GROUPS WITH NORMAL, SOLVABLE HALL p'-SUBGROUPS[1]

BY
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Let P be a p-group which acts faithfully as a group of automorphisms on solvable p'-group H. In this paper we discuss the existence of an element $h \in H$ having a "small" centralizer in P. We give two sufficient conditions for the strongest possible result, that is the existence of an element h centralized in P only by the identity. The first is that the pair $(p, \pi(H))$, where $\pi(H)$ denotes the set of prime divisors of $|H|$, be nonexceptional or essentially that it involve no Fermat or Mersenne primes. The second is that the orbits in H under action of P all have size smaller than p^p. In any case we show that for some element h, $|\mathfrak{C}_P(h)| \leqq |P|^{1/2}$.

We apply these results to two distinct subjects. First we generalize a theorem of Ito which concerns the minimum number of Sylow p-subgroups of a solvable group G which intersect to $\mathfrak{O}_p(G)$, the intersection of all of them. We show that if $(p, \pi(G))$ is nonexceptional then there exists two such Sylow subgroups which intersect to $\mathfrak{O}_p(G)$. In any case there always exist three which work.

Second, we obtain reduction theorems for the study of groups G all of whose absolutely irreducible representations have degrees which are powers of a prime p. In [3] certain relationships between the biggest of these degrees and the existence of "large" abelian subgroups of G were studied. We show here how in most cases these problems can be reduced to a study of p-groups.

1. Nonexceptional prime pairs.

Definition. Let p be a prime and π a set of primes. We say the pair (p, π) is nonexceptional provided it is not one of the following three types.

$(m, 2)$: p is a Mersenne prime and $2 \in \pi$.

$(2, m)$: $p = 2$ and π contains a Mersenne prime.

$(2, f)$: $p = 2$ and π contains a Fermat prime.

In this section we prove the following result.

Theorem 1.1. *Let p-group P act faithfully on solvable p'-group H.*

(i) *If $(p, \pi(H))$ is nonexceptional then there exists $h \in H$ with $\mathfrak{C}_P(h) = \{1\}$.*

(ii) *If $(p, \pi(H))$ is not type $(2, f)$ then there exists $h_1, h_2, \in H$ with $\mathfrak{C}_P(h_1) \cap \mathfrak{C}_P(h_2) = \{1\}$ and the product $\mathfrak{C}_P(h_1)\mathfrak{C}_P(h_2)$ being an abelian group.*

(iii) *In any case there exists $h_1, h_2 \in H$ with $\mathfrak{C}_P(h_1) \cap \mathfrak{C}_P(h_2) = \{1\}$.*

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We can of course view this problem slightly differently. Let $G = H \times_\sigma P$ be the semidirect product of H by P. Then H is equal to $\mathfrak{H}(G)$, the Hall p'-subgroup of G, and P, a Sylow p-subgroup, acts faithfully on H. The above theorem therefore has an obvious restatement in this context. It is convenient in the course of the proof to take this point of view. We discuss first several lemmas which allow us to reduce the proof to a study of certain special cases. In the following all groups are assumed to have normal solvable Hall p'-subgroups. This property is of course inherited by quotient groups and subgroups.

LEMMA 1.2. *Let $\mathscr{P}$ be a property of groups which is inherited by subgroups and quotient groups. Suppose we wish to prove that if $P = \mathfrak{S}_p(G)$ acts faithfully on $\mathfrak{H}(G)$ and if G has property $\mathscr{P}$ then one particular conclusion of Theorem* 1.1 *holds. Then it suffices to assume that $\mathfrak{H}(G)$ is an elementary abelian q-group and P acts irreducibly on it.*

Proof. Since $H = \mathfrak{H}(G)$ is solvable so is G. Moreover P acts faithfully on H so that $\mathfrak{F}(G)$, the Fitting subgroup of G, is contained in H. Since $\mathfrak{F}(G)$ contains its centralizer [5, Theorem 7.4.7], P acts faithfully on $\mathfrak{F}(G)$. Therefore it suffices to prove the result for the group $P\mathfrak{F}(G)$ or equivalently we can assume that $\mathfrak{H}(G)$ is nilpotent.

Let H be nilpotent and let Φ denote its Frattini subgroup. By Theorem 7.3.12 of [5], P acts faithfully on H/Φ. If $\bar{h}$ is the image of h under the homomorphism $H \to H/\Phi$, then $\mathfrak{C}_P(\bar{h}) \supseteq \mathfrak{C}_P(h)$. Hence it suffices to assume that $\Phi = \{1\}$. Now P acts on each Sylow subgroup of H and so by complete reducibility $H = R_1 \times R_2 \times \cdots \times R_m$ where each R_i is an elementary abelian q_i-group acted upon irreducibly by P. For each i we write $H = R_i \times R_i^*$ where R_i^* is the product of the remaining R_j. Let P_i be a Sylow p-subgroup of $\mathfrak{C}_G(R_i)$. Then $\mathfrak{C}_G(R_i) = HP_i = R_iN_i$ where $N_i = R_i^*P_i$. We see easily that $R_i \cap N_i = \{1\}$ and that N_i is normal in G.

Set $G_i = G/N_i$. Then G_i is a homomorphic image of G having the structure described in the statement of the lemma. Since $\mathfrak{C}_G(H) = H$ we see that G is contained isomorphically in $G_1 \times G_2 \times \cdots \times G_m$ in such a way that $\mathfrak{H}(G) = \mathfrak{H}(G_1) \times \mathfrak{H}(G_2) \times \cdots \times \mathfrak{H}(G_m)$. If the result is true for each G_i, it is clearly true for G.

DEFINITION. Let P be a Sylow p-subgroup of G and let $H = \mathfrak{H}(G)$ be abelian.

(i) Let $k_1(G)$ denote the number of conjugacy classes α of H under the action of P which contain an element h with $\mathfrak{C}_P(h) = \{1\}$.

(ii) Let $k_2(G)$ denote the number of conjugacy classes α of H which contain elements h_1 and h_2 with $\mathfrak{C}_P(h_1) \cap \mathfrak{C}_P(h_2) = \{1\}$ and $\mathfrak{C}_P(h_1)\mathfrak{C}(h_2)$ abelian.

(iii) Let $k_3(G)$ denote the number of conjugacy classes α of H which contain elements h_1, h_2 and h_3 with $\mathfrak{C}_P(h_1) \cap \mathfrak{C}_P(h_2) = \{1\}$, $\mathfrak{C}_P(h_1) \cap \mathfrak{C}_P(h_3) = \{1\}$, $\mathfrak{C}_P(h_1) \cap Z\mathfrak{C}_P(h_2) \neq \{1\}$, $\mathfrak{C}_P(h_1) \cap Z\mathfrak{C}_P(h_3) = \{1\}$. Here Z is the center of P.

