SIMPLE AND WEAKLY ALMOST PERIODIC TRANSFORMATION GROUPS[1]

BY
ROBERT N. RICH[2]

1. **Introduction.** In §2 of this paper we study the invariant measures on a compact transformation group, (X, T). We assume that if Y is a closed invariant nonempty subset of X then (Y, T) possesses an invariant measure (complete measurability assumption). Our principal result then is that, under this assumption, $C(X)$ decomposes into a direct sum of invariant functions and functions which have integral 0 for all invariant measures (we then call (X, T) simple) if and only if there exists an upper semicontinuous decomposition Φ' of X into closed invariant sets such that (i) each $M' \in \Phi'$ contains a unique minimal set M and (ii) (M', T) has a unique invariant measure, m. Moreover, in this case m is ergodic and $m(M) = 1$. Some of the theorems of this section generalize results of Oxtoby [12] and of Auslander [1].

In §3 we study weakly almost periodic (w.a.p.) transformation groups. They are defined to be those transformation groups (X, T) such that if $f \in C(X)$ then the set of T-translates of f have a compact closure in the weak topology of $C(X)$. In studying w.a.p. transformation groups, we make extensive use of the enveloping semigroups, $E(X, T)$, of Ellis [8]. Our principal result here is that if (X, T) is w.a.p. and if $E(X, T)$ possesses an invariant mean then (X, T) is simple and completely measurable.

The author gratefully acknowledges the invaluable assistance and advice of Dr. Robert Ellis in the preparation of this work.

2. **Simple transformation groups.** Throughout this section (X, T) will denote a transformation group with X compact Hausdorff. By that we mean that T is a group, and for every $x \in X$ and $t \in T$ there is defined an element $xt \in X$ such that

(i) the map $x \to xt$ is continuous for every $t \in T$;

(ii) $(xt)s = x(ts)$ for all $x \in X$, $t, s \in T$;

(iii) $xe = x$ for all $x \in X$ ($e =$ identity of T).

In the terminology of [10], (X, T) is a discrete transformation group. If f is a function on X, we define the t-translate of f, denoted ft, by $ft(x) = f(xt)$.

Received by the editors November 22, 1965.

(1) This paper is part of a Ph.D. thesis submitted to the University of Pennsylvania in 1961.

(2) This research was supported in part by the U.S. Air Force.

We denote by $C(X)$ the space of all bounded continuous real valued functions defined on X and by $C^*(X)$ the topological dual of $C(X)$. We denote by $\Delta(X, T)$, or simply by Δ, the set of all normed regular T-invariant measures. By a T-invariant measure m, we mean a measure defined on the Borel sets of X satisfying $m(At) = m(A)$.

We will often make use of the Riesz representation theorem to identify regular countably additive set functions on X with elements of $C^*(X)$ and regular measures with positive elements of $C^*(X)$. For a statement and proof of the Riesz theorem see [4, p. 265]. With this identification we have $\Delta \subset C^*(X)$.

2.1. Definition. (X, T) is said to be measurable if Δ is not empty. Otherwise, (X, T) is called nonmeasurable. If Δ contains exactly one element then (X, T) is called uniquely measurable. (X, T) is called completely measurable if for every closed invariant nonempty subset Y of X, (Y, T) is measurable.

2.2. Lemma. *If T is amenable (see 3.6 for definition) then (X, T) is completely measurable.*

Proof. Since T is amenable there exists a positive normed linear T-invariant functional J on $B(T)$ (the space of bounded real valued functions on T). Let x be any point of X and define for every $f \in C(X)$ an element $\bar{f} \in B(T)$ by $\bar{f}(t) = f(xt)$. Then it is easily seen that the measure m defined by $\int f\,dm = J(\bar{f})$ is a normed regular T-invariant measure on X. This argument applies to closed invariant nonempty subsets also and so the lemma follows.

