DECOMPOSITIONS OF E^3 WITH A COMPACT 0-DIMENSIONAL SET OF NONDEGENERATE ELEMENTS⁽¹⁾

BY
STEVE ARMENTROUT

1. Introduction. The purpose of this paper is to study monotone upper semicontinuous decompositions G of E^3 such that the image, under the projection map, of the union of all the nondegenerate elements of G is contained in a compact 0-dimensional set. Such decompositions are of interest since a number of examples that have been studied [4], [6], [7], [9], [15] satisfy the conditions imposed above on G .

Suppose then that G is a monotone upper semicontinuous decomposition of E^3 . Let E^3/G denote the associated decomposition space and let P denote the projection map from E^3 onto E^3/G . Let H_G denote the union of all the nondegenerate elements of G . Suppose that $P[H_G]$ is contained in a compact 0-dimensional set.

In §§3 and 4, point-like decompositions are considered. In §4, we prove that if G is point-like and E^3/G is a 3-manifold, then E^3/G is homeomorphic to E^3 . This settles a special case of a question raised by Bing in [7]. Other special cases of this question have been settled in [1], [2], [14], and [19]. Also in §4, we give a condition which is both necessary and sufficient in order that E^3/G be homeomorphic to E^3 in case G is a point-like decomposition satisfying the conditions above. This condition is in terms of the existence of homeomorphisms from E^3 onto E^3 that shrink the nondegenerate elements of G to small size.

In §§5 and 6, we study the following question: Is it true that if G satisfies the conditions imposed above and E^3/G is homeomorphic to E^3 , then each element of G is point-like? It is known [12] that if the set of nondegenerate elements is countable, the question has an affirmative answer. Furthermore, there is an example due to Bing [10, p. 7] of a monotone decomposition B of E^3 into non-point-like sets such that E^3/B is homeomorphic to E^3 . In this case, $P[H_B]$ is an arc. Consequently, the condition imposed above on $P[H_G]$ cannot be omitted completely if an affirmative answer to the question is to be obtained. Although we do not settle this question, we give some partial affirmative solutions. We

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give a solution in case each nondegenerate element of G is an arc, or, in fact, any tree-like continuum. Additional partial solutions of this question are given in [3] and [13].

In §7, we give a number of special results concerning point-like decompositions G of E^3 such that E^3/G is homeomorphic to E^3 .

2. Notation and terminology. If X is a topological space and G is an upper semi-continuous decomposition of X , then X/G denotes the associated decomposition space, P denotes the projection from X onto X/G , and H_G denotes the union of all the nondegenerate elements of G .

The statement that the upper semicontinuous decomposition G of E^3 is *monotone* means that each set of G is a compact continuum. A compact continuum K in E^3 is *point-like* if and only if $E^3 - K$ is homeomorphic to $E^3 - \{0\}$. By a *point-like decomposition* of E^3 is meant a monotone decomposition of E^3 into point-like sets.

If n is a positive integer, the statement that M is an *n -manifold* means that M is a separable metric space, each point of which has a neighborhood U in M such that U is an open n -cell. The statement that M is an *n -manifold-with-boundary* means that M is a separable metric space such that each point of M has a neighborhood U in M such that U is an n -cell. If M is an *n -manifold-with-boundary*, then the *boundary* of M , denoted by $\text{Bd } M$, is the set of all points p of M which do not have open n -cell neighborhoods in M , and the *interior* of M , $\text{Int } M$, is $M - \text{Bd } M$.

3. Preliminary results. The main result of this section, Theorem 1, will be used in §4 to construct homeomorphisms from E^3 onto E^3 having certain properties. Lemmas 1 and 2 are preliminary results for Theorem 1.

Suppose that C is a polyhedral 3-cell in E^3 . If x and y are distinct points of $\text{Bd } C$, the statement that α is an *unknotted chord* of C from x to y means that α is a polygonal arc with endpoints x and y such that (1) $(\text{Int } \alpha) \subset \text{Int } C$ and (2) if β is any polygonal arc on $\text{Bd } C$ from x to y , then $\alpha \cup \beta$ is the boundary of a polyhedral disc D such that $(\text{Int } D) \subset \text{Int } C$. It may be shown [20] that if α and α' are two unknotted chords of the polyhedral 3-cell C such that α and α' have the same endpoints, there is a homeomorphism h from C onto C such that if $p \in \text{Bd } C$, $h(p) = p$ and $h[\alpha] = \alpha'$.

LEMMA 1. *If C is a polyhedral 3-cell in E^3 , x and y are distinct points of $\text{Bd } C$, and B is a compact 0-dimensional subset of C containing neither x nor y , then there is an unknotted chord α of C such that α has endpoints x and y and is disjoint from B .*

Proof. We may assume that C is the cube

$$\{(x, y, z): (x, y, z) \in E^3, |x| \leq 1, |y| \leq 1, \text{ and } |z| \leq 1\},$$

that B intersects neither the top nor bottom face of C , and that x and y belong to the top and bottom faces, respectively, of C . Let D be the common part of C and the XZ -plane. D is a disc in C such that $\text{Bd } D \subset \text{Bd } C$, and x and y lie on $\text{Bd } D$.

Since B is a compact 0-dimensional set, $B \cap D$ does not separate x and y in D . Hence there is polygonal arc α on D such that α has endpoints x and y , α is disjoint from B , and $(\text{Int } \alpha) \subset \text{Int } D$. It is easy to see that α is an unknotted chord of C .

The proof of Lemma 1 given above was shown to the author by R. H. Bing and is simpler than the author's original argument.

The following lemma is proved in [19].

