ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS OF AN nTH ORDER NONHOMOGENEOUS ORDINARY DIFFERENTIAL EQUATION(1)

BY

THOMAS G. HALLAM

1. **Introduction.** The asymptotic behavior of the solutions of the nth order nonhomogeneous differential equation

$$y^{(n)} + f(t, y, y^{(1)}, \cdots, y^{(n-1)}) = h(t) \tag{1.1}$$

will be considered. Conditions will be established in order that the solutions of (1.1) essentially behave asymptotically as those of $y^{(n)} = h(t)$. The part of the forcing function $h(t)$ which is dominant for large values of t will be called the primary part of the forcing term. The forcing functions considered here will, generally, have primary part $t^m e^{bt}$, m and b real. Sufficient conditions on f and h in order to guarantee the existence of solutions of (1.1) for large values of t will be tacitly assumed.

In [1, §§3.7 and 3.11], Cesari discusses known asymptotic results for equation (1.1) when the equation is linear and the primary part of $h(t)$ is a constant. Recent results on the asymptotic behavior of the solutions of a homogeneous equation are given in [2]–[4], [6], [7].

§2 of this paper is concerned with integrable forcing terms; that is, $\int^\infty h(t)dt$ is finite. The results of the remainder of the paper are developed for $h(t)$ of the form

$$h(t) = H(t)t^m e^{bt} + R(t),$$

where $\lim_{t\to\infty} H(t) = A \neq 0$, and $R(t) = o(t^m e^{bt})$ as t approaches infinity. The main results give sufficient and in some instances, also necessary conditions for the solutions of the equation (1.1) to possess an asymptotic behavior of the type $y(t)/a(t) \sim k \neq 0$, where $a(t)$ is determined by a primitive of $h(t)$.

The following theorem due to Viswanatham [5] as utilized by Waltman [6], [7] is useful here.

THEOREM 1.1. *If $y(t) \leqq \rho + \int_T^t f(s, y(s))ds$, where $f(t, y)$ is continuous and monotonic increasing in y in the region R defined by $|t - T| < a$; $|y - \rho| < b$*

Presented to the Society, April 9, 1965 under the title *Asymptotic behavior of the solutions of an ordinary nonlinear differential equation;* received by the editors July 9, 1965.

(1) Parts of this paper are included in [3], which was directed by Professor W. R. Utz, to whom the author express his gratitude.

where a and b are positive real numbers; then $y(t) \leqq z(t)$ *where* $z(t)$ *is the maximal solution of the differential equation* $z' = f(t,z)$ *through* (T,ρ) *for* $t \geqq T$.

An inequality which will be used in the following results is given in

LEMMA 1.1. *Let* $a_i \geqq 0$, $b_i \geqq 0$, $r_i > 0$, *and* $r = \max_i r_i$ *where* $i = 1,2,\cdots,n$. *If* $b_i > 1$ *for some* i, *then*

$$\sum_{i=1}^{n} a_i b_i^{r_i} \leqq \left[\sum_{i=1}^{n} a_i\right]\left[\sum_{i=1}^{n} b_i\right]^r.$$

2. **Integrable forcing terms.** The differential equation (1.1) is considered subject to the following set of hypotheses.

$$|f(t,y,y^{(1)},\cdots,y^{(n-1)})| \leqq \sum_{i=0}^{n-1} g_i(t)|y^{(i)}|^{r_i}, \tag{2.1}$$

where (2.1.1) $r_i > 0$, $i = 0,1,\cdots,n-1$;
(2.1.2) $g_i(t)$ are continuous, $i = 0,1,\cdots,n-1$.

$$\int^{\infty} |h(t)|\, dt < \infty. \tag{2.2}$$

The inequality in (2.1) need hold only for large values of t.

THEOREM 2.1. *If conditions* (2.1) *and* (2.2) *above are satisfied, and* $\int^\infty t^{(n-i-1)r_i} g_i(t)dt < \infty$, $i = 0,1,\cdots,n-1$, *then there exist solutions* $y(t)$ *of* (1.1) *which have the asymptotic behavior* $y^{(i)}(t)/t^{n-i-1} \sim a_i \neq 0$, $i = 0,1,\cdots,n-1$.

Proof. It will be established that the asymptotic condition is applicable to a given solution $y(t)$ of (1.1) provided the initial conditions of the solution satisfy a certain inequality.

Integrating (1.1) $n-k$ times, yields for $t \geqq t_0 \geqq 1$,

$$\begin{aligned} y^{(k)}(t) = {} & \theta_0 + \theta_1 t + \cdots + \theta_{n-k-1}t^{n-k-1} \\ & + \int_{t_0}^{t} \frac{(t-s)^{n-k-1}}{(n-k-1)!} h(s)\, ds \\ & - \int_{t_0}^{t} \frac{(t-s)^{n-k-1}}{(n-k-1)!} f(s,y(s),\cdots,y^{(n-1)}(s))\, ds. \end{aligned} \tag{2.3}$$

From this, the inequality

$$\left|\frac{y^{(k)}(t)}{t^{n-k-1}}\right| \leqq A_k + B_k \int_{t_0}^{t} |f(s,y(s),\cdots,y^{(n-1)}(s))|\, ds$$

may be obtained, where

$$A_k = [\theta_0 + \cdots + \theta_{n-k-1} t_0^{n-k-1}] t_0^{k+1-n} + \frac{1}{(n-k-1)!} \int_{t_0}^{\infty} |h(s)| \, ds,$$

and

$$B_k = \frac{1}{(n-k-1)!}.$$

From (2.1), we have

$$\left| \frac{y^{(k)}(t)}{t^{n-k-1}} \right| \leqq A_k + B_k \int_{t_0}^{t} \sum_{i=0}^{n-1} s^{(n-i-1)r_i} g_i(s) \left| \frac{y^{(i)}(s)}{s^{n-k-1}} \right|^{r_i} ds \tag{2.4}$$

for all $k = 0, 1, \cdots, n-1$. Thus, using Lemma 1.1,

$$\sum_{k=0}^{n-1} \left| \frac{y^{(k)}(t)}{t^{n-k-1}} \right| \leqq A + B \int_{t_0}^{t} \left[\sum_{i=0}^{n-1} s^{(n-i-1)r_i} g_i(s) \right] \left[\sum_{i=0}^{n-1} \left| \frac{y^{(i)}(s)}{s^{n-i-1}} \right| \right]^r ds,$$

where

$$A = \sum_{k=0}^{n-1} A_k,$$

$$B = \sum_{k=0}^{n-1} B_k,$$

and

$$r = \max_i r_i.$$

It should be observed that the above inequality does not hold if

$$|y^{(i)}(t)/t^{n-i-1}| < 1$$

for all $i = 0, 1, \cdots, n-1$ and all large values of t. However, if this is the case, $\sum_{i=0}^{n-1} |y^{(i)}(t)/t^{n-i-1}|$ is bounded and this is the conclusion which is ultimately desired.