2.3. Definition. Let $N(X)$ be the set of $f \in C(X)$ such that $\int f\,dm = 0$ for all $m \in \Delta$. The elements of $N(X)$ are called the null functions of (X, T). Let $N'(X)$ be the subspace (not necessarily closed) of $C(X)$ generated by all functions of the form $ft - f$ for all $f \in C(X)$ and $t \in T$. Let $N''(X)$ denote the closure of $N'(X)$ in $C(X)$. Here $C(X)$ is provided with topology of uniform convergence.

2.4. Definition. For $J \in C^*(X)$, we let $J^+(f) = \sup[J(g) \mid g \in C(X), g \leqq f]$ and let $J^- = J^+ - J$. We remark that both $J^+, J^- \in C^*(X)$ and both are positive in the sense that if $f \geqq 0$ then $J^+(f)$, $J^-(f) \geqq 0$.

2.5. Lemma. *Let $J \in C^*(X)$. Then J is T-invariant if and only if J^+ and J^- are T-invariant.*

Proof. By J being T-invariant is meant $J(ft) = J(f)$ for all $f \in C(X)$ and $t \in T$. The proof is obvious.

2.6. Lemma. *Let $f \in C(X) - N''(X)$. Then there exists $J \in C^*(X)$ such that J is positive, $J(f) \neq 0$, and J is invariant.*

Proof. Since $N''(X)$ is a closed subspace of $C(X)$, we have $B = C(X)/N''(X)$ is a Banach space. Denote the coset of B containing f by $\bar{f}$. Then there exists $L \in B^*$ such that $L(\bar{f}) \neq 0$. By composing L with the projection of $C(X)$ onto B we can obtain a functional $J \in C^*(X)$. Since $J(N'(X)) = 0$, it follows that J is invariant.

Now consider J^+ and J^-. They are T-invariant and positive. Moreover, they cannot both be 0 at f for then $L(\bar{f})$ would be 0.

2.7. Corollary. $N(X) = N''(X)$.

2.8. Theorem. *(X, T) is uniquely measurable if and only if $B = C(X)/N(X)$ is one dimensional.*

Proof. Assume (X, T) is uniquely measurable and suppose B is not one dimensional. Since (X, T) is measurable, it follows that the dimension of B is not 0. Therefore, we can find $f \in C(X)$ such that the cosets $\bar{f}$ and $\bar{l}$ of f and l in B are linearly independent. We can then find $L \in B^*$ such that $L(a\bar{l} + b\bar{f}) = b$ for all real numbers a and b. Let J be the element of $C^*(X)$ obtained by composing L with the natural projection of $C(X)$ onto B. Then J is invariant since $J(N(X)) = 0$. Now since $J = J^+ - J^-$, it follows that $J^+(1) = J^-(1) > 0$ also $J^+(f) \neq J^-(f)$ since $J(f) \neq 0$. Now if we normalize J^+ and J^- we can obtain two distinct invariant measures on X. This proves half the theorem. The other half is trivial.

2.9. Definition. Let $I(X)$ denote the invariant functions of $C(X)$. If $C(X)$ is the direct sum of $I(X)$ and $N(X)$ then (X, T) is said to be simple. That is, if $f \in C(X)$ then there is a unique representation $f = f^* + f'$ where $f^* \in I(X)$ and $f' \in N(X)$. We call f^* and f' the invariant and null parts of f respectively.

2.10. Lemma. *If (X, T) is completely measurable then $I(X) \cap N(X) = 0$.*

Proof. Let $f \in I(X)$ and suppose $f \neq 0$. Then there exists $a \neq 0$ such that $A = f^{-1}(a)$ is not empty. Now A is closed and invariant, and so we can find $m \in \Delta$ such that $m(A) = 1$. Therefore, $\int f\,dm = a$ and so $f \notin N(X)$.

In view of the above lemma if (X, T) is completely measurable then to show that (X, T) is simple it is sufficient to show that for every $f \in C(X)$ there exists $g \in I(X)$ such that $\int f\,dm = \int g\,dm$ for every $m \in \Delta$.

Before studying simple transformation groups in general, we obtain a characterization of simplicity when the group $T =$ the group of integers Z. In this case we denote the translate of $f \in C(X)$ by f_n rather than fn.

