ON COMPLEX QUADRATIC FIELDS WITH CLASS NUMBER EQUAL TO ONE⁽¹⁾

BY
HAROLD STARK

Let $R(\sqrt{-p})$ be a quadratic extension of the rationals, where p is a positive square free integer. For nine values of p , namely 1, 2, 3, 7, 11, 19, 43, 67, 163, the integers of $R(\sqrt{-p})$ form a unique factorization domain. Heilbronn and Linfoot [1] have shown that there is at most one more such value of p , and Lehmer [2] has shown that p must be a prime greater than $5 \cdot 10^9$. In the present paper we verify and extend the lower bound of $5 \cdot 10^9$ for p . The result is

THEOREM 1. *If the ring of integers of $R(\sqrt{-p})$ (p square free) forms a unique factorization domain, and $p > 10^4$, then $p > \exp(2.2 \cdot 10^7)$.*

It will be assumed throughout that p is an integer satisfying the hypothesis of Theorem 1. We start with a formula equivalent to that given by Lemma 2 of [1].

$$(1) \quad \zeta(s)L(s) - \zeta(2s) = 2^{2s-1} p^{(1/2)-s} \zeta(2s-1) \sqrt{\pi} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} + h(s),$$

valid for $\sigma > \frac{1}{2}$, where $\zeta(s)$ is the Riemann zeta function, $L(s)$ is the Dirichlet L -series formed with the quadratic character (mod p), and

$$h(s) = \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} \left(x - [x] - \frac{1}{2} \right) \frac{d}{dx} \left\{ \left[\left(x + \frac{j}{2} \right)^2 + \frac{pj^2}{4} \right]^{-s} \right\} dx.$$

Let $x + (j/2) = u(j\sqrt{p})/2$ give a change of variable from x to u and integrate by parts $2m-1$ times; we get, for $m = 1, 2, \dots$,

$$(2) \quad h(s) = - \sum_{j=1}^{\infty} \left(\frac{2}{j\sqrt{p}} \right)^{2m+2s-1} \int_{-\infty}^{\infty} \frac{B_{2m}(x - [x])}{(2m)!} \frac{d^{(2m)}}{du^{(2m)}} \{(u^2 + 1)^{-s}\} du,$$

where $B_k(x)$ is the k th Bernoulli polynomial ($B_0(x) = 1$, $B_1(x) = x - \frac{1}{2}$, $B_2(x) = x^2 - x + \frac{1}{6}$, ...) and the series converges for $\text{Res} > 1 - m$.

It is well known ([3, p. 245], Jordan's $\varphi_n(x) = B_n(x)/n!$) that for $0 \leq x \leq 1$, $k > 1$,

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$$(3) \quad \left| \frac{B_k(x)}{(k)!} \right| \leq \frac{2}{(2\pi)^k} \zeta(k),$$

so that to estimate $h(s)$, we need only estimate $d^{(2m)}/du^{(2m)}\{(u^2 + 1)^{-s}\}$. Using the fact that

$$\frac{d^2}{du^2} \{(u^2 + 1)^{-s}\} = 2s(2s + 1)(u^2 + 1)^{-s-1} - 2s(2s + 2)(u^2 + 1)^{-s-2},$$

we see inductively that we can write

$$(4) \quad \frac{d^{(2m)}}{du^{(2m)}} \{(u^2 + 1)^{-s}\} = \sum_{k=0}^m c_{mk}(s)(u^2 + 1)^{-s-m-k}.$$

Again, by induction we have

$$(5) \quad |c_{mk}(s)| \leq \binom{m}{k} \prod_{j=0}^{2m-1} (2|s| + 2j).$$

Thus

$$\begin{aligned} \left| \frac{d^{(2m)}}{du^{(2m)}} \{(u^2 + 1)^{-s}\} \right| &\leq \sum_{k=0}^m \binom{m}{k} (u^2 + 1)^{-\sigma-m-k} \prod_{j=0}^{2m-1} (2|s| + 2j) \\ (6) \quad &\leq 2^m (u^2 + 1)^{-\sigma-m} \prod_{j=0}^{m-1} (2|s| + 2j)(2|s| \\ &\quad + 2(2m - 1 - j)) \\ &\leq 2^m (u^2 + 1)^{-\sigma-m} (2|s| + 2m - 1)^{2m}. \end{aligned}$$