Applying Theorem 1.1 to the above inequality, we obtain as the associated differential equation

$$z' = B \sum_{i=0}^{n-1} t^{(n-i-1)r_i} g_i(t) z^r. \tag{2.5}$$

The solutions of (2.5) are given by

$$z = A \exp \int_{t_0}^{t} B \sum_{i=0}^{n-1} s^{(n-i-1)r_i} g_i(s) \, ds, \qquad r = 1; \tag{2.6}$$

$$z^{1-r} = A^{1-r} + (1-r) \int_{t_0}^{t} B \sum_{i=0}^{n-1} s^{(n-i-1)r_i} g_i(s) \, ds, \qquad r \neq 1. \tag{2.7}$$

If $0 < r \leqq 1$, then $z(t)$ is bounded independent of initial conditions since the integrals in (2.6) and (2.7) are convergent. If $r > 1$, any solution $z(t)$ as given in (2.7) will be bounded provided

$$A^{1-r} > (r-1) \int_{t_0}^{\infty} B \sum_{i=0}^{n-1} s^{(n-i-1)r_i} g_i(s)\, ds.$$

Since $z(t_0) = A$, this corresponds to an appropriate choice of initial conditions for the solutions of (1.1). For such a choice of initial conditions, any solution $y(t)$ of (1.1) may be continued to all $t \geqq t_0$. This follows from the fact that

$$\sum_{k=0}^{n-1} \left| y^{(k)}(t)/t^{n-k-1} \right| \leqq z(t),$$

where $z(t)$ is bounded.

From (2.3), with $k = n-1$, we obtain the equation

$$y^{(n-1)}(t) = c_{n-1} + \int_{t_0}^{t} h(s)\, ds - \int_{t_0}^{t} f(s, y(s), \cdots, y^{(n-1)}(s))\, ds.$$

Noting that $\int_{t_0}^{\infty} h(s)\, ds$ converges and that the integral $\int_{t_0}^{\infty} f(s, y(s), \cdots, y^{(n-1)}(s))\, ds$ is majorized by the integral $\int_{t_0}^{\infty} \sum_{i=0}^{n-1} s^{(n-i-1)r_i} g_i(s)\, ds$, it is clear that $\lim_{t\to\infty} y^{(n-1)}$ is finite, say a_{n-1}. Therefore, by L'Hospital's Rule, $\lim_{t\to\infty} y^{(i)}(t)/t^{n-i-1}$ exists for all $i = 0, 1, \cdots, n-1$. It remains to show these limits may be chosen different from zero. Clearly, it suffices to show a_{n-1} may be chosen distinct from zero.

Select $A > 0$ and t_0 sufficiently large such that

$$A^{1-r} > (r-1) \int_{t_0}^{\infty} B \sum_{i=0}^{n-1} s^{(n-i-1)r_i} g_i(s)\, ds.$$

This condition on A guarantees that $\sum_{i=0}^{n-1} \left| y^{(i)}(t)/t^{n-i-1} \right|$ is bounded for all r. Choose t_1 so that for $0 < \varepsilon < A$,

$$\left| \int_{t_1}^{\infty} h(s)\, ds - \int_{t_1}^{\infty} f(s, y(s), \cdots, y^{(n-1)}(s))\, ds \right| < \varepsilon.$$

The solution $y(t)$ of (1.1) having the initial conditions

$$y(t_1) = y^{(1)}(t_1) = \cdots = y^{(n-2)}(t_1) = 0, \quad y^{(n-1)}(t_1) = A,$$

has the desired type of asymptotic behavior.

REMARK 2.1. For $0 < r \leqq 1$, the proof as given in the above theorem shows all solutions $y(t)$ of (1.1) have the asymptotic behavior $y^{(i)}(t)/t^{n-i-1} \sim a_i$. This follows from the fact that $z(t)$, a solution of (2.5), is always bounded independent of initial conditions.

In the next theorem, a special case of equation (1.1) will be considered. The differential equation which will be considered is

$$y^{(n)} + \sum_{i=0}^{n-1} g_i(t)[y^{(i)}(t)]^{r_i} = h(t). \tag{2.8}$$

Conditions (2.1.1) and (2.1.2) will still be imposed. Further restrictions which will be imposed are

(2.9) $r_i = u_i/w_i$ where u_i and w_i are odd integers, $i = 0, 1, \cdots, n-1$.

(2.10) If $g_i(t)$ is not identically zero, then $\lim_{t\to\infty} t^{p_i} g_i(t) = c_i \neq 0$ for some p_i and all $i = 0, 1, \cdots, n-1$. Furthermore, it is required that the signs of the c_i agree for all $i = 0, 1, \cdots, n-1$ where $g_i(t)$ is not identically zero.

THEOREM 2.2. *Let conditions* (2.1.1), (2.1.2), (2.9), *and* (2.10) *be satisfied. A necessary and sufficient condition for some solution* $y(t)$ *of* (2.8) *to have the asymptotic behavior* $y^{(i)}(t)/t^{n-i-1} \sim a_i \neq 0$ *is that* $p_i > r_i(n-i-1)+1$ *for all* $i = 0, 1, \cdots, n-1$ *where* $g_i(t)$ *is not identically zero.*

Proof. Since (2.8) is a special case of the equation (1.1) of Theorem 2.1, the sufficiency of the condition follows provided $\int^{\infty} t^{(n-i-1)r_i} g_i(t)\, dt < \infty$. This follows immediately from (2.10).

In order to prove the converse, suppose some solution $y(t)$ of (2.8) has the asymptotic behavior $y^{(i)}(t)/t^{n-i-1} \sim a_i \neq 0$, and for some j where $g_j(t)$ is not identically zero, $p_j \leqq r_j(n-j-1)+1$. By hypothesis, r_i is odd, $i = 0,1,\cdots, n-1$; thus if $y(t)$ is a solution of (2.8), $-y(t)$ satisfies equation (2.8) with $h(t)$ replaced by $h_1(t) = -h(t)$. Since $h_1(t)$ is integrable if and only if $h(t)$ is integrable, the nature of the forcing term is similar with respect to the property of being integrable. Thus, we may assume that $y(t)$ is eventually positive and consequently $a_i > 0$, $i = 0, 1, \cdots, n-1$.

Select constants A and C in the following manner: $0 < A < a_i$ for all $i = 0, 1, \cdots, n-1$; $0 < C < |c_i|$ for all $i = 0, 1, \cdots, n-1$ such that $g_i(t)$ is not identically zero. Choose $\varepsilon > 0$ and satisfying the inequalities $0 < A < a_i - \varepsilon$ and $0 < C < |c_i| - \varepsilon$ with i as above. There exist $T_{a_i,\varepsilon}$ and $T_{c_i,\varepsilon}$ such that if $t \geqq T_{a_i,\varepsilon}$ then

$$y^{(i)}(t)/t^{n-i-1} > a_i - \varepsilon > A, \qquad i = 0, 1, \cdots n-1; \tag{2.11}$$

and if $t \geqq T_{c_i,\varepsilon}$ then

$$t^{p_i}|g_i(t)| > |c_i| - \varepsilon > C \tag{2.12}$$

for all $i = 0, 1, \cdots, n-1$ where $g_i(t)$ is not identically zero.

