INTERSECTIONS OF COMBINATORIAL BALLS AND OF EUCLIDEAN SPACES

BY
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1. **Introduction.** Poenaru [5] and Mazur [4] gave the first examples of contractible compact combinatorial 4-manifolds with boundary which were not topological 4-cells, but whose products with the unit interval were combinatorial 5-cells. Curtis [1] and Glaser [3] gave similar examples for $n \geqq 5$. In the latter result the product of the pseudo n-cell M^n with an interval was shown to be a combinatorial $(n + 1)$-cell rather than just merely topological. In addition, it was shown in [3], that for $n \geqq 5$, M^n was a compact combinatorial n-manifold with boundary not topologically I^n, but could be expressed as the union of two combinatorial n-cells whose intersection is also a combinatorial n-cell. Unfortunately the techniques used in [3] gave no hope of lowering the result to $n = 4$.

The purpose of this paper is to give another example of a pseudo 4-cell W with the property that $W \times I \approx I^5$, but in addition W also can be expressed as the union of two combinatorial 4-cells whose intersection is also a combinatorial 4-cell. This also gives an example of two Euclidean 4-spaces intersecting in an Euclidean 4-space so that the union is not topologically E^4.

2. **Definitions.** We will use the standard terminology I^n, E^n, and S^n for the unit n-cell, Euclidean n-space and the n- sphere respectively. If M is an n-manifold, then int M and BdM will denote the interior and boundary of M, respectively. All manifolds and all mappings or homeomorphisms will be considered in the combinatorial sense. Topological equivalence will be denoted by $=$, and we will use $\approx$ to denote combinatorial equivalence. We will use technique of collapsing polyhedra, denoted by $\searrow$, and the notion of regular neighborhoods as in Whitehead [7] or Zeeman [8].

3. **Construction.** In this section we will give an example of a certain contractible 2-complex K and an embedding of K in a combinatorial 4-manifold W with boundary so that $\pi_1(\text{Bd } W) \neq 1$ and W can be considered as a regular neighborhood of K. W will be the pseudo 4-cell promised in the introduction.

K is obtained by attaching two disks along a figure eight. Let us consider the figure eight as four line segments α, β, γ and δ and three vertices a, b, and c as indicated in Figure 1. The two disks are attached by the formula $\beta\gamma\gamma^{-1}\delta^{-1}\delta\alpha$

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and $\delta\alpha\alpha^{-1}\beta^{-1}\beta\gamma$. The resulting 2-complex K is also indicated in Figure 1. We observe that K is a contractible noncollapsible 2-complex by noting that we can easily get K as a deformation retract of a 3-cell and that K has no free edges.

Let T be a solid two-holed 3-dimensional torus in E^3. Let us consider two simple closed curves Γ_1 and Γ_2 embedded in $\operatorname{int}(T \times \{1\}) \subset T \times [0,1]$ as indicated in Figure 2. W will be formed by attaching two 2-handles to the boundary of $T \times [0,1]$ along the curves Γ_1 and Γ_2.

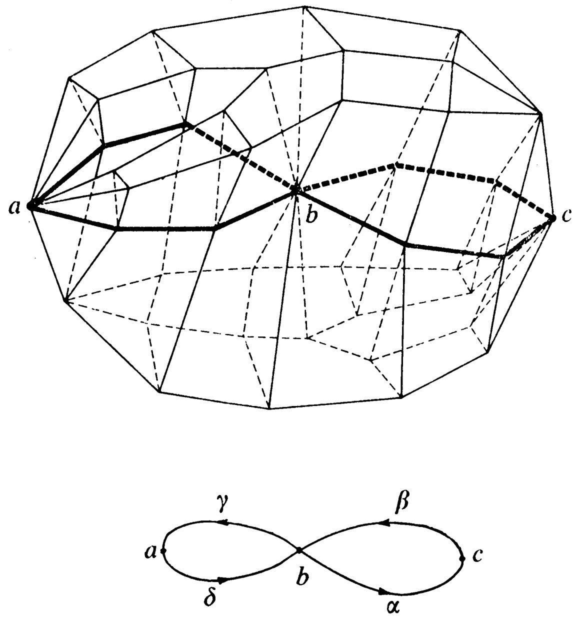

FIGURE 1

More precisely, let j be an embedding of $\operatorname{Bd} I^2 \times I^2 \to \operatorname{int}(T \times \{1\})$ such tha $j(\operatorname{Bd} I^2 \times 0) = \Gamma_1$ and k an embedding of $\operatorname{Bd} I^2 \times I^2 \to \operatorname{int}(T \times \{1\}) - j(\operatorname{Bd} I^2 \times I^2)$ such that $k(\operatorname{Bd} I^2 \times 0) = \Gamma_2$, where $0 \in \operatorname{int} I^2$. Also let us choose j and k, so that in forming the tubular neighborhoods $j(\operatorname{Bd} I^2 \times I^2)$ and $k(\operatorname{Bd} I^2 \times I^2)$ we do not have any twisting as we go around each of Γ_1 and Γ_2 respectively.

