SOME NORMAL SUBGROUPS OF HOMEOMORPHISMS

BY

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1. **Introduction.** In a recent paper [1], Brown and Gluck have considered the question of which homeomorphisms $h \in H(X)$ of a space X onto itself have the property (a) that for given subsets $A, B \subset X$, there is an $f \in H(X)$ which agrees with the identity on A and with h on B. They have confined themselves to the case where X is locally euclidean. In this case, a natural choice for A is the complement of a euclidean neighborhood, and for B the closure of a euclidean neighborhood. They are able to show that the set of h which satisfy (a) is just the group $P(X)$ generated by the elements of $H(X)$ which agree with the identity on some open set, provided X has what they call a stable structure. In case X is a combinatorial manifold, the existence of a stable structure is equivalent to orientability. Their methods rely heavily on recent results concerning locally flat embeddings of spheres and the generalized Schoenflies theorem.

In the present paper, we extend some of their results to a more general class of topological spaces which includes locally normed linear spaces. The definition of stable structure readily extends to the latter class of spaces, and we are able to show that the existence of a stable structure is equivalent to the condition that some h other than the identity should satisfy (a). However, we are unable to identify $P(X)$ as the set of h satisfying (a) in the infinite dimensional case except when X is something like a sphere in a normed linear space.

2. **Bridging homeomorphisms.** Let X be a topological space, and $H(X)$ the group of all homeomorphisms h of X onto itself. The restriction of h to a subset A of X will be denoted by $h \mid A$, and $g \mid A = h \mid A$ will be shortened to $g = h \mid A$. The identity of $H(X)$ is e. We define $P(X)$ to be the set of elements $g \in H(X)$ satisfying the following condition:

(i) For each $x \in X$, there is a finite subset $\Delta(g, x) \subset X$ such that, for every $y \in X - \Delta(g, x)$, we can find neighborhoods U of x and V of y and an $f \in H(X)$ satisfying $f = e \mid U$ and $f = g \mid V$.

Figuratively speaking, f is a bridge between e and g. It may happen that $\Delta(g, x)$ is empty, but if $g(x) \neq x$, then clearly $x, g^{-1}(x) \in \Delta(g, x)$.

LEMMA 1. *$P(X)$ is a normal subgroup of $H(X)$.*

Received the editors January 11, 1965.

Proof. Evidently $e \in P(X)$. Let $g_1, g_2 \in P(X)$, $x \in X$ be given, and choose $y \in X - \Delta$, where

$$\Delta = \Delta(g_1, x) \cup g_1^{-1} g_2(\Delta(g_2, x)).$$

Then we can find neighborhoods U_1 of x, V_1 of y, and $f_1 \in H(X)$ satisfying $f_1 = e \mid U_1$ and $f_1 = g_1 \mid V_1$. In addition, $y \in X - \Delta$ implies $g_2^{-1} g_1(y) \in X - \Delta(g_2, x)$, and there are neighborhoods U_2 of x, V_2 of $g_2^{-1} g_1(y)$ and $f_2 \in H(X)$ with the property that $f_2 = e \mid U_2$ and $f_2 = g_2 \mid V_2$. If we set $U = U_1 \cap U_2$ and $V = V_1 \cap g_1^{-1} g_2(V_2)$, then $f_1 = e \mid U$ and $f_2^{-1} f_1 = f_2^{-1} = e \mid U$. Since $f_2^{-1} = g_2^{-1} \mid g_2(V_2)$ and $g_2(V_2) \supset g_1(V)$, we have $f_1 = g_1 \mid V$ and $f_2^{-1} f_1 = f_2^{-1} g_1 = g_2^{-1} g_1 \mid V$. Thus, $f_2^{-1} f_1$ is a bridge between e and $g_2^{-1} g_1$, where $\Delta = \Delta(g_2^{-1} g_1, x)$ is finite, and $g_2^{-1} g_1 \in P(X)$, so that $P(X)$ is a subgroup of $H(X)$. Now let $g \in P(X)$, $h \in H(X)$, $x \in X$ be given, and set $\Delta' = h^{-1}(\Delta(g, h(x)))$. Since $y \in X - \Delta'$ implies $h(y) \in X - \Delta(g, h(x))$, we can find neighborhoods U_3 of $h(x)$, V_3 of $h(y)$ and an $f \in H(X)$ satisfying $f = e \mid U_3$ and $f = g \mid V_3$. Setting $U = h^{-1}(U_3)$ and $V = h^{-1}(V_3)$, we have $h^{-1} f h = e \mid U$ and $h^{-1} f h = h^{-1} g h \mid V$. Consequently, $h^{-1} f h$ bridges e and $h^{-1} g h$, where $\Delta' = \Delta(h^{-1} g h, x)$ is finite, and $h^{-1} g h \in P(X)$, so that $P(X)$ is normal in $H(X)$.

For each $A \subset X$, we set $S(A) = \{h \in H(X) : h = e \mid X - A\}$. We denote by $\mathscr{K}(X)$ the set of nonempty, connected, open subsets U of X with the property that, for every $x, y \in U$, we can find $f \in S(U)$ such that $f(x) = y$. Evidently $h \in H(X)$ and $U \in \mathscr{K}(X)$ implies $h(U) \in \mathscr{K}(X)$, for $S(h(U)) = h S(U) h^{-1}$. If we let

$$K(X) = \bigcup \{U \in \mathscr{K}(X)\},$$

then $K(X)$ is open, and $h(K(X)) = K(X)$. From this and the proof of Lemma 1, it is clear that if we amend the definition of $P(X)$ by requiring in (i) that x and y belong only to $K(X)$, then the lemma remains valid. We will assume the amendment to be in effect from now on.