2.11. Lemma. *Let $T = Z$ then $N'(X) = [f_1 - f \mid f \in C(X)]$.*

Proof. Let $B = [f_1 - f \mid f \in C(X)]$. It is easily verified that B is a subspace (not necessarily closed) of $C(X)$ and that $B \subset N'(X)$. Since

$$f_k - f = (f_{k-1} + f_{k-2} + \cdots + f)_1 - (f_{k-1} + \cdots + f),$$

it follows that $B \supset N'(X)$ and so the lemma is proved.

2.12. Theorem. *Let $T = Z$. Then (X, T) is simple if and only if $1/n \sum_{i=1}^{n} f_i$ is uniformly convergent for every $f \in C(X)$.*

Proof. Assume (X, T) is simple and let $f \in N'(X)$. By the previous lemma we have $f = g_1 - g$ for some $g \in C(X)$. Then

$$1/n\ \Sigma_1^n f_i = 1/n\ \Sigma_1^n (g_1 - g)_i = 1/n(g_{n+1} - g_1)$$

which clearly converges to 0 uniformly. It is easily seen that the same applies to $N(X)$ since $N'(X)$ is a dense subset. Now for $f \in C(X)$ we have $f = f^* + f'$ with $f^* \in I(X)$ and $f' \in N(X)$. Therefore, $1/n\ \Sigma_1^n f_i = f^* + 1/n\ \Sigma_1^n f_i'$ which clearly converges uniformly to f^*.

To prove the converse, observe that T is amenable (since it is abelian [3]), and so by 2.2 it follows that (X, T) is completely measurable. The conclusion now follows from the remarks following 2.10.

Compare this with 5.3 of Oxtoby [12].

We now proceed to generalize the preceding theorem to arbitrary transformation groups (X, T) with T abelian. To do this we need the following.

2.13. Definition. For $f \in C(X)$ we define $H(f)$ to be the convex closure (in $C(X)$) of the set $[ft \mid t \in T]$.

2.14. Theorem. *If T is abelian then (X, T) is simple if and only if $H(f) \cap I(X) \neq \varnothing$ for all $f \in C(X)$.*

Proof. Suppose $g \in H(f) \cap I(X)$. Since we can uniformly approximate g by convex combinations of elements of $[ft \mid t \in T]$, it follows that $\int g\,dm = \int f\,dm$ for all $m \in \Delta$. Since (X, T) is completely measurable, it follows that (X, T) is simple.

Conversely, suppose that (X, T) is simple. Let us assume that there exists $f \in C(X)$ such that $f^* \notin H(f)$. Then we can find $J \in C^*(X)$ and real numbers a and b such that

$$J(f^*) = a < b \leqq J(H(f)) \tag{1}$$

(see, for example, [4, p. 417]).

We may assume the norm of J is 1. Let W be the set of elements of norm 1 in $C^*(X)$ which satisfy (1) and provide W with the weak topology for $C^*(X)$. Since unit sphere of $C^*(X)$ is compact in the weak topology and since W is easily seen to be closed we have that W is compact. Also it is clear that W is convex. Now for $t \in T$ and $L \in C^*(X)$ define $L_t(h) = L(ht)$ for $h \in C(X)$. It follows that the maps $L \to L_t$ of $C^*(X)$ into $C^*(X)$ are continuous with respect to the weak topology. Moreover, these maps commute, are linear, and carry W into W. Therefore, by the Markov-Kakutani fixed point theorem [4, p. 456], there exists $L \in W$ such that $L(ht) = L(h)$ for all $h \in C(X)$ and all $t \in T$. This implies that L is constant on $H(f)$. Now by 2.5, we have that L^+ and L^- are both invariant and hence constant on $H(f)$. Now by (1) we have either $L^+(f^*) < L^+(H(f))$ or $L^-(f^*) < L^-(H(f))$. Suppose $L^+(f^*) < L^+(H(f))$. Now by normalizing L^+ and passing to the induced measure on X, we obtain $m \in \Delta$ such that $\int f\,dm \neq \int f^*dm$. This contradiction completes the proof.