From (2.8), for $t \geqq T = \max(T_{a_i,\varepsilon}, T_{c_i,\varepsilon})$, consider

$$\left| y^{(n)} + \sum_k g_k(t)[y^{(k)}(t)]^{r_k} - h(t) \right| = \left| \sum_j g_j(t)[y^{(j)}(t)]^{r_j} \right|. \tag{2.13}$$

The k-index set consists of those k whose associated p_k have the property that $p_k > r_k(n-k-1)+1$. The j-index set contains those j whose associated p_j satisfy the inequality $p_j \leqq r_j(n-j-1)+1$. By assumption, the j-index set is nonempty so that the right side of (2.13) is different from zero. In the $\sum_j$ sum, the $g_i(t)$ may be assumed positive since they are all positive or all negative. Therefore,

$$\left|\sum_j g_j(t)[y^{(j)}(t)]^{r_j}\right| = \sum_j t^{p_j} g_j(t)\left[\frac{y^{(j)}(t)}{t^{n-j-1}}\right]^{r_j} t^{r_j(n-j-1)-p_j}$$
$$> CA^{r_q} t^{r_q(n-q-1)-p_q} > 0,$$

where q is in the j-index set.

Using the above inequality in (2.13) yields the inequality

$$y^{(n)} + \sum_k g_k(t)[y^{(k)}(t)]^{r_k} - h(t) > 0.$$

Thus, $y^{(n)} + \sum_k g_k(t)[y^{(k)}(t)]^{r_k} - h(t)$ is of one sign for all large t. Suppose it is positive for $t \geqq T$, then

$$y^{(n)} + \sum_k g_k(t)[y^{(k)}(t)]^{r_k} - h(t) > CA^{r_q} t^{r_q(n-q-1)-p_q}. \tag{2.14}$$

Integration of (2.14) gives the inequality

$$\begin{aligned} y^{(n-1)}(t) - y^{(n-1)}(T) + \sum_k \int_T^t g_k(s)[y^{(k)}(s)]^{r_k}\,ds - \int_T^t h(s)\,ds \\ > CA^{r_\cdot} \int_T^t s^{r_q(n-q-1)-p_q} ds. \end{aligned} \tag{2.15}$$

For any k, $r_k(n-k-1)-p_k < -1$, thus

$$\int_T^\infty s^{p_k} g_k(s)\left[\frac{y^{(k)}(s)}{s^{n-k-1}}\right]^{r_k} s^{r_k(n-k-1)-p_k}\,ds$$
$$< M \int_T^\infty s^{r_k(n-k-1)-p_k}\,ds < \infty.$$

Therefore, the integrals on the left in (2.15) are finite as t approaches infinity

Now, if $r_q(n-q-1)-p_q > -1$,

$$\begin{aligned} &\int_T^t s^{r_\cdot(n-q-1)-p}\,ds \\ &= [r_q(n-q-1)-p_q+1]^{-1}[t^{r_q(n-q-1)-p+1} - T^{r_q(n-q-1)-p_q+1}]; \end{aligned} \tag{2.16}$$

if $r_q(n-q-1)-p_q = -1$,

(2.17) $$\int_T^t s^{r_q(n-q-1)-p_q}\, ds = \ln t - \ln T.$$

Substitution of (2.16) or (2.17), the choice determined by the value of $r_q(n-q-1)-p_q$, into (2.15) leads to $\lim_{t\to\infty} y^{(n-1)}(t) = \infty$. However, this contradicts the hypothesis that $\lim_{t\to\infty} y^{(n-1)}(t) = a_{n-1}$.

If the expression $y^{(n)} + \Sigma_k g_k(t)[y^{(k)}(t)]^{r_k} - h(t)$ is negative for $t \geqq T$, an inequality similar to (2.14) is obtained. Proceeding with an analogous argument as above leads to a similar contradiction and the theorem is proved.

REMARK 2.2. A simple example shows that the theorem cannot be proved, even in the linear case, when the signs of the c_i are different. The differential equation

$$y'' - y' + t^{-1}y = 0$$

has $y(t) = t$ as a solution; however, the condition in the theorem is violated.

3. **Forcing terms with primary part $t^m, m > -1$.** In this section, differential equation (1.1) is considered with condition (2.1) imposed.Further hypotheses which will be required are

(3.1) If $g_i(t)$ is not identically zero then $\lim_{t\to\infty} t^{p_i} g_i(t) = c_i \neq 0$ for some p_i.

(3.2) $$h(t) = h_m(t)t^m + R_m(t)$$

where (3.2.1) $m > -1$;

(3.2.2) $h(t)$ is continuous for $t \geqq t_0$;

(3.2.3) $\lim_{t\to\infty} h_m(t) = b_m \neq 0$;

(3.2.4) $R_m(t) = o(t^m)$ as $t \to \infty$.

Again, the inequality in (2.1) need hold only for all large values of t.

THEOREM 3.1. *If conditions* (2.1), (3.1), *and* (3.2) *are satisfied; and for each* $i = 0,1,\cdots,n-1$, $r_i \leqq 1$, $p_i > r_i(m+n-i) - m$, *then all solutions* $y(t)$ *of* (1.1) *have the asymptotic behavior* $y^{(i)}(t)/t^{m+n-i} \sim a_i \neq 0$, $i = 0,1,\cdots,n-1$. *If* $r_i > 1$ *for some* i, $i = 0,1,\cdots,n-1$, *and the above conditions hold, then there exist solutions* $y(t)$ *of* (1.1) *which have the asymptotic behavior* $y^{(i)}(t)/t^{m+n-i} \sim a_i \neq 0$, *for all* $i = 0,1,\cdots,n-1$.

Proof. From (2.3), dividing by t^{m+n-k}, gives the equation

$$\begin{aligned} \frac{y^{(k)}(t)}{t^{m+n-k}} &= [\theta_0 + \theta_1 t + \cdots + \theta_{n-k-1} t^{n-k-1}]t^{k-m-n} \\ &+ t^{k-m-n}\int_{t_0}^t \frac{(t-s)^{n-k-1}}{(n-k-1)!}\, h(s)\, ds \qquad (3.3) \\ &- t^{k-m-n}\int_{t_0}^t \frac{(t-s)^{n-k-1}}{(n-k-1)!} f(s, y(s), \cdots, y^{(n-1)}(s))\, sd. \end{aligned}$$

If $r_i \leqq 1$ for all $i = 0, 1, \cdots, n-1$, then it will be shown that $y^{(k)}(t)/t^{m+n-k}$ is bounded. If $r_i > 1$ for some i, it will be necessary to select proper initial conditions for $y(t)$ in order to show the existence of a bound for $y^{(k)}(t)/t^{m+n-k}$.

