^8$, $\psi(t)\tau(B)=\phi(t)\tau(B)\,\phi\,(t)^{-1}$ which equals, by (4.1), $\tau(\phi(t)B\,\phi(t)^{-1})=\tau\psi(t)(B)$. Hence, for all $\tau\in G_2$, and all real t, $\psi(t)\tau=\tau\psi(t)$, and (b) is satisfied.

Having proved the existence of a 1-parameter subgroup ψ of F_4 such that
(a) $\psi(1)\tau\psi(1)^{-1}=\sigma(\tau)$, $\tau\in\text{Spin } 8$, and
(b) $\psi(t)\tau=\tau\psi(t)$, $\tau\in G_2$, t real,
we can construct a map $F\colon s(\text{Spin } 8/G_2)\to F_4/\text{Spin } 8$. Here $s(\text{Spin } 8/G_2)$ denotes the suspension of Spin $8/G_2$, i.e., the space obtained from $(\text{Spin } 8/G_2)\times[0,1]$ by collapsing $(\text{Spin } 8/G_2)\times 1$ and $(\text{Spin } 8/G_2)\times 0$ to points. F is defined by $F(\tau G_2,t)=\tau\psi(t)$ Spin 8. Following Harris [8], if E denotes the composition $\pi_j(\text{Spin } 8/G_2)\to\pi_{j+1}(s(\text{Spin } 8/G_2))\xrightarrow{F^*}\pi_{j+1}(F_4/\text{Spin } 8)$, where the first map is ordinary suspension, and if ∂ is the boundary operator of the homotopy sequence of the fibration, $\partial\colon\pi_{j+1}(F_4/\text{Spin } 8)\to\pi_j(\text{Spin } 8/G_2)$, then $T_*=\partial E$, where, as in the preceding section, $T=pq$.

But in §3, we saw that after tensoring with Q_3, T_* is an automorphism of $\pi_j(\text{Spin } 8/G_2)$, hence E provides an inverse to ∂ and the exact homotopy sequence

$$\cdots\to\pi_{j+1}(F_4/G_2)\to\pi_{j+1}(F_4/\text{Spin } 8)\xrightarrow{\partial}\pi_j(\text{Spin } 8/G_2)\to\cdots$$

is split when tensored with Q_3.

Finally, we remark that F_4 is, in a sense, the smallest group containing Spin 8 in which σ becomes inner (in the sense of (4.2)). Evidently, if G is a Lie group. and Spin $8\subset G\subset F_4$, then G is contained in a maximal subgroup of maximal rank of F_4. These subgroups are known by [5] to be SU(2) × Sp(3). Spin 9 and SU(3) × SU(3). The first and last are impossible for dimension reasons, and it is not difficult to check directly that σ does not become inner in any representation of Spin 9 of the form (2.11).

5. **The fibration** $F_4\to E_6\to E_6/F_4$. Throughout this section, let G denote a compact, connected, simply connected Lie group, $\sigma\colon G\to G$ an automorphism of

order 2, K the identity component of the fixed point set of σ. We will give a proof of Theorem 1.5, and we show that this theorem applies to the pair (E_6, F_4). For the proof of (1.5) we use certain results of Harris [8] which we summarize here.

THEOREM 5.1 (HARRIS). *Let G, σ, K be as above. Assume $H^*(G;R) \to H^*(K;R)$ is an epimorphism. Then primitive generators x_i may be chosen for $H^*(G;R)$. so that $\sigma^*(x_i) = \pm x_i$. Let U be the subalgebra generated by the x_i left fixed by σ^*, and let V be the subalgebra generated by the other x_i's. Then:*

(a) *$H^*(G;R) \approx U \otimes V$ as algebra,*

(b) *if $i: K \to G$ is inclusion, i^* maps U isomorphically onto $H^*(K;R)$ and is zero on the positive degree elements of V,*

(c) *if $q: G/K \to G$ is the map $q(gK) = g\sigma(g)^{-1}$ then q^* maps V isomorphically onto $H^*(G/K;R)$ and is zero on the positive degree elements of U,*

(d) *if $l: G \to G/K$ is projection, then* im *$l^* = V$ and l^* is 1-1; moreover $l^*q^*: H^*(G;R) \to H^*(G;R)$ is given by $l^*q^*(x) = x - \sigma^*(x)$, x primitive,*

(e) *$H^*(G/K;R)$ has generators $y_1, \cdots, y_t$ such that $\sigma^*(y_i) = -y_i$ and $q^*l^*(y_i) = 2y_i$, $i = 1, \cdots, t$.*

Using (5.1), we proceed to show that the map $lq: G/K \to G/K$ is an isomorphism in homology with coefficients Q_2. Once this is accomplished, an application of the J. H. C. Whitehead theorem (see Serre [14, p. 276]), yields that

$$l_*q_*: \pi_j(G/K) \to \pi_j(G/K)$$

is a $\mathscr{C}_2$-isomorphism and hence q_* gives the splitting described in (1.5).

