TOPOLOGY OF QUATERNIONIC MANIFOLDS

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Introduction. The holonomy groups of manifolds having affine connection with zero torsion have been classified by M. Berger [1]. The possible restricted holonomy groups for irreducible Riemannian manifolds which are not symmetric spaces are the following:

$\mathrm{SO}(n)$, $U(n)$ ($= T^1 \times \mathrm{SU}(n)$), $\mathrm{SU}(n)$, $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$,

$\mathrm{Sp}(n)$ (all for $n \geqq 2$) and the special groups G_2,

$\mathrm{Spin}(7)$ and $\mathrm{Spin}(9)$ (see also Simons [12]).

Manifolds with holonomy groups in $\mathrm{SO}(n)$ are the oriented Riemannian manifolds. Only general results may be obtained about the topology of this large class. The cohomology of Riemannian manifolds with holonomy groups in $U(n)$ (Kähler manifolds), has been extensively studied (see [3], [6], [13]). The existence of compact Riemannian manifolds with holonomy groups in $\mathrm{SU}(n)$ or $\mathrm{Sp}(n)$ is not known for $n \neq 1$.

Hence, for the general groups, the most interesting cases left are those manifolds whose holonomy groups form subgroups of $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$. These manifolds are called quaternionic manifolds.

In the first part of this paper (§§1–3), a decomposition analogous to the Hodge Decomposition for Kähler manifolds is given for quaternionic manifolds (Theorem 3.5). Using a theorem of Chern, we get an increasing sequence of Betti numbers (Theorem 3.6). In the last part (§§4 and 5), we define a quaternionic pinching. Using it, we give a quaternionic analogue (Theorem 5.5) to Klingenberg's Kähler pinching in [7] and [8].

1. **Definitions and algebra.** Let K^n be the n-dimensional right module over the quaternions K. We define a bilinear form on K^n as follows: if $P = (p_1, \cdots, p_n)$ and $Q = (q_1, \cdots, q_n) \in K^n$, then

$$\langle P, Q \rangle = \frac{1}{2} \sum_{i=1}^{n} (p_i q_i + q_i \bar{p}_i).$$

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Then $\langle P, Q\rangle$ is an inner product of $\boldsymbol{K}^n$ considered as a $4n$-dimensional real vector space.

LEMMA 1.1. *$\langle P, Q\rangle$ is invariant under the action of* Sp(n).

Proof. Sp(n) is defined as the set of all endomorphisms of $\boldsymbol{K}^n$ which preserves the "symplectic product" $(P, Q) = \sum_{i=1}^{n} p_i \bar{q}_i$ (see Chevalley [4]). Now our inner product is $\langle P, Q\rangle = \frac{1}{2}((P,Q) + (Q,P))$. Hence it is clearly invariant. Q.E.D.

REMARK 1.2. As defined above, Sp(1) is the set of all unit quaternions. Hence for $\lambda \in \mathrm{Sp}(1)$ (i.e. $\lambda \in \boldsymbol{K}$, $|\lambda| = 1$), $\langle P\lambda, Q\lambda\rangle = \langle \lambda P, \lambda Q\rangle = \langle P, Q\rangle$.

Now for $\boldsymbol{q} \in \boldsymbol{K}$, write $\boldsymbol{q} = q^0 + q^1\boldsymbol{i} + q^2\boldsymbol{j} + q^3\boldsymbol{k}$, where q^i are real for $i = 0$, 1, 2 or 3 and 1, $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ form the usual basis of $\boldsymbol{K}$ over $\boldsymbol{R}$ (the reals).

DEFINITION 1.3. Considering the three complex structures defined by $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ on $\boldsymbol{K}^n$, we define the following three skew symmetric, bilinear forms:

$$\Omega_I(P,Q) = \langle P\boldsymbol{i}, Q\rangle,$$

$$\Omega_J(P,Q) = \langle P\boldsymbol{j}, Q\rangle$$

and

$$\Omega_K(P,Q) = \langle P\boldsymbol{k}, Q\rangle.$$

By a simple calculation, we have

LEMMA 1.4. (1) $\Omega_I(P\boldsymbol{i}, Q\boldsymbol{i}) = -\Omega_I(P\boldsymbol{j}, Q\boldsymbol{j}) = -\Omega_I(P\boldsymbol{k}, Q\boldsymbol{k}) = \Omega_I(P,Q)$.
(2) $\Omega_J(P\boldsymbol{j}, Q\boldsymbol{j}) = -\Omega_J(P\boldsymbol{k}, Q\boldsymbol{k}) = -\Omega_J(P\boldsymbol{i}, Q\boldsymbol{i}) = \Omega_J(P,Q)$.
(3) $\Omega_K(P\boldsymbol{k}, Q\boldsymbol{k}) = -\Omega_K(P\boldsymbol{i}, Q\boldsymbol{i}) = -\Omega_K(P\boldsymbol{j}, Q\boldsymbol{j}) = \Omega_K(P,Q)$.