Since all Sylow p-subgroups of G are conjugate by elements of $\mathfrak{H}(G)$ and $\mathfrak{H}(G)$ is abelian, it follows that $k_i(G)$ are indeed functions of G.

Let $G \sim C$ denote the wreath product of G by a cyclic group C of order p. If $\mathfrak{H}(G)$ is normal and abelian then so is $\mathfrak{H}(G \sim C)$.

LEMMA 1.3. *Let* $p > 2$.
(i) $k_1(G) = 1$ *implies* $k_1(G \sim C) = 0$,
(ii) $k_1(G) \geqq 2$ *implies* $k_1(G \sim C) \geqq 2$, *and*
(iii) $k_2(G) \geqq 2$ *implies* $k_2(G \sim C) \geqq 2$.

LEMMA 1.4. *Let* $p = 2$.
(i) $k_1(G) = 1$ *implies* $k_1(G \sim C) = 0$,
(ii) $k_1(G) = 2$ *implies* $k_1(G \sim C) = 1$,
(iii) $k_1(G) \geqq 3$ *implies* $k_1(G \sim C) \geqq 3$,
(iv) $k_2(G) \geqq 3$ *implies* $k_2(G \sim C) \geqq 3$, *and*
(v) $k_3(G) \geqq 1$ *implies* $k_3(G \sim C) \geqq 1$

Proof. Set $G^* = G \sim C$. Then G^* has a normal subgroup $N^* = G_1 \times \cdots \times G_p$ the direct product of p copies of G, and $G^* = N^* \times_\sigma C$ where C acts on N^* by cyclically permuting the G_i. Clearly $H^* = \mathfrak{H}(G^*) = H_1 \times H_2 \times \cdots \times H_p$ where $H_i = \mathfrak{H}(G_i)$. Let α_i denote a conjugacy class of H_i, under the action of G_i. Then clearly the conjugacy classes of H^* under the action of N^* are all of the form $\alpha^* = \alpha_1 \times \alpha_2 \times \cdots \times \alpha_p$. If $h^* = h_1 h_2 \cdots h_p \in \alpha^*$ then

$$\mathfrak{C}_{N^*}(h^*) = \mathfrak{C}_{G_1}(h_1) \times \mathfrak{C}_{G_2}(h_2) \times \cdots \times \mathfrak{C}_{G_p}(h_p).$$

Let $\mathfrak{L}_i$ denote the set of conjugacy classes α_i of H_i such that for some $h_i \in \alpha_i, \mathfrak{C}_{G_i}(h_i) = H_i$. If $k = k_1(G)$, then $\mathfrak{L}_i$ contains k members. Clearly C permutes the set $\mathfrak{L}_1 \times \mathfrak{L}_2 \times \cdots \times \mathfrak{L}_p$ which contains k^p elements. If C fixes an element in this set, then this element is uniquely determined by its $\mathfrak{L}_1$ component. Hence C fixes precisely k elements and moves the remaining $k^p - k$ in orbits of size p. These moved orbits then yield $(k^p - k)/p$ conjugacy classes α^* under the action of G^* such that for some $h^* \in \alpha^*$, $\mathfrak{C}_{G^*}(h^*) = \mathfrak{C}_{N^*}(h^*) = H^*$. On the other hand, if α^* has an element h^* with $\mathfrak{C}_{G^*}(h^*) = H^*$, then certainly $\alpha^* \in \mathfrak{L}_1 \times \mathfrak{L}_2 \times \cdots \times \mathfrak{L}_p$. Hence we have $k_1(G \sim C) = (k^p - k)/p$. This yields Lemma 1.3(i), (ii) and Lemma 1.4(i), (ii), (iii).

Using the same proof as above, we show that $k_2(G \sim C) \geqq \{(k_2(G))^p - k_2(G)\}/p$. This yields Lemma 1.3(iii) and Lemma 1.4(iv).

We need only consider part (v) of Lemma 1.4. Let a, b, c be the three given conjugate elements of $H = \mathfrak{H}(G)$. Since H is abelian there exists $w, u \in P$ with $b = a^w$, $c = a^u$. Let $T = \mathfrak{C}_P(a)$ so that $T^w = \mathfrak{C}_P(b)$ and $T^u = \mathfrak{C}_P(c)$. Let z be a central element of P of order 2. By assumption $T^w T$ contains a nonidentity central element but $T^u T$ does not.

Now $P^* = P \sim C$ is a Sylow 2-subgroup of $G \sim C$. This group has as a normal subgroup of index 2 the direct product $P_1 \times P_2 = M^*$ of two copies of P. In dealing with the groups P_i we will use subscripts so that for example $T_i = \mathfrak{C}_{P_i}(a_i)$. Let $C = \langle v \rangle$. Set $a^* = a_1 a_2$, $b^* = c_1^{z_1} c_2$ and $c^* = c_1 b_2$. Then $b^* = (a^*)^{u_1 z_1 u_2}$ and $c^* = (a^*)^{u_1 w_2}$. Thus we set $w^* = u_1 z_1 u_2$ and $u^* = u_1 w_2$. Let $T^* = \mathfrak{C}_{P^*}(a^*)$ so that clearly $T^* = \langle T_1, T_2, v \rangle$. We show now that $T^{*w^*} \cap T^* Z^* \neq \{1\}$, $T^{*w^*} \cap T^* = \{1\}$ and $T^{*u^*} \cap T^* Z^* = \{1\}$, where Z^* is the center of P^*. This will yield the result.

Now $z^* = z_1 z_1^v = v^{u_1 z_1 u_2} v = v^{w^*} v \in T^{*w^*} T^*$ since z_1 and v have order 2 and $u_1 u_2$ centralizes v. Clearly z^* is central and not equal to 1. Now we consider $T^* \cap T^{*w^*}$. Since $T^* \cap M^* = T_1 \times T_2$ and $T^{*w^*} \cap M^* = T_1^{u_1} \times T_2^{u_2}$ we see that if $T^* \cap T^{*w^*} > \{1\}$, then this intersection must contain an element not in M^*. If x is such an element, then we can write $x = t_1 t_2 v \in T^*$ with $t_i \in T_i$. Also $x = \bar{t}_1 \bar{t}_2 z_1 z_1^v v \in T^{*w^*}$ with $\bar{t}_i \in T_i^{u_1}$, since $v^{w^*} = z_1 z_1^v v$. But then $z_1 = t_1 \bar{t}^{-1} \in T_1 T_1^{u_1}$, a contradiction. Thus $T^* \cap T^{*w^*} = \{1\}$.