In view of (2), (3) and (6), we have for $\sigma \geq \frac{1}{2}$ and $m \geq 1$:

$$\begin{aligned} |h(s)| &\leq \sum_{y=1}^{\infty} \left(\frac{2}{y\sqrt{p}} \right)^{2m+2\sigma-1} \cdot \frac{2}{(2\pi)^{2m}} \zeta(2m) \cdot 2^m (2|s| + 2m - 1)^{2m} \int_{-\infty}^{\infty} (u^2 + 1)^{-1} du \\ &\leq 2\pi \zeta^2(2m) \left(\frac{4|s| + 4m - 2}{\pi\sqrt{(2p)}} \right)^{2m} \\ (7) \quad &< 2\pi \left(\frac{2m}{2m - 1} \right)^2 \left(\frac{4|s| + 4m - 2}{\pi\sqrt{(2p)}} \right)^{2m}. \end{aligned}$$

Letting $m = 30$, we see that if $|s| \leq 22$ and $\sigma \geq \frac{1}{2}$ (and of course $p > 10,000$), then

$$|h(s)| < 10^{-19}.$$

Let θ denote a number, complex or real, not necessarily the same each time it occurs, which satisfies $|\theta| \leq 1$. We find that for $|s| < 22$ and $\sigma \geq \frac{1}{2}$, (1) becomes

$$(8) \quad \zeta(s)L(s) - \zeta(2s) = \zeta(2-2s) \frac{\Gamma(1-s)}{\Gamma(s)} \left(\frac{\sqrt{p}}{2\pi} \right)^{1-2s} + 10^{-19}\theta,$$

where the functional equation for $\zeta(s)$ was used to obtain the first term on the right.

Let

$$(9) \quad s_n = \frac{1}{2} + i\gamma_n$$

denote the n th zero of $\zeta(s)$ above the real axis. It is known that $\gamma_1 \approx 14$ and $\gamma_2 \approx 21$ (see Appendix); in particular, $|s_1| < |s_2| < 22$ and thus

$$(10) \quad \zeta(2s_n) = -\zeta(2-2s_n) \frac{\Gamma(1-s_n)}{\Gamma(s_n)} \left(\frac{p}{4\pi^2} \right)^{-i\gamma_n} + 10^{-19}\theta, \quad (n=1,2).$$

Multiplying both sides by $(p/4\pi^2)^{i\gamma_n}(1/\zeta(2s_n))$ and using the fact that $|\zeta(2s_n)| > \frac{1}{2}$ for $n=1,2$ (see Appendix), we get

$$(11) \quad \begin{aligned} \left(\frac{p}{4\pi^2} \right)^{i\gamma_n} &= -\frac{\zeta(2-2s_n)}{\zeta(2s_n)} \frac{\Gamma(1-s_n)}{\Gamma(s_n)} + 2 \cdot 10^{-19}\theta \\ &= -\frac{\zeta(1-2i\gamma_n)\Gamma\left(\frac{1}{2}-i\gamma_n\right)}{\zeta(1+2i\gamma_n)\Gamma\left(\frac{1}{2}+i\gamma_n\right)} (1+2 \cdot 10^{-19}\theta), \quad (n=1,2). \end{aligned}$$

Taking arguments of both sides of (11) gives

$$(12) \quad \gamma_n \log \left(\frac{p}{4\pi^2} \right) = a_n + 2\pi x_n + 3 \cdot 10^{-19}\theta, \quad (n=1,2)$$

where x_n is an integer and

$$(13) \quad \begin{aligned} a_n &\equiv \pi - 2 \arg \zeta(2s_n) - 2 \arg \Gamma(s_n) \pmod{2\pi}, \\ 0 &\leq a_n < 2\pi. \end{aligned}$$