The first term on the right in (3.3) is bounded since $m > -1$. L'Hospital's Rule for evaluating indeterminant forms may be applied $n-k$ times to the second term in (3.3); this yields the result

$$\lim_{t\to\infty} \int_{t_0}^{t} \frac{(t-s)^{n-k-1}}{(n-k-1)!} h(s)\, ds / t^{m+n-k} = \frac{b_m}{(m+n-k)\cdots(m+1)} \neq 0.$$

Thus, the second term on the right in (3.3) is bounded for all large t. Let $C_k > 0$ be an upper bound on the absolute value of these two terms in (3.3); suppose this bound is valid for $t \geqq T_k$. Note that the C_k depend upon the initial conditions of the particular solution in question. For $\varepsilon > 0$, by virtue of (3.1), there exists an L_k such that if $t \geqq L_k$ then $t^{p_k} g_k(t) < c_k + \varepsilon = d_k$. From (3.3), with $T = \max_k(t_0, T_k, L_k)$, we have

$$\left|\frac{y^{(k)}(t)}{t^{m+n-k}}\right| \leqq C_k + B_k \int_T^t \sum_{i=0}^{n-1} s^{r_i(m+n-i)-p_i-m-1} \left|\frac{y^{(i)}(s)}{s^{m+n-i}}\right|^{r_i} ds,$$

where $B_k = \sum_{i=0}^{n-1} d_i/(n-k-1)!$, $k = 0, 1, \cdots, n-1$. Therefore,

$$\sum_{k=0}^{n-1} \left|\frac{y^{(k)}(t)}{t^{m+n-k}}\right| \leqq C + B \int_T^t \left[\sum_{i=0}^{n-1} s^{r_i(m+n-i)-p_i-m-i}\right] \left[\sum_{i=0}^{n-1} \left|\frac{y^{(i)}(s)}{s^{m+n-1}}\right|\right]^r ds,$$

with

$$C = \sum_{k=0}^{n-1} C_k,$$

$$B = \sum_{k=0}^{n-1} B_k,$$

and

$$r = \max_i r_i.$$

Again, the above inequality need not hold if the inequality $|y^{(i)}(t)/t^{m+n-i}| < 1$ for all $i = 0, 1, \cdots, n-1$; however, the objective is to show that

$$\sum_{i=0}^{n-1} \left| y^{(i)}(t)/t^{m+n-i} \right|$$

is bounded and $y(t)$ which possess the above property clearly satisfy this condition.

The differential equation associated with the above integral inequality (see Theorem 1.1) is

$$z' = B \sum_{i=0}^{n-1} t^{r_i(m+n-i)-m-1-p_i} z^r.$$

The solutions of the above differential equation are given by

$$z(t) = C \exp \int_T^t B \sum_{i=0}^{n-1} s^{r_i(m+n-i)-m-1-p_i} ds, \qquad r = 1; \tag{3.4}$$

$$z(t)^{1-r} = C^{1-r} + (1-r) \int_T B \sum_{i=0}^{n-1} s^{r_i(m+n-i)-m-1-p_i} ds, \qquad r \neq 1. \tag{3.5}$$

If $r \leqq 1$, then all solutions $z(t)$ as given in (3.4) or (3.5) are bounded since the integrals in both expressions are convergent as t approaches infinity. If $r > 1$, in order to determine boundedness of the solutions as given by (3.5), C and t_0 must be properly chosen in order to guarantee the boundedness of $z(t)$. For such $z(t)$, it follows from Theorem 1.1 that

$$\sum_{i=0}^{n-1} \left| y^{(i)}(t) / t^{m+n-i} \right| \leqq z(t) \leqq D.$$

It remains to show that $\lim_{t\to\infty} y^{(k)}(t)/t^{m+n-k} = a_k \neq 0$ for all solutions $y(t)$ of (1.1) which satisfy the previous inequality. As remarked before, $m > -1$ gives

$$\lim_{t\to\infty} [\theta_0 + \theta_1 t + \cdots + \theta_{n-1} t^{n-1}]/t^{m+n} = 0; \tag{3.6}$$

and

$$\lim_{t\to\infty} \int_T^t \frac{(t-s)^{n-k-1}}{(n-k-1)!} h(s)\, ds / t^{m+n-k} = \frac{b_m}{(m+n-k)\cdots(m+1)} \neq 0. \tag{3.7}$$

To evaluate the limit of the third term in (3.3), consider

$$\begin{aligned}
&\left| t^{k-m-n} \int_T^t \frac{(t-s)^{n-k-1}}{(n-k-1)!} f(s, y(s), \cdots, y^{(n-1)}(s))\, ds \right| \\
&\quad \leqq \frac{t^{-m-1}}{(n-k-1)!} \int_T^t \sum_{i=0}^{n-1} s^{p_i} g_i(s) \left| \frac{y^{(i)}(s)}{s^{m+n-i}} \right|^{r_i} s^{r_i(m+n-i)-p_i} ds \\
&\quad \leqq \sum_k d_k D_k^{r_k} t^{-m-1} \left[\frac{t^{r_k(m+n-k)-p_k+1} - T^{r_k(m+n-k)-p_k+1}}{r_k(m+n-k)-p_k+1} \right] \\
&\qquad + \sum_j d_j D_j^{r_j} t^{-m-1} [\ln t - \ln T],
\end{aligned}$$

where the Σ_k sum has as an index set those k such that $p_k \neq r_k(m+n-k)+1$; and the Σ_j summation is summed over those j where $p_j = r_j(m+n-j)+1$. Using the hypotheses that $p_k > r_k(m+n-k) - m$ in the Σ_k term and $m > -1$ in the Σ_j sum, we obtain

$$\lim_{t\to\infty} t^{k-m-n} \int_T^t \frac{(t-s)^{n-k-1}}{(n-k-1)!} f(s, y(s), \cdots, y^{(n-1)}(s))\, ds = 0. \tag{3.8}$$

Combining the results of (3.6), (3.7), and (3.8) in (3.3), yields

$$\lim_{t\to\infty} y^{(k)}(t)/t^{m+n-k} = a_k \neq 0.$$

This concludes the proof of the theorem.

In the following theorem a partial converse of Theorem 3.1 is given for the differential equation (2.8).

THEOREM 3.2. *Let the hypotheses* (2.1.1), (2.1.2), (2.9), (2.10), *and* (3.2) *be satisfied. If any solution* $y(t)$ *of* (2.8) *has the asymptotic behavior*

$$y^{(i)}(t)/t^{m+n-i} \sim a_i \neq 0, \qquad i = 0, 1, \cdots, n-1,$$

then $p_i \geqq r_i(m+n-i) - m$ *for all* $i = 0, 1, \cdots, n-1$.