THEOREM 1. *Suppose X is a Hausdorff space, each open subset contains a member of $\mathscr{K}(X)$, and $K(X)$ can not be separated by any two points. Then $P(X)$ coincides with the set of all $h \in H(X)$ of the form $h = f_1 f_2$, where $f_i \in S(X - U_i)$ for some open $U_i \neq \varnothing$ $(i = 1, 2)$.*

Proof. Let $g \in P(X)$ be given, and choose $x, y \in K(X)$, neighborhoods U_1 of x and U_2 of y, and $f_1 \in H(X)$ so as to satisfy (i). Then $f_1 \in S(X - U_1)$, $f_2 = f_1^{-1} g \in S(X - U_2)$, and $g = f_1 f_2$. The theorem will be proved if we can show that $S(X - U) \subset P(X)$ for every open $U \neq \varnothing$, for Lemma 1 implies that the product of two such elements must lie in $P(X)$. Let $f \in S(X - U)$, $x \in K(X)$, and $y \in K(X) - \Delta$ be given, where $\Delta = \{x, f(x)\}$. We know that $K(X)$ is connected and dense in X, so that there is a chain $U_1, \cdots, U_n$ of elements of $\mathscr{K}(X)$ connecting x and some $u \in U \cap K(X)$ which lies entirely in $K(X) - \{y, f(y)\}$. Since X is a Hausdorff space, we can even assume that

$$A = \bar{U}_1 \cup \cdots \cup \bar{U}_n \subset K(X) - \{y, f(y)\}.$$

By a standard construction, we can find $h_i \in S(U_i)$, $1 \leqq i \leqq n$, such that $h(x) = h_n \cdots h_2 h_1(x) = u$. If we set $V = h^{-1}(U)$ and

$$W = (K(X) - A) \cap f^{-1}(K(X) - A),$$

then V, W are neighborhoods of x, y, respectively. Now $f = e \mid h(V)$, whence $h^{-1}fh = e \mid V$, and $h = e \mid W \cup f(W)$, whence $h^{-1}fh = f \mid W$. Thus $h^{-1}fh$ bridges e and f, where $\Delta = \Delta(f, x)$ is finite, and $f \in P(X)$.

The assumption that $K(X)$ should not be separated by a pair of points can not be omitted entirely, for if X is the line E^1, then $P(X) = \{e\}$, while the group F generated by $S(X - U)$, as U runs through the open sets, is the group of order-preserving homeomorphisms. On the other hand, if X is the circle S^1, then the two groups coincide. For F contains all conjugates of $S(X - U)$ and is thus a normal subgroup of $H(S^1)$, whereas $P(X)$ contains the rotations and is thus different from $\{e\}$, while $H(S^1)$ has only one normal subgroup [2]. But here our separation hypothesis is not satisfied.

We define $R(X)$ to be the set of $g \in H(X)$ with the following property:

(ii) For every $x \in K(X)$ and every connected, open subset U of $K(X)$ containing x and $g(x)$, there is a neighborhood V of x and an $f \in S(U)$ satisfying $f = g \mid V$.

Lemma 2. *Suppose X satisfies the same conditions as in Theorem* 1. *Then $R(X)$ is a normal subgroup of $H(X)$.*

Proof. Clearly $e \in R(X)$. Let $g_1, g_2 \in R(X), x \in K(X)$, and $U \subset K(X)$ be given, where U is a connected neighborhood of $\{x, y\}$, and $y = g_2^{-1}g_1(x)$. As in the proof of Theorem 1, we can find an $h_0 \in H(X)$ such that $h_0(g_1(x)) \in U$ and $h_0 = e \mid V_0$ for some neighborhood V_0 of $\{x, y\}$. Then $W = h_0^{-1}(U)$ is a connected neighborhood of $\{x, g_1(x)\}$ and also of $\{y, g_2(y)\}$, so that there are neighborhoods V_1 of x, V_2 of y, and elements $f_1, f_2 \in H(X)$ satisfying

$$f_1 = e \mid X - W,\ f_1 = g_1 \mid V_1,\ f_2 = e \mid X - W,\ f_2 = g_2 \mid V_2.$$

Thus $f_2^{-1}f_1 = e \mid X - W$, $f_2^{-1} = g_2^{-1} \mid g_2(V_2)$, and

$$f_2^{-1}f_1 = f_2^{-1}g_1 = g_2^{-1}g_1 \mid V_1 \cap g_1^{-1}g_2(V_2).$$

If we set

$$V = V_1 \cap g_1^{-1}g_2(V_2) \cap V_0 \cap g_1^{-1}g_2(V_0),$$

then V is a neighborhood of x, $h_0 f_2^{-1} f_1 h_0^{-1} = e \mid X - U$, and $h_0 f_2^{-1} f_1 h_0^{-1} = g_2^{-1}g_1 \mid V$ so that $h_0 f_2^{-1} f_1 h_0^{-1}$ is a bridge between e and $g_2^{-1}g_1$. Thus $g_2^{-1}g_1 \in R(X)$, and $R(X)$ is a group. To show that $R(X)$ is normal in $H(X)$, let $g \in R(X)$, $h \in H(X)$, $x \in K(X)$, and $U \subset K(X)$ be given, where U is a connected neighborhood of $\{x, h^{-1}gh(x)\}$. Then $h(U)$ is a connected neighborhood of $\{h(x), g(h(x))\}$, and we can find a neighborhood V of $h(x)$ and an $f \in H(X)$ satisfying $f = e \mid X - h(U)$ and $f = g \mid V$. Thus $h^{-1}fh = e \mid X - U$, $h^{-1}fh = h^{-1}gh \mid h^{-1}(V)$, and $h^{-1}(V)$ is a neighborhood of x, whence $h^{-1}fh$ is a bridge between e and $h^{-1}gh$. Consequently, $h^{-1}gh \in R(X)$, and $R(X)$ is normal in $H(X)$.

