

ON PRODUCT MEASURES AND FUBINI'S THEOREM IN LOCALLY COMPACT SPACES⁽¹⁾

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1. Introduction. The problems treated in this paper derive from the viewpoint of measure and integration developed in the book of P. R. Halmos [4]. We are concerned, above all, with the formulation of Fubini's theorem in the product of two locally compact spaces, assuming we are given a Borel measure on each of the factor spaces. Our basic tools, treated elegantly in [4], are (1) the theory of the product of two σ -finite measure spaces, and (2) the theory of a single Borel measure on a locally compact space. But these tools alone fail to yield a satisfactory Fubini theorem in the context of locally compact spaces. The reason for this failure is that the domain of definition of the product of two Borel measures, as defined in [4], may not be large enough. (Examples are given in §7 to illustrate insufficient domain. However, a case is given in §8 in which the domain is sufficient.) To explain this circumstance in greater detail, let us introduce some notations.

For the rest of the paper, μ and ν denote Borel measures on the locally compact spaces X and Y , respectively. (At times we shall assume that μ or ν is regular, or that $X = Y$.) For the definitions of Borel measure and regular Borel measure, the reader is referred to [4, Chapter X]. Specifically, the *Borel sets* of X , Y , and $X \times Y$ are the σ -ring generated by the compact subsets of X , Y , and $X \times Y$, respectively; we denote this class by $\mathcal{B}(X)$, $\mathcal{B}(Y)$, and $\mathcal{B}(X \times Y)$, respectively.

By the product σ -ring of $\mathcal{B}(X)$ and $\mathcal{B}(Y)$, denoted

$$\mathcal{B}(X) \times \mathcal{B}(Y),$$

we mean the σ -ring generated by sets of the form $E \times F$, where $E \in \mathcal{B}(X)$ and $F \in \mathcal{B}(Y)$ [4, p. 140]. Since $\mathcal{B}(X) \times \mathcal{B}(Y)$ is in fact generated by rectangles with compact sides [2, 35.2], we have

$$\mathcal{B}(X) \times \mathcal{B}(Y) \subset \mathcal{B}(X \times Y).$$

In general, this inclusion is proper. But if equality holds, we say that *the Borel sets in X and Y multiply* or simply that *the Borel sets multiply*.

The product of μ and ν , $\mu \times \nu$ [4, 35.B], may thus fail to be a Borel measure

Received by the editors March 24, 1965.

⁽¹⁾ This paper is based on the author's doctoral dissertation at the University of Iowa, written under the direction of Professor Sterling K. Berberian.

for want of sufficient domain; that is the case whenever the Borel sets fail to multiply. In §2 we see, however, that if μ and ν are regular, then there exists one and only one regular Borel measure on $X \times Y$ which extends $\mu \times \nu$. We may denote, in this case, the regular extension of $\mu \times \nu$ by $\mu \otimes \nu$.

Now we have a Fubini theorem for $\mu \times \nu$; but if Borel sets do not multiply, we would like to have a Fubini theorem for the regular Borel measure $\mu \otimes \nu$. This is given in Theorem 5.8. The necessary machinery is developed in §§3 and 4, and in §4 we develop extensions of $\mu \times \nu$, under less restrictive conditions than the regularity of both μ and ν .

2. The tensor product of two regular Borel measures. In this section we suppose that μ and ν are regular Borel measures, and we find an extension of $\mu \times \nu$ to a regular Borel measure on the product space. To this end, we make use of Baire sets and Baire measures.

The *Baire sets* in a locally compact space are the σ -ring generated by the compact G_δ 's. If λ is a Borel measure, its restriction to the class of Baire sets is a Baire measure λ_0 , which we call the *Baire restriction* of λ . A Baire measure always has an extension to a regular Borel measure, and such an extension is unique [4, 54.D]. In the proof of Theorem 2.1, we shall appeal to the following elementary criterion for regularity (cf. [2, 59.1]):

CRITERION. *In order that a Borel measure λ be regular, it is necessary and sufficient that each compact set C be contained in a compact G_δ D such that $\lambda(C) = \lambda(D)$.*

What is the product of the regular Borel measures μ and ν ? According to [4, 35.B], it is the unique measure $\mu \times \nu$ on the σ -ring $\mathcal{B}(X) \times \mathcal{B}(Y)$ such that

$$(\mu \times \nu)(E \times F) = \mu(E)\nu(F)$$

for all Borel sets E in X and F in Y .

Now if μ_0 and ν_0 are the Baire restrictions of μ and ν , respectively, then the product of μ_0 and ν_0 , namely $\mu_0 \times \nu_0$, is readily seen to be a Baire measure on $X \times Y$; the reason is that the Baire sets in X and Y always multiply [4, 51.E].

If Borel sets do not multiply, $\mu \times \nu$ fails to be a Borel measure on the simple grounds that its domain is not large enough. Nevertheless, there is a regular Borel measure ρ on $X \times Y$ ready at hand; namely, the unique extension of $\mu_0 \times \nu_0$ to a regular Borel measure [4, 54.D]. We are allowed to hope that ρ is an extension of $\mu \times \nu$, and this hope is indeed fulfilled:

THEOREM 2.1. *If μ and ν are regular Borel measures on the locally compact spaces X and Y , respectively, then there exists one and only one regular Borel measure on $X \times Y$ which extends $\mu \times \nu$. This measure, which we may denote $\mu \otimes \nu$, and call the tensor product of μ and ν , is simply the measure ρ described above.*

Proof. We show that the regular Borel measure ρ , described above, extends $\mu \times \nu$. It is sufficient, for this, to show that

$$(1) \quad \rho(C \times D) = (\mu \times \nu)(C \times D)$$

for every rectangle with compact sides. Indeed, the ring \mathcal{R} generated by such rectangles is the class of all finite disjoint unions of rectangles of the form $(C_1 - C_2) \times (D_1 - D_2)$, where the C_i and D_i are compact sets, and $C_2 \subset C_1$, $D_2 \subset D_1$ [4, 51.F and 33. E]. Since such a rectangle can be written in the form

$$(2) \quad [(C_1 \times D_1) - (C_2 \times D_1)] - [(C_1 \times D_2) - (C_2 \times D_2)],$$

where all of the differences are proper, it is clear from subtractivity that the validity of (1) implies that $\rho = \mu \times \nu$ on rectangles of the form (2). It then follows from additivity that $\rho = \mu \times \nu$ on \mathcal{R} ; but then $\rho = \mu \times \nu$ on the σ -ring generated by \mathcal{R} [4, 13.A], that is, on the σ -ring $\mathcal{B}(X) \times \mathcal{B}(Y)$ on which $\mu \times \nu$ is defined.