Proof of (1.5). From (5.1.e), $H^*(G/K;R) = \Lambda(y_1, \cdots, y_t)$, an exterior algebra on generators y_i of odd degree such that

$$\sigma^*(y_i) = -y_i, \quad i = 1, \cdots, t,$$

and

$$q^*l^*(y_1 \cdots y_t) = 2^t y_1 \cdots y_t.$$

Applying Poincaré duality to the simply-connected compact orientable manifold G/K, we obtain $H^n(G/K;Z) \approx H_0(G/K;Z) \approx Z$.

Let α be a generator of $H^n(G/K;Z)$. Then $\alpha \otimes 1$ generates

$$H^n(G/K;R) \approx H^n(G/K;Z) \otimes R.$$

Let $q^*l^*(\alpha) = d\alpha$, $d \in Z$. Then if $I: R \to R$ is the identity map,

$$(q^*l^* \otimes I)\,(\alpha \otimes 1) = d(\alpha \otimes 1).$$

But we know that $(q^*l^* \otimes I)(y_1 \cdots y_t) = 2^t y_1 \cdots y_t$ and $\alpha \otimes 1 = r y_1 \cdots y_t$, some $r \in R$, hence $q^*l^*(\alpha \otimes 1) = q^*l^*(r y_1 \cdots y_t) = r 2^t y_1 \cdots y_t = 2^t \alpha \otimes 1$. Thus, $q^*l^*(\alpha) = 2^t \alpha$.

Next consider $q^*l^*: H^*(G/K;Q_2) \to H^*(G/K;Q_2)$. Since division by 2 is possible, q^*l^* is an automorphism in dimension n.

Using the fact that real homology and cohomology are dual vector spaces, the above arguments hold for homology as well, and we obtain that

$$l_*q_*: H_n(G/K; Q_2) \to H_n(G/K; Q_2)$$

is an automorphism.

Let u_n be a generator of $H_n(G/K; Q_2)$. The Poincaré duality isomorphism $H^i(G/K; Q_2) \to H_{n-i}(G/K; Q_2)$ is given by $x \to u_n \cap x$. We show that l_*q_* is an automorphism with coefficients Q_2.

Let $f = lq$. Recall the formula $f_*(a) \cap b = f_*(a \cap f_*(b))$, $a \in H_*(G/K; Q_2)$, $b \in H^*(G/K; Q_2)$. (See Hilton-Wylie [10 p. 155].) We have $f^*(u_n) = 2^t u_n$. Take $a = 2^{-t} u_n$ in the formula. Then

$$2^{-t} f_*(u_n) \cap b = 2^{-t} f_*(u_n \cap f^*(b))$$

i.e.,

$$u_n \cap b = 2^{-t} f_*(u_n \cap f^*(b)).$$

From Poincaré duality, $u_n \cap b$ (and hence $2^t u_n \cap b$) ranges over $H_*(G/K; Q_2)$ as b takes on values in $H^*(G/K; Q_2)$. Therefore f_* is onto. It is known that a map of a finitely generated module over a Noetherian ring *onto* itself is necessarily 1-1. Hence f_* is an automorphism. An application of the J. H. C. Whitehead theorem completes the proof of (1.5).

Next we show that (1.5) applies to the pair (E_6, F_4) and hence that (1.3) holds. To do this we establish the following

PROPOSITION 5.2. *Let G be a compact, connected, simple Lie group, $A(G)$ the automorphism group of G, $I(G)$ the subgroup of inner automorphisms. If*

$$A(G)/I(G) \approx Z_2$$

and $\sigma \in A(G)$ is of order 2 and not inner, and if the exterior algebra $H^*(G; R)$ has generators x_{2k_i-1}, at least one k_i odd, then

(a) $\sigma^*(x_{2k_i-1}) = x_{2k_i-1}$, *if k_i even.*

(b) $\sigma^*(x_{2k_i-1}) = -x_{2k_i-1}$, *if k_i odd.*

Proof. Let B_G be the classifying space of G. Then $H^*(B_G; R)$ is a polynomial algebra on generators y_i, $\deg y_i = 2k_i$, which suspend to the x_i. Let T be a maximal torus of G, and let $\alpha \in I(G)$ such that $\alpha \circ \sigma: T \to T$. Since α^* is the identity on cohomology, we obtain induced maps $\sigma^*: H^*(B_G; R) \to H^*(B_G; R)$ and $\sigma^*: H^*(B_T; R) \to H^*(B_T; R)$. There is an inclusion $B_T \subset B_G$ which induces a monomorphism $H^*(B_G; R) \to H^*(B_T; R)$. But $H^*(B_T; R)$ is a polynomial algebra on generators $z_1, \cdots, z_l$ ($l =$ rank G), $\deg z_i = 2$. Hence the generators of $H^*(B_G; R)$ are homogeneous polynomials P_j of degree k_j in the z's.