DEFINITION 1.5. Let $\lambda \in \mathrm{Sp}(1)$ (i.e. λ is a unit quaternion), write $\lambda = a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k}$. Define λ^* on the bilinear forms $\Omega_I, \Omega_J, \Omega_K$ by:

$$\lambda^*\Omega_I(P,Q) = \Omega_I(P\lambda, Q\lambda),$$

$$\lambda^*\Omega_J(P,Q) = \Omega_J(P\lambda, Q\lambda)$$

and

$$\lambda^*\Omega_K(P,Q) = \Omega_K(P\lambda, Q\lambda).$$

LEMMA 1.6.

$$\lambda^*\Omega_I = (a^2 + b^2 - c^2 - d^2)\Omega_I + 2(ad + bc)\Omega_J + 2(bd - ac)\Omega_K.$$

$$\lambda^*\Omega_J = 2(bc - ad)\Omega_I + (a^2 - b^2 + c^2 - d^2)\Omega_J + 2(ab + cd)\Omega_K.$$

$$\lambda^*\Omega_K = 2(ac + bd)\Omega_I + 2(cd - ab)\Omega_J + (a^2 - b^2 - c^2 + d^2)\Omega_K.$$

Proof. This is straight calculation, noting the following equalities: $\langle P\boldsymbol{i}, Q\rangle = -\langle P, Q\boldsymbol{i}\rangle$, $\langle P\boldsymbol{i}, Q\boldsymbol{j}\rangle = \langle P, Q\boldsymbol{k}\rangle$, $\langle P\boldsymbol{i}, Q\boldsymbol{k}\rangle = \langle P\boldsymbol{j}, Q\rangle$ and similarly for other combinations of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$. Q.E.D.

DEFINITION 1.7. Define a 4-form Ω on K^n by

$$\Omega = \Omega_I \wedge \Omega_I + \Omega_J \wedge \Omega_J + \Omega_K \wedge \Omega_K.$$

DEFINITION 1.8. Define the action of the group $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$ on K^n as follows: let $P \in K^n$ and $(A,\lambda) \in \mathrm{Sp}(n) \times \mathrm{Sp}(1)$, then $(A,\lambda)P = AP\lambda$, i.e. apply A to P and multiply on the right by the unit quaternion λ.

THEOREM 1.9. *Ω is invariant under the action of* $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$.

Proof. By Lemma 1.1, Ω is invariant under the action of $\mathrm{Sp}(n)$ on the left. Now let $\lambda \in K, |\lambda| = 1$, i.e. λ represents an element of $\mathrm{Sp}(1)$, then

$$\lambda^*\Omega = \lambda^*\Omega_I \wedge \lambda^*\Omega_I + \lambda^*\Omega_J \wedge \lambda^*\Omega_J + \lambda^*\Omega_K \wedge \lambda^*\Omega_K.$$

By substituting the values of each term on the right from Lemma 1.6, we get $\lambda^*\Omega = \Omega$, hence Ω is invariant under the action of $\mathrm{Sp}(1)$ on the right. Q.E.D.

Let $(K^n)'$ be the dual space of K^n over K and $z_1, \cdots, z_n$ be a basis of $(K^n)'$. We may write $z_\alpha = u_\alpha + v_\alpha i + x_\alpha j + y_\alpha k$, so that $u_1, v_1, x_1, y_1, \cdots, u_n, v_n, x_n, y_n$ form a basis of $(K^n)'$ over R.

There is a complex structure on $(K^n)'$ defined by the endomorphism $P \to Pi$, for $P \in (K^n)'$. The elements

$$z'_\alpha = u_\alpha + v_\alpha i \text{ and } z''_\alpha = (x_\alpha j) - (y_\alpha j)i$$

form a basis of $(K^n)'$ as a $2n$-dimensional complex vector space. Then, by [13, p. 17],

$$\Omega_I = \sum_{\alpha=1}^{n} u_\alpha \wedge v_\alpha - \sum_{\alpha=1}^{n} x_\alpha j \wedge y_\alpha j = \sum_{\alpha=1}^{n} (u_\alpha \wedge v_\alpha + x_\alpha \wedge y_\alpha).$$

Similarly, using the complex structure $P \to Pj$,

$$z'_\alpha = u_\alpha + x_\alpha j \text{ and } z''_\alpha = (v_\alpha i) + (y_\alpha i)j$$

form a basis of $(K^n)'$ over C (the complex field) and

$$\Omega_J = \sum_{\alpha=1}^{n} (u_\alpha \wedge x_\alpha + y_\alpha \wedge v_\alpha).$$

Finally, using the complex structure $P \to Pk$, we have

$$\Omega_K = \sum_{\alpha=1}^{n} (u_\alpha \wedge y_\alpha + v_\alpha \wedge x_\alpha).$$

From the above expression of Ω_I, Ω_J and Ω_K, we can express the exterior 4-form Ω as a linear sum of the basis elements $u_\alpha \wedge v_\beta \wedge x_\gamma \wedge y_\delta$, where $1 \leqq \alpha, \beta, \gamma, \delta \leqq n$.