We have as an immediate consequence of this theorem the following

2.15. COROLLARY. *If* (X,T) *is simple with* T *abelian then for every* $f \in C(X)$, f^* *is the unique invariant function in* $H(f)$.

2.16. LEMMA. *Let* (X,T) *be simple and completely measurable and let* Y *be a closed invariant subset of* X. *Then* (Y,T) *is simple and completely measurable.*

Proof. Clearly, (Y,T) is completely measurable. Let $f \in C(Y)$ and let $g \in C(X)$ be any extension of f. Now $g = g^* + g'$. Let f^* and f' be the restrictions of g^* and g' to Y. Clearly, $f^* \in I(Y)$ and $f' \in N(Y)$ and $f = f^* + f'$. Since

$$I(Y) \cap N(Y) = 0,$$

we have that (Y,T) is simple.

2.17. DEFINITION. A closed nonempty subset A of X is said to be minimal if $x \in A$ implies that xT is dense in A.

2.18. DEFINITION. The kernel of X, denoted by kX, is defined to be the intersection of all closed sets $E \subset X$ which satisfy $m(E) = 1$ for all $m \in \Delta$.

The following lemma is easily proved using the regularity of measures in Δ.

2.19. LEMMA. kX *is closed, invariant and* $m(kX) = 1$ *for all* $m \in \Delta$.

2.20. THEOREM. *Let* (X,T) *be completely measurable. Then the following are equivalent.*

(1) (X,T) *is uniquely measurable*;

(2) $C(X) = R \oplus N(X)$ $(R = \text{reals})$;

(3) (X,T) *is simple and contains exactly one minimal set*;

(4) (X,T) *is simple and* kX *is minimal.*

Proof. In the statement of (1) we have identified R, in the natural way, with the constant functions.

Clearly, (1) is equivalent to (2). We show (1) is equivalent to (3). Assume (1). It is well known [10, p. 15] that X contains a minimal set. If X contained more than one, then by complete measurability X would not be uniquely measurable. Moreover, the only invariant functions on X are constants for otherwise by taking inverse images of distinct values we could produce two disjoint invariant closed sets. Each of these sets would contain a minimal set. Since $C(X)/N(X)$ is one dimensional and $I(X) \cap N(X) = 0$, we have (X,T) is simple.

Now assume (3). Then the only invariant functions are again constants. Therefore, $C(X) = R \oplus N(X)$ and so (X,T) is uniquely measurable.

We now show (1) is equivalent to (4). Assume (1). Then by (3) we have a unique minimal set A in X. Since (X,T) is completely measurable, we have a T-invariant measure m on X such that $m(A) = 1$ and so $A = kX$.

Assume (4). We know then that (kX,T) is simple and completely measurable. Since kX is minimal, $I(kX)$ consists of only constant functions. Hence, kX is uniquely measurable. Now by the definition of kX, it follows that X is uniquely measurable. This completes the proof.

We now proceed to study the structure of simple, completely measurable transformation groups in detail.

2.21. DEFINITION. If $A \subset X$ we define A', the attraction of A, to be

$$[x \mid \mathrm{cl}(xT) \cap A \neq \varnothing].$$

(Here cl = closure.)

2.22. DEFINITION. We denote by Φ the class of all minimal subsets of X and by Φ' the class $[M' \mid M \in \Phi]$.

2.23. LEMMA. *Let (X, T) be simple and completely measurable, A a closed invariant subset of X, and $f \in C(X)$. Then*

(1) *if $f \mid A = c$ then $f^* \mid A = c$;*
(2) *if $f \mid A \geqq c$ then $f^* \mid A \geqq c$.*

Proof. We prove (1). Let $x \in A$ and let B be the closure of xT. Then $f \mid B = c$. There exists $m \in \Delta$ such that $m(B) = 1$. We know that $\int f^* dm = \int f dm = c$ and that $f^* \mid B = f^*(x)$. Therefore, $f^*(x) = c$. The proof of (2) is similar.