Since σ has order 2, we may assume that $\sigma^*(z_i) = \pm z_i$, $i = 1, \cdots, l$. Consider the map $\gamma: T \to T$, $\gamma(t) = t^{-1}$, $t \in T$. Let $\mathscr{G}$ be the complexification of the Lie algebra of G. Then the Lie algebra $\mathscr{T}$ of T (complexified) is a Cartan subalgebra of $\mathscr{G}$, and the map induced by γ on $\mathscr{T}$ is $\gamma(h) = -h$, $h \in \mathscr{T}$.

Jacobson [12, p. 127] shows that the automorphism $\gamma: \mathscr{T} \to \mathscr{T}$ extends to an automorphism of order 2 of $\mathscr{G}$. Thus, passing to the compact Lie groups, the automorphism $\gamma: T \to T$ extends to an automorphism γ of order 2, $\gamma: G \to G$.

On $H^*(B_T; R), \gamma^*(z_i) = -z_i$, $i = 1, \cdots, l$. Hence on $H^*(B_G; R), \gamma^* P_j = -P_j$ if and only if k_j is odd. But there is an odd k_j, hence γ is not the identity on $H^*(G; R)$, hence γ is not inner. Thus by the hypothesis, $\gamma = \tau \circ \sigma, \tau \in I(G)$. Hence $\gamma^* = \sigma^*$, so if k_j is odd, $\sigma^*(x_{2k_j-1}) = \gamma^*(x_{2k_j-1})$, and if k_j is even,

$$\sigma^*(x_{2k_j-1}) = x_{2k_j-1}.$$

This proves (5.2).

Finally we point out that by Theorem 4, Jacobson [12, p. 281], E_6 has only one outer automorphism (up to inner automorphisms), thus E_6 satisfies the hypotheses of (5.2). But $H^*(E_6; R) = \Lambda(x_3, x_9, x_{11}, x_{15}, x_{17}, x_{23})$ (see [4]), so if σ is the involution of E_6 having F_4 as fixed point set, $\sigma^*(x_9) = -x_9$ and $\sigma^*(x_{17}) = -x_{17}$ since $9 = 4 \cdot 2 + 1$ and $17 = 4 \cdot 4 + 1$.

The following result of Harris [8] completes the proof.

Proposition 5.3 (Harris). *If G is a compact, connected, simply connected Lie group, $\sigma: G \to G$ an automorphism of order 2, K the identity component of the fixed point set of σ, we may write $H^*(G; R) = U \otimes V$, where U and V are subalgebras such that σ^* is the identity on U and σ^* multipltes the generators of V by -1. Moreover, if V contains a nonzero element whose degree equals the dimension of G/K, then the map $H^*(G; R) \to H^*(K; R)$ induced by inclusion is an epimorphism.*

Recall that $\sigma^*(x_9) = -x_9$ and $\sigma^*(x_{17}) = -x_{17}$, x_9, x_{17} generators of $H^*(E_6; R)$. Taking $U = \Lambda(x_3, x_{11}, x_{15}, x_{23})$ and $V = \Lambda(x_9, x_{17})$ and noting that $\dim x_9 x_{17} = 26 = \dim E_6/F_4$, an application of (5.3) shows that (E_6, F_4) satisfies the hypotheses of (1.5).

6. **The fibration** $E_6/F_4 \to E_7/F_4 \to E_7/E_6$. The purpose of this section is to establish (1.4). To do this we show that the homotopy exact sequence of the fibration $E_6/F_4 \to E_7/F_4 \to E_7/E_6$ is split when tensored with Q_2. The splitting is given by the Bott suspension map $E: \pi_j(E_6/F_4) \to \pi_{j+1}(E_7/E_6)$. We begin with some general remarks.

Let $K \subset G \subset L$ be compact connected Lie groups. Let σ be an automorphism of period 2 of G such that K is the identity component of the fixed point set of σ. Harris [8] has shown that the Bott suspension map $E: \pi_j(G/K) \to \pi_{j+1}(L/G)$ may be constructed as follows: Let v be a 1-parameter subgroup of L such that (i) $v(1)gv(1)^{-1} = \sigma(g)$, $g \in G$, (ii) $v(t)k = kv(t)$, $k \in K$, $t \in R$. Let

$$s: \pi_j(G/K) \to \pi_{j+1}(s(G/K))$$

be the ordinary suspension. Define $F: s(G/K) \to L/G$ by $F(gK, t) = gv(t)G$, $g \in G$, $t \in R$. Then E is the composition $F_* s: \pi_j(G/K) \to \pi_{j+1}(L/G)$. It is also shown in [8] that if $l: G \to G/K$ is the projection and $q: G/K \to G$ is the map $q(gK) = g\sigma(g)^{-1}$, $g \in G$, then the composition $l_* q_*: \pi_j(G/K) \to \pi_j(G/K)$ is the same as the composition $\partial E: \pi_j(G/K) \to \pi_j(G/K)$, ∂ the boundary map, $\partial: \pi_{j+1}(L/G) \to \pi_j(G/K)$, in the homotopy exact sequence of the fibration $G/K \to L/K \to L/G$. Thus E serves as an inverse to ∂ in the homotopy sequence of $G/K \to L/K \to L/G$ if and only if q_* serves as an inverse to l_* in the homotopy sequence of $K \to G \to G/K$.