Define W as $I^2 \times I^2_j \cup T \times [0,1] \cup_k I^2 \times I^2$.

LEMMA 1. *W can be considered as a regular neighborhood of a combinatorial embedding of K in W.*

Proof. Divide $T \times \{1\}$ into seven 3-cells $B_1, B_2, \cdots, B_7$ as indicated in Figure 2. Let us denote the figure eight forming the core of $T \times \{\frac{1}{2}\}$ by α, β, γ and δ as we did in defining K. This is also indicated in Figure 2. Let us triangulate $\operatorname{Bd}(T \times [0,1]) \approx 2T$ so that $\Gamma_1, \Gamma_2, j(\operatorname{Bd} I^2 \times I^2), k(\operatorname{Bd} I^2 \times I^2), B_1, B_2, \cdots, B_7$ are subcomplexes of our triangulation. We also triangulate each copy of $I^2 \times I^2$ so that $j^{-1}(j(\operatorname{Bd} I^2 \times I^2))$ and $k^{-1}(k(\operatorname{Bd} I^2 \times I^2))$ are subcomplexes of their respective 4-cells. Next we want to extend the triangulation of $\operatorname{Bd}(T \times [0,1])$

which we now will consider as $2T$ to $T \times [0, 1]$ so that the figure eight $\alpha\beta\gamma\delta$ is a subcomplex of $T \times [0, 1]$, $K \subset T \times [0, 1]$ and so that $W \searrow K$.

In considering Bd $(T \times [0, 1])$ as $2T$ let $B_1', B_2', \cdots, B_7'$ denote the corresponding 3-cells of the other copy of T. Now we triangulate $T \times [0, 1]$ so that the cones $a(B_1 \cup B_1')$, $b(B_4 \cup B_4')$ and $c(B_7 \cup B_7)$ are subcomplexes of $T \times [0, 1]$. Let us denote these cones as C_1, C_4 and C_7' respectively. Each of $B_2 \cup B_2'$, $B_3 \cup B_3'$, $B_5 \cup B_5'$ and $B_6 \cup B_6'$ can be considered as a copy of $[0, 1] \times S^2$. For notational purposes we will denote this as $[0, 1]_i \times S^2$, $i = 2,3,5,6$. Let f_2 be a simplicial homeomorphism taking $[0, 1]_2$ onto γ; similarly, $f_3: [0, 1]_3 \to \delta$, $f_5: [0, 1]_5 \to \beta$