Although the connectedness of U was not used in the proof, some such assumption is needed to ensure that $R(X) \neq \{e\}$, as we shall see later. When X is E^1 or S^1, the hypotheses of the lemma are not fulfilled, but it is easily seen that $R(X)$ is the normal subgroup of orientation-preserving homeomorphisms in either case.

We now define $\mathscr{L}(X)$ to be the set of $U \in \mathscr{K}(X)$ with the following property:

(iii) For every pair of open sets V, W such that $\varnothing \neq V \subset U$ and $\bar{U} \subset W$, there is an $h \in S(W)$ satisfying $h(V) \supset \bar{U}$.

We observe that $g \in H(X)$ implies $ghg^{-1} \in S(g(W))$ and $ghg^{-1}(g(V)) \supset g(\bar{U})$, and by reversing the steps, we infer that $g(U) \in \mathscr{L}(X)$. Moreover, if $K(X)$ is connected and $\mathscr{L}(X) \neq \varnothing$, then $K(X) = \bigcup \{U \in \mathscr{L}(X)\}$, for we have already seen that $H(X)$ acts transitively on $K(X)$ in this case.

Theorem 2. *Suppose X is a topological space in which every open set is infinite, $g \in P(X)$, and U, $V \in \mathscr{L}(X)$ such that $\bar{U} \cap (\bar{V} \cup g(\bar{V})) = \varnothing$. Then there is an $f \in H(X)$ with the properties $f = e \mid \bar{U}$ and $f = g \mid \bar{V}$.*

Proof. Since V is infinite, we can choose $x \in U, y \in V - \Delta(g,x)$, and find neighborhoods $U_1 \subset U$ of x, $V_1 \subset V$ of y, and $f_1 \in H(X)$ satisfying $f_1 = e \mid U_1$ and $f_1 = g \mid V_1$. From the definition of $\mathscr{L}(X)$, there is an $h_1 \in H(X)$ with the properties $h_1 = e \mid \bar{V} \cup g(\bar{V})$ and $h_1(U_1) \supset \bar{U}$. Then $f_2 = h_1 f_1 h_1^{-1} = e \mid h_1(U_1)$ and $f_2 = f_1 \mid V_1$, whence $f_2 = e \mid \bar{U}$ and $f_2 = g \mid V_1$. Now $\bar{U} \cap (\bar{V} \cup g(\bar{V})) = \varnothing$ implies $(g^{-1}f_2(\bar{U}) \cup \bar{U}) \cap \bar{V} = \varnothing$, and we can find $h_2 \in H(X)$ satisfying $h_2 = e \mid \bar{U} \cup g^{-1} f_2(\bar{U})$ and $h_2(V_1) \supset \bar{V}$. Hence, $f_3 = h_2 g^{-1} f_2 h_2^{-1} = e \mid h_2(V_1)$ and $f_3 = g^{-1}f_2 \mid \bar{U}$, so that $f_3 = e \mid \bar{V}$ and $f_3 = g^{-1} \mid \bar{U}$. Finally, $f = gf_3 = e \mid \bar{U}$ and $f = g \mid \bar{V}$.

Theorem 3. *Let X be a topological space, $g \in R(X)$, $U \subset K(X)$ a connected open set, and $V \in \mathscr{L}(X)$ such that $\bar{V} \cup g(\bar{V}) \subset U$. Then there is an $f \in S(U)$ satisfying $f = g \mid \bar{V}$.*

Proof. If $x \in V$ is chosen, then we can find $f_1 \in S(U)$ and a neighborhood $V_1 \subset V$ of x with the property $f_1 = g \mid V_1$. Since $\bar{V} \cap (X - (U \cup g^{-1}(U))) = \varnothing$, there is an $h_1 \in S(U \cup g^{-1}(U))$ satisfying $h_1(V_1) \supset \bar{V}$. Then $f_2 = h_1 g^{-1} f_1 h_1^{-1} = g^{-1} \mid X - U$, and $f_2 = e \mid h_1(V_1)$, whence $f_2 = e \mid \bar{V}$. Finally, $f = gf_2 = e \mid X - U$ and $f = g \mid \bar{V}$.

3. **Bridging over connected sets.** We saw in Theorem 1 that, under fairly weak conditions on $X, P(X) \neq \{e\}$. But, so far, we have not considered the question of when $R(X) \neq \{e\}$. Let $\mathscr{M}(X)$ be the set of $U \in \mathscr{L}(X)$ with the following property:

(iv) For every open subset V of U and every $x \in V$, there is a neighborhood W

of x and an $h \in H(X)$ such that $h(V) \supset \bar{U}$ and $h = e \mid W$. The proof that $g \in H(X)$ implies $g(U) \in \mathscr{M}(X)$ and that $K(X)$ connected and $\mathscr{M}(X) \neq \varnothing$ implies $K(X) = \cup\{U \in \mathscr{M}(X)\}$ follows that of the corresponding statements for $\mathscr{L}(X)$.

We now define $\bar{R}(X)$ to be the set of $g \in H(X)$ with the following property:

(v) For every $x \in K(X)$, there is a $U \in \mathscr{M}(X)$ containing x and $g(x)$, a neighborhood V of x, and an $f \in S(U)$ satisfying $f = g \mid V$.