THEOREM 1.10. $\Omega^n \neq 0$, (*n-fold exterior product*

Proof. Since $\Omega = \Omega_I \wedge \Omega_I + \Omega_J \wedge \Omega_J + \Omega_K \wedge \Omega_K$, Ω^n will be a sum of $4n$-forms, hence will be a sum of

$$(*) \qquad \varepsilon u_1 \wedge v_1 \wedge x_1 \wedge y_1 \wedge \cdots \wedge u_n \wedge v_n \wedge x_n \wedge y_n,$$

where $\varepsilon = \pm 1$. We will show that ε will always be $+1$. Each summand of Ω^n will be a product of the 2-forms

$$(**) \qquad u_\alpha \wedge v_\alpha,\ x_\alpha \wedge y_\alpha,\ u_\alpha \wedge x_\alpha,\ y_\alpha \wedge v_\alpha,\ u_\alpha \wedge y_\alpha \text{ and } v_\alpha \wedge x_\alpha.$$

Now let us take one of the summands and rearrange it so that the subscripts will be in nondecreasing order, i.e. so that the summand will be an exterior product of the $4n$ elements $u_1, v_1, x_1, y_1, \cdots, u_n, v_n, x_n, y_n$, such that the first four elements in the product will have subscript 1, the next four will have subscript 2, etc. Since in the original product, we multiply pairs with the same indices, in order to achieve the new product, we have to permute the elements in the product by an even permutation, hence we do not change the value of the product.

Take the term in the product consisting of the four elements with the index α. Since it is a product of terms in (**), it must be one of the following three forms (else would be 0): $u_\alpha \wedge v_\alpha \wedge x_\alpha \wedge y_\alpha$, $u_\alpha \wedge x_\alpha \wedge y_\alpha \wedge v_\alpha$ or $u_\alpha \wedge y_\alpha \wedge v_\alpha \wedge x_\alpha$, which are all equal to each other. So each summand is equal to (*) with $\varepsilon = +1$ and Ω^n is a nonzero multiple of it. Q.E.D.

2. **Decomposition.** We extend the definition of the star operator $*$ and the operators L and Λ to the quaternionic case. Let $\bigwedge(\boldsymbol{K}^n)'$ be the exterior algebra over $\boldsymbol{R}$, considering $(\boldsymbol{K}^n)'$ as a real $4n$-dimensional vector space. Every element of $\bigwedge(\boldsymbol{K}^n)'$ is a linear combination of simple p-forms $\omega = \omega_1 \wedge \cdots \wedge \omega_p$, where each ω_r is one of u_α, v_α, x_α, or y_α.

Definition 2.1. Define $*$, L and Λ on $\bigwedge(\boldsymbol{K}^n)'$ as follows. If ω is a simple p-form, then $*\omega$ is the simple $(4n-p)$-form such that $\omega \wedge *\omega$ is $u_1 \wedge v_1 \wedge x_1 \wedge y_1 \wedge \cdots \wedge u_n \wedge v_n \wedge x_n \wedge y_n$. Extend $*$ by linearity to $\bigwedge(\boldsymbol{K}^n)'$. On an arbitrary exterior form ω, define $L\omega = \Omega \wedge \omega$ and $\Lambda\omega = *(\Omega \wedge *\omega)$.

Remark. (1) for all $\omega \in \bigwedge(\boldsymbol{K}^n)'$, $**\omega = \omega$.
(2) L: $\bigwedge^p(\boldsymbol{K}^n)' \to \bigwedge^{p+4}(\boldsymbol{K}^n)'$.
(3) Λ: $\bigwedge^p(\boldsymbol{K}^n)' \to \bigwedge^{p-4}(\boldsymbol{K}^n)'$.

Definition 2.2. Define a bilinear form on $\bigwedge^p(\boldsymbol{K}^n)'$ by

$$(\omega, \omega') = *(\omega \wedge *\omega') \text{ for } \omega, \omega' \in \bigwedge^p(\boldsymbol{K}^n)'.$$

Lemma 2.3. *$(L\omega, \omega') = (\omega, \Lambda\omega')$ for $\omega \in \bigwedge^p(\boldsymbol{K}^n)'$ and $\omega' \in \bigwedge^{p+4}(\boldsymbol{K}^n)'$.*

Proof. This is straight substitution. Q.E.D.