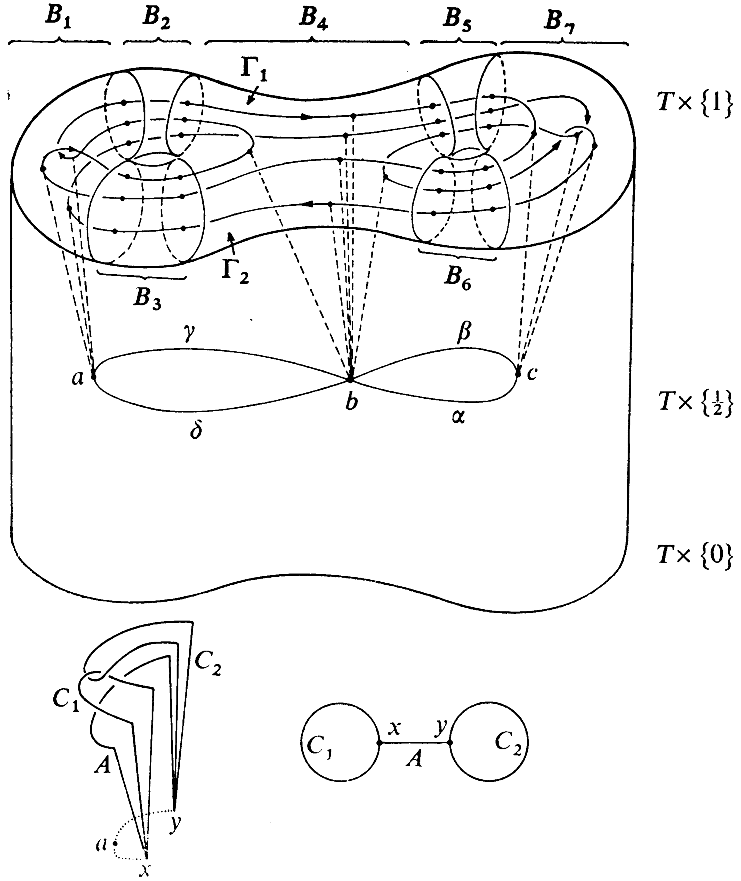

FIGURE 2

and $f_6: [0, 1]_6 \to \alpha$. Let g_i be the simplicial map taking $[0, 1]_i \times S^2$ onto the appropriate segment by taking $[0, 1]_i \times S^2 \to [0, 1]_i$ and then following this by f_i. Let M_i denote the mapping cylinders of g_i, $i = 2,3,5,6$. Now map each $[0, 1]_i \times S^2 \subset M_i$ homeomorphically onto $[0, 1]_i \times S^2 \subset 2T$. Next extend the map so that $a(\{0\}_2 \times S^2)$, $a(\{0\}_3 \times S^2)$, $b(\{1\}_2 \times S^2)$, $b(\{1\}_3 \times S^2)$, $b(\{0\}_5 \times S^2)$,

$b(\{0\}_6 \times S^2)$, $c(\{1\}_5 \times S^2)$ and $c(\{1\}_6 \times S^2)$ agrees with the corresponding complexes in the cones constructed above. Finally extend each homeomorphism so that M_i maps homeomorphically into $\mathrm{Cl}(T \times [0,1] - C_1 - C_4 - C_7)$ in a natural manner. This now gives our desired triangulation of $T \times [0,1]$.

Let F_i be the submapping cylinder of M_i gotten by restricting g_i to $(\Gamma_1 \cup \Gamma_2) \cap ([0,1]_i \times S^2)$, $i = 2,3,5,6$ and L_i the subcone of C_i gotten as the cone over the appropriate vertex on $(B_i \cup B_i') \cap (\Gamma_1 \cup \Gamma_2)$, $i = 1,4,7$. The embedding of K in W is gotten by considering the subcomplex $(I^2 \times 0)_j \cup L_1 \cup F_2 \cup F_3 \cup L_4 \cup F_5 \cup F_6 \cup L_7 \cup_k (I^2 \times 0)$. Since $C_i \searrow L_i$, $i = 1,4,7$, $M_i \searrow F_i$, $i = 2,3,5,6$ and each $I^2 \times I^2 \searrow I^2 \times 0$ and the collapses are such that they match up on the corresponding parts, we get that $W \searrow K$.

THEOREM 1. $\pi_1(\mathrm{Bd}\, W) \neq 1$.

Proof. $\pi_1(\mathrm{Bd}\, W)$ can be obtained by looking at the fundamental group of $E^3 - (K_1 + K_2 + \Gamma_1 + \Gamma_2)$ as indicated in Figure 3 and adding in the relations corresponding to curves slightly above each of K_1, K_2, Γ_1 and Γ_2 respectively.

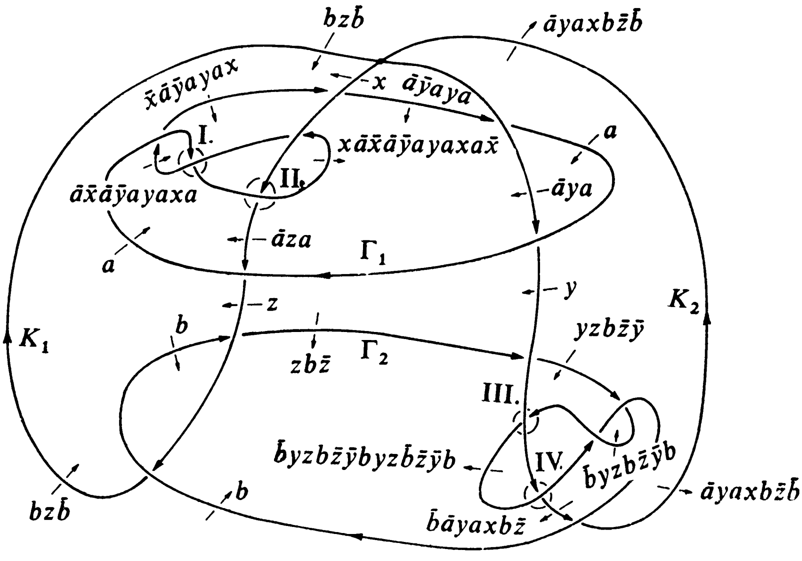

FIGURE 3

The resulting group has the following presentation:

generators: a, b, x, y, and z

relations:

I. $(\bar{a}\,\bar{x}\,\bar{a}\,\bar{y}\,\bar{a}\,y\,a\,x\,a)\,(x\,\bar{a}\,\bar{x}\,\bar{a}\,\bar{y}\,a\,y\,a\,x\,a\,\bar{x})\,(\bar{a}\,\bar{x}\,\bar{a}\,\bar{y}\,a\,y\,a\,x\,a)\bar{a} = 1,$

II. $\bar{x}(x\,\bar{a}\,\bar{x}\,\bar{a}\,\bar{y}\,a\,y\,a\,x\,a\,\bar{x})\bar{a}\,z\,a(x\,\bar{a}\,\bar{x}\,\bar{a}\,\bar{y}\,\bar{a}\,y\,a\,x\,a\,\bar{x}) = 1,$

III. $\bar{y}(\bar{b}\,y\,z\,\bar{b}\,\bar{z}\,\bar{y}\,b)\,y\,(\bar{b}\,y\,z\,b\,\bar{z}\,\bar{y}\,b\,y\,z\,\bar{b}\,\bar{z}\,\bar{y}\,b) = 1,$

IV. $\bar{y}(\bar{b}\, y\, z\, b\, \bar{z}\, \bar{y}\, b\, y\, z\, \bar{b}\, \bar{z}\, \bar{y}\, b)\, (\bar{b}\, \bar{a}\, y\, a\, x\, b\, \bar{z})\, (\bar{b}\, y\, z\, b\, \bar{z}\, \bar{y}\, \bar{b}\, y\, z\, \bar{b}\, \bar{z}\, \bar{y}\, b) = 1$,
Γ_1: $\bar{x}(\bar{a}\, \bar{y}\, a)\, (\bar{a}\, \bar{x}\, \bar{a}\, \bar{y}\, \bar{a}\, y\, a\, x\, a)x\, \bar{a} = 1$,
Γ_2: $\bar{z}\, \bar{y}\, b\, y(\bar{b}\, y\, z\, b\, \bar{z}\, \bar{y}\, b) = 1$,
K_1: $(x\, \bar{a}\, \bar{x}\, \bar{a}\, \bar{y}\, a\, y\, a\, x\, a\, \bar{x})\, \bar{a}\, \bar{b} = 1$,
K_2: $\bar{a}(\bar{b}\, y\, z\, b\, \bar{z}\, \bar{y}\, b\, y\, z\, \bar{b}\, \bar{z}\, \bar{y}\, b)\bar{b} = 1$.

We note that relations I–IV give $\pi_1(E^3 - (\Gamma_1 + \Gamma_2 + K_1 + K_2))$, adding in relations K_1 and K_2 give $\pi_1(2T - (\Gamma_1 + \Gamma_2))$, and adding in relations Γ_1 and Γ_2 gives $\pi_1(\mathrm{Bd}\, W)$.

Now Γ_1 gives that $\bar{a}\, \bar{x}\, \bar{a}\, \bar{y}\, \bar{a}\, y\, a\, x\, a = \bar{a}\, y\, a\, x\, a\, \bar{x}$. This relation applied to I gives $1 = 1$. By K_1 we have $x\, \bar{a}\, \bar{x}\, \bar{a}\, \bar{y}\, a\, y\, a\, x\, a\, \bar{x} = b\, a$. Applying this to II gives that $\bar{x}\, b\, z\, \bar{b} = 1$ or $z = \bar{b}\, x\, b$. Using Γ_2 and K_2 in III we get $\bar{y}(\bar{z}\, \bar{y}\, b\, y)y\, a\, b = 1$. Using K_2 and the fact that $z = \bar{b}\, x\, b$ in IV we get $1 = 1$.

Using Γ_2, $\bar{z}\, \bar{y}\, b\, y = \bar{b}\, y\, z\, \bar{b}\, \bar{z}\, \bar{y}\, b$ in K_2 gives that $\bar{a}(\bar{y}\, \bar{b}\, y\, z)y\, z\, \bar{b}\, \bar{z}\, \bar{y} = 1$. Next applying the new relation III $\bar{y}\, \bar{b}\, y\, z = y\, a\, b\, \bar{y}$ and $z = \bar{b}\, x\, b$ to the preceding relation for K_2 we then get $x\, \bar{b}\, \bar{x}\, b\, \bar{y} = \bar{a}\, \bar{y}\, a$.