Proof. Suppose $p_j < r_j(m+n-j) - m$ for some j where $j = 0, 1, \cdots, n-1$. Write (2.8) in the form

$$y^{(n)}(t) + \sum g_k(t)[y^{(k)}(t)]^{r_k} - h(t) = -\sum_j g_j(t)[y^{(j)}(t)]^{r_j}. \tag{3.9}$$

In (3.9), the Σ_k (Σ_j) sum ranges over those k (j) whose associated p_k (p_j) satisfy the inequality $p_k \geqq r_k(m+n-k) - m$ ($p_j < r_j(m+n-j) - m$). By hypothesis, the j-index set is nonempty.

Dividing (3.9) by t^m and taking the limit as t approaches infinity yields the equation

$$\lim_{t\to\infty} y^{(n)}/t^m + \sum_k c_k a_k^{r_k} \rho_k - b_m = -\sum_j c_j a_j^{r_j} \lim_{t\to\infty} t^{r_j(m+n-j)-m-p_j}$$

where

$$\rho_k = \begin{cases} 0 & \text{if } p_k > r_k(m+n-k) - m, \\ 1 & \text{if } p_k = r_k(m+n-k) - m. \end{cases}$$

Since $p_j < r_j(m+n-j) - m$, and $c_j a_j^{r_j}$ are all of the same sign, $\lim_{t\to\infty} y^{(n)}/t^m = \pm\infty$. This contradicts

$$a_{n-1} = \lim_{t\to\infty} y^{(n-1)}/t^{m+1} = \lim_{t\to\infty} y^{(n)}/(m+1)t^m.$$

REMARK 3.1. The previous theorem may be improved slightly. Suppose for some j's, $p_j = r_j(m+n-j) - m$, where j is in the set $0, 1, \cdots, n-1$. From the differential equation (2.8),

$$\frac{y^{(n)}}{t^m} = \frac{h(t)}{t^m} - \sum_k t^{p_k} g_k(t) \left[\frac{y^{(k)}(t)}{t^{m+n-k}}\right]^{r_k} t^{r_k(m+n-k)-p_k-m}$$
$$- \sum_j t^{p_j} g_j(t) \left[\frac{y^{(j)}(t)}{t^{m+n-j}}\right]^{r_j} t^{r_j(m+n-j)-p_j-m},$$

where the k-index set consists of those k such that $p_k > r_k(m+n-k) - m$ and the j-index set consists of those j such that $p_j = r_j(m+n-j) - m$. Therefore,

$$\lim_{t\to\infty} y^{(n)}/t^m = b_m - \sum_j c_j a_j^{r_j},$$

and necessarily

$$a_{n-1} = \lim_{t\to\infty} y^{(n-1)}/t^{m+1} = \left(b_m - \sum_j c_j a_j^{r_j}\right)/(m+1). \tag{3.10}$$

But,

$$a_0 = \lim_{t\to\infty} y(t)/t^{m+n} = \lim_{t\to\infty} y^{(k)}/(m+n)\cdots(m+n-k+1)t^{m+n-k}$$

therefore,

$$a_k = (m+n)\cdots(m+n-k+1)a_0, \qquad k = 0,1,\cdots,n-1.$$

Using this result in (3.10), we obtain the equation

$$\begin{aligned}(m+n)(m+n-1)\cdots(m+1)^2 a_0 \\ = b_m - \sum_j c_j[(m+n)\cdots(m+n-j+1)a_0]^{r_j}.\end{aligned} \tag{3.11}$$

Equation (3.11) improves Theorem 3.2 for the case where $p_i = r_i(m+n-i) - m$ for some i, $i = 0,1,\cdots,n-1$, by placing an additional restriction on a_0. For if a_0 is not a root of (3.11), then a contradiction is obtained.

REMARK 3.2. The example given in this remark shows that Remark 3.1 yields the best possible result subject to our hypotheses. Consider the differential equation

$$y^{(n)} - ct^{m-mn-rn}y^r = b_m t^m, \tag{3.12}$$

where $c \neq 0$ and $b_m \neq 0$ are determined by the equation

$$(m+n)\cdots(m+1) - c = b_m.$$

Note $y = t^{m+n}$ is a solution of (3.12) having the desired asymptotic property.

REMARK 3.3. An example which shows that the hypothesis concerning the signs of the c_i is necessary is

$$y'' - y' + 3t^{-1}y = 6t.$$

$y = t^3$ is a solution having the asymptotic property $y(t)/t^3 \sim 1$; however, $p_i \geqq r_i(m+n-i) - m$ is false for $i = 0,1$.

REMARK 3.4. The motivation for the previous theorems is due to the reasoning that if the coefficients of y, $y^{(1)},\cdots,y^{(n-1)}$, approach zero sufficiently fast then the solutions of (2.1) should behave like the solutions of the equation $y^n = h(t)$. It is interesting to observe that if the nonlinearity, as determined by the r_i's, is sufficiently small it is actually possible for the coefficients to have limit infinity rather than zero.

REMARK 3.5. Proceeding under the hypotheses (2.1.1), (2.1.2), (2.9), (2.10), and (3.2), we investigate other possible types of asymptotic behavior of solutions of (2.8). The kind of behavior discussed here is $y^{(i)}(t)/t^{\rho-i} \sim a_i \neq 0$, with $\rho < m+n$, $\rho \neq 0,1,\cdots,n-1$, $i = 0,1,\cdots,n-1$. First, we establish the following result. If $p_i \leqq r_i(\rho - i) - \rho + n$ then no solution $y(t)$ of (2.8) has the behavior

$$y^{(i)}(t)/t^{\rho-i} \sim \ a_i \neq 0.$$

Since $\rho \neq 0,1,\cdots,n-1$, then

$$a_{n-1} = \lim_{t\to\infty} y^{(n-1)}(t)/t^{\rho-n+1} = \lim_{t\to\infty} y^{(n)}(t)/(\rho - n + 1)t^{\rho-n}$$

provided the latter limit exists. Suppose $y(t)$ is any solution of (2.8) which has the asymptotic property $y^{(i)}(t)/t^{\rho-i} \sim a_i \neq 0$. From (2.8),

$$\frac{y^{(n)}(t)}{t^{\rho-n}} = \frac{h(t)}{t^{\rho-n}} - \sum_{i=0}^{n-1} t^{p_i} g_i(t) \left[\frac{y^{(i)}(t)}{t^{\rho-i}}\right]^{r_i} t^{r_i(\rho-i)-p_i-\rho+n}.$$

Therefore, $\lim_{t\to\infty} y^{(n)}(t)/t^{\rho-n} = \pm\infty$. This is a contradiction and thus, if any solution $y(t)$ of (2.8) has the asymptotic property $y^{(i)}(t)/t^{\rho-i} \sim a_i \neq 0$, then for all $i = 0,1,\cdots,n-1$, $p_i > r_i(\rho - i) - \rho + n$.