Finally we consider $T^{*u^*} \cap T^* Z^*$. Let x belong to this intersection. If $x \in M^*$ then since $Z^* \subseteq M^*$ we have $x = t_1 t_2 \bar{z}_1 \bar{z}_1^v \in T^* Z^*$ with $t_i \in T_i$ and $\bar{z}_1$ in the center of P. Also $x = \bar{t}_1 \bar{t}_2 \in T^{*u^*}$ with $\bar{t}_1 \in T_1^{u_1}$ and $\bar{t}_2 \in T_2^{w_2}$. Then $\bar{z}_1 = \bar{t}_1 t_1^{-1} \in T_1^{u_1} T_1$ and so $\bar{z}_1 = 1$. This yields $t_1 = \bar{t}_1 = 1$ and $t_2 = \bar{t}_2 = 1$ and so $x = 1$. Now suppose $x \notin M^*$. Then $x = t_1 t_2 v z^* \in T^* Z^*$ with $t_i \in T_i$ and $z^* \in Z^*$. Since x normalizes $T^{*u^*} \cap M^*$ we see that $T_1^{u_1 x} = T_2^{w_2} = T_1^{w_1 v}$. Hence since z^* is central and t_2 centralizes P_1 we have $T_1^{u_1 t_1} = T_1^{w_1}$. But then $(T_1^{u_1} T_1)^{t_1} = T_1^{w_1} T_1$ since $t_1 \in T_1$. Thus if one of these terms contains a nonidentity central element of P_1 so does the other. This is a contradiction so we have $T^{*u^*} \cap T^* Z^* = \{1\}$ and the proof is complete.

We need finally the following number theoretic fact.

LEMMA 1.5. *Let p and q be primes satisfying $p^\varepsilon = q^\delta - 1$. Then we must have*

(i) $q = 2$, $\varepsilon = 1$ *and* $p = 2^\delta - 1$ *is a Mersenne prime,*

(ii) $p = 2$, $\delta = 1$ *and* $q = 2^\varepsilon + 1$ *is a Fermat prime, or*

(iii) $p = 2$, $\varepsilon = 3$, $q = 3$, $\delta = 2$.

Proof. Clearly one of p or q is equal to 2. Suppose first that $q = 2$ so that $2^\delta = p^\varepsilon + 1$. If ε is even then since p is odd, $p^\varepsilon \equiv 1 \bmod 4$. Hence $2^\delta \equiv 2 \bmod 4$. This yields $2^\delta = 2$ and $p^\varepsilon = 1$, a contradiction. Thus ε is odd and $2^\delta = (p+1)(p^{\varepsilon-1} - p^{\varepsilon-2} + \cdots - p + 1)$. Since the second factor contains an odd number of terms, it is odd and thus equal to 1. Hence $\varepsilon = 1$ and $p = 2^\delta - 1$.

Now let $p = 2$ so that $2^\varepsilon = q^\delta - 1$. Suppose that δ is odd. Then $2^\varepsilon = (q-1)(q^{\delta-1} + q^{\delta-2} + \cdots + q + 1)$. Since δ is odd, the second factor here also contains an odd number of terms. Hence as above $\delta = 1$ and $q = 2^\varepsilon + 1$. If $\delta = 2\gamma$ is even then $2^\varepsilon = (q^\gamma - 1)(q^\gamma + 1)$. Thus $q^\gamma - 1 = 2^r, q^\gamma + 1 = 2^s$ and $2 - 2^r = 2$. This yields $2^s = 4, 2^r = 2$ and we have (iii).

We now proceed with the proof of Theorem 1.1. By Lemma 1.2, it suffices to assume that P acts faithfully and irreducibly on H an elementary abelian q-group of order q^n. We extend P to a Sylow p-subgroup of $\mathrm{GL}(n,q)$. It is clear that if the theorem holds for this larger group then it holds for P. Hence it suffices to assume that P is a Sylow p-subgroup of $\mathrm{GL}(n,q)$. There is a slight difference in structure here according to whether $p=2$ or not.

Let Q be an elementary abelian q-group for prime $q \neq p$ and let $|Q| = q^\delta$. Let $\bar{C}$ be a cyclic group of order $p^\varepsilon > 1$ which acts irreducibly on Q. Let $G_1 = Q \times_\sigma \bar{C}$ and for $i > 1$ set $G_i = G_{i-1} \sim C$ where C is cyclic of order p. Since P acts irreducibly on H we see by [1] and [6] that $H \times_\sigma P = G_i$ for some i provided $p > 2$ or $p = 2$ and $q \equiv 1 \bmod 4$. Since $\bar{C}$ acts fixed point free on Q we have in the terminology of Lemmas 1.3 and 1.4, $k_1(G_1) = (q^\delta - 1)/p^\varepsilon$. Suppose first that $p > 2$. If $k_1(G_1) \geqq 2$ then by Lemma 1.3 and induction $k_1(G_i) \geqq 2$ and the result follows. The exception occurs when $q^\delta - 1 = p^\varepsilon$ and by Lemma 1.5, $q = 2, \varepsilon = 1$ and $p = 2^\delta - 1$ is a Mersenne prime. The result will follow if we show in this case that $k_2(G_2) \geqq 2$. We know of course that $k_1(G_2) = 0$. Now G_2 has a normal subgroup N of index p of the form $N = (Q_1 \times Q_2 \times \cdots \times Q_p) \times_\sigma (\bar{C}_1 \times \bar{C}_2 \times \cdots \times \bar{C}_p$ where $Q_i \simeq Q$, $\bar{C}_i \simeq \bar{C}$ and $\bar{C}_i$ acts irreducibly on Q_i and centralizes the remaining Q_j. Let α^* be the conjugacy class containing $h^* = h_1 h_2 \cdots h_p$ with $h_i \in Q_i$ and $h_i \neq 1$. Since no element of $\bar{C}_1 \times \bar{C}_2 \times \cdots \times \bar{C}_p$ fixes this element and $\mathfrak{C}_P(h^*) > \{1\}$ we have $|\mathfrak{C}_P(h^*)| = p$. Let $\mathfrak{C}_P(h^*) = \langle x \rangle$. Then $\mathfrak{C}_P(h^*)$ is abelian and since x commutes with one of its conjugates, α^* yields one class of conjugates corresponding to k_2. A second class β^* is obtained by taking all conjugates of $h^* = h_1 h_2 \cdots h_{p-1}$ with $h_i \in Q_i$ and $h_i \neq 1$. In this case $\mathfrak{C}_P(h^*) = \bar{C}_p$. Since $\bar{C}_p$ is abelian and commutes with its disjoint conjugate $\bar{C}_1$, the result follows here.