Eliminating $\log(p/4\pi^2)$ from the equations (12), and solving for x_2 , we obtain

$$(14) \quad x_2 = \frac{\gamma_2}{\gamma_1} x_1 + a_0 + 10^{-18}\theta,$$

where

$$(15) \quad a_0 = \frac{1}{2\pi} \left(\frac{\gamma_2}{\gamma_1} a_1 - a_2 \right).$$

From the Appendix,

$$(16) \quad \begin{aligned} \frac{\gamma_2}{\gamma_1} &= 1.487\,262\,003\,892\,890\,048 + 10^{-18}\theta, \\ a_0 &= a + .4 \cdot 10^{-9}\theta \quad \text{where } a = -.461\,786\,352. \end{aligned}$$

We can rewrite (14) as

$$(17) \quad x_2 = \frac{\gamma_2}{\gamma_1} x_1 + a + \frac{1}{2} \cdot 10^{-9} \theta.$$

Note that

$$(18) \quad 3.999\ 999\ 660 = \frac{\gamma_2}{\gamma_1} \cdot 3 + a + \frac{1}{2} \cdot 10^{-9} \theta.$$

It is not accidental that $3(\gamma_2/\gamma_1) + a$ should be close to an integer; $x_1 = 3$ corresponds to $p = 163$ (see introduction). In fact

$$\gamma_1 \log \left(\frac{163}{4\pi^2} \right) = 20.042\ 984\ 673\ 072 \dots,$$

$$a_1 + 2\pi \cdot 3 = 20.042\ 984\ 673\ 470 \dots$$

(Compare this with (12), where these numbers would agree to at least 19 decimal places if $p > 10,000$.) From (12), we now see that $p > 10^4$ implies that $x_1 > 3$.

Subtracting (18) from (17) gives:

$$(19) \quad x_2 - 4 = \frac{\gamma_2}{\gamma_1} (x_1 - 3) - b + 10^{-9} \theta, \quad \text{where } b = .000\ 000\ 340.$$

Now let

$$(20) \quad \begin{aligned} p_1 &= 83,532,765, & p_2 &= 12,832,922, \\ q_1 &= 56,165,467, & q_2 &= 8,628,555. \end{aligned}$$

Then $p_1 q_2 - q_1 p_2 = 1$, so that p_1 and q_1 are relatively prime. Also,

$$(21) \quad \left| \left(\frac{\gamma_2}{\gamma_1} - \frac{p_1}{q_1} \right) \right| < 2.3 \cdot 10^{-16}.$$

Let

$$(22) \quad Q + \frac{R}{q_1} = \frac{p_1}{q_1} (x_1 - 3),$$

where $0 \leq R < q_1$ and Q and R are integers. Subtracting (22) from (19) gives

$$(23) \quad x_2 - Q - 4 = \left(\frac{\gamma_2}{\gamma_1} - \frac{p_1}{q_1} \right) (x_1 - 3) + \left(\frac{R}{q_1} - b \right) + 10^{-9} \theta.$$

If $x_1 \leq 5.1 \cdot 10^7$, then

$$(24) \quad \left| \left(\frac{\gamma_2}{\gamma_1} - \frac{p_1}{q_1} \right) (x_1 - 3) \right| < 12 \cdot 10^{-9},$$

and thus $x_1 \leq 5.1 \cdot 10^7$ implies

$$(25) \quad |x_2 - Q - 4| < 12 \cdot 10^{-9} + (1 - 340 \cdot 10^{-9}) + 10^{-9} < 1.$$

On the other hand, since

$$\frac{18}{q_1} < 321 \cdot 10^{-9} < b = 340 \cdot 10^{-9} < 356 \cdot 10^{-9} < \frac{20}{q_1},$$

we find that $x_1 \leq 5.1 \cdot 10^7$ and $R \neq 19$ implies

$$(26) \quad \begin{aligned} |x_2 - Q - 4| &\geq \left| \frac{R}{q_1} - b \right| - \left| \left(\frac{\gamma_2}{\gamma_1} - \frac{p_1}{q_1} \right) (x_1 - 3) \right| - 10^{-9} \\ &> 16 \cdot 10^{-9} - 12 \cdot 10^{-9} - 10^{-9} > 0. \end{aligned}$$

Inequalities (25) and (26) are contradictory, and therefore $x_1 \leq 5.1 \cdot 10^7$ implies $R = 19$.