Comparison of (ii) and (v) shows that $\bar{R}(X) \supset R(X)$. The proof that $g \in \bar{R}(X)$ and $h \in H(X)$ implies $ghg^{-1} \in \bar{R}(X)$ follows that of the corresponding statement for $R(X)$. It is also clear that $g^{-1} \in \bar{R}(X)$, so that $\bar{R}(X)$ is almost a normal subgroup of $H(X)$. We set $T(X) = \{g \in S(U) \colon U \in \mathscr{M}(X)\}$ and observe that if $K(X)$ is connected, then $T(X) \subset \bar{R}(X)$. For if $g \in S(U_0)$ and $U_0 \in \mathscr{M}(X)$, then $U_0 \in \mathscr{L}(X)$ and setting $U = V = U_0$ in (iii), we obtain $h \in H(X)$ satisfying $h(U_0) \supset \bar{U}_0$ and $h(U_0) \in \mathscr{M}(X)$. Now if $x \in \bar{U}_0 \cap K(X)$, then (v) will hold with $U = V = h(U_0)$ and $f = g$. If $x \in K(X) - \bar{U}_0$, then (v) holds with $f = e$ and any choices for U, $V \in \mathscr{M}(X)$ which contain x.

THEOREM 4. *Suppose X is a Hausdorff space, each open subset contains a member of $\mathscr{K}(X)$, $K(X)$ is connected, and $\mathscr{M}(X) \neq \varnothing$. Then the following conditions are equivalent*:

(a) $R(X) \neq \{e\}$,

(b) $R(X) = \bar{R}(X)$,

(c) $g_1, g_2 \in \bar{R}(X)$ *implies* $g_1 g_2 \in \bar{R}(X)$,

(d) $g_1, g_2 \in T(X)$ *implies* $g_1 g_2 \in \bar{R}(X)$,

(e) $T(X) \subset R(X)$.

Proof. (a) implies (b). Choose $g_0 \in R(X)$ and $x_0 \in X$ so that $g_0(x_0) \neq x_0$. Then we can find a neighborhood U_0 of x_0 satisfying $g_0(U_0) \cap U_0 = \varnothing$. Let $g \in \bar{R}(X)$, $x \in K(X)$, and $U \subset K(X)$ be given, where U is a connected nieghborhood of x and $g(x)$. From (v) we obtain $U_1 \in \mathscr{M}(X)$ containing x and $g(x)$, a neighborhood V_1 of x, and an $f_1 \in S(U_1)$ such that $f_1 = g \mid V_1$. Since $K(X) \cap U_0 \neq \varnothing$, and $H(X)$ acts transitively on $K(X)$, we can find $h_0 \in H(X)$ satisfying $h_0(U_1) \cap U_0 \neq \varnothing$. From $h_0(U_1) \in \mathscr{L}(X)$ we infer that there is an $h_1 \in H(X)$ such that $h_1(h_0(U_1) \cap U_0) \supset h_0(\bar{U}_1)$. Setting $h = h_1^{-1} h_0$, we have $h(U_1) \in \mathscr{L}(X)$, $h(\bar{U}_1) \subset U_0$, and $f_2 = h f_1 h^{-1} \in S(U_0)$. Now $g_1 = (f_2 g_0^{-1} f_2^{-1}) g_0 \in R(X)$ from Lemma 2, and we verify directly that $g_1 = f_2 \mid U_0$. Moreover, $h(U)$ is a connected neighborhood of $h(x)$ and $h(g(x)) = h(f_1(x)) = f_2(h(x)) = g_1(h(x))$, and (ii) tells us that there is an $f_3 \in S(h(U))$ and a neighborhood V_2 of $h(x)$ with the property $f_3 = g_1 \mid V_2$. Finally, if we set $f = h^{-1} f_3 h$ and $V = V_1 \cap h^{-1}(V_2)$, then $f \in S(U)$ and

$$f = h^{-1} f_3 h = h^{-1} g_1 h = h^{-1} f_2 h = f_1 = g \mid V.$$

Thus $g \in R(X)$, $\bar{R}(X) \subset R(X)$, and $\bar{R}(X) = R(X)$.

(b) implies (c). This follows from Lemma 2.

(c) implies (d). This follows from $T(X) \subset \bar{R}(X)$.

(d) implies (e). Let $g \in T(X), x_1 \in K(X)$, and $U_1 \subset K(X)$ be given, where U_1 is a connected neighborhood of x_1 and $g(x_1)$. Evidently $\mathscr{M}(X)$ is a base for $K(X)$ in its relative topology, for $K(X) = \bigcup\{U \in \mathscr{M}(X)\}$, and the conclusion of (iv) can be rewritten as $h^{-1}(\bar{U}) \subset V$, where $h^{-1}(U) \in \mathscr{M}(X)$, and $x \in h^{-1}(U)$. Since $K(X)$ is connected, and points are closed, each member of $\mathscr{M}(X)$ has at least two points. Choose $V_1 \in \mathscr{M}(X)$ so that $x_1 \in V_1 \subset U_1$. Then we can find a chain of elements from $\mathscr{M}(X)$ joining x_1 with $g(x_1)$ which lie inside U_1. Hence, there is an $h_1 \in H(X)$ such that $h_1(x_1) = x_1$ and $g(x_1) \in h_1(V_1) \subset U_1$. Since $h_1(V_1) \in \mathscr{M}(X)$, we can find $g_1 \in S(h(V_1))$ satisfying $g_1(x_1) = g(x_1)$. In addition, $g_1^{-1} \in T(X)$, and by hypothesis, $g_1^{-1}g \in \bar{R}(X)$. Now $g_1^{-1}g(x_1) = x_1$, and from (v) we have $V_2 \in \mathscr{M}(X)$ containing x_1, $f_1 \in S(V_2)$, and a neighborhood W_1 of x_1 with the property $f_1 = g_1^{-1}g \mid W_1$. From (iv) we know that there is an $h_2 \in H(X)$ satisfying $h_2(U_1 \cap V_2) \supset \bar{V}_2$ and $h_2 = e \mid W_2$, where W_2 is some neighborhood of x_1. Then $h_2^{-1}f_1h_2 \in S(h_2^{-1}(V_2)) \subset S(U_1)$ and

$$f_2 = h_2^{-1}f_1h_2 = g_1^{-1}g \mid W_1 \cap W_2 \cap f_1^{-1}(W_2).$$

If we set $W = W_1 \cap W_2 \cap f_1^{-1}(W_2)$, then $x_1 \in W$, $g_1f_2 = g \mid W$, and $g_1f_2 \in S(U_1)$ Thus $g \in R(X)$ and $T(X) \subset R(X)$.