Consider (2.8) where the differential equation is linear, that is, $r_i = 1$ for all $i = 0,1,\cdots,n-1$. Suppose some solution $y(t)$ of (2.8) has the asymptotic behavior $y^{(i)}(t)/t^{\rho-i} \sim a_i \neq 0$, $\rho < m + n$, $i = 0,1,\cdots,n-1$. From Theorem 3.1, p_i satisfies the inequality

$$p_i \leqq n - i, \qquad i = 0,1,\cdots,n-1.$$

From the above result,

$$p_i > n - i, \qquad i = 0,1,\cdots,n-1.$$

Since these two statements are incompatible we have shown that for no p_i can the linear equation possess solutions which have the asymptotic behavior

$$y^{(i)}(t)/t^{\rho-i} \sim a_i \neq 0,$$

where $\rho < m + n$.

In the next remark investigation of the asymptotic behavior of solutions of (2.8), under the hypotheses of Remark 3.5, is continued. In particular, the behavior

$$y^{(i)}(t)/t^{\rho-i} \sim a_i \neq 0, \qquad \rho = 0, 1, \cdots, n-1, \ i = 0,1,\cdots,n-1,$$

is discussed.

REMARK 3.6. If some solution $y(t)$ of (2.8) has the asymptotic behavior $y^{(i)}(t)/t^{\rho-i} \sim a_i \neq 0$ then for all $i = 0,1,\cdots,n-1$, $p_i \geqq r_i(\rho - i) - m$.

Proof. From the differential equation (2.8),

$$\frac{y^{(n)}(t)}{t^{m}} = \frac{h(t)}{t^{m}} - \sum_{i=0}^{n-1} t^{p_i} g_i(t) \left[\frac{y^{(i)}(t)}{t^{\rho-i}}\right]^{r_i} t^{r_i(\rho-i)-p_i-m}.$$

Taking the limit as t approaches infinity yields the equation

$$\lim_{t\to\infty} y^{(n)}(t)/t^m = b_m - \sum_{i=0}^{n-1} c_i a_i^{r_i} \lim_{t\to\infty} t^{r_i(\rho-i)-p_i-m}.$$

If $p_j < r_j(\rho-j)-m$ for some j, $\lim_{t\to\infty} t^{r_j(\rho-j)-p_j-m} = \infty$. This implies $\lim_{t\to\infty} y^{(n)}(t)/t^m = \pm\infty$, which yields a contradiction as previously observed. This completes the proof of the remark.

We continue with the investigation of asymptotic behavior of the form $y^{(i)}(t)/t^{\rho-i} \sim a_i \neq 0$; $\rho, i = 0,1,\cdots,n-1$. Suppose $y(t)$ is a solution of (2.8) having the above given asymptotic behavior and $p_i \geqq r_i(\rho-i)-m$, $i=0,1,\cdots,n-1$. From (2.8), we obtain

$$\frac{y^{(n)}}{t^m} + \sum_j t^{p_j} g_j(t) \left[\frac{y^{(j)}(t)}{t^{\rho-j}}\right]^{r_j} t^{r_j(\rho-j)-p_j-m}$$
$$= \frac{h(t)}{t^m} - \sum_k t^{p_k} g_k(t) \left[\frac{y^{(k)}(t)}{t^{\rho-k}}\right]^{r_k} t^{r_k(\rho-k)-p_k-m},$$

where the j-index (k-index) set consists of those j (k) whose associated $p_j > r_j(\rho-j)-m$ ($p_k = r_k(\rho-k)-m$). Taking the limit as t approaches infinity, yields the equation

$$\lim_{t\to\infty} y^{(n)}/t^m = b_m - \sum_k c_k a^{r_k} = 0.$$

Therefore, it is necessary that the equation

$$b_m - \sum_k c_k a_k^{r_k} = 0$$

be satisfied in order that $y^{(i)}(t)/t^{\rho-i} \sim a_i \neq 0$.

For a special case of equation (2.8),

$$y^{(n)} + g(t)y^r = h(t), \tag{3.13}$$

the above remark may be improved. We assume that $\lim_{t\to\infty} t^p g(y) = c \neq 0$ for some p; $h(t)$ is as given by (3.2); $r > 0$ and odd in the sense previously given.

REMARK 3.7. If some solution $y(t)$ of (3.13) has the asymptotic behavior $y(t)/t^\rho \sim a \neq 0$, $\rho = 0,1,\cdots,n-1$, then $p = \rho r - m$. When $p = \rho r - m$ the asymptotic behavior can occur only if $b_m - ca^r = 0$.

Proof. By Remark 3.6, $p \geqq \rho r - m$. If $p > \rho r - m$, then $\lim_{t\to\infty} t^{\rho r - m - p} = 0$. As in the previous remark,

$$0 = \lim_{t\to\infty} y^{(n)}(t)/t^m = b_m \neq 0.$$

Again, we have reached a contradiction and the first statement of the remark is proved.

Suppose that $p = \rho r - m$ and $b_m - ca^r \neq 0$. From the differential equation (3.13),

$$0 = \lim_{t\to\infty} y^{(n)}/t^m = b_m - ca^r \neq 0.$$

This completes the proof of the remark.

4. **Forcing terms with primary part t^{-1}.** Asymptotic properties of the solutions of (1.1) with the forcing term essentially behaving like t^{-1} will be considered in this section. The conditions (2.1) and (3.1) are assumed throughout the section. The explicit hypothesis concerning the forcing term $h(t)$ is given by

$$h(t) = h_{-1}(t)t^{-1} + R_{-1}(t), \tag{4.1}$$

where (4.1.1) $h(t)$ is continuous for $t \geqq t_0$;

(4.1.2) $\lim_{t\to\infty} h_{-1}(t) = b_{-1} \neq 0$;

(4.1.3) $R_{-1}(t) = o(t^{-1})$ as t approaches infinity.

THEOREM 4.1 *If* (2.1), (3.1), *and* (4.1) *are valid,* $\int^{\infty} t^{r_i(n-1)-p_i}(\ln t)^{r_i-1}dt < \infty$, *and* $r_i \leqq 1$ *for* $i = 0, 1, \cdots, n-1$, *then all solutions of* (1.1) *have the asymptotic behavior*

$$y^{(i)}(t)/t^{n-i-1}\ln t \sim a_i \neq 0, \qquad i = 0, 1, \cdots, n-1.$$

If $r_i > 1$ *for some* i *and the above conditions hold, then there exist solutions* $y(t)$ *of* (1.1) *which have the asymptotic behavior* $y^{(i)}(t)/t^{n-i-1}\ln t \sim a_i \neq 0$.