But if $R = 19$, then we see from (22) that

$$19 \equiv p_1(x_1 - 3) \pmod{q_1},$$

and this implies

$$x_1 - 3 \equiv 51,611,611 \pmod{q_1}.$$

Thus under all circumstances, $x_1 > 5.1 \cdot 10^7$. Hence by (12),

$$\begin{aligned} \log \left(\frac{p}{4\pi^2} \right) &= \frac{a_1 + 2\pi x_1 + 3 \cdot 10^{-19}\theta}{\gamma_1} \\ &> \frac{2\pi(5.1 \cdot 10^7) - 3 \cdot 10^{-19}}{14.2} \\ &> 2.2 \cdot 10^7, \end{aligned}$$

and Theorem 1 follows.

APPENDIX

I wish to express my thanks to M.D. Bigg who furnished values of γ_1 and γ_2 to fifty decimal places and to R.S. Lehman who furnished values of $\arg \zeta(2s_n)$ and $|\zeta(2s_n)|$ for $n = 1$ and 2 , with a proved accuracy of $\pm 10^{-10}$. The values of γ_1 and γ_2 were confirmed independently by Robert Spira to fifteen decimal places. Their values are:

$$(A1) \quad \begin{aligned} \gamma_1 &= 14.134\ 725\ 141\ 734\ 693\ 790\ 457 + 10^{-21}\theta, \\ \gamma_2 &= 21.022\ 039\ 638\ 771\ 554\ 992\ 628 + 10^{-21}\theta, \\ \frac{1}{\pi} \arg \zeta(2s_1) &= -.108\ 452\ 737\ 083\ 095 + 10^{-10}\theta \pmod{2}, \\ \frac{1}{\pi} \arg \zeta(2s_2) &= .067\ 103\ 865\ 503\ 910 + 10^{-10}\theta \pmod{2}, \\ |\zeta(2s_1)| &= 1.948\ 757\ 313\ 817\ 40 + 10^{-10}\theta, \\ |\zeta(2s_2)| &= .830\ 962\ 021\ 546\ 955 + 10^{-10}\theta. \end{aligned}$$

All other numbers were calculated on a Monroe desk calculator to fifteen places.

The Euler-Maclaurin sum formula for $\log \Gamma(\frac{1}{2} + i\gamma)$ can be expressed ([4, p. 132]. It should be noted that the exponent $(m+1)$ in Nörlund's remainder term should be replaced by m . See also [5, p. 131]):

$$\begin{aligned} \log \Gamma\left(\frac{1}{2} + i\gamma\right) &= i\gamma \log(i\gamma) - i\gamma + \frac{1}{2} \log(2\pi) \\ &+ \sum_{j=2}^{11} (-1)^j \frac{B_j\left(\frac{1}{2}\right)}{j(j-1)} \cdot \frac{1}{(i\gamma)^{j-1}} \\ &- \int_0^\infty \frac{B_{11}\left(x - \frac{1}{2} - \left[x - \frac{1}{2}\right]\right)}{11(x + i\gamma)^{11}} dx. \end{aligned}$$

From this we see that

$$\begin{aligned} \arg \Gamma\left(\frac{1}{2} + i\gamma\right) &= \gamma(\log \gamma - 1) + \sum_{k=1}^5 \frac{(-1)^k B_{2k}\left(\frac{1}{2}\right)}{2k(2k-1)\gamma^{2k-1}} \\ &- \operatorname{Im} \int_0^\infty \frac{B_{11}\left(x - \frac{1}{2} - \left[x - \frac{1}{2}\right]\right)}{11(x + i\gamma)^{11}} dx. \end{aligned} \quad (\text{A2})$$