Now, let us suppose C and D are arbitrary compact sets in X and Y , respectively. By the regularity criterion listed above, we may choose compact G_δ 's, K , L , and M , in the appropriate spaces, such that

$$\begin{aligned} C &\subset K, & \mu(C) &= \mu(K), \\ D &\subset L, & \nu(D) &= \nu(L), \\ C \times D &\subset M, & \rho(C \times D) &= \rho(M). \end{aligned}$$

Now, ρ and $\mu \times \nu$ agree on the Baire sets of $X \times Y$, for they are both extensions of $\mu_0 \times \nu_0$. Since both $K \times L$ and M are Baire sets containing $C \times D$, we have

$$(\mu \times \nu)(C \times D) \leq (\mu \times \nu)(M) = \rho(M) = \rho(C \times D),$$

and

$$\rho(C \times D) \leq \rho(K \times L) = (\mu \times \nu)(K \times L) = \mu(K)\nu(L) = \mu(C)\nu(D) = (\mu \times \nu)(C \times D);$$

this establishes the relation (1). Uniqueness is clear [4, 52.H].

This raises the following question: If μ and ν are arbitrary Borel measures, not necessarily regular, is it always possible to extend $\mu \times \nu$ to a Borel measure? We do not know. In §4 we shall show that such an extension is possible if at least one of μ or ν is regular. In any case, the next theorem shows that if $\mu \times \nu$ is nonzero, then no *regular* Borel extension of $\mu \times \nu$ is possible, unless μ and ν are both regular to begin with:

THEOREM 2.2. *If ρ is a nonzero regular Borel measure on $X \times Y$ which extends $\mu \times \nu$, then both μ and ν are regular.*

Proof. If ρ is a regular Borel extension of $\mu \times \nu$, it is a regular extension of the Baire measure $\mu_0 \times \nu_0$. It follows from Theorem 2.1 that ρ extends $\mu' \times \nu'$,

where μ' and ν' are the regular Borel extensions of μ_0 and ν_0 , respectively. Hence $\mu' \times \nu' = \mu \times \nu$, and so

$$\mu'(E)\nu'(F) = \mu(E)\nu(F)$$

for all Borel sets E in X and F in Y . Since ρ , and hence $\mu \times \nu$, is nonzero, it follows that $\mu = \mu'$ and $\nu = \nu'$.

Note. Theorem 2.1 will follow as a corollary to Theorems 4.5 and 5.7. Nevertheless, it is pleasant that this result can be established relatively easily using only the tools mentioned in the Introduction.

3. Sections of Borel sets and Borel functions. Following [4, p. 141], if M is a subset of $X \times Y$, we define the y -section [x -section] of M , denoted M^y [M_x], as follows:

$$M^y = \{x \in X : (x, y) \in M\},$$

$$M_x = \{y \in Y : (x, y) \in M\}.$$

If Pr_X and Pr_Y are the projection mappings of $X \times Y$ onto X and Y , respectively, then

$$M^y = \text{Pr}_X[M \cap (X \times \{y\})],$$

$$M_x = \text{Pr}_Y[M \cap (\{x\} \times Y)].$$

If $M \in \mathcal{B}(X) \times \mathcal{B}(Y)$, then $M^y \in \mathcal{B}(X)$ for all $y \in Y$, and $M_x \in \mathcal{B}(Y)$ for all $x \in X$ [4, 34.A]. It may be surprising that the same result holds for *any* Borel set M in $X \times Y$:

THEOREM 3.1. *If $M \in \mathcal{B}(X \times Y)$, then $M^y \in \mathcal{B}(X)$ for all $y \in Y$, and $M_x \in \mathcal{B}(Y)$ for all $x \in X$.*

Proof. Let \mathcal{C} be the class of all $M \subset X \times Y$ such that $M^y \in \mathcal{B}(X)$ for all $y \in Y$, and $M_x \in \mathcal{B}(Y)$ for all $x \in X$. Since sections preserve countable unions and set-theoretic differences, \mathcal{C} is a σ -ring. The theorem will be proved if we show that \mathcal{C} contains the compact sets in $X \times Y$.

Let C be a compact set in $X \times Y$, and let $y \in Y$. Then $X \times \{y\}$ is closed; $C \cap (X \times \{y\})$ is compact; and

$$C^y = \text{Pr}_X[C \cap (X \times \{y\})]$$

is compact by the continuity of Pr_X . Hence $C^y \in \mathcal{B}(X)$ for all $y \in Y$. Similarly, $C_x \in \mathcal{B}(Y)$ for all $x \in X$. Hence $C \in \mathcal{C}$, as we wished to show.

From Theorem 3.1 we conclude the following:

$$\mathcal{B}(X) = \{M^y : M \in \mathcal{B}(X \times Y), y \in Y\},$$

$$\mathcal{B}(Y) = \{M_x : M \in \mathcal{B}(X \times Y), x \in X\}.$$

In particular, if the class $\mathcal{B}(X \times Y)$ is known, then the classes $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ are determined. Conversely, if $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ are known, can one determine $\mathcal{B}(X \times Y)$? We do not know.

We recall that a real valued function f on X is called a *Borel measurable function* if it is measurable with respect to the σ -ring of Borel sets [4, §51]. The briefer term "Borel function" will be used, and we shall favor the letters f , g , and h for functions defined on X , Y , and $X \times Y$, respectively.

Suppose h is a real valued function on $X \times Y$. The sections of h are defined as in [4, p. 141]; thus, if $x \in X$, we define $h_x(y) = h(x, y)$ for all $y \in Y$, and if $y \in Y$, we define $h^y(x) = h(x, y)$ for all $x \in X$. Using Theorem 3.1, we see that if h is a Borel function on $X \times Y$, then all its sections are Borel functions on the appropriate spaces.

COROLLARY. *If h is a Borel function on $X \times Y$, then h_x is a Borel function on Y and h^y is a Borel function on X , for all $x \in X$ and $y \in Y$.*

Proof. If M is any set of real numbers, then $(h_x)^{-1}(M) = (h^{-1}(M))_x$ and $(h^y)^{-1}(M) = (h^{-1}(M))^y$. The corollary now follows directly from Theorem 3.1.

4. Extension of products of not necessarily regular Borel measures. Under suitable conditions we shall construct Borel measures ρ_1 and ρ_2 on $X \times Y$ which extend $\mu \times \nu$. Our procedure is similar to that of [4, Chapter VII] or [1]. The machinery developed here is used to prove the Fubini-Tonelli theorems of §5. The gateway to the results of this section is the conclusion of Theorem 4.1.