Lemma 2.4. *L: $\bigwedge^p(\boldsymbol{K}^n)' \to \bigwedge^{p+4}(\boldsymbol{K}^n)'$ is an isomorphism into for $p + 4 \leqq n + 1$.*

Proof. It is sufficient to prove that for $\omega \in \bigwedge^p(K^n)'$, $p+4 \leqq n+1$, $L\omega = \Omega \wedge \omega = 0$ implies $\omega = 0$.

Assume $\omega \neq 0$ and write $\omega = \sum_{A,B,C,D} \gamma_{ABCD} u_A \wedge v_B \wedge x_C \wedge y_D$, where A, B, C and D are subsets of the index set $\{1,\cdots,n\}$ and if $A = \{\alpha_1,\cdots,\alpha_p\}$, then $u_A = u_{\alpha_1} \wedge \cdots \wedge u_{\alpha_p}$.

In the summation above, consider the terms with the highest total degree, say r, in u's and v's. Let ω' be the sum of these terms,

$$\omega' = \sum \gamma_{ABCD} u_A \wedge v_B \wedge x_C \wedge y_D \neq 0,$$

where the summation is taken over the indices A, B, C and D such that $|A| + |B| = r$ ($|A|$ and $|B|$ denote the cardinalities of A and B respectively). Similarly, express $L\omega = \Omega \wedge \omega$ in u_A, v_B, x_C and y_D and consider the terms with the highest total degree in u's and v's. From the expression for Ω_I, Ω_J and Ω_K given at the end of §1, it follows that the sum of these terms is given by $\sum_{\alpha,\beta=1}^n u_\alpha \wedge v_\alpha \wedge u_\beta \wedge v_\beta \wedge \omega'$. Hence $L\omega = 0$ implies that $\sum_{\alpha,\beta=1}^n u_\alpha \wedge v_\alpha \wedge u_\beta \wedge v_\beta \wedge \omega' = 0$, which means that $\sum_{C,D}(\sum_{\alpha,\beta,A,B} \gamma_{ABCD} u_\alpha \wedge v_\alpha \wedge u_\beta \wedge v_\beta \wedge u_A \wedge v_B) \wedge x_C \wedge y_D = 0$. This implies that

$$\left(\sum_{\alpha=1}^n u_\alpha \wedge v_\alpha\right) \wedge \left(\sum_{\beta=1}^n u_\beta \wedge v_\beta\right) \wedge \left(\sum_{A,B} \gamma_{ABCD} u_A \wedge v_B\right) = 0$$

for each fixed C and D, or $\Omega'^2 \wedge \omega'' = 0$, where

$$\Omega' = \sum_{\alpha=1}^n u_\alpha \wedge v_\alpha \text{ and } \omega'' = \sum_{A,B} \gamma_{ABCD} u_A \wedge v_B \neq 0.$$

Consider the n-dimensional complex vector space with the coordinate system $u_1 + (-1)^{1/2} v_1, \cdots, u_n + (-1)^{1/2} v_n$. Then Ω' is the fundamental 2-form. Applying the Hodge Decomposition Theorem, (since degree of $\omega'' \leqq n-3$) $\Omega' \wedge (\Omega' \wedge \omega'') = 0$ implies $\Omega' \wedge \omega'' = 0$, which in turn implies that $\omega'' = 0$, which is a contradiction. Q.E.D.

Definition 2.5. A p-form ω is said to be *effective* if $\Lambda\omega = 0$. We denote by $\bigwedge_e^p \subset \bigwedge^p (K^n)'$ the set of all effective p-forms.

Theorem 2.6. *There is a direct sum decomposition of* $\bigwedge^p(K^n)'$ *as follows: for* $p \leqq n+1$, $r = [p/4]$,

$$\bigwedge^p(K^n)' = \bigwedge_e^p + L\bigwedge_e^{p-4} + \cdots + L^r \bigwedge_e^{p-4r}.$$

Proof. By Lemma 2.4, L is an isomorphism into. By Lemma 2.3, Λ is the adjoint of L and is therefore onto for $p \leqq n+1$. We will prove the theorem by induction on p. The statement is true for $p = 0$, 1, 2 and 3, since Λ lowers degree by 4 and hence $\bigwedge^p = \bigwedge_e^p$ for these p's.

Assume the theorem true for $k < p$. We shall prove it for $k = p$. We claim that $\bigwedge_e^p$ is the orthogonal compliment of the subspace $L\bigwedge^{p-4}(K^n)'$ in $\bigwedge^p(K^n)'$.