From the formula [5, p. 123],

$$B_k\left(\frac{1}{2}\right) = -(1 - 2^{1-k})B_k,$$

we get the following:

$$\frac{B_2\left(\frac{1}{2}\right)}{2 \cdot 1} = -.041 \ 666 \ 666 \ 666 \ 667 + 10^{-15}\theta,$$

$$\frac{B_4\left(\frac{1}{2}\right)}{4 \cdot 3} = .002 \ 430 \ 555 \ 555 \ 556 + 10^{-15}\theta,$$

$$\frac{B_6\left(\frac{1}{2}\right)}{6 \cdot 5} = -.000 \ 768 \ 849 \ 206 \ 349 + 10^{-15}\theta,$$

$$\frac{B_8\left(\frac{1}{2}\right)}{8 \cdot 7} = .000 \ 590 \ 587 \ 797 \ 619 + 10^{-15}\theta,$$

$$\frac{B_{10}\left(\frac{1}{2}\right)}{10 \cdot 9} = -.000 \ 840 \ 106 \ 797 \ 138 + 10^{-15}\theta.$$

From [6], we get

$$\log \gamma_1 = 2.648\ 634\ 545\ 730\ 790 + 10^{-15}\theta,$$

$$\log \gamma_2 = 3.045\ 571\ 393\ 984\ 561 + 10^{-15}\theta.$$

Finally, for $t > 14$, we get from (3),

$$\begin{aligned} \left| \int_0^\infty \frac{B_{11} \left(x - \frac{1}{2} - \left[x - \frac{1}{2} \right] \right)}{11(x+it)^{11}} dx \right| &\leq \frac{(2\pi)^{\frac{2 \cdot 11!}{(2\pi)^{11}}} \cdot \frac{11}{10}}{11t^9} \int_0^\infty \frac{dx}{x^2 + t^2} \\ &= \frac{11!\pi}{10(2\pi)^{11}t^{10}} \\ &< 10^{-13}. \end{aligned}$$

Putting these numbers into (A2), we obtain

$$\begin{aligned} \frac{1}{\pi} \arg \Gamma(s_1) &= 7.418\ 512\ 651\ 985\ 173 + 2 \cdot 10^{-13}\theta, \\ \frac{1}{\pi} \arg \Gamma(s_2) &= 13.688\ 619\ 111\ 000\ 235 + 2 \cdot 10^{-13}\theta. \end{aligned} \quad (\text{A3})$$

We may rewrite (12) thusly:

$$\begin{aligned} \frac{a_n}{2\pi} &\equiv \frac{1}{2} - \frac{1}{\pi} \arg \zeta(2s_n) - \frac{1}{\pi} \arg \Gamma(s_n) \pmod{1}, \\ 0 &\leq \frac{a_n}{2\pi} < 1. \end{aligned}$$

Using (A1) and (A3), we find that

$$\frac{a_1}{2\pi} \equiv .189\ 940\ 085\ 097\ 922 + 1.002 \cdot 10^{-10}\theta \pmod{1},$$

$$\frac{a_2}{2\pi} \equiv .744\ 277\ 023\ 495\ 855 + 1.002 \cdot 10^{-10}\theta \pmod{1}.$$

Since these numbers are between 0 and 1, the above is actually an equality.

Finally,

$$\frac{\gamma_2}{\gamma_1} = 1.487\ 262\ 003\ 892\ 890\ 048 + 10^{-18}\theta \quad (\text{A4})$$

and

$$\begin{aligned} a_0 &= \frac{\gamma_2}{\gamma_1} \left(\frac{a_1}{2\pi} \right) - \frac{a_2}{2\pi} \\ &= -.461\ 786\ 351\ 913\ 533 + 3 \cdot 10^{-10}\theta. \end{aligned} \quad (\text{A5})$$

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UNIVERSITY OF MICHIGAN
ANN ARBOR, MICHIGAN