Since we have shown in §5 that q_* gives a splitting of the homotopy sequence $F_4 \to E_6 \to E_6/F_4$, when tensored with Q_2, the above remarks make it clear that (1.4) will be proved if we can establish

PROPOSITION 6.1. *There exists a* 1-*parameter subgroup v of E_7 such that* (i) $v(1)gv(1)^{-1} = \sigma(g)$, $g \in E_6$, *where σ is the automorphism of E_6 having F_4 as fixed point set, and* (ii) $v(t)k = kv(t)$, $k \in F_4$, $t \in R$.

The proof of the existence of such a v proceeds as follows. Let $\mathscr{E}_7$ be the compact Lie algebra described by (2.12), $\mathscr{E}_6$ the subalgebra given by (2.16) and $\mathscr{F}_4$ the subalgebra of $\mathscr{E}_6$ consisting of all derivations of M_3^8. Then $\mathscr{F}_4 \subset \mathscr{E}_6 \subset \mathscr{E}_7$ are all compact real forms. There is an automorphism $\sigma: \mathscr{E}_6 \to \mathscr{E}_6$ defined by

$$\sigma\left(d + \frac{ih}{2} \otimes a\right) = d - \frac{ih}{2} \otimes a, \quad d \in \mathscr{F}_4,\ a \in M_3^8(0).$$

The fixed set of σ is $\mathscr{F}_4$, and σ has period 2. We will define an automorphism $v: \mathscr{E}_7 \to \mathscr{E}_7$, $v = \exp \operatorname{ad} x$, x a certain element of $\mathscr{E}_7$, such that v restricted to $\mathscr{E}_6$ is σ. We will show that x may be chosen such that $[x\mathscr{F}_4] = 0$. The exponential map $\operatorname{Exp}: \mathscr{E}_7 \to E_7$ will be used to transfer these data to the simply connected Lie groups $F_4 \subset E_6 \subset E_7$ (we will establish these inclusions later). Taking $v(t) = \operatorname{Exp} tx$, $t \in R$, we will obtain the desired one-parameter subgroup. (i) is satisfied since $v | \mathscr{E}_6 = \sigma$, hence the inner automorphism $g \to (\operatorname{Exp} x) g (\operatorname{Exp} x)^{-1}$, $g \in E_7$, coincides with σ on E_6, and (ii) follows from $[x\mathscr{F}_4] = 0$. We proceed with the details.

PROPOSITION 6.2. *Let* $x = ((e+f)\pi/2) \otimes I \in Y \otimes M_3^8 \subset \mathscr{E}_7$, *$I$ the identity matrix. Then*

(i) $[x\mathscr{F}_4] = 0$ *and*

(ii) *σ is the restriction of the automorphism* exp ad x *of $\mathscr{E}_7$ to the subalgebra $\mathscr{E}_6$.*

Proof of 6.2. If $d \in \mathscr{F}_4$, $[xd] = [((e+f)\pi/2)) \otimes I, d] = -((e+f)\pi/2)) \otimes d(I)$ by (2.13), and $d(I) = 0$ since d is a derivation. Hence (i) is established. To prove (ii), recall from §2, that

$$\begin{aligned} [ih, e+f] &= -2i(e-f), \\ [i(e-f), e+f] &= 2ih \end{aligned}$$

and that

$$(ih, e+f) = 0,$$
$$(i(e-f), e+f) = 0,$$

hence if $a \in M_3^8$, using formula (2.14) we obtain

$$(\operatorname{ad} x)^{2k-1}(ih \otimes a) = (-1)^k \pi^{2k-1} i(e-f) \otimes a,$$
$$(\operatorname{ad} x)^{2k}(ih \otimes a) = (-1)^k \pi^{2k} ih \otimes a.$$

Therefore

$$\begin{aligned}(\exp \operatorname{ad} x)\left(\frac{ih}{2} \otimes a\right) &= \frac{ih}{2}\left(1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \cdots\right) \otimes a \\ &\quad - \frac{i(e-f)}{2}\left(\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \cdots\right) \otimes a \\ &= -\frac{ih}{2} \otimes a.\end{aligned}$$

Since $[x\mathscr{F}_4] = 0$, $(\exp \operatorname{ad} x)d = d$, $d \in \mathscr{F}_4$. Hence $\exp \operatorname{ad} x = \sigma$ and (ii) is proved, and this completes the proof of 6.2.

Next we establish inclusions $F_4 \subset E_6 \subset E_7$ of the simply connected forms.