2.23. LEMMA. *Let (X, T) be simple and completely measurable and let M be a minimal subset of X. Then $M' = \bigcap [f^{-1}(f(M)) \mid f \in I(X)]$; and therefore, M' is closed.*

Proof. Let $E = [f^{-1}(f(M)) \mid f \in I(X)]$. Since an invariant continuous function must be constant on the closure of the set xT for any x, we have $M' \subset E$. Now suppose $x \notin M'$. The closure of xT contains a minimal set, say A. We must have $A \cap M = \varnothing$ since minimal sets are either disjoint or identical. By applying the previous lemma we can find $f \in I(X)$ such that $f = 0$ on M and $f = 1$ on A. Then $x \notin f^{-1}(0)$ and therefore, $x \notin E$. This implies $E \subset M'$ and completes the proof.

2.25. THEOREM. *Let (X, T) be simple and completely measurable. Then Φ' is an upper semicontinuous partition of X into compact sets.*

Proof. Let M and N be distinct minimal sets. Clearly, M and N are disjoint and closed and so by 2.23 we can find $f \in I(X)$ such that $f(M) = 0$ and $f(N) = 1$. Since f is invariant it follows that f is constant on the set xT and so M' and N' are disjoint. Since the closure of xT contains a minimal set, it follows that Φ' is a partition of X. Now it is easily seen that the sets $V = f^{-1}(-\infty, \frac{1}{2})$, $V = f^{-1}(\frac{1}{2}, \infty)$ are disjoint neighborhoods of M' and N' in the quotient topology of Φ'. Therefore, the quotient space is Hausdorff, and this implies the theorem.

We now examine the ergodic measures and find their relation to the invariant measures.

2.26. Definition. If $m \in \Delta$ is such that the only measurable invariant sets of X have measure either 0 or 1, then m is said to be ergodic.

2.27. Definition. Let $m \in \Delta$. The support of m, denoted by $k(m)$, is defined to be the intersection of all closed sets which have m-measure 1. It follows that $k(m)$ is closed, invariant, and $m(k(m)) = 1$.

2.28. Lemma. *Let (X, T) be simple and completely measurable and let m be an ergodic measure. Then $k(m)$ is minimal.*

Proof. Let $K = k(m)$, let $x \in K$ and let A be the closure of xT. We consider the transformation group (K, T). We know by 2.16 that (K, T) is simple and completely measurable. Since m is a regular measure, we have that

$$m(A) = \inf[\ \textstyle\int f\,dm \,|\, f \in C(K) \text{ and } f \supset A] = \inf[\ \textstyle\int f^*dm \,|\, f \in C(K) \text{ and } f \supset A].$$

Here the notation $f \supset A$ means that $f(x) \geqq 1$ for $x \in A$ and $f(x) \geqq 0$ for $x \notin A$. Since m is an ergodic measure, the functions in $I(K)$ are just constants. Therefore, for $f \in C(K)$, we have $\int f\,dm = \int f^*dm = f^*(x)$. Since $f \,|\, A \geqq 1$ implies $f^* \,|\, A \geqq 1$, we have $\int f^*dm \geqq 1$ if $f \supset A$. Therefore, $m(A) = 1$ and so $K = A$ completing the proof.

In what follows we shall consider Δ to be a subset of $C^*(X)$.

2.29. Lemma. *Let $m \in \Delta$.*

(1) If m is an extreme point of Δ then m is ergodic.

(2) If (X, T) is simple then m is an extreme point of Δ if and only if m is ergodic.

Proof. We prove (1). Suppose m is not ergodic. Then we can choose a measurable invariant subset A of X such that $0 < m(A) < 1$. Let B be the complement of A (in X) and define $m_1(C) = m(C \cap A)/m(A)$ and $m_2(C) = m(C \cap B)/m(B)$. It is easily seen that m is a proper convex combination of m_1 and m_2 with $m_1 \neq m_2$ and so m is not extreme. This proves (1).