Our construction of ρ_1 and ρ_2 is based on integration. We now describe the basic theory used. Suppose $(X, \mathcal{S}, \lambda)$ is an abstract measure space. As in [2], we reserve the terms *measurable* and *integrable* for real (finite) valued functions only. If an extended real valued function f is equal a.e. $[\lambda]$ to a measurable or integrable function g , we call f *a.e. measurable* or *a.e. integrable*, respectively. If f is a.e. integrable and g is an integrable function equal to f a.e., we define $\int f d\lambda$ to be $\int g d\lambda$. It is easily seen that the value for $\int f d\lambda$ does not depend on the representative g . Strong use will be made of the following theorem:

MONOTONE CONVERGENCE THEOREM [4, 27.B]. *Let $(X, \mathcal{S}, \lambda)$ be an abstract measure space. Suppose f is an extended real valued function on X , and suppose f_n is an increasing sequence of nonnegative a.e. integrable functions on X such that f_n increases to f a.e. (that is, $f_n(x) \uparrow f(x)$ for almost all $x \in X$). Then f is a.e. integrable if and only if $\int f_n d\lambda$ is bounded. In this case, $\int f_n d\lambda \uparrow \int f d\lambda$.*

If M is a Borel set in $X \times Y$, every x -section of M is a Borel set in Y (Theorem 3.1). We may therefore define a nonnegative extended real valued function f_M on X by the formula

$$f_M(x) = \nu(M_x)$$

for all $x \in X$ (cf. [1]). We propose to integrate f_M for at least bounded Borel sets and hence must check to see that f_M is integrable for such sets. That is the case if f_C is a Borel function, for each compact set C in $X \times Y$.

THEOREM 4.1. *If ν is regular, then f_C is a Borel function, for each compact set C in $X \times Y$.*

Proof. Suppose C is a compact set in $X \times Y$. Since $f_C \geq 0$, it will suffice to show that for each positive number a , the set

$$A = \{x: f_C(x) \geq a\}$$

is a Borel set. We shall in fact show that A is compact. Moreover, since $A \subset \text{Pr}_X C$, a compact set, it will suffice to show that A is closed.

Suppose $z \in X - A$; that is, $\nu(C_z) < a$. Since ν is regular, we may choose an open Borel set V in Y so that $C_z \subset V$ and $\nu(V) < a$. Let $U = X - \text{Pr}_X[C - (X \times V)]$. It is easy to see that U is open and that $x \in U$ if and only if $C_x \subset V$. It follows that U is a neighborhood of z which is disjoint from A . Since z is arbitrary, A is closed.

We now prove a type of sequential upper semicontinuity for f_C , where C is a compact set in $X \times Y$. For first countable spaces (spaces in which each point has a fundamental sequence of neighborhoods) the result implies upper semicontinuity of f_C .

THEOREM 4.2. *Suppose C is a compact set in $X \times Y$ and that x_n is a sequence in X converging to $x \in X$. If $f_C(x_n) \geq a$ for all n , then $f_C(x) \geq a$.*

Proof. Assuming $\nu(C_{x_n}) \geq a$ for all n , we wish to show that $\nu(C_x) \geq a$. To do this, we show that $\limsup C_{x_n}$ (the set of all points which belong to C_{x_n} for infinitely many n) is a subset of C_x having ν -measure greater than or equal to a .

Each C_{x_n} is a subset of a common set of finite measure, $\text{Pr}_Y C$. Hence we may quote the Arzela-Young Theorem [2, 17.2] to assert that $\nu(\limsup C_{x_n}) \geq a$. Now if $E = \limsup C_{x_n}$, we show that E is a subset of C_x . Suppose $y \in E$. Given a positive integer k , there exists an integer $n(k) \geq k$ such that $y \in C_{x_{n(k)}}$. That is, $(x_{n(k)}, y) \in C$. Since x_n converges to x , so does $x_{n(k)}$; hence $(x_{n(k)}, y)$ converges to (x, y) . Because C is closed, $(x, y) \in C$, which means $y \in C_x$. Therefore $E \subset C_x$, and $\nu(C_x) \geq a$, as we wished to show.

THEOREM 4.3. *If X is first countable, then f_C is a Borel function, for each compact set C in $X \times Y$.*

Proof. It suffices to show that for each positive number a , the set

$$A = \{x: f_C(x) \geq a\}$$

is closed. (Cf. proof of Theorem 4.1.) Suppose x is an accumulation point of A .

Since X is first countable, there exists a sequence x_n in A converging to x [5, 2.8]. By Theorem 4.2, $x \in A$, so that A is closed.

We now proceed axiomatically, assuming the following axiom:

AXIOM (M_1): f_C is a Borel function, for each compact set C in $X \times Y$.

We emphasize that in view of Theorems 4.1 and 4.3, Axiom (M_1) is always verified when v is regular or X is first countable. We also emphasize, however, that Axiom (M_1) does not always hold. An example will appear in a future paper.

LEMMA 1. If $M = C - D$, where C and D are compact sets in $X \times Y$ such that $D \subset C$, then $f_M = f_C - f_D$, and in particular, f_M is a Borel function.

LEMMA 2. If M and N are disjoint Borel sets in $X \times Y$, then

$$f_{M \cup N} = f_M + f_N.$$

In particular, if f_M and f_N are Borel functions, then so is $f_{M \cup N}$.

Proof of Lemmas 1 and 2. The proofs follow immediately from subtractivity and additivity of measure.

We denote by \mathcal{R} the class of all finite disjoint unions of proper differences of compact subsets of $X \times Y$. From [4, 51.F], we know that \mathcal{R} is a ring and that the σ -ring generated by \mathcal{R} is $\mathcal{B}(X \times Y)$. We observe from Lemmas 1 and 2 that f_M is Borel measurable for all $M \in \mathcal{R}$.

Let us write $\mathcal{B}_b(X \times Y)$ for the class of all bounded Borel sets in $X \times Y$. These are the members of $\mathcal{B}(X \times Y)$ which have compact closure or, equivalently, are contained in a compact rectangle. We observe that $\mathcal{B}_b(X \times Y)$ is a ring containing \mathcal{R} , and that the σ -ring generated by $\mathcal{B}_b(X \times Y)$ is the class $\mathcal{B}(X \times Y)$ of all Borel sets in $X \times Y$. We now show that f_M is pleasant when M is any bounded Borel set in $X \times Y$:

LEMMA 3. If $M \in \mathcal{B}_b(X \times Y)$, then f_M is a Borel function; indeed, it is μ -integrable.