Orthogonal — let $\omega \in \bigwedge_e^p$ and $L\omega' \in L\bigwedge^{p-4}(K^n)'$, then

$$(\omega, L\omega') = \Lambda(\omega, \omega') = (0, \omega') = 0.$$

COMPLIMENT. Let $\omega \in \bigwedge^p(K^n)'$ be such that $(\omega, L\omega') = 0$ for all $\omega' \in \bigwedge^{p-4}K^n$. Then $(\Lambda\omega, \omega') = 0$ and hence $\Lambda\omega = 0$, since (,) is a nondegenerate bilinear form.

By induction hypothesis, we have

$$\begin{aligned}\textstyle\bigwedge^p(K^n)' &= \textstyle\bigwedge_e^p + L\bigwedge^{p-4}(K^n)' \\ &= \textstyle\bigwedge_e^p + L\bigwedge_e^{p-4} + \cdots + L^r\bigwedge_e^{p-4r} \quad \text{(direct sum).}\end{aligned}$$

Q.E.D.

3. Quaternionic manifolds.

DEFINITION 3.1. A $4n$-dimensional Riemannian manifold M is called a quaternionic manifold if its holonomy group is a subgroup of $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$.

Let M be a $4n$-dimensional quaternionic manifold and $x \in M$. We may identify $T_x(M)$ with K^n. However, this quaternionic structure of $T_x(M)$ may not be invariant under parallel displacement. Using this identification, we may define Ω, which will be invariant under parallel displacement (Theorem 1.9). Hence Ω is independent of the choice of quaternionic structure on $T_x(M)$. From the above discussion and Theorem 1.10, we have

LEMMA 3.2. *Ω as defined above is a closed differential form of degree 4 and of maximal rank.*

THEOREM 3.3. *Let M be a $4n$-dimensional quaternionic manifold and let B^i denote its ith Betti number, then*

$$B^{4i} \neq 0, \text{ for } i = 0, 1, \cdots, n.$$

Proof. By Lemma 3.2, Ω is a closed 4-form of maximal rank, hence Ω^i is a nonzero element of $H^{4i}(M; \boldsymbol{R})$.

Therefore B^{4i} = dimension of $H^{4i}(M; \boldsymbol{R}) \neq 0$. Q.E.D.

DEFINITION 3.4. Define the operators $*$, L and Λ on the space of differential forms Φ^p as follows: If ω is a differential p-form, then $*\omega$ is the $(4n-p)$-form, such that

$$(*\omega)_x = *(\omega_x), \text{ for all } x \in M,$$

$$L\omega = \Omega \wedge \omega, \qquad \Lambda\omega = *(\Omega \wedge *\omega).$$

A differential form ω is said to be effective if $\Lambda\omega = 0$.

THEOREM 3.5. *Let M be a $4n$-dimensional quaternionic manifold and ω a differential form on M of degree $p \leq n+1$. Then*

$$\omega = \sum_{i=0}^{[p/4]} L^i \omega_e^{p-4i}, \text{ where } \omega_e^k \text{ is an effective } k\text{-form.}$$

Proof. Let Φ_e^k denote the space of effective k-forms. By Theorem 2.6, there is a direct sum decomposition for $p \leqq n+1$,

$$\Phi^p = \Phi_e^p + L\Phi_e^{p-4} + \cdots + L^r\Phi_e^{p-4r},$$

where $r = [p/4]$. Q.E.D.

A Theorem of Chern in [2, p. 105] states the following: Let M be a compact Riemannian manifold with structure group G, $W_1, \cdots, W_k$ be the irreducible invariant subspaces of Φ^q under the action of G and P_{W_i} be the projection map of Φ^q into W_i. Then, if a q-form ω is harmonic, so is $P_{W_i}\omega$.

Clearly each of the $L^i\Phi_e^{p-4i}$ is an invariant subspace of Φ^p under the action of the holonomy group G, since Ω is invariant under G. So each $L^i\Phi_e^{p-4i}$ is a sum of the W_i's. Therefore the projection of a harmonic form into $L^i\Phi_e^{p-4i}$ is again harmonic and we have

THEOREM 3.6. *If M is a quaternionic manifold of dimension $4n$, then there is an increasing sequence of Betti numbers $B^i \leqq B^{i+4} \leqq \cdots \leqq B^{i+4r}$, for $i + 4r \leqq n+1$, $i = 0, 1, 2$ or 3.*

4. **Sectional curvature of quaternionic projective space.** A quaternionic projective space has $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$ as its holonomy group, so it is a quaternionic manifold. As a symmetric space, it is represented as $\mathrm{Sp}(n+1)/\mathrm{Sp}(n) \times \mathrm{Sp}(1)$. Now let $P^n(3)$ be the $4n$-dimensional quaternionic projective space. We will first find an explicit representation of the Killing form and then express the sectional curvature of $P^n(3)$ in an invariant form in order to define pinching.