Writing Γ_2 as $\bar{z}\, \bar{y}\, b\, y = \bar{b}\, y\, z\, \bar{b}\, \bar{z}\, \bar{y}\, b$, replacing z by $\bar{b}\, x\, b$ and using the fact that $x\, \bar{b}\, \bar{x}\, b\, \bar{y} = \bar{a}\, \bar{y}\, a$ we get that $\bar{x}\, b\, \bar{y}\, b\, y = y\, \bar{b}\, \bar{a}\, \bar{y}\, a\, b$.

In considering III, $\bar{z}\, \bar{y}\, b\, y = y\, \bar{b}\, \bar{a}\, \bar{y}$, if we replace z by $\bar{b}\, x\, b$ and $\bar{x}\, b\, \bar{y}$ by $b\, \bar{x}\, \bar{a}\, \bar{y}\, a$, we get $\bar{x}\, \bar{a}\, \bar{y}\, a\, b\, y = y\, \bar{b}\, \bar{a}\, \bar{y}$.

Finally, using Γ_1, $\bar{a}\, \bar{x}\, \bar{a}\, \bar{y}\, \bar{a}\, y\, a\, x\, a = \bar{a}\, y\, a\, x\, a\, \bar{x}$ in K_1, gives that $x^2\bar{a}\, \bar{x}\, \bar{a}\, \bar{y}\, a\, \bar{x}\, \bar{a} = b$.

Our group now has the following presentation:

I. $1 = 1$,
II. $z = \bar{b}\, x\, b$,
III. $\bar{x}\, \bar{a}\, \bar{y}\, a\, b\, y = y\, \bar{b}\, \bar{a}\, \bar{y}$,
IV. $1 = 1$,
Γ_1: $\bar{a}\, \bar{x}\, \bar{a}\, \bar{y}\, \bar{a}\, y\, a\, x\, a = \bar{a}\, y\, a\, x\, a\, \bar{x}$,
Γ_2: $\bar{x}\, b\, \bar{y}\, b\, y = y\, \bar{b}\, \bar{a}\, \bar{y}\, a\, b$,
K_1: $x^2\, \bar{a}\, \bar{x}\, \bar{a}\, \bar{y}\, a\, \bar{x}\, \bar{a} = b$,
K_2: $x\, \bar{b}\, \bar{x}\, b\, \bar{y} = \bar{a}\, \bar{y}\, a$.

If we replace the first x in K_2 by using relation III, and replaoe the $\bar{x}$ by using Γ_2 we get that $a\, b\, \bar{y}\, \bar{b}\, y\, \bar{b}\, \bar{a}\, \bar{y}\, a\, b\, y = 1$. Using the fact that the x from III equals the x from Γ_2, that is, that $\bar{a}\, \bar{y}\, a\, b\, y = b\, \bar{y}\, b\, y\, \bar{b}\, \bar{a}$, the preceding relation just becomes $1 = 1$.

Now using III, $x = \bar{a}\, \bar{y}\, a\, b\, y^2\, a\, b\, \bar{y}$ we get:

III $= \Gamma_2$: $\bar{a}\, \bar{y}\, a\, b\, y = b\, \bar{y}\, b\, y\, \bar{b}\, \bar{a}$,
K_1: $[\bar{a}\, \bar{y}\, a(b\, y^2\, a\, b\, \bar{y})]^2\, \bar{a}\, (y\, \bar{b}\, \bar{a}\, \bar{y}^2\, \bar{b})^2\, \bar{a}\, y = b$,
Γ_1: $\bar{a}(y\, \bar{b}\, \bar{a}\, \bar{y}^2\, \bar{b})\, \bar{a}\, (b\, y^2\, a\, b\, \bar{y}) = (b\, y^2\, a\, b\, \bar{y})\, a\, (y\, \bar{b}\, \bar{a}\, \bar{y}^2\, \bar{b})\, \bar{a}\, y$.

Here we have used in K_1 and Γ_1 the fact that III also gives $\bar{a}\, \bar{y}\, a = x\, y\, \bar{b}\, \bar{a}\, \bar{y}^2\, \bar{b}$.

Setting $\bar{a}\,y = \beta$ and $\alpha = y\,\bar{b}\,\bar{a}\,\bar{y}^2\,\bar{b}$ we get:
Γ_1: $\bar{a}\,\alpha\,\bar{a}\,\bar{\alpha} = \bar{\alpha}\,a\,\alpha\,\beta$,
K_1: $(\bar{a}\,\bar{\beta}\,\alpha)^2\bar{a}\,(\alpha^2)\,\beta = b$.