Proof. Suppose $M \in \mathcal{B}_b(X \times Y)$. Choose compact sets G and H such that $M \subset G \times H$. Then M belongs to the class

$$[\mathcal{B}(X \times Y)] \cap (G \times H),$$

which is the σ -ring generated by the ring

$$\mathcal{R} \cap (G \times H) \quad [4, 5.E].$$

Let us write \mathcal{M} for the class of all Borel sets N in $X \times Y$ such that $N \subset G \times H$, and such that f_N is a Borel function. Since $\mathcal{R} \cap (G \times H)$ is a subring of \mathcal{R} , we have

$$\mathcal{R} \cap (G \times H) \subset \mathcal{M}$$

by the remarks following Lemma 2. We assert that \mathcal{M} is a monotone class [4, p. 27]. That \mathcal{M} is closed under increasing sequences follows from the fact that ν is continuous from below [4, p. 39]. That \mathcal{M} is closed under decreasing sequences follows from the finiteness of ν on $\mathcal{B}(Y) \cap H$ and the fact that ν is thus continuous from above on such sets. Hence \mathcal{M} is a monotone class containing the ring $\mathcal{R} \cap (G \times H)$. It follows from [4, 6.B] that \mathcal{M} contains the σ -ring generated by $\mathcal{R} \cap (G \times H)$, namely $[\mathcal{B}(X \times Y)] \cap (G \times H)$. In particular, $M \in \mathcal{M}$, and so f_M is a Borel function by the definition of \mathcal{M} . Finally, since

$$0 \leq f_M \leq \nu(H)\chi_G,$$

where χ_G is the characteristic function of G , we conclude that f_M is μ -integrable [4, 27.A].

LEMMA 4. For each $M \in \mathcal{B}_b(X \times Y)$, define

$$\rho_1(M) = \int f_M d\mu.$$

Then, ρ_1 is a finite measure on the ring $\mathcal{B}_b(X \times Y)$.

Proof. The definition of ρ_1 is of course justified by Lemma 3. It will suffice to show that ρ_1 is additive and continuous from below [4, 9.F].

Additivity follows from the additivity of integration. We show that ρ_1 is continuous from below. Suppose M_n and M are bounded Borel sets in $X \times Y$ such that $M_n \uparrow M$. Then $f_{M_n} \uparrow f_M$, since measure (in this case, ν) is continuous from below. Hence,

$$\rho_1(M_n) \uparrow \rho_1(M)$$

by the Monotone Convergence Theorem.

By the Unique Extension Theorem [4, 13.A], ρ_1 has a unique extension to the σ -ring generated by $\mathcal{B}_b(X \times Y)$, in other words, to the σ -ring $\mathcal{B}(X \times Y)$ of all Borel sets in $X \times Y$. We continue to use ρ_1 to denote the new measure on $\mathcal{B}(X \times Y)$. Summarizing:

THEOREM 4.4. If Axiom (M_1) holds, there exists a unique Borel measure ρ_1 on $X \times Y$ such that

$$\rho_1(M) = \int f_M d\mu$$

or all bounded Borel sets M in $X \times Y$.

THEOREM 4.5. Assuming Axiom (M_1) , let ρ_1 be the unique Borel measure on $X \times Y$, as given by Theorem 4.4, such that

$$\rho_1(M) = \int f_M d\mu$$

for all bounded Borel sets M in $X \times Y$. Then ρ_1 is an extension of $\mu \times \nu$.

Proof. It suffices to show that ρ_1 and $\mu \times \nu$ agree on rectangles whose sides are compact sets (see the proof of Theorem 2.1). Let $E \times F$ be such a rectangle. Then $\int_{E \times F} f_M d\mu = \nu(F) \chi_E$, so that $\rho_1(E \times F) = \int_{E \times F} d\mu = \nu(F) \mu(E) = (\mu \times \nu)(E \times F)$.

Our next result is an analogue of [2, 40.1].

THEOREM 4.6. *Assume Axiom (M_1) , and let ρ_1 be the Borel measure of Theorem 4.4. Suppose M is a Borel set in $X \times Y$. Then, $\rho_1(M) < \infty$ if and only if f_M is a.e. integrable with respect to μ . In this case, $\rho_1(M) = \int f_M d\mu$.*

Proof. Suppose $M \in \mathcal{B}(X \times Y)$. Choose, as is clearly possible, a sequence of bounded Borel sets M_n such that $M_n \uparrow M$. Since measure is continuous from below, we have $\rho_1(M_n) \uparrow \rho_1(M)$ and $f_{M_n} \uparrow f_M$.

Suppose $\rho_1(M) < \infty$. Then, the integrals $\int f_{M_n} d\mu$ are bounded, since $\int f_{M_n} d\mu = \rho_1(M_n) \leq \rho_1(M)$. Thus, f_M is a.e. integrable by the Monotone Convergence Theorem.

Conversely, suppose f_M is a.e. integrable. Then the measures $\rho_1(M_n) = \int f_{M_n} d\mu$ are bounded (by $\int f_M d\mu$), and so their limit, $\rho_1(M)$ is finite.

The preceding theorem yields an analogue of [4, 36.A]:

THEOREM 4.7. *Assume Axiom (M_1) , and let ρ_1 be the Borel measure of Theorem 4.4. Suppose M is a Borel set in $X \times Y$. Then $\rho_1(M) = 0$ if and only if $f_M = 0$ a.e. $[\mu]$.*

Proof. If $\rho_1(M) = 0$, then f_M is a.e. integrable by Theorem 4.6, and $\int f_M d\mu = \rho_1(M) = 0$. Since $f_M \geq 0$, we conclude from [4, 25.B] that $f_M = 0$ a.e. $[\mu]$.

Conversely, if $f_M = 0$ a.e. $[\mu]$, then f_M is obviously a.e. integrable. Citing Theorem 4.6 again, we have $\rho_1(M) = \int f_M d\mu = \int 0 d\mu = 0$.

We now "dualize" the foregoing results. If M is any Borel set in $X \times Y$, we define a nonnegative extended real function g^M on Y by the formula

$$g^M(y) = \mu(M^y)$$

for all $y \in Y$.

Paraphrasing the proofs of Theorems 4.1, 4.2, and 4.3, we have:

THEOREM 4.8. *If μ is regular, then g^C is a Borel function, for each compact set C in $X \times Y$.*

THEOREM 4.9. *Suppose C is a compact set in $X \times Y$ and that y_n is a sequence in Y converging to $y \in Y$. If $g^C(y_n) \geq a$ for all n , then $g^C(y) \geq a$.*

THEOREM 4.10. *If Y is first countable, then g^C is a Borel function, for each compact set C in $X \times Y$.*

Theorems 4.11–4.14 are the duals of Theorems 4.4–4.7, and we may clearly omit their proofs. In these theorems, the following axiom, the dual of Axiom (M_1) , is assumed:

AXIOM (M₂): g^C is a Borel function, for each compact set C in $X \times Y$.