(e) implies (a). We have already seen that the members U of $\mathscr{M}(X)$ have more than one point, so that $S(U) \neq \{e\}$ and $T(X) \neq \{e\}$.

We define $Q(X)$ to be the group generated by the elements of $T(X)$ and note that $Q(X)$ is a normal subgroup of $H(X)$. For if $h \in H(X)$ and $U \in \mathscr{M}(X)$, then $hS(U)h^{-1} = S(h(U)) \subset T(X)$. Also, $Q(X) \subset P(X)$ provided X is Hausdorff, for if $g \in S(U)$ and $U \in \mathscr{M}(X)$, then we can find $h \in H(X)$ so that $X - h(U)$ contains an open set, whence $hgh^{-1} \in S(h(U)) \subset P(X)$, and $g \in P(X)$.

Corollary 1. *With the same hypotheses as in Theorem 4, suppose, for every open set $V \neq \varnothing$ in X, there is a $U \in \mathscr{M}(X)$ such that $U \cup V = X$. Then $P(X) = Q(X) = R(X)$.*

Proof. Let $g \in P(X)$ be given, where $g = e \mid V$ for some open set $V \neq \varnothing$. Then we can find $U \in \mathscr{M}(X)$ satisfying $U \cup V = X$, and observe that $g \in S(U) \subset Q(X)$, so that $P(X) \subset Q(X)$ and $P(X) = Q(X)$. If $g' \in R(X)$ is given, then we can choose a connected neighborhood W of some x and $g'(x)$ so that $X - W$ contains an open set. If $f' \in S(W)$ agrees with g' on a neighborhood of x, then $g' = f'(f'^{-1}g') \in P(X)$, whence $R(X) \subset P(X)$. Finally, if we can show that $Q(X) \subset \bar{R}(X)$, then condition (d) of Theorem 4 will be verified, and $Q(X) = R(X)$. Let $g'' \in Q(X)$ and $x \in K(X)$ be given, and U_0, $V_0 \in \mathscr{M}(X)$ be so chosen that $U_0 \cup V_0 = X$. Now $\bar{U}_0 \neq X$, for otherwise we could let $U = U_0$ and V be a proper subset of X in (iv) and obtain a contradiction. As in the proof of Theorem 4, we can construct $h \in H(X)$ so as to map $X - \bar{U}_0$ over x and $g''(x)$. Choose $W_0 \in \mathscr{M}(X)$ with the properties $x \in W_0$ and $\bar{W}_0 \cup g''(\bar{W}_0) \subset X - h(\bar{U}_0)$ Since

$g'' \in P(X)$ and $h(U_0) \in \mathscr{M}(X)$, Theorem 2 implies the existence of an $f'' \in H(X)$ satisfying $f'' = e \mid h(\bar{U}_0)$ and $f'' = g'' \mid \bar{W}_0$. Moreover, $h(U_0) \cup h(V_0) = X$ implies $f'' \in S(h(V_0))$, whence $g'' \in \bar{R}(X)$ and $Q(X) \subset \bar{R}(X)$.

Corollary 2. *With the same hypotheses as in Theorem* 4, *suppose there is a subset* $\mathscr{M}'(X)$ *of* $\mathscr{M}(X)$ *which forms a base for* $K(X)$ *in its relative topology and satisfies the conditions* (1) *that* $U, V \in \mathscr{M}'(X)$ *implies* $\bar{U} \cup \bar{V}$ *lies in some member of* $\mathscr{M}'(X)$, (2) $U \in \mathscr{M}'(X)$, $V \in \mathscr{M}(X)$, *and* $V \subset U$ *implies* $V \in \mathscr{M}'(X)$. *Then* $R(X) \neq \{e\}$.

Proof. Let $T'(X) = \{f \in S(U) : U \in \mathscr{M}'(X)\}$, and consider the following variants from Theorem 4:

(d′) $g_1, g_2 \in T'(X)$ implies $g_1 g_2 \in \bar{R}(X)$,

(e′) $T'(X) \subset R(X)$.

Clearly (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d′), and (e′) $\Rightarrow$ (a). We will complete the cycle by showing that $(d') \Rightarrow$ (e′). The proof that (d) $\Rightarrow$ (e) needs only to be modified by replacing $\mathscr{M}(X)$ with $\mathscr{M}'(X)$ and $T(X)$ with $T'(X)$, except that $h_1(V_1) \in \mathscr{M}'(X)$ needs to be verified directly. Now $h_1(V_1)$ lies in the union of a finite collection of members of $\mathscr{M}'(X)$, and conditions (1) and (2) imply that $h_1(V_1) \in \mathscr{M}'(X)$. Hence, (d′) $\Rightarrow$ (e′) $\Rightarrow \cdots \Rightarrow$ (e), and (e) implies $Q(X) \subset R(X)$. To complete the proof, we will verify (d′) directly. Let $g_i \in S(U_i)$, $U_i \in \mathscr{M}'(X)$ $(i = 1, 2)$, and $x \in K(X)$ be given. We have already seen that if $x \in K(X) - (\bar{U}_1 \cup \bar{U}_2)$, then (v) holds with $f = e$. If $x \in \bar{U}_1 \cup \bar{U}_2$, then we choose $U_3 \in \mathscr{M}'(X)$ so that $\bar{U}_1 \cup \bar{U}_2 \subset U_3$ and notice that (v) is satisfied with $f = g_1 g_2$ and $U = V = U_3$. Therefore, $g_1 g_2 \in \bar{R}(X)$.