LEMMA 2. *If G is a point-like decomposition of E^3 such that E^3/G is homeomorphic to E^3 and U is a simply connected open set in E^3/G , then $P^{-1}[U]$ is simply connected.*

We may now establish the main result of this section.

THEOREM 1. *Suppose that G is a point-like decomposition E^3 such that E^3/G is homeomorphic to E^3 and $\text{Cl } P[H_G]$ is a compact 0-dimensional set. Suppose that M is a compact polyhedral 3-manifold-with-boundary in E^3 such that $\text{Bd } M$ is connected and is disjoint from $\text{Cl } H_G$. Then there is a homeomorphism h from M onto $P[M]$ such that $h|_{\text{Bd } M} = P|_{\text{Bd } M}$.*

Proof. Let K denote $P[\text{Bd } M]$; K is homeomorphic to $\text{Bd } M$. Since E^3/G is homeomorphic to E^3 , K has exactly one bounded complementary domain U in E^3/G . Let N denote $K \cup U$. It may be shown that $P[M] = N$.

Since $\text{Bd } M$ has a cartesian product neighborhood in E^3 , K has a cartesian product neighborhood in E^3/G . Hence N is a compact 3-manifold-with-boundary, $\text{Bd } N = P[\text{Bd } M]$, and $\text{Int } N = P[\text{Int } M]$.

Since Theorem 1 is easily proved if $\text{Bd } M$ is a 2-sphere, we assume that $\text{Bd } M$ is not a 2-sphere. Let T_0 be a triangulation of N [11], [16] such that if σ is any 3-simplex of T_0 and S_2 is the carrier of the 2-skeleton on T_0 , then $S_2 - \text{Bd } \sigma$ is connected. Let $v_1, v_2, \dots$, and v_k be the vertices of T_0 lying in $\text{Int } N$. Let $B_1, B_2, \dots$, and B_k be mutually disjoint open balls such that if $i = 1, 2, \dots$, or k , $v_i \in B_i$ and $B_i \subset \text{Int } N$. If $i = 1, 2, \dots$, or k , then, since $\text{Cl } P[H_G]$ is 0-dimensional, $B_i - \text{Cl } P[H_G]$ exists, and let v_i^* be a point of $B_i - \text{Cl } P[H_G]$. There is a homeomorphism f_0 from N onto N such that if $x \in N$ and $x \notin \bigcup_{i=1}^k B_i$, then $f_0(x) = x$, and if $i = 1, 2, \dots$, or k , $f_0[B_i] = B_i$ and $f_0(v_i) = v_i^*$. Let T_1 be $\{f_0[\sigma] : \sigma \in T_0\}$. Then T_1 is a triangulation of N , no vertex of T_1 belongs to $\text{Cl } P[H_G]$, and if σ is any 3-simplex of T_1 and S_2' is the carrier of the 2-skeleton of T_1 , then $S_2' - \text{Bd } \sigma$ is connected.

If no 1-simplex of T_1 intersects $\text{Cl } P[H_G]$, let T be T_1 . Otherwise, let $s_1, s_2, \dots$, and s_m be the 1-simplexes of T_1 which intersect $\text{Cl } P[H_G]$. There exist mutually

disjoint polyhedral (relative to T_1) 3-cells $R_1, R_2, \dots$, and R_m such that if $i = 1, 2, \dots$, or m , then (1) $(s_i \cap \text{Cl } P[H_G]) \subset \text{Int } R_i$, (2) $R_i \subset \text{Int } N$, and (3) $s_i \cap R_i$ is an unknotted chord of R_i . Such 3-cells may be constructed in the following way: If $i = 1, 2, \dots$, or m , let s'_i be a subarc of s_i , lying in $\text{Int } s_i$, and such that $(s'_i \cap \text{Cl } P[H_G]) \subset \text{Int } s'_i$. Since s'_i is polygonal (relative to T_1), it may be thickened slightly to give R_i .

If $i = 1, 2, \dots$, or m , let x_i and y_i be the two points of $s_i \cap \text{Bd } R_i$. By Lemma 1, there is an unknotted chord t_i of R_i from x_i to y_i missing $\text{Cl } P[H_G]$. There is, then, a homeomorphism k_i from R_i onto R_i such that if $p \in \text{Bd } R_i$, $k_i(p) = p$ and $k_i[s_i \cap R_i] = t_i$. It is clear that there is a homeomorphism k from N onto N such that (1) if $x \notin \bigcup_{i=1}^m R_i$, $k(x) = x$ and (2) if $i = 1, 2, \dots$, or m , and $x \in R_i$, $k(x) = k_i(x)$. Let T be $\{k[\sigma] : \sigma \in T_1\}$. Then T is a triangulation of N such that the carrier Σ_1 of the 1-skeleton of T is disjoint from $\text{Cl } P[H_G]$. Let Σ_2 denote the carrier of the 2-skeleton of T .

Let $\Delta_1, \Delta_2, \dots$, and Δ_r be the 2-simplexes of T whose interiors lie in $\text{Int } N$. If $i = 1, 2, \dots$, or r , there is an annulus A_i on Δ_i such that (1) A_i is disjoint from $\text{Cl } P[H_G]$ and (2) $\text{Bd } \Delta_i$ is one boundary component of A_i . If $i = 1, 2, \dots$, or r , let J_i be a centerline of A_i and let D_i be the subdisc of Δ_i having J_i as its boundary; $D_i \cap A_i$ is an annulus B_i . Let $V_1, V_2, \dots$, and V_r be mutually disjoint simply connected open sets such that if $i = 1, 2, \dots$, or r , then (1) $\text{Cl } V_i \subset \text{Int } N$, (2) $V_i \cap \Sigma_2 = \text{Int } D_i$, and (3) $\text{Cl } V_i \cap \Sigma_2 = D_i$. Such open sets may be constructed by slight thickenings of $\text{Int } D_1, \text{Int } D_2, \dots$, and $\text{Int } D_r$.