Proof. Consider (2.3) divided by $t^{n-k-1}\ln t$,

$$\begin{aligned}\frac{y^{(k)}(t)}{t^{n-k-1}\ln t} &= [\theta_0 + \theta_1 t + \cdots + \theta_{n-k-1}t^{n-k-1}][t^{n-k-1}\ln t]^{-1} \\ &\quad + [t^{n-k-1}\ln t]^{-1}\int_{t_0}^{t}\frac{(t-s)^{n-k-1}}{(n-k-1)!}h(s)\,ds \\ &\quad - [t^{n-k-1}\ln t]^{-1}\int_{t_0}^{t}\frac{(t-s)^{n-k-1}}{(n-k-1)!}f(s, y(s), \cdots, y^{(n-1)}(s))\,ds.\end{aligned} \tag{4.2}$$

Clearly, the first term in (4.2) is bounded for all large t. Application of L'Hospital's Rule to the second term on the right in (4.2) gives the equation

$$\lim_{t\to\infty}[t^{n-k-1}\ln t]^{-1}\int_{t_0}^{t}\frac{(t-s)^{n-k-1}}{(n-k-1)!}h(s)\,ds = -\frac{b_{-1}}{(n-k-1)!}. \tag{4.3}$$

Proceeding as in the proof of Theorem 3.1, let $C_k > 0$ be an upper bound on the absolute value of the first two terms valid for $t \geqq T_k$. For $\varepsilon > 0$, there exists an L_k such that if $t \geqq L_k$ then $t^{p_k}g_k(t) < c_k + \varepsilon = d_k$. From (4.2), we obtain

$$\begin{aligned}&\sum_{k=0}^{n-1}\left|[y^{(k)}(t)/(t^{n-k-1}\ln t)]\right| \\ &\qquad \leqq C + B\int_T^t\left(\sum_{i=0}^{n-1} s^{r_i(n-i-1)-p_i}(\ln s)^{r_i-1}\right)\left(\sum_{i=0}^{n-1}\left|\frac{y^{(i)}(s)}{s^{n-i-1}\ln s}\right|\right)^r,\end{aligned}$$

where

$$C = \sum_{k=0}^{n-1} C_k,$$

$$B = \sum_{k=0}^{n-1} \sum_{i=0}^{n-1} d_i/(n-k-1)!,$$

$$r = \max_i r_i.$$

The differential equation associated with the above integral inequality is

$$z' = B \sum_{i=0}^{n-1} t^{r_i(n-1-i)-p_i}(\ln t)^{r_i-1} z^r.$$

The remainder of the argument proceeds as in Theorem 3.1, and for this reason will be omitted.

THEOREM 4.2. *Let conditions* (2.1.1), (2.1.2), (2.9), (2.10), *and* (4.1) *be satisfied. A necessary and sufficient condition for some solution of* (2.8) *to have the asymptotic behavior* $y^{(i)}(t)/t^{n-i-1}\ln t \sim a_i \neq 0$, $i = 0, 1, \cdots, n-1$, *is that*

$$p_i > r_i(n-i-1)+1, \qquad i = 0, 1, \cdots, n-1.$$

Proof. In order to show the condition is sufficient, let $p_i = r_i(n-i-1)+1+\varepsilon_i$, where ε_i is positive for all $i = 0, 1, \cdots, n-1$. Consider

$$\int^\infty t^{r_i(n-i-1)-p_i}(\ln t)^{r_i-1} dt < \int^\infty t^{r_i(n-i-1)-p_i-\varepsilon_i/2} dt < \infty .$$

By Theorem 4.1, there exists a solution $y(t)$ of (2.8) such that

$$y^{(i)}(t)/t^{n-i-1}\ln t \sim a_i \neq 0.$$

In order to prove the necessity of the condition, consider

$$ty^{(n)} + t \sum_k g_k(t)[y^{(k)}(t)]^{r_k} = th(t) - t \sum_j g_j(t)[y^{(j)}(t)]^r . \tag{4.8}$$

The k-index (j – index) set is composed of those k (j) whose associated p_k (p_j) satisfy the inequality $p_k > r_k(n-k-1)+1$ ($p_j \leqq r_j(n-j-1)+1$).

Notice that

$$\lim_{t\to\infty} y^{(n-1)}(t)/\ln t = \lim_{t\to\infty} ty^{(n)}(t) = a_{n-1}$$

provided the latter limit exists. Rewriting (4.8) leads to the equation

$$ty^{(n)} + \sum_k t^{p_k} g_k(t) \left[\frac{y^{(k)}(t)}{\ln t} t^{k+1-n}\right]^{r_k} (\ln t)^{r_k} t^{r_k(n-k-1)-p_k+1}$$

$$= th(t) - \sum_j t^{p_j} g_j(t) \left[\frac{y^{(j)}(t)}{\ln t} t^{j+1-n}\right]^{r_j} (\ln t)^{r_j} t^{r_j(n-j-1)-p_j+1} .$$

Taking the limit as t approaches infinity gives $\lim_{t\to\infty} ty^{(n)}(t) = \pm\infty$. This is a contradiction and completes the proof of the theorem.

REMARK 4.1. If $r \leqq 1$, the condition of Theorem 4.2 is necessary and sufficient for all solutions to have the asymptotic behavior $y^{(i)}(t)/t^{n-i-1}\ln t \sim a_i \neq 0$. This follows immediately from the first sentence in the statement of Theorem 4.1 and the result of Theorem 4.2.

A simple example in the linear case may be given to show that the signs of the c_i must all be the same. Asymptotic behavior of the type $y^{(i)}(t)/t^{\rho-i}\ln t \sim a_i \neq 0$, $i = 0, 1, \cdots, n-1$, will now be considered. The r_i will be restricted such that $0 < r_i \leqq 1$, $i = 0, 1, \cdots, n-1$. From Remark 4.1, the only p_i which need be considered are the ones which satisfy the inequality $p_i \leqq r_i(n-i-1)+1$.

COROLLARY 4.1. *Let* (2.1.1), (2.1.2), (2.9), (2.10), *and* (4.1) *be satisfied. If* $\rho \geqq n-1$, *then no solution* $y(t)$ *of* (2.8) *has the asymptotic behavior*

$$y^{(i)}(t)/t^{\rho-i}\ln t \sim a_i \neq 0.$$

Proof. Suppose $p_i \leqq r_j(\rho - j) + 1$ for some j and some solution $y(t)$ of (2.8) has the asymptotic behavior $y^{(i)}(t)/t^{\rho-i}\ln t \sim a_i \neq 0$. Write equation (2.8) as

$$ty^{(n)} + t\sum_k g_k(t)[y^{(k)}(t)]^{r_k} = th(t) - t\sum_j g_j(t)[y^{(j)}(t)]^{r_j}, \tag{4.9}$$

where the k-index (j-index) set consists of those k (j) whose associated p_k (p_j) satisfy the inequality $p_k > r_k(\rho - k)+1$ ($p_j \leqq r_j(\rho - j) + 1$). Taking the limit as t approaches infinity in (4.9) gives

$$\lim_{t\to\infty} ty^{(n)}(t) = b_{-1} - \sum_j c_j a_j^{r_j} \lim_{t\to\infty} t^{r_j(\rho-j)-p_j+1}(\ln t)^{r_j}. \tag{4.10}$$

Since $p_j \leqq r_j(\rho - j) + 1$, the limit on the right side of (4.10) is $\pm\infty$. This is a contradiction and therefore $p_i > r_i(\rho - i) + 1$ for all $i = 0, 1, \cdots, n-1$. But, since $\rho \geqq n-1$,

$$p_i > r_i(\rho - i) + 1 \geqq r_i(n-1-i) + 1.$$

This result contradicts Remark 4.1 and the corollary is proved.