Now let $p=2$. If $k_1(G_1) \geqq 3$ the result follows by Lemma 1.4 and induction. If $k_1(G_1) = 1$ or 2 we have $q^\delta - 1 = 2^\varepsilon$ or $2^{\varepsilon+1}$ and by Lemma 1.5, q is a Fermat prime and $\delta = 1$. Since $\bar{C}$ is a Sylow 2-subgroup of $\mathrm{GL}(1,q)$ we must have $2^\varepsilon = q - 1$. Thus $k_1(G_1) = 1$ and $k_1(G_2) = 0$. We show that $k_3(G_2) \geqq 1$ in this case. Let α^* be the class containing $h^* = h_1 h_2$ with $h_i \in Q_i$ and $h_i \neq 1$. Since $\mathfrak{C}_P(h^*) > \{1\}$ and $\mathfrak{C}_P(h^*) \cap (\bar{C}_1 \times \bar{C}_2) = \{1\}$ it follows that $|\mathfrak{C}_P(h^*)| = 2$. Say $\mathfrak{C}_P(h^*) = \langle x \rangle$. It is clear that we need only find two conjugates of x, say x^w and x^u, with $x^w x \in Z - \{1\}$ and $x^u x \notin Z$ where Z is the center of P. Let w be an element of order 2 of $\bar{C}_1$. Then $x^w x = w w^x \in Z - \{1\}$. Finally since $q \equiv 1 \bmod 4$, $q \geqq 5$ so $|\bar{C}| = 2^\varepsilon \geqq 4$. Thus let u be an element of $\bar{C}_1$ of order 4. Then $x^u x = u^2(u u^x)$. Since $u u^x \in Z$ we see that $x^u x \in Z$ implies $u^2 \in Z$, a contradiction. Thus $k_3(G_2) \geqq 1$ and the result follows by Lemma 1.4.

Now let $p = 2$ and $q \equiv 3 \bmod 4$. Let W be the group generated by x and y satisfying $x^{2^s} = 1$, $y^2 = 1$ and $y^{-1} x y = x^{-1+2^{s-1}}$ and let Q be elementary abelian of order q^2. For suitable s, W is a Sylow 2-subgroup of $\mathrm{GL}(2,q)$. Set $G_1 = Q \times_\sigma W$ and for $i > 1$ let $G_i = G_{i-1} \sim C$ where $|C| = 2$. Then by [1], $H \times_\sigma P = G_t$ for some i. We study G_1. Since $\langle x \rangle$ acts fixed point free on Q and $\langle x \rangle$ has index 2 in W we see that for any $h \in Q - \{1\}$ we have $|\mathfrak{C}_W(h)| = 1$ or 2. Conversely let

$w \in W - \langle x \rangle$ be an element of order 2. Since w is not central its minimal polynomial on Q is $(w+1)(w-1)=0$. Thus $Q^{1+w} = \mathfrak{C}_Q(w)$ is a proper subspace of Q. But $|Q| = q^2$ so $|\mathfrak{C}_Q(w)| = q$. Now W contains 2^{s-1} such elements w. Hence we have easily $k_1(G_1) = \{(q^2-1) - 2^{s-1}(q-1)\}/2^{s+1}$. If $k_1(G_1) \geqq 3$ the result follows by Lemma 1.4. Since $q \equiv 3 \bmod 4$, $k_1(G_1) = 1$ or 2 yields $q = 3$ and $8 = 2^s + 2^{s+1}$ or $8 = 2^s + 2^{s+2}$ which cannot hold. Finally $k_1(G_1) = 0$ yields $q = 2^{s-1} - 1$, a Mersenne prime. In this case we have clearly $k_2(G_1) = (q^2-1)/2^s$. If $k_2(G_1) \geqq 3$ the result again follows. But by Lemma 1.5, $k_2(G_1) = 1$ or 2 yield $q = 3$. Since $q = 2^{s-1} - 1$ we have $2^s = 8$. Note that $q = 3$ is also a Fermat prime. In this case it is quite easy to see that $k_3(G_1) \geqq 1$ and thus the result follows.

2. **Small orbits.** The main result of this section is the following:

THEOREM 2.1. *Let p-group P act faithfully and irreducibly on elementary abelian q-group Q. Let all orbits in Q under the action of P have size at most p^e. Then there exists $h \in Q$ with $|\mathfrak{C}_P(h)| < (p^{1/p-1})^{e^2} \leqq 2^{e^2}$. Moreover if $e \leqq p$ then there exists $h \in Q$ with $\mathfrak{C}_P(h) = \{1\}$ with the following exceptions which occur when there is an orbit of size p^p.*

(i) $p = 2, q = 3$
$Q \times_\sigma P = (V \times_\sigma C) \sim C$ *where* $|V| = 3$, $|C| = 2$

(i.i) $q = 2$, $p = 2^\alpha - 1$ *is a Mersenne prime*
$Q \times_\sigma P = (V \times_\sigma C) \sim C$ *where* $|V| = 2^\alpha$, $|C| = p$.

LEMMA 2.2. *Let A be an abelian p-group acting faithfully on solvable p'-group H. Then there exists $h \in H$ with $\mathfrak{C}_A(h) = \{1\}$.*

Proof. Since the property of having an abelian Sylow p-subgroup is inherited by subgroups and quotient groups, Lemma 1.2 applies. Thus it suffices to assume that H is elementary abelian and A acts irreducibly. But then A is cyclic and acts fixed point free so the result follows.

The following is essentially a restatement of Jordan's theorem for linear p-groups. While the result contains more than we need here, the exact values of the bounds may be of some interest in themselves.

PROPOSITION 2.3. *Let p-group P be a faithful irreducible complex linear group of degree p^n. Then P has a normal abelian subgroup A with $[P:A] \leqq p^{(p^n-1)/(p-1)}$ and the latter bound is possible. Moreover if χ is any irreducible complex character of P then $\deg\chi \leqq p^{(p^n-1)/(p-1)}$ and again this bound is best possible.*

Proof. Let A be an abelian subgroup of P and let χ be an irreducible complex character of P. If λ is a constituent of $\chi|A$, then χ is a constituent of λ^* (induction to P). Thus $\deg\chi \leqq \deg\lambda^* = [P:A]$. We prove this proposition by first obtaining the first bound. This then implies the second bound. Moreover by the above argument, if the second bound is best possible, then so is the first. Set $j(n) = (p^n - 1)/(p - 1)$.