Let E_7' denote the adjoint group of $\mathscr{E}_7$ and E_6' the connected subgroup of E_7' corresponding to the subalgebra $\operatorname{ad}_{\mathscr{E}_7}(\mathscr{E}_6)$ of $\operatorname{ad}\mathscr{E}_7$ (the adjoint Lie algebra). We have the commutative diagram:

$$\begin{array}{ccccc} E_6' & \subset & E_7' & \subset & GL(\mathscr{E}_7; R) \\ \uparrow & & \uparrow & & \uparrow \exp \\ \operatorname{ad}_{\mathscr{E}_7}(\mathscr{E}_6) & \subset & \operatorname{ad}\mathscr{E}_7 & \subset & \mathfrak{gl}(\mathscr{E}_7; R). \\ \approx \uparrow & & \uparrow & & \\ \mathscr{E}_6 & \subset & \mathscr{E}_7 & & \end{array}$$

PROPOSITION 6.3. *E_6' is the simply connected form of this group.*

Proof of 6.3. It suffices to show that the center of E_6' contains a nontrivial element, since the center of the simply connected E_6 is Z_3. Let

$$z = \frac{ih}{2} \otimes \operatorname{diag}(\lambda, \lambda, -2\lambda), \quad \lambda = 4\pi/3.$$

Then $z \in \mathscr{E}_6$. We will show that $\exp \operatorname{ad} z$ is a nontrivial element in the center of E_6. Let $Y' \subset Y$ be the subspace spanned by $e+f$, $i(e-f)$. We first compute the eigenvalues of $\operatorname{ad} z$ acting on $\mathscr{E}_7$. To do this, consider the complexification of $\mathscr{E}_7$. Then $\operatorname{ad} z$ acting on $\mathscr{E}_7 \otimes C$ has the complexified form of the subspaces

$$\mathscr{D}(M_3^8) \oplus ((ih/2) \otimes M_3^8), \quad e \otimes M_3^8, \quad f \otimes M_3^8 \tag{6.4}$$

as invariant subspaces. For from (2.13), (2.14) and (2.17),

$$(\operatorname{ad} z)\ (d + (ih/2) \otimes a) = (ih/2) \otimes d(y) - \langle a, y \rangle,$$

$$(\operatorname{ad} z)\ (e \otimes a) = [e, ih/2] \otimes ay = ie \otimes ay,$$

$$(\operatorname{ad} z)\ (f \otimes a) = [f, ih/2] \otimes ay = -if \otimes ay,$$

where $y = \operatorname{diag}(\lambda, \lambda, -2\lambda)$, $d \in \mathscr{F}_4$, $a \in M_3^8$. We propose to show

LEMMA 6.5. *The eigenvalues of* exp ad z *acting on* $\mathscr{E}_7 \otimes C$ *are as follows*:
(a) *on* $(\mathscr{D}(M_3^8) \oplus (ih/2) \otimes M_3^8) \otimes C$, *all eigenvalues are* 1,
(b) *on* $(e \otimes M_3^8) \otimes C$, *all eigenvalues are* $\exp 4\pi i/3$,
(c) *on* $(f \otimes M_3^8) \otimes C$, *all eigenvalues are* $\exp 2\pi i/3$. *Hence* exp ad z *acts as the identity on* $\mathscr{D}(M_3^8) \oplus (ih/2) \otimes M_3^8) \otimes C$, *as multiplication by* $\exp 4\pi i/3$ *on* $(e \otimes M_3^8) \otimes C$ *and as multiplication by* $\exp 2\pi i/3$ *on* $(f \otimes M_3^8) \otimes C$.

Proof of 6.5. If μ is an eigenvalue of ad z acting on $\mathscr{E}_7 \otimes C$, then μ corresponds to an eigenvector in one of the invariant subspaces (6.4). If $(\operatorname{ad} z)(d + (ih/2) \otimes m) = \mu(d + (ih/2) \otimes m)$, $d \in \mathscr{F}_4$, $m \in M_3^8$, then $(ih/2) \otimes d(y) - \langle m, y \rangle = \mu(d + (ih/2) \otimes m)$ so $d(y) = \mu m$ and $-\langle m, y \rangle = d$, from which we obtain $-\langle m, y \rangle(y) = \mu d(y) = \mu^2 m$.

Thus $-m(yy) + y(my) = \mu^2 m$. But $yy = \lambda^2(\varepsilon_1 + \varepsilon_2 + 4\varepsilon_3)$, and if m is written in the form of (2.3),

$$m = \sum_{i=1}^{3} \alpha_i \varepsilon_i + a_{12} + b_{13} + c_{23}, \quad \alpha_i \text{ real, } a,b,c \text{ Cayley,}$$

then

$$my = \lambda(\alpha_1\varepsilon_1 + \alpha_2\varepsilon_2 - 2\alpha_3\varepsilon_3 + a_{12} - \tfrac{1}{2}b_{13} - \tfrac{1}{2}c_{23})$$

(using (2.6)-(2.8)), hence

$$y(my) = \lambda^2\left(\alpha_1\varepsilon_1 + \alpha_2\varepsilon_2 + 4\alpha_3\varepsilon_3 + a_{12} + \frac{1}{4}\,b_{13} + \frac{1}{4}\,c_{23}\right),$$

$$m(yy) = \lambda^2\left(\alpha_1\varepsilon_1 + \alpha_2\varepsilon_2 + 4\alpha_3\varepsilon_3 + a_{12} + \frac{5}{2}\,b_{13} + \frac{5}{2}\,c_{23}\right),$$

and therefore

$$-\langle m, y \rangle(y) = \lambda^2\left(-\frac{9}{4}\right)(b_{13} + c_{23}) = \mu^2 m.$$

It follows that $(-9/4)\lambda^2 = \mu^2$, and therefore that $\mu = \pm 2\pi i$. The eigenvalues of exp ad z are obtained by exponentiating, hence 6.5(a) is proved.