We now prove (2). If m is not an extreme point then there exists m_1 and m_2 and positive real numbers a and b such that $m_1 \neq m_2$ and $a + b = 1$ and $am_1 + bm_2 = m$. Since $m_1 \neq m_2$ there exists $f \in C(X)$ such that $\int f\,dm_1 \neq \int f\,dm_2$. Therefore, we can find $g \in I(X)$ such that $\int g\,dm_1 \neq \int g\,dm_2$. It can be seen that there exists an interval $[x, y]$ such that $m_1(g^{-1}[x, y]) \neq m_2(g^{-1}[x, y])$. Now $g^{-1}[x, y]$ is an invariant measurable (in fact closed) subset whose m-measure is different from both 0 and 1 hence m is not ergodic. This completes the proof of (2).

2.30. Theorem *Let (X, T) be simple and completely measurable and let $M = \bigcup \Phi$. Then $kX =$ the closure of M.*

Proof. Let N be the closure of M. By complete measurability we have $M \subset kX$ and since kX is closed we have $N \subset kX$. It is sufficient then to show that if $m \in \Delta$ then $m(N) = 1$. Consider $C^*(X)$ to be provided with the weak topology and observe that Δ, as a subset of $C^*(X)$, is compact and convex. Therefore, Δ is the convex closure of the extreme points of Δ [4, p. 440]. Therefore, by the previous lemma we can find a net m_i of convex combinations of ergodic measures such that $\int f dm_i \to \int f dm$ for all $f \in C(X)$. Now let $f \in C(X)$ be such that $f \supset N$. Clearly then by 2.23 we have $\int f dm_i \geqq 1$ and so $\int f dm \geqq 1$ and since $m(N) = \inf [\int f dm \mid f \in C(X),\ f \supset N]$ the theorem follows.

2.31. THEOREM. *Let (X, T) be completely measurable. If there exists an upper semicontinuous decomposition Ψ of X such that each element of Ψ is invariant and uniquely measurable then (X, T) is simple.*

Proof. Let $\Psi = [A_i \mid i \in Q]$ where Q is an index set. Let m_i be the unique element of $\Delta(A_i, T)$ and provide $\Delta(X, T)$ with the weak topology of $C^*(X)$. Throughout this proof we use m_i to denote both the measure on A_i and its natural extension to X.

Let $f \in C(X)$ and define $g(A_i) = \int f dm_i$. We now prove that g is a continuous function of Ψ. We are assuming that Ψ is provided with its quotient topology. Let A_α be a net in Ψ converging to A_i. Consider the net m_α in Δ. Since Δ is compact we may assume $m_\alpha \to n$ for some $n \in \Delta$. Let $h \in C(A_i)$ and define $J(h) = \int h_1 dn$ where h_1 is any continuous extension of h to all of X. J is well defined for if $h = 0$ and $\varepsilon > 0$, then there exists α_0 such that if $\alpha > \alpha_0$ then $\sup[|h_1(x)| : x \in A_\alpha] < \varepsilon$. This is a consequence of the fact that Ψ is upper semicontinuous. Therefore, if $\alpha > \alpha_0$ then $|\int h_1 dm_\alpha| < \varepsilon$ and this implies $\int h_1 dm_\alpha \to 0$ and so $\int h_1 dn = 0$. Now, in general, if k and l are extensions of h then $k - l$ is an extension of 0, and it follows that $\int k dn = \int l dn$ and so J is well defined. Since the measures are positive and invariant we may conclude that J is also and moreover $J(1) = 1$. Therefore, by unique measurability we see that $J(h) = \int h dm_i$ and so $n = m_i$. This implies $m_\alpha \to m_i$ from which we conclude $\int f dm_\alpha \to \int f dm_i$ and so g is continuous.

Now let p denote the natural projection of X onto Ψ and let $f^* = gp$. Let m be an ergodic measure on X. Then the measure mp^{-1} is atomic on Ψ. Therefore, $\int_\Psi g dmp^{-1} = g(A_i)$ for some $A_i \in \Psi$. Therefore, $k(m) = A_i$ and $m = m_i$. Now $\int_X f^* dm_i = \int_X gp dm_i = \int_\Psi g dm_i p^{-1} = g(A_i) = \int_X f dm_i$. Since this holds for ergodic measure we can use the facts that the extreme points of $\Delta(X, T)$ are ergodic measures and that every element of $\Delta(X, T)$ is the limit of convex combinations of extreme points.