THEOREM 4.11. Assuming Axiom (M₂), there exists one and only one Borel measure ρ_2 on $X \times Y$ such that

$$\rho_2(M) = \int g^M dv$$

for all bounded Borel sets M in $X \times Y$.

THEOREM 4.12. ρ_2 is an extension of $\mu \times \nu$.

THEOREM 4.13. If M is a Borel set in $X \times Y$, then $\rho_2(M) < \infty$ if and only if g^M is a.e. integrable with respect to ν . In this case $\rho_2(M) = \int g^M dv$.

THEOREM 4.14. If M is a Borel set in $X \times Y$, then $\rho_2(M) = 0$ if and only if $g^M = 0$ a.e. $[\nu]$.

5. Iterated integration, and the Fubini-Tonelli theorems in locally compact spaces.

Let $(X_1, \mathcal{S}_1, \lambda_1)$ and $(X_2, \mathcal{S}_2, \lambda_2)$ be abstract measure spaces. Suppose h is an extended real valued function on $X_1 \times X_2$. We say that the *iterated integral* $\int \int h d\lambda_2 d\lambda_1$ exists if and only if there exists a set E in \mathcal{S}_1 of measure zero (briefly, a λ_1 -null set), and a λ_1 -integrable function f , such that $x \in X_1 - E$ implies h_x is λ_2 -integrable, and $\int h_x d\lambda_2 = f(x)$. The value of $\int \int h d\lambda_2 d\lambda_1$ is then defined to be $\int f d\lambda_1$, and is clearly independent of the particular E and f selected to exhibit the existence of the iterated integral. Iterated integrals $\int \int h d\lambda_1 d\lambda_2$ are treated dually. We cite, for reference, the classical theorems of Fubini and Tonelli concerning iterated integration in σ -finite measure spaces:

FUBINI'S THEOREM [cf. 2, 41.1]. If $(X_1, \mathcal{S}_1, \lambda_1)$ and $(X_2, \mathcal{S}_2, \lambda_2)$ are σ -finite measure spaces, and if h is a $(\lambda_1 \times \lambda_2)$ -integrable function on $X_1 \times X_2$, then both of the iterated integrals of h exist, and

$$\int \int h d\lambda_2 d\lambda_1 = \int \int h d\lambda_1 d\lambda_2 = \int h d(\lambda_1 \times \lambda_2).$$

TONELLI'S THEOREM [cf. 2, 41.2]. If $(X_1, \mathcal{S}_1, \lambda_1)$ and $(X_2, \mathcal{S}_2, \lambda_2)$ are σ -finite measure spaces, and if h is a nonnegative measurable function on $X_1 \times X_2$ such that at least one of the iterated integrals of h exists, then h is $(\lambda_1 \times \lambda_2)$ -integrable.

We emphasize that the functions h which occur in the Fubini-Tonelli theorems are required to be measurable with respect to the σ -ring $\mathcal{S}_1 \times \mathcal{S}_2$.

We turn now to the problem of iterated integration in the product of two locally compact spaces. What is the problem? Suppose μ and ν are Borel measures, not necessarily regular, on the locally compact spaces X and Y . The measures μ and ν are obviously σ -finite [4, Exercise 52.1]. The classical Fubini-Tonelli

theorems are concerned with the product measure $\mu \times \nu$. Since the domain of definition of $\mu \times \nu$ is the ring $\mathcal{B}(X) \times \mathcal{B}(Y)$, we see that the classical theory deals with functions h which are measurable with respect to $\mathcal{B}(X) \times \mathcal{B}(Y)$; such functions are of course Borel functions on $X \times Y$, in view of the inclusion

$$\mathcal{B}(X) \times \mathcal{B}(Y) \subset \mathcal{B}(X \times Y).$$

But when this inclusion is proper, $X \times Y$ admits Borel functions which are not measurable with respect to the σ -ring $\mathcal{B}(X) \times \mathcal{B}(Y)$. We may now state our problem very simply, as follows. How are the Fubini-Tonelli theorems to be formulated, let alone proved, for *Borel functions* on $X \times Y$? At the same time, how can we define a *Borel measure* on $X \times Y$ which will play the role of a "product measure"? We have answers to these questions in the case that either μ or ν is regular; a fully satisfying answer is obtained in the case that μ and ν are *both* regular. We will not, however, assume regularity until Theorem 5.7. It will be convenient to adopt the following convention:

CONVENTION. *When we make a statement concerning ρ_i ($i = 1, 2$), we are taking for granted that Axiom (M_i) is verified. That is, we are taking for granted the existence of ρ_i .*

We now turn to the problem of defining iterated integrals for Borel functions on $X \times Y$. A function is said to have *compact support* if it vanishes outside some compact set.

LEMMA. *If h is a simple Borel function with compact support, then each h_x is ν -integrable. If $f(x) = \int h_x d\nu$, then f is μ -integrable and $\int f d\mu = \int h d\rho_1$.*

Proof. The case that h is the characteristic function of a bounded Borel set is essentially Theorem 4.4. The lemma then follows by the linearity of integration.

THEOREM 5.1. *If h is a ρ_1 -integrable Borel function on $X \times Y$, then the iterated integral $\int \int h d\nu d\mu$ exists and is equal to $\int h d\rho_1$.*

Proof [cf. 2, 40.2]. Writing $h = h^+ - h^-$, where $h^+ = h \cup 0$ and $h^- = -(h \cap 0)$, we are clearly reduced, by linearity, to the case that $h \geq 0$.

Suppose h is a nonnegative ρ_1 -integrable function. Choose a sequence of simple Borel functions with compact support such that $0 \leq h_n \uparrow h$. By the Lemma, we may define f_n by $f_n(x) = \int (h_n)_x d\nu$. Then each f_n is μ -integrable, and

$$(*) \quad \int f_n d\mu = \int h_n d\rho_1 \uparrow \int h d\rho_1.$$

For each $x \in X$, we have $(h_n)_x \uparrow$, so that $f_n(x) \uparrow$. Moreover, it is clear from (*) that the sequence $\int f_n d\mu$ is bounded (by $\int h d\rho_1$). Then by the Monotone Convergence Theorem applied to μ , it follows that there exists a μ -integrable Borel function f such that $f_n \uparrow f$ a.e.

Choose a Borel set E in X of measure zero, such that $f_n(x) \uparrow f(x)$ whenever $x \in X - E$. Then if $x \in X - E$, we have

$$\int (h_n)_x dv = f_n(x) \uparrow f(x);$$

since $(h_n)_x \uparrow h_x$, we conclude from the Monotone Convergence Theorem for v that h_x is v -integrable, and $\int h_x dv = \lim \int (h_n)_x dv = f(x)$.