Let θ be the character of the given faithful irreducible representation of P. We obtain the first result by induction on $\deg\theta = p^n$. If $p^n = 1$, then P is abelian and the result follows. Let $p^n > 1$. Since p-groups are monomial, there exists a normal subgroup N of P of index p and an irreducible character ψ of N with $\theta = \psi^*$. Then $\deg\psi = p^{n-1}$ and $\theta \mid N = \psi_1 + \psi_2 + \cdots + \psi_p$, the sum of p conjugates of ψ. Let K_i be the kernel of ψ_i so that K_i is normal in N, the K_i are conjugate in P, and $\bigcap K_i = \{1\}$. By induction N/K_1 has a normal abelian subgroup A_1/K_1 with $[N/K_1 : A_1/K_1] \leqq p^{j(n-1)}$. Let A be the intersection of the at most p conjugates in P of A_1. Then A is normal in P and $A' \subseteq \bigcap K_i$ so A is abelian. Also $[P:A] \leqq p^{pj(n-1)+1} = p^{j(n)}$ so the first result follows.

We show now by example that the second bound is best possible. Let C denote the cyclic group of order p. Set $P_0 = C$ if $p > 2$ and let P_0 be the cyclic group of order 4 if $p = 2$. Define P_n inductively for $n \geqq 1$ $P_n = P_{n-1} \sim C$. We show inductively that P_n is a faithful irreducible linear group of degree p^n and that P_n has at least 3 irreducible characters with degree at least $p^{j(n)}$. This is trivially true for $n = 0$. Let $n \geqq 1$ so that P_n has a normal subgroup N of index p with $N = R_1 \times R_2 \times \cdots \times R_p$, the direct product of p copies of P_{n-1}. Let θ be the given irreducible faithful character of degree p^{n-1} of R_1 viewed as a character of N. Then we see easily that θ^* is a faithful irreducible character of P_n of degree p^n. Now let $\mathfrak{L}_i$ be the set of irreducible characters of R_i of degree at least $p^{j(n-1)}$ and set $|\mathfrak{L}_i| = k \geqq 3$. Then $\mathfrak{L} = \mathfrak{L}_1 \times \mathfrak{L}_2 \times \cdots \times \mathfrak{L}_p$ is a set of size k^p of characters of N of degree at least $p^{pj(n-1)}$. As in the proof of Lemma 1.3, P_n/N permutes the elements of $\mathfrak{L}$, fixing k characters and moving the remaining ones in orbits of size p. Thus there are $(k^p - k)/p \geqq (3^p - 3)/p \geqq 3$ orbits of size p. Each of these orbits yields an induced character of P_n which is irreducible and of degree at least $p^{pj(n-1)+1} = p^{j(n)}$. This completes the proof.

We now proceed to prove Theorem 2.1. We can of course assume the maximal orbit size to be p^e. If $|P| = p^e$, then certainly for some $h \in Q$, $\mathfrak{C}_P(h) = \{1\}$. Thus we assume that $|P| > p^e$. Let N be a normal subgroup of P of order p^{e+1}. Then N has at most $(p^{e+1} - 1)/(p - 1)$ subgroups of order p. Let $h \in Q$. Then $\mathfrak{C}_N(h) > \{1\}$ and therefore $h \in \mathfrak{C}_Q(\langle x\rangle)$ where $\langle x\rangle$ is a subgroup of N of order p. Thus $Q = \cup\mathfrak{C}_Q(\langle x\rangle)$. If $\mathfrak{C}_Q(\langle x\rangle)$ denotes one of the right-hand groups of maximal order, then counting nonidentity elements we have $|Q| - 1 \leqq (|\mathfrak{C}_Q(x)| - 1)$ $\cdot(p^{e+1} - 1)/p - 1)$. Since P acts faithfully $|\mathfrak{C}_Q(x)| < |Q|$ and thus

$$|Q|/|\mathfrak{C}_Q(x)| < (|Q|-1)/(|\mathfrak{C}_Q(x)| - 1) \leqq (p^{e+1} - 1)/(p - 1).$$

Let M be the normal subgroup of P generated by all conjugates of x. Clearly $M \subseteq N$. If $M < N$, then $|M| \leqq p^e$ so M is generated by a set of at most e conjugates of x. If $M = N$, then since N cannot be abelian by Lemma 2.3, the same result holds. Thus $M = \langle x_1, x_2, \cdots, x_j \rangle$ where the x_i are $j \leqq e$ conjugates of x. Clearly $[Q:\mathfrak{C}_Q(x_i)] < (p^{e+1}-1)/(p-1)$. Thus $[Q:\mathfrak{C}_Q(M)] < \{(p^{e+1}-1)/(p-1)\}^e$. Now M is normal in P and P acts irreducibly on Q. Thus $\mathfrak{C}_Q(M) = \{1\}$ and $|Q| < \{(p^{e+1}-1)/(p-1)\}^e$.

Let θ be an absolutely irreducible constituent of the action of P on Q. Let $\mathrm{GF}(q)(\theta) = \mathrm{GF}(q^r)$. Then the representation of P on Q must contain the r conjugates of θ under the Galois group of $\mathrm{GF}(q^r)/\mathrm{GF}(q)$. Hence $|Q| \geqq q^{r \deg \theta}$. Since the representation associated with θ is realizable over $\mathrm{GF}(q^r)$ (since we are dealing with representations over finite fields) it follows that $\mathrm{GF}(q^r)$ contains a pth root of unity. Hence p divides $q^r - 1$ and $q^r \geqq p+1$. This yields $|Q| \geqq (p+1)^{\deg \theta}$. If $\deg \theta \geqq e^2$ we obtain

$$\{(p^{e+1}-1)/(p-1)\}^e > (p+1)^{e^2}$$

or

$$\sum_0^e p^i > \sum_0^e \binom{c}{i} p^i,$$

a contradiction. Hence $\deg \theta < e^2$.

Since all the other absolutely irreducible constituents of the representation of P on Q are algebraic conjugates of θ under the above mentioned Galois group, it follows that θ must be faithful. Let $\deg \theta = p^n$. Then by Proposition 2.3, P has an abelian subgroup A with $[P:A] \leqq p^{(p^n-1)/(p-1)} < (p^{1/(p-1)})^{e^2} \leqq 2^{e^2}$. Note we used the fact that θ is equivalent to a complex character of P. Since A is abelian, by Lemma 2.2, there exists $h \in Q$ with $\mathfrak{C}_P(h) \cap A = \mathfrak{C}_A(h) = \{1\}$. Thus $|\mathfrak{C}_P(h)| \leqq [P:A]$ and the first result follows.