2.32. COROLLARY. *Let (X, T) be uniformly equicontinuous. Then (X, T) is simple and completely measurable.*

Proof. In [10, p. 18] it is proved that Φ is an upper semicontinuous decomposition of X. Clearly, the elements of Φ are invariant. In [13] it is proved that each (M, T) for $M \in \Phi$ is uniquely measurable.

We can now combine some of the previous results and form the following

2.33. THEOREM. *Let (X, T) be completely measurable. Then (X, T) is simple if and only if there exists an upper semicontinuous decomposition of X into closed invariant uniquely measurable sets. In this case the decomposition is unique and is in fact Φ'. The unique measure on any element M' of Φ' is ergodic and its support is M. Moreover, kX is the closure of $\bigcup \Phi$.*

If we let Φ' be provided with the quotient topology, we can show that if (X, T) is simple and completely measurable then $C(\Phi')$ is naturally isomorphic to $I(X)$; this isomorphism induces a natural isomorphism between regular measures on Φ' and regular T-invariant measures on X. The correspondence is such that atomic measures on Φ' correspond to ergodic measures on X.

3. **Weakly almost periodic transformation groups.** In this section weakly almost periodic transformation groups are studied and a connection between them and simple transformation groups is obtained. Using this, some of the theorems of §2 are sharpened. Throughout this section (X, T) denotes a transformation group with X compact Hausdorff.

3.1. DEFINITION. Let $f \in C(X)$ and let $C(X)$ be provided with the weak topology. If $[ft \mid t \in T]$ has a compact closure in $C(X)$ then f is said to be weakly almost periodic (w.a.p.). If every $f \in C(X)$ is w.a.p. then (X, T) is called w.a.p.

This notion was defined and studied for topological groups in [5].

3.2. DEFINITION. Let $F(X)$ denote the space of all (not just continuous) functions of X into X and let $F(X)$ be provided with the topology of pointwise convergence. Now, every element of T defines a homeomorphism of X into X, and so we may consider the closure, $E(X, T)$, in $F(X)$ of this set of homeomorphisms. $E(X, T)$ is called the enveloping semigroup of (X, T) and, as it implies, is a semigroup under composition. It was originally defined and studied in [8]. When no confusion is likely, $E(X, T)$ will be denoted by E. The image of x under $p \in E$ will be denoted by xp and the composition of a function $f \in C(X)$ and a map $p \in E$ will be denoted by fp, i.e., $fp(x) = f(xp)$.

At times we will identify elements of T with the homeomorphisms they define and so will write expressions like $T \subset E$. With this understanding we say that T is a dense subset of E.

The following result, proved in [7], will be used extensively in this section.

3.3. THEOREM. *Every element of $E(X, T)$ is continuous if and only if (X, T) is w.a.p.*

3.4. DEFINITION. By a topological semigroup we mean a topological space, S, endowed with a semigroup structure such that the maps $x \to ax$ and $x \to xa$ of S into S are continuous for all $a \in S$. A left (right) ideal of S is a nonempty subset A of S such that $SA \subset A (AS \subset A)$. A minimal left (right) ideal is a left (right) ideal which contains no proper subset which is a left (right) ideal.

3.5. Remark. If E consists of only continuous maps then it is easily seen that E is a compact topological semigroup. Moreover, in this case, if T is abelian then so is E.

We will use the following notions of invariant mean and amenable semigroup extensively in what follows.

3.6. Definition. Let S be a topological semigroup. For $f \in C(S)$ and $s \in S$ we define f^s and f_s by $f^s(x) = f(sx)$ and $f_s(x) = f(xs)$ for $x \in S$. Clearly, f_s and f^s are in $C(S)$. By a left (right) invariant mean on S we mean an element $J \in C^*(S)$ such that J is positive, $J(1) = 1$, and $J(f_s) = J(f)$ $(J(f^s) = J(f))$ for all $f \in C(S)$ and $s \in S$. An invariant mean is an element of $C^*(S)$ which is both a left and right invariant mean. If an invariant mean exists, we say that S is amenable.