Corollary 3. *With the same hypotheses as in Theorem* 4, $R(X) \neq \{e\}$ *if, and only if, there is a subfamily* $\mathscr{M}^*(X)$ *of* $\mathscr{M}(X)$ *which is a base for* $K(X)$ *and has the property that if* $U, U_1, \cdots, U_n \in \mathscr{M}^*(X)$, $x \in U$, $f_i \in S(U_i)$, $f_i f_{i-1} \cdots f_1(x) \in U_i$ $(1 \leqq i \leqq n)$, *and* $f_n f_{n-1} \cdots f_1(x) \in U$, *then there is a neighborhood* V *of* x *and an* $f \in S(U)$ *such that* $f = f_n \cdots f_2 f_1 \mid V$.

Proof. If $R(X) \neq \{e\}$, then we can evidently set $\mathscr{M}^*(X) = \mathscr{M}(X)$. Suppose, conversely, that $\mathscr{M}^*(X)$ is given, and choose $U \in \mathscr{M}^*(X)$ and $g \in S(U)$. To show that $g \in R(X)$, let $x \in U$ and $W \subset K(X)$ a connected neighborhood of x and $g(x)$ be given. Since $\mathscr{M}^*(X)$ is a base for $K(X)$, we can find a chain $U_1, \cdots, U_n \in \mathscr{M}^*(X)$ satisfying $x \in U_1$, $g(x) \in U_n$, and $\bigcup\{U_i : 1 \leqq i \leqq n\} \subset W$. Choose $x_i \in U_i \cap U_{i+1}$ and $f_i \in S(U_i)$ so that $f_i(x_{i-1}) = x_i$ $(1 \leqq i \leqq n-1)$, where we set $x_0 = x$. Now $x_{n-1} \in U_n$,

$$g f_1^{-1} f_2^{-1} \cdots f_{n-1}^{-1}(x_{n-1}) = g(x) \in U_n,$$

and from the definition of $\mathscr{M}^*(X)$, we can find $f_n \in S(U_n)$ and a neighborhood V of x_{n-1} such that $f_n = g f_1^{-1} f_2^{-1} \cdots f_{n-1}^{-1} \mid V$. Then $g = f_n \cdots f_2 f_1 \mid V'$, where $V' = f_1^{-1} f_2^{-1} \cdots f_{n-1}^{-1}(V)$ is a neighborhood of x, while $f_n \cdots f_2 f_1 \in S(W)$. Hence, $g \in R(X)$.

4. **Normed linear neighborhoods.** An open subset U of a topological space X will be called a normed linear neighborhood if U, in its relative topology, is homeomorphic to an open ball of positive radius in a normed linear space N over the real numbers. We will call X a regional normed linear space if every open subset contains a normed linear neighborhood. The choice of N may vary for different neighborhoods. Finally, we denote by $\mathcal{N}(X)$ the family of all open subsets of X with the following property:

(vi) There is a neighborhood V of $\bar{U}$ and a homeomorphism θ which maps V onto an open subset of a normed linear space N in such a a way that $\theta(\bar{U})$ is a closed ball of positive radius in N.

This evidently implies that $\theta(U)$ is the interior of $\theta(\bar{U})$. We note that in a regular, regional normed linear space, $\mathcal{N}(X)$ is not empty. For if W is a normed linear neighborhood and $\theta(W) \subset N$, then we can find an open V satisfying $\bar{V} \subset W$, and an open ball $B \subset N$ such that $\bar{B} \subset \theta(V)$, whence $U = \theta^{-1}(B)$ and V fulfill the conditions of (vi). If $h \in H(X)$ and $U \in \mathcal{N}(X)$, then clearly $h(U) \in \mathcal{N}(X)$.

LEMMA 3. *Suppose A, B, and C are concentric, open balls in a normed linear space N, where $\bar{A} \subset B$ and $\bar{B} \subset C$, and let $W \neq N$ be a neighborhood of $\bar{C}$. Then there is an $f \in S(W)$ such that $f = e \mid A$ and $f(B) \supset \bar{C}$.*

Proof. We will assume, for convenience, that the common center of the balls is the origin of N, and their radii are $0 < \alpha < \beta < \gamma$. For each $x \in N$, let $\rho(x)$ be the distance from x to $N - W$, and $\pi(x) = x / \| x \|$ for $x \neq 0$. If u is a unit vector and $\lambda \geqq 0$ a scalar, then we will define $f(\lambda u) = \phi(\lambda)u$, where $\phi(\lambda)$ is a scalar, and ϕ is a piecewise linear function joining the points $(0,0)$, (α,α), (β',γ), and $(\rho(\gamma u), \rho(\gamma u))$, in which β' is chosen so that $\alpha < \beta' < \beta$. Specifically, we define

$$f(x) = \begin{cases} x & \text{for } \| x \| \leqq \alpha, \\ \pi(x) \left\{\alpha + \dfrac{\gamma - \alpha}{\beta' - \alpha} (\| x \| - \alpha)\right\} & \text{for } \alpha \leqq \| x \| \leqq \beta', \\ \pi(x) \left\{\gamma + \dfrac{\rho(\gamma\pi(x))}{\rho(\gamma\pi(x)) + \gamma - \beta'} (\|x\| - \beta')\right\} & \text{for } \beta' \leqq \|x\| \leqq \gamma + \rho(\gamma\pi(x)), \\ x & \text{for } \| x \| \geqq \gamma + \rho(\gamma\pi(x)). \end{cases}$$

The definitions clearly agree for $\| x \| = \alpha$, β', and $\gamma + \rho(\gamma\pi(x))$, if we recall that $x = \| x \| \pi(x)$. The continuity of ρ on N and of π on $N - A$ implies that f is continuous on each of four closed subsets of N, whence f is continuous on all of N. The expressions for f^{-1} are easily found to be of the same sort as those for f, and we conclude that $f \in H(N)$. Setting $D = \{x \in N : \| x \| \leqq \beta'\}$ and

$$V = \{x \in N : \| x \| < \gamma + \rho(\gamma\pi(x))\},$$

we have $f(D) = \bar{C}$, and $D \subset B$ implies $f(B) \supset \bar{C}$. Moreover, $x \in N - W$ implies $\rho(\gamma\pi(x)) \leqq \| x \| - \gamma$, whence $x \in N - V$ and $V \subset W$. Since $f \in S(V)$, we have $f \in S(W)$. By construction, $f = e \mid A$.