If $i = 1, 2, \dots$, or r , let U_i be $P^{-1}[V_i]$. By Lemma 2, each of $U_1, U_2, \dots$, and U_r is simply connected. It is easily seen, with the aid of Dehn's lemma [18] that if $i = 1, 2, \dots$, or r , there is a polyhedral disc E_i such that (1) $\text{Int } E_i \subset U_i$ and (2) $\text{Bd } E_i = P^{-1}[J_i]$. Note that $E_i \subset M$. If $i = 1, 2, \dots$, or r , let Y_i be $E_i \cup P^{-1}[A_i - B_i]$; Y_i is a disc with boundary $P^{-1}[\Delta_i]$, and $Y_i \subset M$.

If i and j are distinct positive integers, neither greater than r , then $\text{Int } Y_i$ and $\text{Int } Y_j$ are disjoint, for (1) $\text{Int } A_i$ and $\text{Int } A_j$ are disjoint and (2) U_i and U_j are disjoint.

If $i = 1, 2, \dots$, or r , let L_i denote the closure of $(A_i - B_i)$. It is clear that $P^{-1}[(\text{Bd } N) \cup (\bigcup_{i=1}^r L_i)]$ is a homeomorphism h_0 from $(\text{Bd } N) \cup (\bigcup_{i=1}^r L_i)$ onto $(\text{Bd } M) \cup (\bigcup_{i=1}^r P^{-1}[L_i])$ such that $h_0|_{\text{Bd } N} = P^{-1}|_{\text{Bd } N}$. It is easily seen that there is a homeomorphism g from Σ_2 onto $(\text{Bd } M) \cup (\bigcup_{i=1}^r Y_i)$ such that (1) $g|_{\text{Bd } N} = P^{-1}|_{\text{Bd } N}$ and (2) if $i = 1, 2, \dots$, or r , $g[\Delta_i] = Y_i$. It is clear that $g[\Sigma_2] \subset M$.

Suppose that σ is a 3-simplex of T and let S be $\text{Bd } \sigma$. We shall show now that $g[\Sigma_2]$ is disjoint from $\text{Int } g[S]$, the interior, in E^3 , of the 2-sphere $g[S]$.

Suppose that there is a point p of Σ_2 such that $g(p) \in \text{Int } g[S]$. Clearly $p \in \Sigma_2 - S$. Now T has the property that if σ is any 3-simplex of T , then $\Sigma_2 - \text{Bd } \sigma$ is connected. Therefore $\Sigma_2 - S$ is connected. Consequently $g[\Sigma_2] \subset (g[S] \cup \text{Int } g[S])$, and thus $\text{Bd } M \subset (g[S] \cup \text{Int } g[S])$. Since $\text{Bd } M$ is not a 2-sphere, there is a point

of $g[S]$ not in M . This is a contradiction, for $g[\Sigma_2] \subset M$. Hence, no point of $g[\Sigma_2]$ lies in $\text{Int } g[S]$.

Let $K_1, K_2, \dots$, and K_n be the distinct 2-spheres in N which are boundaries of 3-simplexes of T . Let W be $\{g[K_i] \cup \text{Int } g[K_i] : i = 1, 2, \dots, \text{ or } n\}$. We shall show now that W is a triangulation of M .

First, W is a 3-complex. For suppose that i and j are distinct positive integers, neither greater than n . Then $\text{Int } g[K_i]$ and $\text{Int } g[K_j]$ are disjoint. For, it was shown that $g[K_i]$ and $\text{Int } g[K_j]$ and $\text{Int } g[K_j]$ are disjoint, and so are $g[K_j]$ and $\text{Int } g[K_i]$. If $\text{Int } g[K_i]$ and $\text{Int } g[K_j]$ intersect, then one contains the other, say $\text{Int } g[K_i] \subset \text{Int } g[K_j]$. Since $g[K_i] \neq g[K_j]$, then $g[K_i]$ intersects $\text{Int } g[K_j]$. This is a contradiction, and hence $\text{Int } g[K_i]$ and $\text{Int } g[K_j]$ are disjoint. Since $g[K_i] \cap g[K_j] = g[K_i \cap K_j]$, it is clear that W is a 3-complex.

Second, $\bigcup \{w : w \in W\} = M$. Since $g[\Sigma_2] \subset M$, it is clear that

$$\bigcup \{w : w \in W\} \subset M.$$

Hence we need only to show that $M \subset \bigcup \{w : w \in W\}$.

Suppose that $M = \bigcup \{w : w \in W\}$ exists. Since $\bigcup \{w : w \in W\}$ is closed, there is a point p of $\text{Int } M$ such that $p \notin \bigcup \{w : w \in W\}$. Let q be a point such that for some 3-simplex w of W , $q \in \text{Int } w$. Then $q \in \text{Int } M$ and there is an arc pq in $\text{Int } M$; we may assume that pq is disjoint from the carrier of the 1-skeleton of W . Let b be the first point of $\bigcup \{w : w \in W\}$ on pq in the order from p to q ; $pb - \{b\}$ is disjoint from $\bigcup \{w : w \in W\}$. There is a 2-simplex τ of W such that $b \in \text{Int } \tau$, and there is a 3-simplex σ of W such that $\tau \subset \text{Bd } \sigma$. Now $\tau \not\subset \text{Bd } M$. For if $\tau \subset \text{Bd } M$, then clearly the arc pb intersects $E^3 - M$, which is impossible. Thus $\tau \not\subset \text{Bd } M$. It follows that $g^{-1}[\tau] \not\subset \text{Bd } N$.