Checking the properties of E and f , we see that the iterated integral $\int \int h dv d\mu$ exists, and is equal to $\int f d\mu$. Finally, $\int \int h dv d\mu = \int f d\mu = \lim \int f_n d\mu = \lim \int h_n d\rho_1 = \int h d\rho_1$.

We may regard Theorem 5.1 as the analogue, for locally compact spaces, of Fubini's theorem. It is a lopsided analogue, in that it refers to only one of the iterated integrals of h , but this is unavoidable. Indeed, there exists an example of a ρ_1 -integrable Borel function on $X \times Y$ for which the other iterated integral fails to exist. We now prove the corresponding analogue of the Tonelli theorem:

THEOREM 5.2. *If h is a nonnegative Borel function on $X \times Y$ such that the iterated integral $\int \int h dv d\mu$ exists, then h is ρ_1 -integrable, and consequently $\int h d\rho_1 = \int \int h dv d\mu$ by Theorem 5.1.*

Proof. By assumption, there exists a μ -integrable function f , and a μ -null Borel set E , with the property that $x \in X - E$ implies h_x is v -integrable and $\int h_x dv = f(x)$.

Choose a sequence of simple Borel functions h_n with compact support and such that $0 \leq h_n \uparrow h$. By the Lemma to Theorem 5.1, we may define $f_n(x) = \int (h_n)_x dv$. Then each f_n is μ -integrable and $\int f_n d\mu = \int h_n d\rho_1$.

Suppose $x \in X - E$. Then h_x is v -integrable, and $\int h_x dv = f(x)$. Since $(h_n)_x \leq h_x$, we have $\int (h_n)_x dv \leq \int h_x dv$, and so $f_n(x) \leq f(x)$. Thus, $f_n \leq f$ a.e. $[\mu]$, so that $\int f_n d\mu \leq \int f d\mu$. Since $\int h_n d\rho_1 = \int f_n d\mu \leq \int f d\mu$, we see that the sequence $\int h_n d\rho_1$ is bounded. Since $h_n \uparrow h$, we conclude from the Monotone Convergence Theorem for ρ_1 that h is ρ_1 -integrable.

Dually, we have:

THEOREM 5.3. *If h is a ρ_2 -integrable Borel function on $X \times Y$, then the iterated integral $\int \int h d\mu dv$ exists and is equal to $\int h d\rho_2$.*

THEOREM 5.4. *If h is a nonnegative Borel function on $X \times Y$ such that the iterated integral $\int \int h d\mu dv$ exists, then h is ρ_2 -integrable, and consequently $\int h d\rho_2 = \int \int h d\mu dv$ by Theorem 5.3.*

Combining Theorems 5.1–5.4, we have:

THEOREM 5.5. *Assume that ρ_1 and ρ_2 are equal. Let us write $\rho = \rho_1 = \rho_2$. Let h be a Borel function on $X \times Y$.*

(i) If h is ρ -integrable, then both iterated integrals of h exist, and

$$\int \int h d\nu d\mu = \int \int h d\mu d\nu = \int h d\rho.$$

(ii) Conversely, if h is nonnegative, and if one of the iterated integrals of h exists, then h is ρ -integrable.

In the "reverse" direction, we have the following:

THEOREM 5.6. Suppose that for any nonnegative Borel function h on $X \times Y$, $\int \int h d\nu d\mu$ and $\int \int h d\mu d\nu$ are equal whenever they both exist. Then $\rho_1 = \rho_2$.

Proof. It suffices to show that ρ_1 and ρ_2 agree on compact sets. Suppose C is a compact set in $X \times Y$, and let $h = \chi_C$. It follows from Theorems 5.1 and 5.3 that the iterated integrals $\int \int h d\nu d\mu$ and $\int \int h d\mu d\nu$ exist and are equal to $\rho_1(C)$ and $\rho_2(C)$, respectively. By hypothesis, the iterated integrals are equal, so that $\rho_1(C) = \rho_2(C)$.

We may summarize Theorems 5.5 and 5.6 as follows:

SUMMARY. In order that the Fubini-Tonelli theorems hold for Borel functions on $X \times Y$, it is necessary and sufficient that $\rho_1 = \rho_2$.

Finally, we consider the case that both μ and ν are regular. In our key theorem, Theorem 5.7, we show that the simultaneous regularity of μ and ν implies that ρ_1 and ρ_2 are themselves regular Borel measures. Since both ρ_1 and ρ_2 extend $\mu \times \nu$ (Theorems 4.5 and 4.12), it will then follow from Theorem 2.1 that $\rho_1 = \rho_2 = \mu \otimes \nu$.

THEOREM 5.7. If μ and ν are both regular, then so are ρ_1 and ρ_2 , and indeed

$$\rho_1 = \rho_2 = \mu \otimes \nu.$$

Proof. We show for instance, that ρ_1 is regular. If U is any bounded open set in $X \times Y$, it will suffice to show that U is inner regular with respect to ρ_1 [4, 52.F]. That is, given $\varepsilon > 0$, it will suffice to find a compact set F in $X \times Y$ such that $F \subset U$ and $\rho_1(U - F) < 2\varepsilon$. Since U is bounded, we have

$$U \subset G \times H$$

for suitable compact sets G and H . If $\nu(H) = 0$, then $\rho_1(U) \leq \rho_1(G \times H) = (\mu \times \nu)(G \times H) = \mu(G)\nu(H) = 0$, where we have used the fact that ρ_1 extends $\mu \times \nu$ (Theorem 4.5); thus $\rho_1(U) = 0$ in this case, and we may simply take F to be the empty set.

From now on we assume $\nu(H) > 0$. Since f_U is a Borel function (Theorem 4.4, Lemma 3), by Lusin's theorem [4, Exercise 55.3] we may choose a compact set C in X so that

$$(1) \quad C \subset G, \quad \mu(G - C) < \varepsilon/\nu(H),$$

and such that the restriction of f_U to C is continuous.

Let $U' = U \cap (C \times Y)$. It is easily seen that $U - U' \subset (G - C) \times H$, so that $\rho_1(U - U') < \varepsilon$ by (1). We shall find a compact set F such that

$$(2) \quad F \subset U, \text{ and } \rho_1(U' - F) \leq \varepsilon.$$

Then the relation

$$\rho_1(U - F) \leq \rho_1(U - U') + \rho_1(U' - F)$$

will show that $\rho_1(U - F) < 2\varepsilon$, and the proof will be complete.

Now, if $\mu(C) = 0$, it is clear from the definition of U' that $\rho_1(U') = 0$, so in this case we could simply take F to be the empty set. From now on we assume that $\mu(C) > 0$.