THEOREM 5. *If X is a normal, regional normed linear space, then $\varnothing \neq \mathcal{N}(X) \subset \mathcal{M}(X)$.*

Proof. We have already seen that $\mathcal{N}(X) \neq \varnothing$. Let $U \in \mathcal{N}(X)$ be given, and suppose N, V, and θ are as in (vi), but with $\theta(U) = C$. Since C is connected, so is U. It was shown in [3, Lemma 4] that $S(C)$ acts transitively on C, so the same is true of $S(U)$ and U, whence $U \in \mathcal{K}(X)$. We will verify (iii) and (iv) together in the following form:

(vii) For every pair of open subsets W_1, W_2 of X and every $x \in U$ such that $x \in W_1 \subset U$ and $\bar{U} \subset W_2$, there is an $h \in S(W_2)$ and a neighborhood W_3 of x with the properties $h = e \mid W_3$ and $h(W_1) \supset \bar{U}$.

Choose $g \in S(C)$ so as to map $\theta(x)$ onto the center of C, and let A and B be open balls concentric with C satisfying $\bar{A} \subset B \subset g\theta(W_1)$ and $\bar{B} \subset C$. Let D be a neighborhood of $\bar{C}$ such that $\bar{D} \subset \theta(V)$, and choose a neighborhood V_1 of $\bar{U}$ so that $\bar{V}_1 \subset V \cap W_2 \cap \theta^{-1}(D)$. Evidently $(\theta(V_1))^- \subset \theta(V)$, whence $\theta(\bar{V}_1) = (\theta(V_1))^-$. By Lemma 3, we can find $f \in S(\theta(V_1))$ with the properties $f = e \mid A$ and $f(B) \supset \bar{C}$. If we define $h = \theta^{-1}g^{-1}fg\theta \mid \bar{V}_1$ and $h = e \mid X - V_1$, then clearly $h \in H(X)$ and, indeed, $h \in S(W_2)$. In addition, $h = e \mid \theta^{-1}g^{-1}(A)$, where $\theta^{-1}g^{-1}(A)$ is a neighborhood of x, and

$$h(W_1) = \theta^{-1}g^{-1}fg\theta(W_1) \supset \theta^{-1}g^{-1}f(B) \supset \theta^{-1}g^{-1}(\bar{C}) = \bar{U}.$$

Therefore, (vii) is verified, $U \in \mathcal{M}(X)$, and $\mathcal{N}(X) \subset \mathcal{M}(X)$.

Following an earlier result, we observe that if $K(X)$ is connected and $\mathcal{N}(X) \neq \varnothing$, then $K(X) = \bigcup\{U \in \mathcal{N}(X)\}$. Theorem 5 tells us that if we assume X to be a normal, regional normed linear space with $K(X)$ connected, then Theorems 2–4 apply to X, and Theorem 1 applies with the additional assumption that the dimension of N is at least 2.

We will adopt the terminology of Brown and Gluck [1] and say that a regional normed linear space X has a stable structure if $K(X) = \bigcup\{U_\alpha : \alpha \in A\}$, where A may be an infinite set, and U_α is a normed linear neighborhood which is mapped by θ_α homeomorphically onto the open unit ball B with center at the origin of a normed linear space N which is the same for all α. The θ_α are further assumed to satisfy the following condition:

(viii) If $U_\alpha \cap U_\beta \neq \varnothing$ and x lies in the domain of $\theta_\alpha\theta_\beta^{-1}$, then there is a neighborhood V of x and an $f \in S(B)$ such that $f = \theta_\alpha\theta_\beta^{-1} \mid V$.

THEOREM 6. *Let X be a normal, regional normed linear space with $K(X)$ connected. Then X has a stable structure if, and only if, $R(X) \neq \{e\}$.*

Proof. Suppose that $R(X) \neq \{e\}$. Choose $U \in \mathcal{N}(X)$, and let θ be the corresponding homeomorphism into N, where $\theta(U)$ is the unit ball B with center at 0. We will show that $\{h(U) : h \in Q(X)\}$ satisfies (viii), where θh^{-1} maps $h(U)$ onto B, and the domain of θh^{-1} is restricted to $h(U)$. Suppose $h_1(U) \cap h_2(U) \neq \varnothing$ and lies in the domain of $\theta h_1^{-1} h_2 \theta^{-1}$. Since

$$h_1^{-1}h_2 \in Q(X) \subset R(X),\ \theta^{-1}(x),\ h_1^{-1}h_2(\theta^{-1}(x)) \in U,$$

and U is connected, we can find $g \in S(U)$ and a neighborhood $W \subset U$ of $\theta^{-1}(x)$ with the property $g = h_1^{-1}h_2 \,|\, W$. If we set $f = \theta g \theta^{-1} \,|\bar{B}$ and $f = e \,|\, N - B$, then $f \in S(B)$ follows from the fact that θ is defined on $\bar{U}$. Evidently (viii) is satisfied with $V = \theta(W)$. Since $Q(X)$ acts transitively on $K(X)$, we have $K(X) = \bigcup\{h(U) : h \in Q(X)\}$, and X has a stable structure.