There is exactly one 3-simplex t_1 of T such that $\text{Bd } \sigma = g[\text{Bd } t_1]$. There is a 3-simplex t_2 of T such that $\text{Int } t_1$ and $\text{Int } t_2$ are disjoint and $\text{Bd } t_1 \cap \text{Bd } t_2 = g^{-1}[\tau]$. Let σ' be $g[\text{Bd } t_2] \cup \text{Int } g[\text{Bd } t_2]$. Then $\sigma' \in W$ and $\sigma \cap \sigma' = \tau$. Since $\text{Int } \sigma$ and $\text{Int } \sigma'$ are disjoint and $\text{Bd } \sigma \cap \text{Bd } \sigma' = \tau$, it follows that $(pb - \{b\}) \subset \text{Int } \sigma'$. This is a contradiction, and it follows that $M = \bigcup \{w : w \in W\}$.

Consequently, W is a triangulation of M . It is clear that there is an extension f of g such that (1) f is a homeomorphism from N onto M and (2) $f|_{\text{Bd } N} = P^{-1}|_{\text{Bd } N}$. It follows that there is a homeomorphism h from M onto N such that $h|_{\text{Bd } M} = P|_{\text{Bd } M}$. This concludes the proof of Theorem 1.

4. Results on point-like decompositions. In this section, we establish the main results of the paper relative to point-like decompositions G of E^3 such that E^3/G either is homeomorphic to E^3 or is a 3-manifold.

THEOREM 2. *Suppose that G is a point-like decomposition of E^3 such that (1) E^3/G is homeomorphic to E^3 and (2) $\text{Cl } P[H_G]$ is a compact 0-dimensional set. Suppose that U is an open set in E^3 containing $\text{Cl } H_G$ and ε is a positive*

number. Then there is a homeomorphism h from E^3 onto E^3 such that (1) if $x \notin U$, $h(x) = x$ and (2) if g is any nondegenerate element of G , $(\text{diam } h[g]) < \varepsilon$.

Proof. Since $\text{Cl}P[H_G]$ is a compact 0-dimensional subset of E^3/G and E^3/G is homeomorphic to E^3 , there exists a sequence $N_1, N_2, N_3, \dots$ such that (1) for each positive integer i , N_i is a compact polyhedral 3-manifold-with-boundary such that (a) each component of N_i has connected boundary and diameter less than $1/i$, and (b) $N_{i+1} \subset \text{Int } N_i$, and (2) $\text{Cl}P[H_G] = \bigcup_{i=1}^{\infty} N_i$. For each positive integer i , let M_i denote $P^{-1}[N_i]$; M_i is a compact 3-manifold-with-boundary. Since, for each positive integer i , N_i is polyhedral, then M_i is tame in E^3 .

Since $\text{Cl}H_G \subset U$, $P[U]$ is open in E^3/G and there is, accordingly, a positive integer k such that $N_k \subset P[U]$. It follows that $M_k \subset U$. Let $M_{k1}, M_{k2}, \dots$, and M_{km} be the components of M_k . If $j = 1, 2, \dots$, or m , there is, by Theorem 1, a homeomorphism f_j from M_{kj} onto $P[M_{kj}]$ such that $f_j|_{\text{Bd } M_{kj}} = P|_{\text{Bd } M_{kj}}$. Let f be the homeomorphism from M_k onto $P[M_k]$ such that if $j = 1, 2, \dots$, or m , $f|_{M_{kj}} = f_j$. Now f^{-1} is uniformly continuous and there exists a positive integer r greater than k such that if L is any component of $P[M_r]$, then $(\text{diam } f^{-1}[L]) < \varepsilon$. Let $M_{r1}, M_{r2}, \dots$, and M_{rn} be the components of M_r . With the aid of Theorem 1, it follows that if $i = 1, 2, \dots$, or n , there is a homeomorphism k_i from M_{ri} onto $f^{-1}[P[M_{ri}]]$ such that $k_i|_{\text{Bd } M_{ri}} = f^{-1}P|_{\text{Bd } M_{ri}}$. Let k be the homeomorphism from M_r onto $f^{-1}[P[M_r]]$ such that if $i = 1, 2, \dots$, or n , $k|_{M_{ri}} = k_i$.

Define a homeomorphism h as follows: (1) If $x \notin M_k$, then $h(x) = x$. (2) If $x \in (M_k - M_r)$, then $h(x) = f^{-1}(P(x))$. (3) If $x \in M_r$, then $h(x) = k(x)$. Then h is a homeomorphism from E^3 onto E^3 . If $x \notin U$, then since $x \notin M_k$, $h(x) = x$. If g is any nondegenerate element of G , then for some positive integer i not greater than n , $g \subset M_{ri}$. Hence $h[g] \subset h[M_{ri}]$; but $h[M_{ri}] = f^{-1}[P[M_{ri}]]$, and $(\text{diam } f^{-1}[P[M_{ri}]]) < \varepsilon$. Therefore $(\text{diam } h[g]) < \varepsilon$ and the proof of Theorem 2 is complete.

The following theorem may be compared with Theorem 2 of [1].

THEOREM 3. Suppose that G is a point-like decomposition of E^3 such that $P[H_G]$ is a compact 0-dimensional set. Then E^3/G is homeomorphic to E^3 if and only if for each open set U in E^3 containing H_G and each positive number ε , there is a homeomorphism h from E^3 onto E^3 such that (1) if $x \notin U$, $h(x) = x$ and (2) if g is any nondegenerate element of G , $(\text{diam } h[g]) < \varepsilon$.