5. **Forcing terms with primary part $t^m e^{bt}$.** In this section forcing functions having primary part $t^m e^{bt}$ will be considered. Specifically, the forcing term is to satisfy the following properties.

$$h(t) = H(t)t^m e^{bt} + R(t), \tag{5.1}$$

where $h(t)$ is continuous for $t \geqq t_0$;

$$\lim_{t\to\infty} H(t) = A \neq 0;$$

and

$$R(t) = o(t^m e^{bt}) \text{ as } t \to \infty.$$

If $b < 0$, $h(t)$ is integrable and the forcing function is included in the class of forcing terms of §2. If $b = 0$, the primary part of the forcing function is a power of t. §§2, 3, and 4 give results for forcing functions of this type. The purpose of this section is to consider forcing terms with primary part $t^m e^{bt}$ where $b > 0$.

THEOREM 5.1. *Let conditions* (2.1), (3.1), *and* (5.1) *be satisfied*; $b > 0$, $0 < r_i \leqq 1$ *for all* $i = 0, 1, \cdots, n-1$; *furthermore, if* $r_i = 1$, *it is required that* $n < p_i$. *Under these hypotheses, all solutions* $y(t)$ *of* (1.1) *have the asymptotic behavior* $y^{(i)}(t)/t^m e^{bt} \sim a_i \neq 0$, $i = 0, 1, \cdots, n-1$.

Proof. The argument proceeds as in the previous Theorems 2.1, 3.1, and 4.1; for this reason it will only be sketched here. The form of equation (1.1) which is used is

$$\begin{aligned} \frac{y^{(k)}(t)}{t^m e^{bt}} &= \frac{\theta_0 + \theta_1 t + \cdots + \theta_{n-k-1} t^{n-k-1}}{t^m e^{bt}} \\ &\quad + e^{-bt} t^{-m} \int_{t_0}^{t} \frac{(t-s)^{n-k-1}}{(n-k-1)!} h(s)\, ds \\ &\quad - e^{-bt} t^{-m} \int_{t_0}^{t} \frac{(t-s)^{n-k-1}}{(n-k-1)!} f(s, y(s), \cdots, y^{(n-1)}(s))\, ds. \end{aligned} \tag{5.2}$$

The boundedness of each term will be investigated. It is clear that the first two terms on the right in (5.2) are bounded. Proceeding as before with T as previously chosen and with the additional requirement that the function

$$X(t) = [t^{m-n+k+1} e^{bt}]^{-1}$$

is decreasing for $t \geqq T$, we obtain

$$\left| \frac{y^{(k)}(t)}{t^m e^{bt}} \right| \leqq C_k + B_k \int_T^t \sum_{i=0}^{n-1} s^{n-m-k-1-p_i+r_i m} e^{(r_i - 1)bs} \left| \frac{y^{(i)}(s)}{s^m e^{bs}} \right|^{r_i} ds.$$

If the sum $\sum_{i=0}^{n-1} | y^{(i)}(t)/t^m e^{bt}| \leqq 1$ for all large t, then clearly $\sum_{i=0}^{n-1} | y^{(i)}(t)/t^m e^{bt}|$ is bounded. If $\sum_{i=0}^{n-1} | y^{(i)}(t)/t^m e^{bt}|^{r_i} > 1$, then $\sum_{i=0}^{n-1} | y^{(i)}(t)/t^m e^{bt}|^{r_i} \leqq \sum_{i=0}^{n-1} | y^{(i)}(t)/t^m e^{bt}|$. Therefore,

$$\begin{aligned} &\sum_{k=0}^{n-1} \left| \frac{y^{(k)}(t)}{t^m e^{bt}} \right| \\ &\quad \leqq \sum_{k=0}^{n-1} C_k + \sum_{k=0}^{n-1} b_k \int_T^t \sum_{i=0}^{n-1} s^{n-m-k-1-p_i+r_i m} e^{(r_i-1)bs} \sum_{i=0}^{n-1} \left| \frac{y^{(i)}(s)}{s^m e^{bs}} \right| ds. \end{aligned}$$

Applying Theorem 1.1, we obtain the inequality

$$\sum_{k=0}^{n-1} \left| \frac{y^{(k)}(t)}{t^m e^{bt}} \right| \leqq C \exp \int_T^t \sum_{k=0}^{n-1} B \sum_{i=0}^{n-1} s^{n-k-1-m-p_i+r_i m} e^{(r_i-1)bs}\, ds.$$

By hypothesis, the above integrals converge and thus, $\sum_{i=0}^{n-1} |y^{(k)}(t)/t^m e^{bt}|$ is bounded. Proceeding as in the previous proofs, the result

$$\lim_{t\to\infty} y^{(i)}(t)/t^m e^{bt} = a_i \neq 0, \qquad i = 0, 1, \cdots, n-1,$$

may be obtained.

REMARK 5.1. This theorem or theorems determining existence of solutions having the above type of asymptotic behavior may not be demonstrated for nonlinearity involving higher powers. Consider the following example:

$$y^{(n)} + t^p y^r = e^t,$$

where p is arbitrary and $r > 1$. Suppose some solution $y(t)$ has the asymptotic behavior $y(t)/e^t \sim a \neq 0$. If $\lim_{t\to\infty} y^{(n)}/e^t$ exists, necessarily, $\lim_{t\to\infty} y^{(n)}/e^t = a$. Thus, from the equation,

$$\frac{y^{(n)}}{e^t} + t^p \frac{y^r}{e^t} e^{(r-1)t} = 1;$$

we obtain $\lim_{t\to\infty} y^{(n)} e^{-t} = \pm\infty$. This is a contradiction and the desired asymptotic behavior is impossible.

BIBLIOGRAPHY

1. L. Cesari, *Asymptotic behavior and stability problems in ordinary differential equations*, 2nd ed., Academic Press, New York, 1963.

2. J. K. Hale and N. Onuchic, *On the asymptotic behavior of solutions of a class of differential equations*, Contributions to Differential Equations **2** (1963), 61–75.

3. T. G. Hallam, *Asymptotic behavior and stability problems in ordinary differential equations* Doctoral Dissertation, University of Missouri, Columbia, Mo., 1965.

4. R. A. Moore and Z. Nehari, *Nonoscillation theorems for a class of nonlinear differential equations*, Trans. Amer. Math. Soc. **93** (1959), 30–52.

5. B. Viswanatham, *A generalization of Bellman's lemma*, Proc. Amer. Math. Soc. **14** (1963), 15–18.

6. P. Waltman, *On the asymptotic behavior of a nonlinear equation*, Proc. Amer. Math. Soc. **15** (1964), 918–923.

7. P. Waltman, *On the asymptotic behavior of solutions of an nth order equation*, Monatsch. Math. **69** (1965), 427–430.

FLORIDA STATE UNIVERSITY,
TALLAHASSEE, FLORIDA