Also $x = \bar{a}\bar{\beta}\,\bar{\alpha}$.

Using Γ_1 to solve for β and applying this to K_1 we have $b = \bar{a}\,\alpha\,a\;\bar{\alpha}\,a\,\alpha\,\bar{a}\,\bar{\alpha}$. We now have b and β in terms of a and α; and hence y in terms of a and α also. Thus we now have only two relations to consider. Namely, $\alpha = y\,\bar{b}\,\bar{a}\,\bar{y}^2\,\bar{b}$ and $\bar{a}\,\bar{y}\,a\,b\,y = b\,\bar{y}\,b\,y\,\bar{b}\,\bar{a}$. Writing y and b in terms of a and α we get the following group presentation:

generators: a, α,
relations:

$$\bar{\alpha}\,a\,\bar{\alpha}\,\bar{a}\,\alpha\,\bar{a}\,\bar{\alpha}\,a\,\alpha\,\bar{a}\,\alpha\,a\;\bar{\alpha} = \bar{a}\,\alpha\,a\,\bar{\alpha}\,a\,\alpha\,\bar{a}\,\bar{\alpha}\,a\,\bar{\alpha}\,\bar{a}\,\alpha\,\bar{a}$$
$$= a\,\bar{\alpha}\,\bar{a}\,\alpha\,\bar{a}\,\bar{\alpha}\,a^2\,\bar{\alpha}\,\bar{a}\;\alpha\;\bar{a}\,\bar{\alpha}\,a\,\alpha\,\bar{a}\,\alpha\,a\,\bar{\alpha}\,a.$$

Now if we add the relation that $\bar{\alpha}\;a = \bar{a}\;\alpha$, the first equality becomes $\alpha^5 = a^5$ and the second $\bar{a}\;\alpha^3\;\bar{a}^2\;\bar{\alpha} = a\;\bar{\alpha}^3\;a^2\;\bar{\alpha}^3\;a^2\;\alpha$. Adding the relation $\alpha^5 = a^5 = (a^2\alpha^2)^2 = 1$, we get the group

$$\{a, \alpha \mid \bar{\alpha}a = \bar{a}\alpha, \alpha^5 = a^5 = (a^2\alpha^2)^2 = 1\}.$$

Replacing a^2 by u and α^2 by v we get

$$\{u, v \mid v^2u^3v^2 = u^2, u^5 = v^5 = (uv)^2 = 1\}.$$

This group can be shown to have a nontrivial representation in P_5 by letting $u \to (12345)$ and $v \to (12354)$. If we desire to check that this does indeed give a nontrivial representation of the original group, we have the following:

$$\alpha \to (15243), \quad \beta \to (254), \quad a \to (14253), \quad b \to (12543),$$
$$x \to (14352), \quad y \to (12453) \quad \text{and} \quad z \to (14523).$$

4. **Main results.** In this section we will discuss some additional properties of the pseudo 4-cell W and show how the particular chosen 2-complex K leads to the desired results.

LEMMA 2. *Suppose K is a contractible -subcomplex in the interior of a combinatorial 4-manifold M and W is a regular neighborhood of K in M. If K can be combinatorially embedded in E^3, then W can be embedded in E^4 and $W \times I \approx I^5$.*

Proof. By [1, Proposition 2] $W \times I^2 = I^6$. Since Bd$(W \times I^2)$ is homeomorphic to S^5 triangulated as a combinatorial 5-manifold and $2(W \times I) \approx$ Bd$(W \times I^2)$, $W \times I$ can be combinatorially embedded in a combinatorial triangulation of

E^5. Let K' be a combinatorial embedding of K in $E^3 \subset E^5$. Since the regular neighborhood of K' in E^3 is necessarily a combinatorial 3-cell, the regular neighborhood N of K' in E^5 is a combinatorial 5-cell. By the corollary of [6], this implies that $W \times I \approx N \approx I^5$. The fact that $2W \approx \mathrm{Bd}(W \times I) \approx S^4$ gives that W can be combinatorially embedded in E^4.

THEOREM 2. *There exists a pseudo 4-cell $W \neq I^4$ such that $W \subset E^4$, $W \times I \approx I^5$ and $W \approx X \cup Y$, where $X \approx Y \approx X \cap Y \approx I^4$.*

Proof. W is the pseudo 4-cell of §3. Since $\pi_1(\mathrm{Bd}\, W) \neq 1$ we have $W \neq I^4$. Since $W \searrow K$ and K can be embedded in E^3, the fact that $W \subset E^4$ and $W \times I \approx I^5$ follows from Lemma 2.