Proof. If, for each open set U in E^3 containing H_G and each positive number ε , there exists a homeomorphism h from E^3 onto E^3 having the properties stated above, then it follows from the proof of Theorem 1 of [4] that E^3/G is homeomorphic to E^3 . Hence the condition stated is sufficient. By Theorem 2, it is necessary.

The following theorem may be compared with Theorem 5 of [1].

THEOREM 4. *Suppose that G is a point-like decomposition of E^3 such that $P[H_G]$ is a compact 0-dimensional set. If E^3/G is a 3-manifold, then E^3/G is homeomorphic to E^3 .*

Proof. If E^3/G is a 3-manifold, then a local version of Theorem 1 holds. Hence the proof of Theorem 2 is valid and thus the conclusion of that theorem holds. Then by Theorem 3, E^3/G is homeomorphic to E^3 .

COROLLARY 1. *If G is a point-like decomposition of S^3 such that $P[H_G]$ is a compact 0-dimensional set and S^3/G is a 3-manifold, then S^3/G is a 3-sphere.*

5. Monotone decompositions of E^3 . In this section and the next, we study monotone decompositions G of E^3 such that (1) $P[H_G]$ is contained in a compact 0-dimensional set and (2) E^3/G is homeomorphic to E^3 . We want to determine conditions under which it can be concluded that each element of G is point-like.

A compact continuum K in E^3 is *cellular* if and only if there exists a sequence $C_1, C_2, C_3, \dots$ of 3-cells in E^3 such that (1) if n is any positive integer, $C_{n+1} \subset \text{Int } C_n$, and (2) $K = \bigcap_{i=1}^{\infty} C_i$. It is well known that for compact continua in E^3 , "point-like" and "cellular" are equivalent; see [21].

THEOREM 5. *Suppose that G is a monotone decomposition of E^3 such that $P[H_G]$ is contained in a compact 0-dimensional set and E^3/G is homeomorphic to E^3 . Suppose that if U is any open set in E^3 containing $\text{Cl } H_G$ and ε is any positive number, there exists a homeomorphism f from E^3 onto E^3 such that (1) if $x \notin U$, $f(x) = x$, and (2) if g is any nondegenerate element of G , then $(\text{diam } f[g]) < \varepsilon$. Then each element of G is point-like.*

Proof. Since $\text{Cl } P[H_G]$ is compact and 0-dimensional, there exists a sequence $N_1, N_2, N_3, \dots$ of compact 3-manifolds-with-boundary in E^3/G such that (1) if i is any positive integer, $N_{i+1} \subset \text{Int } N_i$, and each component of N_i has diameter less than $1/i$, and (2) $\bigcap_{i=1}^{\infty} N_i = P[H_G]$. For each positive integer i , let M_i be $P^{-1}[N_i]$. Then for each positive integer i , M_i is a compact 3-manifold-with-boundary and $M_{i+1} \subset \text{Int } M_i$. Further, $\bigcap_{i=1}^{\infty} M_i = \text{Cl } H_G$.

Suppose that g is a nondegenerate element of G . For each positive integer i , let K_i be the component of M_i containing g . Then for each positive integer i , K_i is a compact 3-manifold-with-boundary, and $K_{i+1} \subset \text{Int } K_i$. Further $\bigcap_{i=1}^{\infty} K_i = g$.

Suppose now that j is a positive integer. We shall show that there is a 3-cell C such that $g \subset \text{Int } C$ and $C \subset \text{Int } K_j$. Let $\{R_1, R_2, \dots, R_m\}$ be a finite set of 3-cells, each contained in $\text{Int } K_j$ and such that $\{\text{Int } R_1, \text{Int } R_2, \dots, \text{Int } R_m\}$ covers K_{j+1} . There exists a positive number ε such that if A is any subset of K_{j+1} of diameter less than ε , then A is contained in some one of $\text{Int } R_1, \text{Int } R_2, \dots$, and $\text{Int } R_m$. By hypothesis, there is a homeomorphism f from E^3 onto E^3 such that (1) if $x \notin \text{Int } M_{j+1}$, then $f(x) = x$ and (2) $(\text{diam } f[g]) < \varepsilon$. It follows that $f[K_{j+1}] \subset K_{j+1}$, and hence there is a positive integer k not greater than m such that $f[g] \subset \text{Int } R_k$. Let C be $f^{-1}[R_k]$; clearly C is a 3-cell such that $g \subset \text{Int } C$ and $C \subset \text{Int } K_j$.

It now follows that g is cellular. There exists a 3-cell C_1 such that $g \subset \text{Int } C_1$ and $C_1 \subset \text{Int } K_1$. Let n_1 be 1. There is a positive integer n_2 such that $K_{n_2} \subset \text{Int } C_1$. There is a 3-cell C_2 such that $g \subset \text{Int } C_2$ and $C_2 \subset \text{Int } K_{n_2}$. A continuation of this process yields an increasing sequence $n_1, n_2, n_3, \dots$ of positive integers and a sequence $C_1, C_2, C_3, \dots$ of 3-cells such that for each positive integer i , $g \subset \text{Int } C_i$ and $C_i \subset \text{Int } K_{n_i}$. It is clear that for each positive integer i , $C_{i+1} \subset \text{Int } C_i$, and that $\bigcap_{i=1}^{\infty} C_i = g$. Hence g is cellular. Hence g is point-like and Theorem 5 is established.

The following theorem is closely related to Theorem 1. In the proof of Theorem 6 we construct triangulations in the order opposite to that used in the proof of Theorem 1.