If $(\operatorname{ad} z)(Ae \otimes m) = \mu Ae \otimes m$, $0 \neq A \in C, 0 \neq m \in M_3^8$, then

$$iAe \otimes my = \mu Ae \otimes m,$$

and by writing m in the form of (2.3), and using (2.6)-(2.8) as above, we obtain

$$iA\lambda\left(\alpha_1\varepsilon_1 + \alpha_2\varepsilon_2 - 2\alpha_3\varepsilon_3 + a_{12} - \frac{1}{2}b_{13} - \frac{1}{2}c_{23}\right)$$
$$= \mu A(\alpha_1\varepsilon_1 + \alpha_2\varepsilon_2 + \alpha_3\varepsilon_3 + a_{12} + b_{13} + c_{23}).$$

Since one of α_i, $a,b,c \neq 0$, we have either

$$i\lambda = \mu, \quad -2i\lambda = \mu \quad \text{or} -\frac{1}{2}\, i\lambda = \mu,$$

hence $\mu = 4\pi i/3, -8\pi i/3$ or $-2\pi i/3$. This proves 6.5(b)

Part (c) of 6.5 is proved in the same way.

Next we observe that the subspaces (6.4) are invariant under elements of $\mathrm{ad}_{\mathscr{E}_7}\mathscr{E}_6$. This follows immediately from (2.13) and (2.14). Thus the transformation ad $(d + (ih/2) \otimes m)$, $d \in \mathscr{F}_4, m \in M_3^8(0)$, has matrix representation

$$\begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix},$$

where A, B, C are square matrices representing the action of $\mathrm{ad}(d+(ih/2)\otimes m)$ on the subspaces (6.4). Thus exp ad $(d + (\mathrm{ih}/2) \otimes m)$ has matrix

$$\begin{bmatrix} \exp A & 0 & 0 \\ 0 & \exp B & 0 \\ 0 & 0 & \exp C \end{bmatrix}$$

and from 6.5, exp ad z commutes with all such elements of E_7', i.e. with all elements of E_6'. Thus exp ad $z \in Z(E_6')$, is not the identity, and has order 3. This completes the proof of 6.3.

Finally if E_7' is not simply connected, the inclusion $E_6' \subset E_7'$ can be lifted to an inclusion $E_6' \subset E_7$, where E_7 is the simply connected form. We obtain

$$F_4 \subset E_6' \approx E_6 \subset E_7.$$

This, together with the remarks at the beginning of this section, yields the proof of 6.1, hence the isomorphisms (1.4) are established.

7. **Proof of Theorem 1.6.** In this section we prove Theorem 1.6 by making use of É. Cartan's classification of symmetric spaces. To begin, we recall that Harris [8] has proved this result for the following pairs (G, K):

$$(\mathrm{SU}_{2n+1}, \mathrm{SO}_{2n+1}) \text{ with } L = \mathrm{Sp}_{2n+1},$$
$$(\mathrm{SU}_{2n}, \mathrm{Sp}_n) \text{ with } L = \mathrm{SO}_{4n},$$
$$(\mathrm{Spin}_{2n}, \mathrm{Spin}_{2n-1}) \text{ with } L = \mathrm{Spin}_{2n+1},$$

and by the main result (1.4) of the last section, the theorem holds for the pair

$$(E_6, F_4) \text{ with } L = E_7.$$

The fact that in each of the above cases the inequality rank $L-$ rank $G \leqq 1$ holds is easily checked. Moreover,

$$\mathrm{Sp}_{2n+1}/U_{2n+1}, \qquad \mathrm{SO}_{4n}/U_{2n},$$

$$\mathrm{Spin}_{2n+1}/\mathrm{Spin}_{2n}, \qquad E_7/E_6 \times S^1,$$

are symmetric spaces, and $U_k \approx \mathrm{SU}_k \times S^1$, hence in each of these cases L/G or L/M is a symmetric space, where M is locally isomorphic to $G \times S^1$. This shows that (1.6) is satisfied for these pairs (G, K).

Now the pairs (G, K), where K is the identity component of the fixed point set of an involution σ of C, are classified by É. Cartan (see Helgason [9, p. 354]). In addition to those mentioned above, we have the pairs (G, K) listed in the following table (7.1). (We drop the assumption of simple connectedness for the purposes of the table.)