The Lie algebra $\mathrm{sp}(n+1)$ of $\mathrm{Sp}(n+1)$ is the set of all $(n+1) \times (n+1)$ skew-quaternionic matrices, i.e. matrices $(\boldsymbol{a}_{ij})$, where each $\boldsymbol{a}_{ij}$ is a quaternion satisfying $\boldsymbol{a}_{ji} = -\bar{\boldsymbol{a}}_{ij}$, with $\bar{\boldsymbol{a}}$ the quaternionic conjugate of $\boldsymbol{a}$.

LEMMA 4.1. *Let $B(X, Y)$ = real part of the trace of XY, where X and $Y \in \mathrm{sp}(n+1)$. Then $B(X, Y)$ is the Killing form of $\mathrm{sp}(n+1)$ up to a constant factor.*

Proof. Since for any quaternions $\boldsymbol{p}$ and $\boldsymbol{q}$, $\mathrm{Re}(\boldsymbol{pq}) = \mathrm{Re}(\boldsymbol{qp})$, B is clearly symmetric. Since $\mathrm{sp}(n+1)$ is simple, we need only to show that B is invariant under the action of $\mathrm{Sp}(n+1)$.

If we represent X and Y as real $4n$-dimensional square matrices $\tilde{X}$ and $\tilde{Y}$, then

$$\mathrm{Re}\,\mathrm{Tr}(XY) = \mathrm{Tr}(\tilde{X}\tilde{Y}) = \sum_{i,j=1}^{n+1} \sum_{k=0}^{3} X_{ij}^k Y_{ij}^k,$$

where $X = (X_{ij})$ and $X_j = X_{ij}^0 + X_{ij}^1 \boldsymbol{i} + X_{ij}^2 \boldsymbol{j} + X_{ij}^3 \boldsymbol{k}$, similarly for Y.

Since $\mathrm{Tr}(\tilde{X}, \tilde{Y})$ is invariant under $O(4n+4) \supset \mathrm{Sp}(n+1)$, we have our result. Q.E.D.

Let $P = (\boldsymbol{p}_1, \cdots, \boldsymbol{p}_n)$ and $Q = (\boldsymbol{q}_1, \cdots, \boldsymbol{q}_n) \in \boldsymbol{K}^n$, write

$$\boldsymbol{p}_i = p_i^0 + p_i^1 \boldsymbol{i} + p_i^2 \boldsymbol{j} + p_i^3 \boldsymbol{k} \text{ and } \boldsymbol{q}_i = q_i^0 + q_i^1 \boldsymbol{i} + q_i^2 \boldsymbol{j} + q_i^3 \boldsymbol{k}.$$

Recall that in §1 we defined two products in K^n as follows: Considering K^n as a real $4n$-space, we have

$$\langle P,Q\rangle = \frac{1}{2}\sum_{i=1}^{n}(p_i\bar{q}_i + q_i\bar{p}_i) = \sum_{i=1}^{n}\sum_{j=0}^{3} p_i^j q_i^j.$$

Considering K^n as a quaternionic n-space, we have the "symplectic product":

$$(P,Q) = \sum_{i=1}^{n} p_i\bar{q}_i.$$

We have the following relation,

$$(P,Q) = \langle P,Q\rangle + i\langle P,Qi\rangle + j\langle P,Qj\rangle + k\langle P,Qk\rangle.$$

By a calculation similar to that in the proof of Theorem 1.9, it can be shown that $\langle P,Qi\rangle^2 + \langle P,Qj\rangle^2 + \langle P,Qk\rangle^2$ is invariant under the action of $\mathrm{Sp}(n)\times\mathrm{Sp}(1)$. $\mathrm{Sp}(n)\times\mathrm{Sp}(1)$ acts transitively on the set of all unit vectors in K^n, hence, in the above sum, we may assume that $P=(p_1,0,\cdots,0)$. Then, by a straight calculation, it follows that if P and Q are unit vectors, $\langle P,Qi\rangle^2 + \langle P,Qj\rangle^2 + \langle P,Qk\rangle^2 \leqq 1$. These results will enable us to make the following definition.

Definition 4.2. Let M be the quaternionic projective space and $x\in M$. For each pair of unit vectors X and Y in $T_x(M)$, define the "angle" function $\alpha(X,Y)$, $0\leqq\alpha(X,Y)\leqq\pi/2$ by the equality

$$\mathrm{Cos}^2\alpha(X,Y) = \langle X,Yi\rangle^2 + \langle X,Yj\rangle^2 + \langle X,Yk\rangle^2.$$

Remark. $\alpha(X,Y)$ is well defined since it is independent of the choice of a quaternionic structure on $T_x(M)$.