THEOREM 6. *Suppose that G is a monotone decomposition of E^3 such that (1) E^3/G is homeomorphic to E^3 and (2) $\text{Cl } P[H_G]$ is a compact 0-dimensional set. Suppose that M is a compact polyhedral 3-manifold-with-boundary in E^3 such that $\text{Bd } M$ is a connected and is disjoint from H_G . Suppose that M has a triangulation T such that the carrier Σ_1 of the 1-skeleton of T is disjoint from $\text{Cl } H_G$. Then there is a homeomorphism h from M onto $P[M]$ such that $h|_{\text{Bd } M} = P|_{\text{Bd } M}$.*

Proof. Let $\Delta_1, \Delta_2, \dots$, and Δ_n be the 2-simplexes of T that intersect $\text{Int } M$. If $i = 1, 2, \dots$, or n , let F_i be a disc in $\text{Int } \Delta_i$ and such that $\text{Int } F_i$ contains $\Delta_i \cap \text{Cl } H_G$. If $i = 1, 2, \dots$, or n , there is an annulus A_i on Δ_i such that

$$\text{Bd } A_i = (\text{Bd } \Delta_i) \cup (\text{Bd } F_i);$$

let γ_i be a centerline of A_i and let D_i be the subdisc of Δ_i bounded by γ_i . Note that $D_i \cap A_i$ is an annulus B_i , and that A_i is disjoint from $\text{Cl } H_G$.

Let N be $P[M]$; it may be shown that N is a compact 3-manifold-with-boundary, $\text{Bd } N = P[\text{Bd } M]$, and $\text{Int } N = P[\text{Int } M]$. If $i = 1, 2, \dots$, or n , then $P[D_i]$ is a singular disc lying in $\text{Int } N$, and having no singularities on $P[B_i]$. Further, there is a neighborhood U of Σ_1 such that (1) $P|_{[U \cup (\bigcup_{i=1}^n A_i)]}$ is a homeomorphism and (2) if $x \notin U \cup (\bigcup_{i=1}^n A_i)$, then $P(x) \notin P[U \cup (\bigcup_{i=1}^n A_i)]$.

With the aid of Dehn's lemma [18], it may be shown that there exists discs $K_1, K_2, \dots$, and K_n such that (1) if $i = 1, 2, \dots$, or n , (a) $K_i \subset \text{Int } N$, (b) $\text{Bd } K_i = P[\text{Bd } D_i]$, and (c) $K_i \cap P[U \cup (\bigcup_{j=1}^n A_j)] \subset P[B_i]$, and (2) if i and j are distinct positive integers, neither greater than n , then K_i and $\text{Bd } K_j$ are disjoint.

The discs $K_1, K_2, \dots$, and K_n are not necessarily mutually disjoint. By an argument similar to that given in the proof of Theorem 9 of [7], it may be shown that there exist mutually disjoint discs $K'_1, K'_2, \dots$, and K'_n such that if $i = 1, 2, \dots$, or n , $\text{Bd } K'_i = \text{Bd } K_i$, $K'_i \subset \text{Int } N$, and K'_i is disjoint from $P[\bigcup_{j=1}^n (A_j - B_j)]$. Then if $i = 1, 2, \dots$, or n , $K'_i \cup P[A_i - B_i]$ is a disc Δ'_i .

By an argument similar to that used in the proof of Theorem 1, it may be

shown that there is a homeomorphism h from M onto $P[M]$ such that $h|_{\text{Bd } M} = P|_{\text{Bd } M}$. This completes the proof of Theorem 6.

COROLLARY 2. *Suppose that G is a monotone decomposition of E^3 such that (1) E^3/G is homeomorphic to E^3 and (2) $\text{Cl } P[H_G]$ is a compact 0-dimensional set. Suppose that there exists a sequence $M_1, M_2, M_3, \dots$ of compact 3-manifolds-with-boundary in E^3 such that (1) if n is any positive integer, $M_n \subset M_{n+1}$ and M_n has a triangulation such that the carrier of its 1-skeleton is disjoint from $\text{Cl } H_G$, and (2) $\bigcap_{i=1}^{\infty} M_i = \text{Cl } H_G$. Then each element of G is point-like.*

Proof. By an argument similar to that given to prove Theorem 2, but using Theorem 6 in place of Theorem 1, we may show that under the hypothesis of the corollary, the following holds: If U is any open set in E^3 containing H_G and ε is any positive number, there exists a homeomorphism h from E^3 onto E^3 such that (1) if $x \notin U$, $h(x) = x$, and (2) if g is any nondegenerate element of G , then $(\text{diam } h[g]) < \varepsilon$. It then follows by Theorem 5 that each element of G is point-like.

6. Decompositions of E^3 into continua of type T . The statement that a compact metric continuum M is of type T means that if K is any subcontinuum of M that can be embedded in a 2-sphere and f is an embedding of K in some 2-sphere S , then $f[K]$ does not separate S . Since for planar continua, separating the plane is a topological invariant, the criterion used is meaningful. Continua of type T include tree-like continua [8], snake-like continua [8], dendrons, and arcs.

THEOREM 7. *If G is a monotone decomposition of E^3 such that (1) E^3/G is homeomorphic to E^3 , (2) $P[H_G]$ is contained in a compact 0-dimensional set, and (3) each element of G is a continuum of type T , then each element of G is point-like.*

Proof. First it will be shown that if M is a compact polyhedral 3-manifold-with-boundary in E^3 such that $\text{Bd } M$ is connected and is disjoint from $\text{Cl } H_G$, then M has a triangulation L such that the carrier of the 1-skeleton of L is disjoint from $\text{Cl } H_G$.