Now let $p \geqq e$. Since P is nonabelian $\deg \theta > 1$. But $\deg \theta < e^2 < p^2$. Hence $\deg \theta = p$. By Proposition 2.3, P has a normal abelian subgroup A of index p. By Lemma 2.2, $|A| \leqq p^e$ so $|A| = p^e$ and $|P| = p^{e+1}$.

Let R be an irreducible A-submodule of Q. Then by Clifford's Theorem [2, Theorem 49.2], $Q = \sum_1^p R_i$ where $R_i = R^{x^i}$ and $P = \langle A, x \rangle$. If two of these are inequivalent A-modules then they are all inequivalent and the sum $Q = \sum R_i$ is direct. We consider this case first. Since A is abelian, if $a \in A$ then $\mathfrak{C}_{R_i}(a)$ is an A-submodule. Hence either a centralizes R_i or it moves all nonidentity elements of R_i. Let $h^* = h_1 h_2 \cdots h_p$ with $h_i \in R_i$ and $h_i \neq 1$. If $a \in \mathfrak{C}_P(h^*) \cap A$ then a centralizes each h_i. Hence a centralizes Q and $a = 1$. Since $\mathfrak{C}_P(h^*) > \{1\}$ it follows that P contains an element, say x, of order p with $x \notin A$. Now let $h^* = h_2 h_3 \cdots h_p$ with $h_i \in R_i$ and $h_i \neq 1$. Since every element of $P - A$ cyclically permutes the R_i, it follows that $\mathfrak{C}_P(h^*) \subseteq A$. Since $\mathfrak{C}_P(h^*) > \{1\}$ we can choose $y \in \mathfrak{C}_P(h^*)$ with y of order p. Then y centralizes each $h_i (i \geqq 2)$ so y centralizes each R_i with $i \geqq 2$. Clearly y does not centralize R_1. Let $y_j = y^{x^{j-1}}$. Then y_j centralizes all R_i with $i \neq j$ and does not centralize R_j. With this we see that the p elements $y_1, \cdots, y_p$ are independent and thus generate an elementary abelian subgroup of A of order p^p. But $|A| = p^e$ and $p \geqq e$ so $p = e$ and $A = \langle y_1, \cdots, y_p \rangle$. Thus $G = Q \times_\sigma P$ is the group $G = (R \times_\sigma C) \sim C$ where $|C| = p$. Moreover for this group $k_1(G) = 0$. Now if $|R| = q^\alpha$, then $k_1(R \times_\sigma C) = (q^\alpha - 1)/p$. Thus by Lemmas 1.3 and 1.4, $k_1(R \times_\sigma C) = 1$ or equivalently $p = q^\alpha - 1$. By Lemma 1.5 this occurs for $q = 2$ and $p = 2^\alpha - 1$ or $p = 2$ and $q = 3$. These are groups (i) and (ii).

We need to consider the case now where all the A-modules R_i are equivalent. Since the action of A on Q is faithful we see that R_1 is a faithful irreducible A-module. Thus since A is abelian, it is cyclic and acts fixed point free. Let $h \in Q$ with $h \neq 1$. Then $\mathfrak{C}_P(h) > \{1\}$ and $\mathfrak{C}_P(h) \cap A = \{1\}$. Thus P has an element of order p not contained in A. Note that if $p = 2$ then $|P| = 8$. With this we see that P has the following structure. $P = \langle x, a \rangle$ with $x^p = 1$, $a^{p^e} = 1$ and $x^{-1}ax = a^{1+p^{e-1}}$. But then $B = \langle x, a^p \rangle$ is a noncyclic abelian subgroup of P of index p. Using B instead of A we obtain the first case already discussed. With this the result follows.

COROLLARY 2.4. *Let p-group P act faithfully on solvable p'-group H.*

(i) *If $(p, \pi(H))$ is nonexceptional or if all the orbits have size less than p^p, then there exists $h \in H$ with $\mathfrak{C}_P(h) = \{1\}$.*

(ii) *If $(p, \pi(H))$ is not $(2,f)$ then there exists $h \in H$ with $|\mathfrak{C}_P(h)| \leqq |P|^{1/3}$.*

(iii) *In any case there exists $h \in H$ with $|\mathfrak{C}_P(h)| \leqq |P|^{1/2}$.*

Proof. The first part of (i) follows from Theorem 1.1(i). Since the property of having small orbits is inherited by subgroups and quotient groups, we conclude by Lemma 1.2 that the second part of (i) follows from Theorem 2.1. For (ii) let h_1 and h_2 be given as in Theorem 1.1(ii). Then $\mathfrak{C}_P(h_1) \cap \mathfrak{C}_P(h_2) = \{1\}$ and $A = \mathfrak{C}_P(h_1)\mathfrak{C}_P(h_2)$ is abelian. By Lemma 2.2 there exists $h_3 \in H$ with $\mathfrak{C}_A(h_3) = \{1\}$ or $\mathfrak{C}_P(h_3) \cap A = \{1\}$. Thus

$$|\mathfrak{C}_P(h_1)|\,|\mathfrak{C}_P(h_2)|\,|\mathfrak{C}_P(h_3)| = |\mathfrak{C}_P(h_1)\mathfrak{C}_P(h_2)\mathfrak{C}_P(h_3)| \leqq |P|.$$

Choose $h = h_1$, h_2 or h_3 so that $|\mathfrak{C}_P(h)| \leqq |\mathfrak{C}_P(h_i)|$. Then (ii) follows immediately. Part (iii) is clear from Theorem 1.1(iii).

We remark that the second part of Corollary 2.4(i) follows also from Satz (5) of [4] and properties of regular p-groups.

COROLLARY 2.5. *Let G be a solvable group, P a Sylow p-subgroup of G and $\mathfrak{D}_p(G)$ the intersection of all Sylow p-subgroup of G.*

(i) *If $(p, \pi(G))$ is nonexceptional then there exists $a \in G$ with $P \cap P^a = \mathfrak{D}_p(G)$.*

(ii) *In any case there exists $a, b \in G$ with $P \cap P^a \cap P^b = \mathfrak{D}_p(G)$.*

Proof. Clearly we can assume $\mathfrak{D}_p(G) = \{1\}$. Let $H = \mathfrak{F}(G)$, the Fitting subgroup of G. Since H is nilpotent $P \cap H = \{1\}$. Also since H contains its centralizer, it follows that $\mathfrak{D}_p(PH) = \{1\}$. Hence it suffices to assume that $G = PH$, that is that $H = \mathfrak{H}(G)$ is normal and solvable and P acts faithfully on H. Let

$a \in H$. Clearly $P \cap P^a \supseteq \mathfrak{C}_P(a)$. Conversely let $x \in P \cap P^a$. Then $x = y^a$ with $y \in P$. Thus $(y, a) = y^{-1}y^a = y^{-1}x \in P$. But $a \in H$ and H is normal in G so $(y, a) \in H$. Hence $(y, a) \in P \cap H = \{1\}$ and $x = y \in \mathfrak{C}_P(a)$.