We shall find, for each point p in C , a pair of compact sets D and E such that (i) D is a neighborhood of p , relative to C , (ii) $D \times E \subset U$, and (iii) $f_{U-(D \times E)}(x) < \varepsilon/\mu(C)$ whenever $x \in D$. We describe the process for finding such pairs.

Suppose $p \in C$. Then the section U_p of U is a Borel set in Y by Theorem 3.1. (Indeed, U_p is a bounded open set.) Since $U_p \subset H$, $v(U_p) < \infty$. Since v is regular, there exists a compact set $E \subset U_p$ such that $v(U_p) < v(E) + \varepsilon/\mu(C)$. That is, $f_U(p) < v(E) + \varepsilon/\mu(C)$. Since f_U is continuous on C , there exists an open neighborhood V of p , relative to the subspace C , such that

$$(3) \quad f_U(x) < v(E) + \varepsilon/\mu(C) \quad \text{for all } x \in V.$$

Since $C \times E - U$ is compact, so is $\text{Pr}_X(C \times E - U)$. Thus, $V' = V - \text{Pr}_X(C \times E - U)$ is open, relative to C . If $x \in V$, then $x \in V'$ if and only if $E \subset U_x$. Hence, V' is an open neighborhood of p , relative to C , such that $V' \times E \subset U$. Let D be a compact neighborhood of p , relative to C , such that $D \subset V'$. Then $D \times E \subset U$, and $D \subset V$. In view of (3), we have

$$(4) \quad f_U(x) < v(E) + \varepsilon/\mu(C) \quad \text{for all } x \in D.$$

It follows that

$$(5) \quad f_{U-(D \times E)}(x) < \varepsilon/\mu(C) \quad \text{for all } x \in D.$$

For each $p \in C$, we choose, by the foregoing construction, compact sets $D(p)$ and $E(p)$ such that:

- (i) $D(p)$ is a neighborhood of p , relative to C ;
- (ii) $D(p) \times E(p) \subset U$;
- (iii) $f_{U-D(p) \times E(p)}(x) < \varepsilon/\mu(C)$ for all $x \in D(p)$.

Varying p , the interiors of the $D(p)$'s cover C ; we pick out a finite subcover, say

$$C \subset D(p_1) \cup \dots \cup D(p_n),$$

and we define

$$F = \bigcup_1^n D(p_i) \times E(p_i).$$

Clearly F is compact. Moreover, $F \subset U$ by (ii). We now assert that

$$(6) \quad f_{U'-F} \leq [\varepsilon/\mu(C)]\chi_C.$$

Since $f_{U'-F} \leq f_{U'}$ and $f_{U'}$ vanishes on $X - C$, the inequality (6) is trivially verified at the points of $X - C$. On the other hand, if $x \in C$, then $x \in D(p_j)$ for some index j , and so

$$f_{U-D(p_j) \times E(p_j)}(x) < \varepsilon/\mu(C)$$

by (iii); since $U' - F \subset U - D(p_j) \times E(p_j)$, we have $f_{U'-F}(x) < \varepsilon/\mu(C)$, and thus the inequality (6) is verified at the point $x \in C$.

From (6), and the definition of ρ_1 , we see that

$$\rho_1(U' - F) = \int f_{U'-F} d\mu \leq [\varepsilon/\mu(C)]\mu(C) = \varepsilon;$$

this is the promised relation (2), and the proof grinds to a halt.

Combining Theorem 5.7 with the Summary preceding it, we obtain a theorem of Fubini-Tonelli type for regular Borel measures:

THEOREM 5.8. *Assume that μ and ν are regular Borel measures on the locally compact spaces X and Y , respectively, and let $\mu \otimes \nu$ be the unique regular Borel measure on $X \times Y$ which extends $\mu \times \nu$, as given by Theorem 2.1. Let h be a Borel function on $X \times Y$.*

(1) *If h is $(\mu \otimes \nu)$ -integrable, then both iterated integrals of h exist, and*

$$\iint h d\nu d\mu = \iint h d\mu d\nu = \int h d(\mu \otimes \nu).$$

(2) *Conversely, if h is nonnegative, and if one of the iterated integrals of h exists, then h is $(\mu \otimes \nu)$ -integrable.*

6. An example for which $\rho_1 \neq \rho_2$. We now show that the measures ρ_1 and ρ_2 of §4 need not be equal. Let Ω' be the set of ordinals less than or equal to the first uncountable ordinal Ω . Let $\Omega_0 = \Omega' - \{\Omega\}$. Let μ be Dieudonné's non-regular measure on $\mathcal{B}(\Omega')$, namely the measure μ such that for each $E \in \mathcal{B}(\Omega')$, $\mu(E) = 1$ if E contains a closed unbounded subset of Ω_0 , and $\mu(E) = 0$ otherwise [4, Exercise 52.10].

Suppose C is a compact set in $\Omega' \times \Omega'$. If a is a positive number, we wish to show that the set

$$A = \{x \in \Omega': f_C(x) \geq a\}$$

is a Borel set in Ω' . It suffices to show that $A \cap \Omega_0$ is closed in Ω_0 . Suppose x is an accumulation point in Ω_0 of A . Since Ω_0 is first countable, there exists a sequence $x_n \in A$ such that x_n converges to x [5, 2.8]. By Theorem 4.2, $f_C(x) \geq a$, so that $x \in A$. Thus, $A \cap \Omega_0$ is closed in Ω_0 .

Accordingly, the Axiom (M_1) of §4 is verified, and by symmetry so is Axiom (M_2) . Thus the measures ρ_1 and ρ_2 are defined.

To show that $\rho_1 \neq \rho_2$, consider, for example, the set Z defined by

$$Z = \{(x, y): x < y < \Omega \text{ or } x = \Omega\}.$$

We have $f_Z \equiv 1$ and $g^Z \equiv 0$. Then, $\rho_1(Z) = \int f_Z d\mu = 1$, and $\rho_2(Z) = \int g^Z dv = 0$. Thus $\rho_1(Z) \neq \rho_2(Z)$.

7. Examples for which the Borel sets do not multiply.

EXAMPLE 1. Let Ω' be the set of ordinals less than or equal to the first uncountable ordinal. With the order topology, Ω' is a compact Hausdorff space. §6 shows that the Borel sets do not multiply.