Let L_0 be any triangulation of M . Let $v_1, v_2, \dots$, and v_n denote the vertices of L_0 . Since $\text{Cl } P[H_G]$ is compact and 0-dimensional, there are points $v'_1, v'_2, \dots$, and v'_n of $\text{Int } M$ not in $\text{Cl } H_G$. There is a homeomorphism h from M onto M such that $h|_{\text{Bd } M}$ is the identity and if $i = 1, 2, \dots$, or n , $h(v_i) = v'_i$. Let L_1 denote $\{h[\sigma] : \sigma \in L_0\}$. L_1 is a triangulation of M and no vertex of L_1 belongs to $\text{Cl } H_G$.

If s is a 1-simplex of L_1 , let C_s be a 3-cell containing $s \cap \text{Cl } H_G$, obtained by a slight thickening of a subarc s' of s and such that $C_s \cap s = s'$. It is to be true that if s and t are distinct 1-simplexes of L_1 , C_s and C_t are disjoint. Now by hypothesis, if g_0 is any subcontinuum of an element of G , g_0 does not separate any

2-sphere containing g_0 . In addition, each component of $(\text{Bd } C_s) \cap \text{Cl } H_G$ is a subcontinuum of some element of G . Hence no component of $(\text{Bd } C_s) \cap \text{Cl } H_G$ separates $\text{Bd } C_s$. By unicoherence, $(\text{Bd } C_s) - (\text{Cl } C_s)$ is connected. There is, therefore, an arc s'' on $\text{Bd } C_s$, disjoint from $\text{Cl } H_G$ and having as endpoints the endpoints of s' . It is now easy to construct a triangulation L of M such that the carrier of the 1-skeleton of L is disjoint from $\text{Cl } H_G$.

By using Theorem 6 in place of Theorem 1, the argument used to establish Theorem 2 shows that the hypothesis of Theorem 5 is satisfied. Hence by Theorem 5, each element of G is point-like.

COROLLARY 3. *If G is a monotone decomposition of E^3 such that (1) E^3/G is homeomorphic to E^3 , (2) $P[H_G]$ is contained in a compact 0-dimensional set, and (3) each element of G is a tree-like continuum, then each element of G is point-like.*

COROLLARY 4. *If G is a monotone decomposition of E^3 into arcs and one-point sets such that E^3/G is homeomorphic to E^3 and $\text{Cl } P[H_G]$ is a compact 0-dimensional set, then each element of G is point-like ⁽²⁾.*

7. Results on point-like decompositions. In this section, we present some results on point-like decompositions of E^3 . We begin with two results on 3-cells-with-handles.

The statement that K is a 3-cell-with-handles means that K is an orientable 3-manifold-with-boundary such that there exist 3-cells $C_0, C_2, \dots$, and C_n such that (1) any two of $C_1, C_2, \dots$ and C_n are disjoint, (2) if $i = 1, 2, \dots$, or n , $C_0 \cap C_i = (\text{Bd } C_0) \cap (\text{Bd } C_i)$ and $C_0 \cap C_i$ is the union of two disjoint discs, and (3) $K = \bigcup_{i=0}^n C_i$.

LEMMA 3. *If K is a polyhedral 3-cell-with-handles and L is a polyhedral 3-cell such that $K \cap L = (\text{Bd } K) \cap (\text{Bd } L)$ and $K \cap L = D_1 \cup D_2 \cup \dots \cup D_n$ where $D_1, D_2, \dots$, and D_n are mutually disjoint polyhedral discs, then $K \cup L$ is a polyhedral 3-cell-with-handles.*

LEMMA 4. *If B is a compact 0-dimensional subset of E^3 and U is any open set containing B , there exists a polyhedral 3-manifold-with-boundary M such that $B \subset \text{Int } M$, $M \subset U$, and each component of M is a 3-cell-with-handles.*

Proof. There exists a polyhedral 3-manifold-with-boundary N such that $B \subset N$ and $N \subset U$. It follows with the aid of Lemma 1 that there is a triangulation T of N such that the carrier of the 1-skeleton of T is disjoint from B .

Let N_0 be a component of N , and let S_0 be the carrier of the 1-skeleton of the triangulation T_0 of N_0 induced by T . There exists a polyhedral tubular neighborhood S_0^* of S_0 such that (1) S_0^* is disjoint from B , (2) if σ and σ' are distinct

(2) Joseph M. Martin has recently established this result without requiring that $\text{Cl } P[H_G]$ be compact and 0-dimensional.

3-simplexes of T_0 and $\sigma - (\text{Int } S_0^*)$ and $\sigma' - (\text{Int } S_0^*)$ intersect, then their common part is a disc lying in the interior of some 2-simplex of T_0 , and (3) $N_0 - \text{Int } S_0^*$ is connected. With the aid of Lemma 3, it follows that $N_0 - \text{Int } S_0^*$ is a 3-cell-with-handles. Lemma 4 now follows easily.

Suppose that G is an upper semicontinuous decomposition of E^3 . The statement that H_G is *definable by 3-cell-with-handles* means that there exists a sequence $M_1, M_2, M_3, \dots$ such that (1) for each positive integer n , M_n is a polyhedral 3-manifold-with-boundary such that each component of M_n is a 3-cell-with-handles and $M_{n+1} \subset \text{Int } M_n$, (2) $\text{Cl } H_G = \bigcap_{i=1}^{\infty} M_i$, and (3) g is a nondegenerate element of G if and only if g is a nondegenerate component of $\bigcap_{i=1}^{\infty} M_i$. It is clear that if H_G is definable by 3-cells-with-handles, then $\text{Cl } P[H_G]$ is a compact 0-dimensional set.