TABLE 7.1

	G	K	rank G	rank K	
1	SU_{2n}	SO_{2n}	$2n-1$	n	
2	SU_{p+q}	$S(U_p \times U_q)$	$p+q-1$	$p+q-1$	
3	SO_{p+q}	$\mathrm{SO}_p \times \mathrm{SO}_q$	$r+s$	$r+s$	$(p=2r,\ q=2s)$
4			$r+s$	$r+s$	$(p=2r,\ q=2s+1)$
5			$r+s$	$r+s$	$(p=2r+1, q=2s)$
6			$r+s+1$	$r+s$	$(p=2r+1, q=2s+1)$
7	SO_{2n}	U_n	n	n	
8	Sp_n	U_n	n	n	
9	Sp_{p+q}	$\mathrm{Sp}_p \times \mathrm{Sp}_q$	$p+q$	$p+q$	
10	E_6	Sp_4	6	4	
11	E_6	$\mathrm{SU}_6 \times \mathrm{SU}_2$	6	$5+1$	
12	E_6	$\mathrm{SO}_{10} \times S^1$	6	$5+1$	
13	E_7	SU_8	7	7	
14	E_7	$\mathrm{SO}_{12} \times \mathrm{SU}_2$	7	$6+1$	
15	E_7	$E_6 \times S^1$	7	$6+1$	
16	E_8	SO_{16}	8	8	
17	E_8	$E_7 \times \mathrm{SU}_2$	8	$7+1$	
18	F_4	$\mathrm{Sp}_3 \times \mathrm{SU}_2$	4	$3+1$	
19	F_4	SO_9	4	4	
20	G_2	$\mathrm{SU}_2 \times \mathrm{SU}_2$	2	$1+1$	

Observe that the case $(\mathrm{Spin}_{2n}, \mathrm{Spin}_{2n-1})$ is listed as type 6 in the table where $r = n-1$, $s = 0$, (hence $p+q = (2r+1)+(2s+1) = 2(n-1)+1+1 = 2n$), and SO_1 is a point.

PROPOSITION 7.2. *The only pairs* (G,K) *satisfying the hypotheses of Theorem 1.6 are*:

$$(\mathrm{SU}_{2n+1},\mathrm{SO}_{2n+1}),\quad (\mathrm{SU}_{2n},\mathrm{Sp}_n),\quad (\mathrm{Spin}_{2n},\mathrm{Spin}_{2n-1}),\quad (E_6,F_4).$$

To prove this we use the following:

LEMMA 7.3. *If* K *has maximal rank in* G, *then* $H^*(G;R)\to H^*(K;R)$ *induced by inclusion, is not an epimorphism.*

Proof of 7.3. If K has maximal rank in G, the Euler-Poincaré characteristic $\chi(G/K)$ is positive. (See Hopf-Samelson [11].) If $H^*(G;R)\to H^*(K;R)$ is onto, then by Harris' result (Theorem 5.1(c)), G/K would have the homology of a product of odd spheres. Since $\chi(\text{odd sphere})=0$, $\chi(G/K)=0$, a contradiction.

In the table (7.1) we have noted the ranks of G and K. All but the following pairs are therefore eliminated by (7.3):

(a) $(\mathrm{SU}_{2n},\mathrm{SO}_{2n})$;

(b) $(\mathrm{SO}_{2r+1+2s+1},\mathrm{SO}_{2r+1}\times \mathrm{SO}_{2s+1})\qquad (r,s\neq 0)$;

(c) (E_6,Sp_4).

To complete the proof of (7.2) we show that in each of the cases (a)-(c), the map $H^*(G;R)\to H^*(K;R)$ is not an epimorphism. (Equivalently, $H_*(K;R)\to H_*(G;R)$ is not 1-1.)

The real cohomology of the simple groups is known (see Borel [3], Borel and Chevalley [4]).

Recall that $H^*(E_6;R) = \Lambda(x_3,x_9,x_{11},x_{15},x_{17},x_{23})$ and $H_*(\mathrm{Sp}_4;R) =\Lambda(x_3,x_7,x_{11},x_{15})$, hence the inclusion $\mathrm{Sp}_4\subset E_6$ cannot be 1-1 in homology, for $H^*(E_6;R)$ contains nothing in degree 7. This eliminates (c).

Similarly, if $r,s\neq 0$,

$$H_*(\mathrm{SO}_{2(r+s+1)};R)=\Lambda(x_3,x_7,\cdots,x_{4(r+s+1)-5},Y_{2(r+s+1)-1}),$$

$$H_*(\mathrm{SO}_{2r+1};R)=\Lambda(x_3,x_7,\cdots,x_{4r-1}),$$

hence $H_*(\mathrm{SO}_{2r+1}\times\mathrm{SO}_{2s+1};R)$ has two independent generators in degree 3. Thus the inclusion $\mathrm{SO}_{2r+1}\times\mathrm{SO}_{2s+1}\to\mathrm{SO}_{2(r+s+1)}$ is not 1-1 in homology. This eliminates (b).