If $(p, \pi(G))$ is nonexceptional then there exists $a \in H$ with $\mathfrak{C}_P(a) = P \cap P^a = \{1\}$. Thus (i) follows. By Theorem 1.1 (iii) there exists $a, b \in H$ with $\{1\} = \mathfrak{C}_P(a) \cap \mathfrak{C}_P(b) = P \cap P^a \cap P \cap P^b = P \cap P^a \cap P^b$ and (ii) follows.

3. **Reduction theorems.** Let G be a finite group and p a fixed prime. If $H = \mathfrak{H}(G)$, the Hall p'-subgroup of G, is normal and abelian then the degrees of the irreducible complex characters of G are all powers of p. If the biggest such degree is equal to p^e, then we let $e = e(G)$ be the character exponent of G. We say G has r.x.e (representation exponent e) if $e(G) \leqq e$ and G has r.x.(e, s) if $e(G) \leqq e$ and $e(\mathfrak{S}_p(G)) \leqq s$. We apply the results of the previous sections to obtain relations between certain functions studied in [3]. In the following, all groups are assumed to have normal abelian Hall p'-subgroups.

Definition. (i) Let f be the smallest function with the following property. If G has r.x.e, then G has a subinvariant abelian subgroup whose index in G divides $p^{f(e)}$.

(ii) Let f_p be the smallest function with the following property. If G is a p-group with r.x.e then G has an abelian subgroup whose index in G divides $p^{f_p(e)}$

(iii) Let g be the smallest function with the following property. If group G has r.x.(e, s), then G has a subinvariant abelian subgroup whose index in G divides $p^{g(e,s)}$.

Theorem 3.1. *We have $f(e) = f_p(e)$. Moreover for $p \neq 2$, if G is a group with r.x.e having no subinvariant abelian subgroup with index dividing $p^{f(e)-1}$, then $\mathfrak{H}(G)$ is central.*

Theorem 3.2. *If either $p \neq 2$ and p is not a Mersenne prime or $p > e$, then for $e \geqq s$ we have $g(e, s) = e + f_p(s) - s$. Moreover under these assumptions, if G is a group with r.x.(e, s) which has no subinvariant abelian subgroup whose index divides $p^{g(e,s)-1}$, then $G/\mathfrak{C}(\mathfrak{H}(G))$ is abelian.*

Proof. Let P be a Sylow p-subgroup of G and $H = \mathfrak{H}(G)$. If P_1 is a Sylow p-subgroup of $\mathfrak{C}(H)$ then $\mathfrak{C}(H) = P_1 \times H$ and P_1 is normal in G. Thus p-group P/P_1 acts faithfully on abelian p'-group H. Therefore p-group P/P_1 acts faithfully on $\hat{H}$, the abelian p'-group of linear characters of H. We apply Corollary 2.4 to this action. We show first that if G has r.x.e then all orbits have size at most p^e. Let λ be a linear character of H and χ a constituent of λ^* (induction to G). Since H is normal in G we have by Clifford's Theorem, $\chi \mid H = a \sum_1^t \lambda_i$ where the λ_i are the t distinct conjugates of λ. Thus $t \leqq at = \deg \chi \leqq p^e$ and this fact is proved. Define δ to be 0 if $p \neq 2$ and p is not a Mersenne prime or if $p > e$. Let $\delta = 1/3$ if p is Mersenne and $p \leqq e$ and let $\delta = \frac{1}{2}$ if $p = 2 \leqq e$. Let $[P : P_1] = p^m$. Then by Corollary 2.4, H has a linear character λ whose inertia group $T(\lambda)$ (the analog of the centralizer) satisfies $T(\lambda) \supseteq \mathfrak{C}(H)$ and $[T(\lambda):\mathfrak{C}(H)] \leqq p^{\delta m}$.

Let $e_1 = e(P_1)$ and let θ be an irreducible character of P_1 of degree p^{e_1}. Then $\theta\lambda$ is an irreducible character of $\mathfrak{C}(H)$. Let χ be a constituent of $(\theta\lambda)^*$ so that $\chi \,|\, \mathfrak{C}(H) = a \sum_1^t (\theta\lambda)_i$. Since both H and P_1 are normal in G, we have clearly $T(\theta\lambda) \subseteq T(\theta)$. Thus $t = [G:T(\theta\lambda)] \geqq [G:T(\lambda)] \geqq p^{(1-\delta)m}$. Since $p^e \geqq \deg \chi = at$ $\deg \theta \geqq p^{(1-\delta)m} p^{e_1}$ we have $e_1 \leqq e - (1-\delta)m$.

We now consider Theorem 3.1. Clearly $f(e) \geqq f_p(e)$. We show the reverse inequality below. Since P_1 has r.x.$(e-(1-\delta)m)$, by definition of f_p, P_1 has an abelian subgroup A with $[P_1:A] = p^a$ and $a \leqq f_p(e-(1-\delta)m)$. Then $A \times H$ is a subinvariant abelian subgroup of G whose index equals p^{a+m}. By Lemma 3.6 of [3], which applies equally well to f_p, we have for any $r, s, f_p(r+s) \geqq f_p(r) + 2s$. Thus

$$\begin{aligned} a + m &\leqq f_p(e-(1-\delta)m) + m \\ &\leqq f_p(e) - 2(1-\delta)m + m = f_p(e) - (1-2\delta)m. \end{aligned}$$

Since $\delta \leqq \frac{1}{2}$ we have $a + m \leqq f_p(e)$ and by definition of f as the smallest such function with the appropriate property, $f(e) \leqq f_p(e)$. Thus $f(e) = f_p(e)$. Finally $a + m = f_p(e)$ implies $(1-2\delta)m = 0$ and if $p \neq 2$ then $\delta < \frac{1}{2}$ so $m = 0$. Thus $\mathfrak{H}(G)$ is central in this case.