THEOREM 8. *Suppose that G is a point-like decomposition of E^3 such that (1) E^3/G is homeomorphic to E^3 and (2) $\text{Cl } P[H_G]$ is a compact 0-dimensional set. Then H_G is definable by 3-cells-with-handles.*

Proof. Since E^3/G is homeomorphic to E^3 , there is, by Lemma 4, a sequence $M_1, M_2, M_3, \dots$ such that (1) for each positive integer i , M_i is a compact polyhedral 3-manifold-with-boundary, $M_{i+1} \subset \text{Int } M_i$, and each component of M_i is a 3-cell-with-handles, and (2) $\text{Cl } P[H_G] = \bigcap_{i=1}^{\infty} M_i$. It is clear that $\text{Cl } H_G = \bigcap_{i=1}^{\infty} P^{-1}[M_i]$.

Suppose that j is a positive integer and L is a component of M_j . Since L is a 3-cell-with-handles, there exist polyhedral 3-cells $C_0, C_1, \dots$, and C_r such that (1) any two of $C_1, C_2, \dots$, and C_r are disjoint, (2) if $i = 1, 2, \dots$, or r ,

$$C_0 \cap C_i = (\text{Bd } C_0) \cap (\text{Bd } C_i)$$

and $C_0 \cap C_1$ is the union of two disjoint polyhedral discs D'_i and D_i , and (3) $L = \bigcup_{i=0}^r C_i$. An argument similar to that given in proving Theorem 1 may be used to show that there exists 3-cells $C'_0, C'_1, \dots$, and C'_r such that $\bigcup_{i=0}^r C'_i = P^{-1}[L]$ and $C'_0, C'_1, \dots$, and C'_r satisfy conditions relative to $P^{-1}[L]$ analogous to those satisfied by $C_0, C_1, \dots$, and C_r relative to L . It follows that $P^{-1}[L]$ is a 3-cell-with-handles. It is easily seen that H_G is definable by 3-cells-with-handles.

The following result, which was announced in [3], provides a converse to Theorem 8. It may be proved by line of argument similar to that used to establish Theorem 6 and Corollary 2.

THEOREM 9. *Suppose that G is a monotone decomposition of E^3 such that (1) E^3/G is homeomorphic to E^3 and (2) $P[H_G]$ is a compact 0-dimensional set. If H_G is definable by 3-cells-with-handles, then each element of G is point-like.*

Our last result concerns the construction of a point-like decomposition of E^3 for which the associated decomposition space has certain given properties.

THEOREM 10. *If K is any compact 0-dimensional subset of E^3 , there exist a point-like decomposition G of E^3 and a homeomorphism h from E^3/G onto E^3 such that $h[P[H_G]] = K$.*

Proof. Let $M_1, M_2, M_3, \dots$ be a sequence of polyhedral 3-manifolds-with-boundary such that (1) if i is any positive integer, $M_{i+1} \subset \text{Int } M_i$, and (2) $\bigcap_{i=1}^{\infty} M_i = K$. Let P_1 and P_2 be disjoint planes in E^3 .

There is a homeomorphism h_1 from E^3 onto E^3 such that if M'_1 is any component of M_1 , $h_1[\text{Int } M'_1]$ intersects both P_1 and P_2 . There is a homeomorphism h_2 from E^3 onto E^3 such that if $x \in E^3 - \text{Int } M_1$, $h_2(x) = x$ and if M'_2 is any component of M_2 , then $h_2 h_1[\text{Int } M'_2]$ intersects both P_1 and P_2 . Suppose that i is a positive integer and $h_1, h_2, \dots$, and h_i have been defined. There is a homeomorphism h_{i+1} from E^3 onto E^3 such that if $x \in E^3 - \text{Int } M_i$, $h_{i+1}(x) = x$, and if M'_{i+1} is any component of M_{i+1} , then $h_{i+1} h_i \cdots h_1[\text{Int } M'_{i+1}]$ intersects both P_1 and P_2 . Therefore there exists a sequence $h_1, h_2, h_3, \dots$ of homeomorphisms from E^3 onto E^3 having properties indicated above. For each positive integer i , let N_i be $h_i h_{i-1} \cdots h_1[M_i]$.

Let G be the decomposition of E^3 whose nondegenerate elements are the components of $\bigcap_{i=1}^{\infty} N_i$. It may be proved, using [17, Chapter V, Theorem 20], that G is upper semicontinuous. Note that $H_G = \bigcap_{i=1}^{\infty} N_i$.

Now we shall show that if U is any open set in E^3 containing H_G and ε is any positive number, there is a homeomorphism f from E^3 onto E^3 such that if $x \notin U$, $f(x) = x$ and if g is any nondegenerate element of G , $(\text{diam } f[g]) < \varepsilon$. Suppose U is open in E^3 and ε is any positive number. There is a positive integer j such that $N_j \subset U$. Since (1) for each positive integer n , M_n is compact, (2) $h_j h_{j-1} \cdots h_1$ is a homeomorphism, and (3) the maximum of the diameters of the components of M_n approaches 0 as n increases without bound, there is a positive integer k greater than j such that if M'_k is any component of M'_k , $(\text{diam } h_j h_{j-1} \cdots h_1[M'_k]) < \varepsilon$. Let f be $(h_k h_{k-1} \cdots h_{j+1})^{-1}$. Then $f[N_k] = h_j h_{j-1} \cdots h_1[M_k]$ and if $x \in E^3 - N_j$, then $f(x) = x$. Hence f is a homeomorphism from E^3 onto E^3 having the specified properties.

It may now be seen, from a consideration of the proof of Theorem 1 of [4], that there is a homeomorphism h from E^3/G onto E^3 such that $h[P[H_G]] = K$. With the aid of Theorem 5, it may be proved that each element of G is point-like.

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UNIVERSITY OF IOWA,
 IOWA CITY, IOWA
 INSTITUTE FOR ADVANCED STUDY,
 PRINCETON, NEW JERSEY