We shall now calculate the sectional curvature K of the quaternionic projective space M in terms of α. Choose a quaternionic structure on $T_x(M)$, for $x\in M$, then given an element X in $T_x(M)$, write $X=(x_1,\cdots,x_n)$ as an element of K^n. Then there is a representation of X as an element in $\mathrm{sp}(n+1)$ (see Nomizu [11]) by the skew quaternionic matrix (a_{ij}), where $a_{1i} = -\bar{a}_{i1} = x_{i-1}$ for $i\neq 1$ and $a_{ij}=0$ otherwise.

Lemma 4.3. *For X and $Y\in T_x(M)$,*

$$B([X,Y],[X,Y]) = 2((X,Y)^2 - (X,Y)(Y,X) + (Y,X)^2 - (X,X)(Y,Y)).$$

Proof. This is straight calculation, using Lemma 4.1. Q.E.D.

Lemma 4.4. *If X and Y are orthonormal (as real vectors) in $T_x(M)$, then*

$$B([X,Y],[X,Y]) = 2(3(X,Y)^2 - 1).$$

Proof. Using the fact that $\langle X,Xi\rangle = \langle X,Xj\rangle = \langle X,Xk\rangle = 0$ for any $X\in K^n$, the lemma follows immediately from Lemma 4.3. Q.E.D.

Theorem 4.5. *Let M be the quaternionic projective n-space $P^n(3)$,*

$x \in M$, X and Y be two orthonormal vectors in $T_x(M)$. Furthermore, let K denote the sectional curvature of M. Then,

$$0 < K(X,Y) = \frac{1}{4}(1 + 3\operatorname{Cos}^2\alpha(X,Y)) < 1.$$

Proof. For a symmetric space, up to a positive constant factor, $K(X,Y)$ is

$$-B([X,Y],[X,Y]) = 2(1 - 3(X,Y)^2).$$

Now, for orthonormal vectors X and Y, it is a straight calculation to show that

$$(X,Y)^2 = -\langle X, Y\boldsymbol{i}\rangle^2 - \langle X, Y\boldsymbol{j}\rangle^2 - \langle X, Y\boldsymbol{k}\rangle^2 = -\operatorname{Cos}^2\alpha(X,Y).$$

Hence $K(X,Y)$ is a positive multiple of

$$(1 + 3\operatorname{Cos}^2\alpha(X,Y)).$$

The latter function attains a maximum of 4 when $X = Y\boldsymbol{i}$. Since the sectional curvature of M with the usual Riemannian metric attains a maximum of 1, the constant factor must be $\frac{1}{4}$. Q.E.D.

5. **Pinching.** We first state some general results of Klingenberg [7]. Let $P^n(1)$ and $P^n(3)$ denote the complex and quaternionic projective space of real dimensions $2n$ and $4n$ respectively, $P^n(7)$, for $n = 2$, denotes the Cayley plane, each endowed with the usual Riemannian metric for which the curvature varies between $\frac{1}{4}$ and 1. Let M be an m-dimensional complete and simply connected Riemannian manifold and let $G = (p(s))$, $0 \leqq s \leqq \infty$ be a geodesic ray in M, parametrized by the arc length.

DEFINITION 5.1. For $k = 1$, 3 or 7, we say G satisfies (π, k) if

(1) there are no conjugate points in $[0, \pi[$,

(2) there are k conjugate points in $[\pi, 4\pi/3[$,

(3) there are no conjugate points in $[4\pi/3, 2\pi[$ and

(4) there are λ conjugate points in $[2\pi, 8\pi/3[$, $\lambda > k + 1$.

THEOREM 5.2 (KLINGENBERG). *Let M be as above of dimension $(k+1)n$ with $n \geqq 2$. Assume that there is a point o in M such that (π, k) holds for all geodesic rays starting from o. For $k = 1$, assume also that the distance between o and its cut locus $C(o)$ is greater or equal to π. Furthermore, assume $k + \lambda \geqq m = \dim M$. Then M has the same integral cohomology ring as the symmetric space $P^n(k)$. For $k = 1$, M actually has the same homotopy type as $P^n(1)$ [8, p. 338].*

For a Kähler manifold M of dimension $2n \geqq 4$, if σ is a 2-plane tangent to M and X is in σ, then $\alpha(\sigma)$ is defined to be the angle between the plane σ and the plane $\bar{\sigma}$ spanned by X and JX, where J defines the almost complex structure in M. Define $K'(\alpha(\sigma)) = \frac{1}{4}(1 + 3\operatorname{Cos}^2\alpha(\sigma))$.