Let A be the middle polyhedral arc going from the vertex b to the vertex c in the top disk used in the construction of K. Similarly, let B be the middle polyhedral arc going from the vertex a to the vertex b in thebottom disk. If we separate K along the polyhedral arc $B \cup A$ we end up with two collapsible complexes which we will denote as K_1 and K_2. Hence $K \equiv K_1 \cup K_2$, $K_1 \cap K_2 \equiv B \cup A$ and each of K_1, K_2, and $K_1 \cap K_2$ collapses to a point. Let W' be a regular neighborhood of K in W under the secondary centric subdivision of W. Let X' be the regular neighborhood of K_1 and Y' the regular neighborhood of K_2 under this subdivision. Now $X' \cap Y'$ is combinatorially equivalent to the regular neighborhood of $K_1 \cap K_2 \equiv B \cup A$. Since $X' \searrow K_1 \searrow 0$, $Y' \searrow K_2 \searrow 0$, and $X' \cap Y' \searrow B \cup A \searrow 0$ we have $X' \approx Y' \approx X' \cap Y' \approx I^4$ by the results of Whitehead [7]. Again using [7] we have that $W \approx W'$ and hence the conclusion to the theorem.

COROLLARY 1. *For $n \geqq 4$ there exist pseudo n-cells $W^n \neq I^n$ such that*

$$W^n \times I \approx I^{n+1}$$

and $W^n \approx X^n \cup Y^n$, where $X^n \approx Y^n \approx X^n \cap Y^n \approx I^n$.

Proof. The result for $n = 4$ is just Theorem 2 and for $n \geqq 5$ follows from [3].

COROLLARY 2. *For $n \geqq 3$ there exists open contractible combinatorial n-manifolds $O^n \neq E^n$ such that $O^n \approx U^n \cup V^n$, where $U^n \approx V^n \approx U^n \cap V^n \approx E^n$.*

Proof. The result for $n \geqq 5$ follows from [3]. For $n = 4$ we use $U \approx \mathrm{int}\, X$, $V \approx \mathrm{int}\, Y$ and $O^4 \approx \mathrm{int}\, W$ of Theorem 2. We have that $O^4 \neq E^4$ since $\pi_1(\mathrm{Bd}\, W) \neq 1$. That is, if $O^4 = E^4$ then simple closed curves near "infinity" could be shrunk near "infinity", but the collar of Bd W is not simply connected.

For $n = 3$, the result has been known for some time, but apparently is not too well known. Hence for completeness, the example will be included here. Consider the double Fox-Artin arc A in S^3 intersecting the 2-sphere S^2 in the point p as indicated in Figure 4. Taking U' and V' as the two components of $S^3 - S^2$, one can easily express each of $U' - A$ and $V' - A$ as a monotone increasing sequence of open 3-cells. Let C' be a small double collar of S^2 so that $C' \cap A$ is an open straight line segment and let $C = C' - A$. Then taking $U = (U' - A) \cup C$ and $V = (V' - A) \cup C$ we have $S^3 - A = U \cup V$ where $U \approx V \approx E^3$ and $U \cap V \approx E^3$ since $U \cap V \approx C \approx \{S^2 - p\} \times (-1, 1)$. We get that $S^3 - A \neq E^3$ since $S^3 - (A + B)$ is not simply connected, where B is the simple closed curve in dicated in Figure 4. That is, if $S^3 - A = E^3$, then simple closed curve near "infinity" (here this means curves in an arbitrarily small neighborhood of A in S^3) could be shrunk missing B and this will not always be possible.

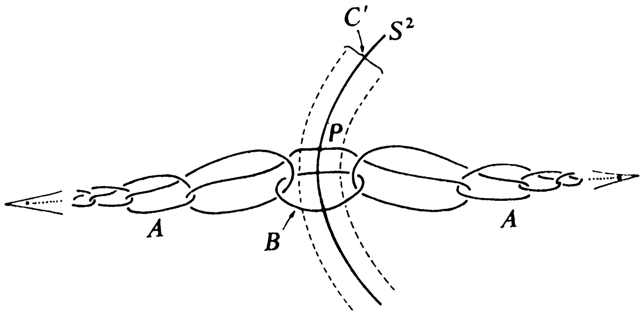

FIGURE 4

Clearly in the construction of W we could have altered slightly our embeddings of Γ_1 and Γ_2 in int T, say link Γ_1 or Γ_2 with itself differently, add local knots, or link Γ_1 with Γ_2, and still get a contractible 4-manifold with boundary which also collapses to K. Also, it is interesting to note that in some sense the given embeddings are the simplest possible in order to get an example where $\pi_1(\mathrm{Bd}\, W) \neq 1$. In fact the crucial part of the construction is the linking of Γ_1 over a and the linking of Γ_2 over c. Moreover, our next result says that as long as $\mathrm{lk}(a, K)$ and $\mathrm{lk}(c, K)$ are "nice", no matter how badly Γ_1 and Γ_2 are locally knotted or linked together in the middle section of T, if we repeat the same construction the resulting W is indeed $\approx I^4$.