Conversely, suppose that X has a stable structure, where $K(X) = \bigcup\{U_\alpha : \alpha \in A\}$ and $\theta_\alpha(U_\alpha) = B$. We define $\mathscr{N}^*(X)$ to be the set of neighborhoods $U \subset U_\alpha$, for some α, with the properties $\bar{U} \subset U_\alpha$, $\theta_\alpha(U)$ is a ball, and $(\theta_\alpha(U))^- \subset B$. Clearly $\mathscr{N}^*(X) \subset \mathscr{N}(X) \subset \mathscr{M}(X)$, and $\mathscr{N}^*(X)$ is a base for $K(X)$. We will show that the hypotheses of Corollary 3 of Theorem 4 are satisfied with $\mathscr{M}^*(X) = \mathscr{N}^*(X)$. Let $U, U_1, \cdots, U_n \in \mathscr{N}^*(X)$, $f_i \in S(U_i)$, and $x \in U$ be given, where $f_i \cdots f_2 f_1(x) \in U_i$ and $f_n \cdots f_2 f_1(x) \in U$. If we set $x_i = f_i \cdots f_2 f_1(x)$ $(1 \leqq i \leqq n)$, then our hypothesis implies $x_i \in U_i \cap U_{i+1}$ $(0 \leqq i \leqq n)$, where we set $x_0 = x$, $U_0 = U$, and the subscripts are reduced modulo $n+1$, so that $U_{n+1} = U_0$. Now $\bar{U}_i \subset U_{\alpha_i}$ for some $\alpha_i \in A$, and we will abbreviate θ_{α_i} to θ_i. From (viii) we know that there is a neighborhood V_i of $(\theta_{i+1}\theta_i^{-1}(\theta_i)(x_i))$ and a $g_i \in S(B)$ satisfying $g_i = \theta_{i+1}\theta_i^{-1} \,|\, V_i$. Let $\theta_i(U_i) = B_i$, and choose $h \in H(N)$ so that $h(B) \subset B_0$. If we extend $\theta_i f_i \theta_i^{-1}$ to be e in $N - B_i$, then $h\theta_i f_i \theta_i^{-1} h^{-1}$, $hg_i h^{-1} \in S(B_0)$. We now define

$$\begin{aligned} f = \theta_0^{-1} h^{-1}\{&(hg_n h^{-1})(h\theta_n f_n \theta_n^{-1} h^{-1}) \cdots (hg_2 h^{-1})(h\theta_2 f_2 \theta_2^{-1} h^{-1}) \\ &\cdot (hg_1 h^{-1})(h\theta_1 f_1 \theta_1^{-1} h^{-1})(hg_0 h^{-1})\} h_0 \theta \end{aligned}$$

in $\bar{U}_0$ and extend f to be e in $X - U_0$, so that $f \in S(U)$. Moreover, we can rewrite the equation as

$$\begin{aligned} f &= \theta_0^{-1}(\theta_0\theta_n^{-1})\theta_n f_n \theta_n^{-1} \cdots (\theta_3\theta_2^{-1})\theta_2 f_2 \theta_2^{-1}(\theta_2\theta_1^{-1})\theta_1 f_1 \theta_1^{-1}(\theta_1\theta_0^{-1})\theta_0 \\ &= f_n \cdots f_2 f_1 \,|\, W, \end{aligned}$$

where

$$W = \theta_0^{-1}(V_0) \cap f_1^{-1}\theta_1^{-1}(V_1) \cap f_1^{-1}f_2^{-1}\theta_2^{-1}(V_2) \cap \cdots \cap f_1^{-1}f_2^{-1} \cdots f_n^{-1}\theta_n^{-1}(V_n)$$

is a neighborhood of x. Hence, $R(X) \neq \{e\}$.

COROLLARY. *Let X be a normed linear space N with an additional point p at infinity, where the neighborhoods of p are complements of closed, bounded sets in N. Then X has a stable structure.*

Proof. Since an inversion h in the boundary of an open ball $B \subset N$ with respect to its center is clearly a homeomorphism of X, and since $B \in \mathscr{M}(X)$, we have $p \in h(B) \in \mathscr{M}(X)$, and $K(X) = X$. Also, X is clearly normal. If we can show that the hypotheses of Corollary 1 of Theorem 4 are satisfied, then $R(X) \neq \{e\}$ and Theorem 6 gives the desired result. Now if $V \subset X$ is open, we can choose for U either an open ball in N or the image of such a ball under an inversion, depending on whether $p \in V$ or $p \in X - V$. We also note that $P(X) = Q(X) = R(X)$.

The assumption that $K(X)$ be connected can not be dropped in Theorem 6. As a counterexample, let $X = Y \cup Z$, where Y is a 2-sphere, Z is real projective 2-space, and $Y \cap Z = \varnothing$. Brown and Gluck [1] have shown that the existence of a stable structure for a combinatorial manifold is equivalent to its orientability, so that X has no stable structure. However, $R(X) = R(Y) = Q(Y) = P(Y) \neq \{e\}$.

Finally, we consider the question of whether, in general, $R(X) = Q(X)$ or $R(X) = P(X)$, assuming $R(X) \neq \{e\}$. If X is a closed ball in euclidean n-space, then we know from [3] that $Q(X)$ consists of all $h \in H(X)$ such that $h = e$ on a neighborhood of the boundary of X. But any rotation about the center of X evidently belongs to $\bar{R}(X)$, hence to $R(X)$, so that $R(X) \neq Q(X)$. Brown and Gluck [1] have shown, in effect, that if X is a locally euclidean manifold with stable structure, then $\bar{R}(X) = P(X)$, whence, by Theorem 4, $R(X) = P(X)$. When X is infinite dimensional, the only case in which we have determined $R(X)$ is that of the preceding corollary.

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