We now consider Theorem 3.2. Here $\delta = 0$ and P_1 has r.x.$(e-m)$. By Lemma 3.6 of [3], if $e \geqq s$ then $g(e,s) \geqq e - s + f_p(s)$. We show the reverse inequality. Now if $e(P_1) = e_1$ then $e_1 \leqq e - m$ and $e_1 \leqq s$. By definition of f_p, P has an abelian subgroup A of index p^a with $a \leqq f_p(e_1)$. Then $A \times H$ is a subinvariant abelian subgroup of G of index p^{a+m}. Since $m \leqq e - e_1$ we have

$$a + m \leqq f_p(e_1) + m \leqq e + f_p(e_1) - e_1.$$

But $e_1 \leqq s$ so $f_p(e_1) \leqq f_p(s) - 2(s - e_1)$ and

$$a + m \leqq e + f_p(s) - s - (s - e_1).$$

Hence $g(e,s) = e + f_p(s) - s$. If $a + m = e + f_p(s) - s$ then $s = e_1$. By Lemma 5.5 of [3] this implies that $P/P_1 \simeq G/\mathfrak{C}(H)$ is abelian. Thus the result follows.

There is good reason to believe that $g(e,s) = e + f_p(s) - s$ in all cases. However the last statement of Theorem 3.2 is definitely not true without the additional assumptions. For example the exceptional groups of Theorem 2.1 have r.x.$(p,1)$ and $G/\mathfrak{C}(Q)$ is nonabelian. Since $g(p,1) = p + 1$ these groups are counterexamples to that statement.

We conclude by discussing two conjectures of the nature of a Chinese Remainder Theorem. They make sense only for $p > e$.

CONJECTURE 3.3. Let G be a p-group with $e(G) = e$. Let $x_1, x_2, \cdots, x_v$ be any $v = p - e$ nonidentity elements of G. Then there exists an irreducible character χ of G of degree p^e such that for all i, x_i is not in the kernel of χ.

CONJECTURE 3.4. Let G satisfy $e(G) = e$. Let $x_1, x_2, \cdots, x_v$ be any $v = p - e$ elements not contained in $\mathfrak{H}(G)$. Then there exists an irreducible character χ of G of degree p^e such that for all i, x_i is not in the kernel of χ.

The latter is of course a generalization of the former. We note that the condition $x_i \notin \mathfrak{H}(G)$ cannot be replaced by $x_i \neq 1$. For example, let P be an abelian p-group and let H be an abelian noncyclic p'-group. Set $G = P \times H$ so that $e(G) = 0$. If $p > |H|$ we get an easy counterexample by choosing $\{x_i\} = H - \{1\}$. It is not hard to show by example that if Conjecture 3.3 is true then $v = p - e$ is best possible. At present the validity of this conjecture is known only for small values of e.

PROPOSITION 3.5. *Conjectures* 3.3 *and* 3.4 *are equivalent.*

Proof. Certainly Conjecture 3.4 implies Conjecture 3.3. We assume the latter now. Let $e(G) = e$ and let $x_1, \cdots, x_v$ be $v = p - e$ elements of G not in $H = \mathfrak{H}(G)$. Since $v \geqq 1$ we have $p > e$. Assume the numbering so chosen that $x_1, x_2, \cdots, x_w \in \mathfrak{C}(H)$ and the remainder do not. Let P_1 be a Sylow p-subgroup of $\mathfrak{C}(H)$ so that $\mathfrak{C}(H) = P_1 \times H$. For each $x_i \in \mathfrak{C}(H)$, write $x_i = y_i h_i$ with $y_i \in P_1$, $h_i \in H$. Since $x_i \notin H$, $y_i \neq 1$.

Let P be a Sylow p-subgroup of G. Then P/P_1 acts faithfully on H and hence on $\hat{H}$, the group of linear characters of H. Since $p > e$, Corollary 2.4 implies that there exists a linear character λ of H whose inertia group $T(\lambda)$ is equal to $\mathfrak{C}(H)$. Let θ be any irreducible character of P_1. Then $\theta\lambda$ is an irreducible character of $\mathfrak{C}(H)$. We show that $(\theta\lambda)^*$ is irreducible. Let χ be a constituent of $(\theta\lambda)^*$. Then $\chi \mid \mathfrak{C}(H) = a \sum_1^t (\theta\lambda)_i$ where the $(\theta\lambda)_i$ are the t conjugates of $\theta\lambda$. Now $t = [G: T(\theta\lambda)] \geqq [G: T(\lambda)] = [G: \mathfrak{C}(H)]$. Thus $[G: \mathfrak{C}(H)] \deg\theta \leqq at \deg\theta = \deg\chi$ and $\deg\chi \leqq \deg(\theta\lambda)^* = [G: \mathfrak{C}(H)] \deg\theta$. Hence $\chi = (\theta\lambda)^*$. In particular, if $e(G) = e$, $e(P_1) = e_1$ and $[G: \mathfrak{C}(H)] = p^m$, then $e \geqq e_1 + m$. This follows by choosing θ to have degree p^{e_1}.

Now let χ be a character of G of degree p^e and let ψ be a constituent of $\chi \mid \mathfrak{C}(H)$. Then χ is a constituent of ψ^* so $p^e = \deg\chi \leqq p^m \deg\psi$. Thus $\deg\psi \geqq p^{e-m}$ and $e(\mathfrak{C}(H)) \geqq e - m$. Since $e(\mathfrak{C}(H)) = e(P_1) = e_1$, this yields $e_1 = e - m$. Now we apply Conjecture 3.3 to the set $\{y_i\}$ of size $w \leqq v \leqq p - e \leqq p - e_1$ and obtain an irreducible character θ of P_1 of degree p^{e_1} with all y_i not in the kernel of θ. Set $\chi = (\theta\lambda)^*$. Then χ is an irreducible character of G of degree $p^{e_1+m} = p^e$. Since $\theta\lambda$ is a constituent of $\chi \mid \mathfrak{C}(H)$ we see that $x_1, \cdots, x_w \notin \ker\chi$. Finally χ vanishes off $\mathfrak{C}(H)$ so that none of the remaining x_i are in the kernel either. This completes the proof.

Added in proof. By modifying slightly the techniques of §1 we can obtain the following results which yield better bounds in the $(m,2)$ case.

THEOREM. *Let p-group P act faithfully on solvable p'-group H. Then there exists elements* $h_1, h_2, \cdots, h_p \in H$ *such that for* $i = 1, 2, \cdots, p-1$

$$\langle \mathfrak{C}_P(h_1), \mathfrak{C}_P(h_2), \cdots, \mathfrak{C}_P(h_i) \rangle \cap \mathfrak{C}_P(h_{i+1}) = \{1\}.$$

COROLLARY. *Let p-group P act faithfully on solvable p'-group H. Then there exist element* $h \in H$ *with* $|\mathfrak{C}_P(h)| \leqq |P|^{1/p}$.

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