THEOREM 5.3 (KLINGENBERG). *Let M be a Kähler manifold of dimension $2n \geqq 4$. Assume that for all 2-planes σ tangent to M, the curvature $K(\sigma)$ satisfies the inequality*

$$9/16 < K(\sigma)/K'(\alpha(\sigma)) \leqq 1 .$$

Then M is compact and has the homotopy type of the complex projective n-space $P^n(1)$ [8, p. 339].

Using his method, we will obtain a similar result for quaternionic manifolds. Let M be a quaternionic manifold of dimension $4n$. For some $x \in M$, let X and Y be orthonormal vectors in $T_x(M)$. Then X and Y span a 2-plane σ. We may define the angle function $\alpha(\sigma) = \alpha(X, Y)$ by the equality

$$\mathrm{Cos}^2\alpha(X, Y) = \langle X, Y\boldsymbol{i}\rangle^2 + \langle X, Y\boldsymbol{j}\rangle^2 + \langle X, Y\boldsymbol{k}\rangle^2, \qquad 0 \leqq \alpha(\sigma) \leqq \pi/2,$$

and a function $K'(\alpha(\sigma))$ by

$$K'(\alpha(\sigma)) = \frac{1}{4}(1 + 3\ \mathrm{Cos}^2\alpha(\sigma)) .$$

$K'(\alpha(\sigma))$ is well defined since $\alpha(\sigma)$ is invariant under the action of $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$. If M is a quaternionic projective space, then $K'(\alpha(\sigma))$ reduces to the sectional curvature $K(\sigma)$.

THEOREM 5.4. *Let M be a quaternionic manifold of dimension $4n$, G a geodesic ray on M, G_0 the initial geodesic segment of length $2\pi/\sqrt{\delta}$, with $\delta = 9/16$. Assume that the sectional curvature $K(\sigma)$ of each plane section σ tangent to G_0 satisfies the inequality*

$$\delta < K(\sigma)/K'(\alpha(\sigma)) \leqq 1 .$$

Then G satisfies $(\pi, 3)$.

Proof. The proof proceeds in the same way as that of Proposition 3.3 of [7]. We may rewrite the inequality as

$$(*) \qquad \delta/4 \leqq \delta K'(\alpha(\sigma)) < K(\sigma) \leqq K'(\alpha(\sigma)) \leqq 1 .$$

Let G_0' be a geodesic segment of length $2\pi/\sqrt{\delta}$ in $M' = P^n(3)$. There exists an isometry I, compatible with $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$, mapping the tangent space of the initial point of G_0 onto the tangent space of the initial point of G_0', sending the initial direction of G_0 onto the initial direction of G_0'. I gives a 1-1 correspondence between plane section σ tangent to plane section $\sigma' = I\sigma$ tangent to G_0'. Since $\alpha(\sigma)$ and $\alpha(\sigma')$ are invariant under the action of $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$, they are invariant under parallel translation along G and G' respectively. Hence $K(I\sigma) = K'(\alpha(\sigma))$ and from (*) we see that $K(\sigma) \leqq K(I\sigma)$. By Lemma 3.1 of [7], this means that index $G' \geqq$ index G.

Since G_0' has no conjugate points in $[0, \pi[$, hence has index 0, G_0 has index 0, hence no conjugate points in that interval. Also since G_0' has 3 conjugate points in $[\pi, \pi/(\delta)^{1/2}[$, G_0 has at most 3 conjugate points in $[\pi, \pi/(\delta)^{1/2}[$.

Now let M'' be the space obtained from M' by multiplying the usual metric by $1/(\delta)^{1/2} > 1$ and let K'' be its curvature. Let G''_0 be a geodesic of length $2\pi/(\delta)^{1/2}$ in M'' and introduce the isometry I as before. Then from (*), we see that $K''(I\sigma) < K(\sigma)$. Now G''_0 has 3 conjugate points in $[0, \pi/(\delta)^{1/2}[$, so G_0 has exactly 3 conjugate points in $[\pi, \pi/(\delta)^{1/2}[$. By a similar argument, we conclude that G_0 has no conjugate points in $[\pi/(\delta)^{1/2}, 2\pi[$ and $4n-3$ conjugate points in $[2\pi, 2\pi/(\delta)^{1/2}[$. Letting $\delta = 9/16$, the Theorem follows. Q.E.D.

THEOREM 5.5. *Let M be a compact quaternionic manifold of dimension $4n \geqq 8$. Assume that for all 2-planes σ tangent to M, the curvature $K(\sigma)$ satisfies the inequality*

$$9/16 < K(\sigma)/K'(\alpha(\sigma)) \leqq 1 .$$

Then M has the same integral cohomology ring as $P^n(3)$.

Proof. This Theorem follows from Theorems 5.2 and 5.4 (noting from the proof of Theorem 5.4 that $\lambda = 4n-3$). Q.E.D.

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