EXAMPLE 2. It is possible for Borel sets and Baire sets (the σ -ring generated by compact G_δ 's) to coincide in X but for this phenomenon to fail in $X \times X$. For example, let X be a compact nonmetrizable space such that every closed (and hence compact) subset is a G_δ . (The space Z of Example 3 is such a space.) The diagonal D of $X \times X$ is compact and is thus a Borel set. We show that D is not a Baire set. If it were, it would be a G_δ [4, 51.D]. Now, the neighborhoods of D define a uniform structure which yields the given topology of X ; if D were a G_δ , it would have a fundamental sequence of neighborhoods, so that X would be metrizable [5, 6.13 and 6.30], contrary to our assumption on X .

Since Baire sets and Borel sets coincide in X but not in $X \times X$, the Borel sets cannot multiply in this case.

The spaces listed in Examples 1 and 2 cannot be homeomorphic with any topological group. The space of Example 1 lacks homogeneity in that every point but one is a G_δ . In the space of Example 2 every point is a G_δ . It follows that the space is first countable; that is, every point has a fundamental sequence of neighborhoods. Thus, if a topological group structure could be assigned to the space, the space would be metrizable [3]. One may ask then if $\mathcal{B}(G \times G) = \mathcal{B}(G) \times \mathcal{B}(G)$ whenever G is a locally compact topological group? Example 3 will show that the answer is in general no. We first notice the following fact:

THEOREM 7.1. *Suppose X' and Y' are locally compact subspaces of the locally compact spaces X and Y , respectively. If $\mathcal{B}(X \times Y) = \mathcal{B}(X) \times \mathcal{B}(Y)$, then $\mathcal{B}(X' \times Y') = \mathcal{B}(X') \times \mathcal{B}(Y')$.*

Proof. Suppose C is a compact set in $X' \times Y'$. It suffices to show that $C \in \mathcal{B}(X') \times \mathcal{B}(Y')$. Now C is also compact in $X \times Y$, and we have $C \in \mathcal{B}(X) \times \mathcal{B}(Y)$ by assumption. Since C is compact in $X' \times Y'$, there exist compact sets D and E such that $C \subset D \times E \subset X' \times Y'$. Then,

$$C \in \mathcal{B}(X) \times \mathcal{B}(Y) \cap (D \times E) = \mathcal{B}(D) \times \mathcal{B}(E) \subset \mathcal{B}(X') \times \mathcal{B}(Y').$$

EXAMPLE 3. Let G be the set of functions from the closed unit interval into the discrete topological group with two elements. Let G have the product group structure and the product topology. Then G is a compact Hausdorff topological group.

Let Z be the class of nonconstant increasing functions in G . It can be shown that Z is a compact nonmetrizable space such that every closed subset is a G_δ . By the reasoning of Example 2, $\mathcal{B}(Z \times Z) \neq \mathcal{B}(Z) \times \mathcal{B}(Z)$. Citing Theorem 7.1, we see that $\mathcal{B}(G \times G) \neq \mathcal{B}(G) \times \mathcal{B}(G)$.

EXAMPLE 4. Let X be the 1-point compactification of a discrete space having cardinality greater than that of the continuum. The Borel sets in X are the class $\mathcal{P}(X)$ of all subsets of X . Indeed, it can be seen that every subset of the compact space X is either open or closed. It is almost as easy to see that

$$\mathcal{B}(X \times X) = \mathcal{P}(X \times X).$$

For, every subset of $X \times X$ is the difference of two compact subsets.

Now by the reasoning of [4, Exercise 59.2] we see that $\mathcal{P}(X \times X) \neq \mathcal{P}(X) \times \mathcal{P}(X)$ since the diagonal, D , of $X \times X$ is not a member of $\mathcal{P}(X) \times \mathcal{P}(X)$. Therefore $\mathcal{B}(X \times X) \neq \mathcal{B}(X) \times \mathcal{B}(X)$.

Note. If X is the 1-point compactification of a discrete space having the cardinality of the continuum, then it is still true that $\mathcal{B}(X \times X) = \mathcal{B}(X) \times \mathcal{B}(X)$ if and only if $\mathcal{P}(X \times X) = \mathcal{P}(X) \times \mathcal{P}(X)$. Does $\mathcal{P}(X \times X) = \mathcal{P}(X) \times \mathcal{P}(X)$ in this case? We do not know.

8. A case in which Borel sets multiply.

THEOREM 8.1. *Suppose that each bounded subspace of Y is second countable (equivalently, in view of the Urysohn theorem, each bounded subspace of Y is metrizable). Then the Borel sets in X and Y multiply.*

Proof. Since, in a locally compact space, every bounded open set is the difference of two compact sets, and since every compact set is the difference of two bounded open sets, the Borel sets are precisely the σ -ring generated by the bounded open sets. Hence it is sufficient to show that if W is a bounded open set in $X \times Y$, then

$$W \in \mathcal{B}(X) \times \mathcal{B}(Y).$$

Let $F = \text{Pr}_Y W$. Evidently F is a bounded open set in Y . By hypothesis, the subspace F of Y has a countable base for open sets, say \mathcal{V} . Since F is a bounded open set in Y , we notice that the members of \mathcal{V} are also bounded open sets in Y . Since W is an open subset of $X \times F$, it follows that

$W = \bigcup \{U \times V : U \text{ open in } X, V \in \mathcal{V}, U \times V \subset W\}$. For each $V \in \mathcal{V}$, define

$$U_V = \bigcup \{U : U \text{ open in } X, U \times V \subset W\}.$$

For each $V \in \mathcal{V}$, U_V is open in X ; U_V is bounded since it is a subset of $\text{Pr}_X W$. Obviously $U_V \times V \in \mathcal{B}(X) \times \mathcal{B}(Y)$. Since

$$\begin{aligned} W &= \bigcup \{U \times V : U \text{ open in } X, V \in \mathcal{V}, U \times V \subset W\} \\ &= \bigcup_{V \in \mathcal{V}} \bigcup \{U \times V : U \text{ open in } X, U \times V \subset W\} \\ &= \bigcup_{V \in \mathcal{V}} [\bigcup \{U : U \text{ open in } X, U \times V \subset W\}] \times V \\ &= \bigcup_{V \in \mathcal{V}} U_V \times V, \end{aligned}$$

and since the index family \mathcal{V} for the last union is countable, we see that W is the union of a sequence of sets $U_V \times V$, each of which belongs to the σ -ring $\mathcal{B}(X) \times \mathcal{B}(Y)$; consequently, we conclude that $W \in \mathcal{B}(X) \times \mathcal{B}(Y)$, as we wished to show.

COROLLARY. *Let X and Y be locally compact spaces, and assume that Y is metrizable. Then*

$$\mathcal{B}(X \times Y) = \mathcal{B}(X) \times \mathcal{B}(Y).$$

Paraphrasing Theorem 8.1, a sufficient condition for Borel sets to multiply is that the bounded subspaces of one of the factor spaces be second countable. Is this condition necessary? We do not know.

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