Finally, recall that $H_*(\mathrm{SU}_{2n};R) = \Lambda(x_3,x_5,\cdots,x_{4n-1})$ and $H_*(\mathrm{SO}_{2n};R) =\Lambda(w_3,w_7,\cdots,w_{4n-5},y_{2n-1})$. Let σ be the automorphism of SU_{2n} with SO_{2n} as fixed point set.

If $n=2k$, then $H_*(\mathrm{SO}_{2n};R)$ has two generators of degree $2n-1=4k-1$. Thus, the inclusion $i\colon \mathrm{SO}_{2n}\to\mathrm{SU}_{2n}$ is not 1-1 in homology.

If $n=2k+1$, and $i^*\colon H^*(\mathrm{SU}_{2n};R)\to H^*(\mathrm{SO}_{2n};R)$ is an epimorphism, then by Harris' result (5.1), $H^*(\mathrm{SU}_{2n};R)\approx U\otimes V$, U the subalgebra generated by those x_i such that $\sigma^*(x_i)=x_i$, and V the subalgebra generated by those x_i such that $\sigma^*(x_i)=-x_i$. Moreover, i^* is zero on the positive degree elements of V and maps U isomorphically onto $H^*(\mathrm{SO}_{2n};R)$.

From Jacobson [12, p. 281], $A(\mathrm{SU}_{2n})/I(\mathrm{SU}_{2n}) = Z_2$, and σ is of order 2 and not inner. Hence (5.2) applies to SU_{2n}, and we see that if deg $x_i = 4k_i + 1$, then $\sigma^*(x_i) = -x_i$.

Now deg $y_{2n-1} = 4k+1$, and i^* is onto, so $y_{4k+1} = y_{2n-1} = i^*(tx_{4k+1} + d)$, $t \in R$, d decomposable. Since $i^*(d) = y_{4k+1} - ti^*(x_{4k+1})$ is primitive and decomposable, we have $i^*(d) = 0$. Thus $i^*(tx_{4k+1}) = y_{4k+1}$. But $\sigma^*(x_{4k+1}) = -x_{4k+1}$, by the remark of the preceding paragraph, hence

$$\sigma^*(y_{4k+1}) = \sigma^* i^*(tx_{4k+1}) = i^*\sigma^*(tx_{4k+1}) = -i^*(tx_{4k+1}) = -y_{4k+1},$$

a contradiction, since σ^* is the identity on $H^*(\mathrm{SO}_{2n}: R)$. Thus i^* is not onto, and case (a) is eliminated.

The proof of Theorem 1.6 is therefore complete.

References

1. S. Araki, *On the Brouwer degrees of some maps of compact symmetric spaces*, Topology Vol. 3, Pergamon, New York, 1965, pp. 281–290.

2. E. Artin, *Geometric algebra*, Interscience Tract No. 3, Wiley, New York, 1957.

3. A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. **57** (1953), 115–207.

4. A. Borel and C. Chevalley, *The Betti numbers of the exceptional groups*, Mem. Amer. Math. Soc. No. 14 (1955), 1–9.

5. A. Borel and J. de Siebenthal, *Les sous-groupes fermés connexes de rang maximum des groupes de Lie clos*, Comment. Math. Helv. **23** (1949–1950), 200–221.

6. R. Bott, *The stable homotopy of the classical groups*, Ann. of Math. **70** (1959), 313–337.

7. E. B. Dynkin and A. L. Oniščik, *Compact global Lie groups*, Uspehi. Mat. Nauk (N.S.) **10** (1955), 3–74; English transl., Amer. Math. Soc. Transl. (2) **21** (1962), 119–192.

8. B. Harris, *Suspensions and characteristic maps for symmetric spaces*, Ann. of Math. **76** (1962), 295–305.

9. S. Helgason, *Differential geometry and symmetric spaces*, Academic Press, New York, 1962.

10. P. Hilton and S. Wiley, *Homology theory*, Cambridge Univ. Press, New York, 1960.

11. H. Hopf and H. Samelson, *Ein Satz über die Wirkungsräume geschlossener Leischer Gruppen*, Comment. Math. Helv. **13** (1940–1941), 240–251.

12. N. Jacobson, *Lie algebras*, Interscience Tract No. 10, Wiley, New York, 1962.

13. ———, *Some groups of transformations defined by Jordan algebras*. I, II, III, J. Reine Angew. Math. I: **201** (1959), 178–195, II: **204** (1960), 74–98, III: **207** (1961), 61–85.

14. J.-P. Serre, *Groupes d'homotopie et classes de groupes abéliens*, Ann. of Math. **58** (1953), 258–294.

15. N. E. Steenrod, *The topology of fibre bundles*, Princeton Univ. Press, Princeton, N. J., 1951.

16. J. Tits, *Une classe d'algèbres de Lie en relation avec les algèbres de Jordan*, Nederl. Akad. Wetensch. Proc. Ser. A **65** (1962), 530–535.

Brown University,
Providence, Rhode Island
State University of New York at Stony Brook,
Stony Brook, New York