In the following we apply some of the techniques of [8]. It is easy to see that each of $\mathrm{lk}(a, K)$ and $\mathrm{lk}(c, K)$ is merely two circles, C_1 and C_2 say, joined by an arc $A = xy$ (refer to Figure 2). We will say that the embedding of K in the interior of a combinatorial 4-manifold M^4 is nice at a if $\mathrm{lk}(a, K)$ in $\mathrm{lk}(a, M^4) \approx S^3$ is such that there exist a 2-sphere S^2 in $\mathrm{lk}(a, M^4)$ separating C_1 and C_2 and meeting A in a single point $z \in \mathrm{int}\, A$. Similarly for the vertex c. We note in the given construction that we have embedded K in W so that the circles corresponding to C_1 and C_2 in each of $\mathrm{lk}(a, K)$ and $\mathrm{lk}(c, K)$ are linked in $\mathrm{lk}(a, W)$ and $\mathrm{lk}(c, W)$ respectively.

THEOREM 3. *Let $K \subset \mathrm{int}\, M^4$ and suppose $M^4 \searrow K$. If the embedding of K is nice at a and c, then $M^4 \approx I^4$.*

Proof. Let us write $\mathrm{lk}(a, K) = C_1 \cup A \cup C_2$ and $\mathrm{lk}(c, K) = C_1' \cup A' \cup C_2'$. There exists a 2-dimensional polyhedron P such that:

(i) $C_1 \subset P \subset \mathrm{lk}(a, M^4)$;
(ii) $P \searrow x$;
(iii) $P \cap A = x$;
(iv) $P \cap C_2 = \varnothing$.

Such a P is not difficult to get and the actual construction of such a polyhedron is given in the proof of Theorem 8 of [8]. Similarly there exists a P' such that:

(i) $C_1' \subset P' \subset \mathrm{lk}(c, M^4)$;
(ii) $P' \searrow x'$;
(iii) $P' \cap A' = x'$;
(iv) $P' \cap C_2' = \varnothing$.

Now C_1 intersects either γ or δ in a single point x and C_1' intersects one of α or β in x'. Recall we used α,β,γ and δ in defining K (refer back to Figure 1). For notational purposes let us suppose that $C_1 \cap \gamma \neq \varnothing$ and $C_1' \cap \alpha \neq \varnothing$. Now we have the following: $M^4 \searrow K \nearrow K \cup aP \nearrow K \cup aP \cup cP'$. Since $P \searrow x$ and $P' \searrow x'$ we have $aP \searrow ax \cup P$ and $cP' \searrow cx' \cup P'$. Therefore,

$$K \cup aP \cup cP' \searrow \mathrm{Cl}(K - aC_1 - cC_1') \cup P \cup P'$$

which we will denote by K'.

Let us consider the top half of K'. cx' is now a free edge and hence we can collapse the right half and back part of the top half to the remainder $\cup P$. Then we can collapse $P \searrow x$ and the remaining complex of the top half to δ. Similarly, in considering the bottom half of K', we have that ax is a free edge on this half and hence we can collapse this half to β. Hence we have $K' \searrow \delta \cup \beta \searrow b$. We now have obtained a sequence of elementary collapses and expansions going from M^4 to b; hence by Lemma 3 of [8], $M^4 \approx I^4$.

COROLLARY 3. *If $K \subset \mathrm{int}\ M^n$ $(n \geqq 5)$ and $M^n \searrow K$, then $M^n \approx I^n$.*

Proof. Since $n \geqq 5$ we can get C_1 to bound a disk P in $\mathrm{lk}(a, M^n)$ and C_1 to bound a disk P' in $\mathrm{lk}(c, M^n)$ with the same properties as the P and P' of Theorem 3.

We can also prove Corollary 3 by making use of [6]. That is $M^n \times I = I^{n+1}$ and hence M^n can be embedded in a combinatorial triangulation of E^n. Since K can be embedded in E^3, say as K', and $n \geqq 5$, the corollary of [6] says that the regular neighborhoods of K' and K in E^n are combinatorially equivalent and hence $M^n \approx I^n$.

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