3.7. Remark. If S is an abelian topological semigroup then S is amenable [3]. In view of this, 3.3 and 3.5 we have that if (X, T) is w.a.p. and T is abelian then E is amenable.

3.8. Lemma. *If (X,T) is w.a.p. and if Y is a closed invariant nonempty subset of X, then (Y,T) is w.a.p. Moreover, if $E(X,T)$ possesses a right invariant mean then so does $E(Y,T)$.*

Proof. The first part of the lemma is an application of 3.3. To prove the second part, let h be the map of $E(X,T)$ onto $E(Y,T)$ obtained by restricting the maps of $E(X,T)$ to Y. It is easily verified that h is a continuous semigroup homomorphism. Now given a mean on $E(X,T)$ we use h in the obvious way to define one for $E(Y,T)$.

3.9. Lemma. *Let (X,T) be w.a.p., let $f \in C(X)$, and let $g: E \to C(X)$ be defined by $g(p) = fp$. Then g is weakly continuous.*

Proof. We can easily see that if $C(X)$ is given the topology of pointwise convergence then g is continuous. Therefore, $g(E)$ is compact in this topology. Moreover, $g(E)$ is bounded in norm. Now we can apply a theorem of Grothendieck [11] which shows that the topology of pointwise convergence and the weak topology agree on $g(E)$.

3.10. Lemma. *Let (X,T) be w.a.p., let $m \in \Delta$, and let $p \in E$. Then*

(1) *if $f \in C(X)$, then $\int f\,dm = \int fp\,dm$;*

(2) *if A is a measurable subset of X, then $m(Ap^{-1}) = m(A)$;*

(3) *if A is a closed subset of X, then $m(Ap) \geqq m(A)$.*

Proof. We prove (1). Let t_i be a net in T converging to p. Then by the previous lemma, we have $\int ft_i\,dm \to \int fp\,dm$. Since m is T-invariant, we have $\int fp\,dm = \int f\,dm$. To prove (2) consider the measure $n = mp^{-1}$. By (1) we have $n(A) = m(A)$. To prove (3) we have by (2) that $m(Ap) = m(App^{-1})$; and since $App^{-1} \supset A$, (3) follows. Note that here Ap is closed and hence measurable.

3.11. LEMMA. *Let I be a minimal right ideal of $E(X,T)$. Then*
(1) xI is a minimal subset of X for every $x \in X$;
(2) if u is an idempotent (i.e., $u^2 = u$) of I and $p \in I$, then $up = p$.

Proof. See [8].

3.12. LEMMA. *Let (X,T) be w.a.p. and completely measurable. Then*
(1) if M is a minimal subset of X and u is an idempotent of E, $xu = x$ for all $x \in M$;
(2) if I is a minimal right ideal of E, then I contains exactly one idempotent.

Proof. We prove (1). Consider $A = \bigcap [Mp \mid p \in E]$. Since (X,T) is completely measurable, we can find $m \in \Delta$ such that $m(M) = 1$. Therefore, by (3.10 we have $m(Mp) = 1$ for all $p \in E$ and so $m(A) = 1$. Therefore, A is not empty. Since A is closed and invariant, it follows that $A = M$. Therefore, $Mu = M$ and so u is the identity on M.

We now prove (2). It is known (see [8]) that I contains at least one idempotent. Suppose it contained two, say u and v. Then by 3.11, we have $uv = v$ and xI is minimal for every $x \in X$. Now by (1), $xv = xuv = xu$ for every $x \in X$. Therefore, $u = v$.

3.13. LEMMA. *Let S be a compact topological semigroup with a left identity e. Then if the only idempotent of S is e, S is a group.*

Proof. The proof is a trivial modification of Lemma 3 of [9] and is left to the reader.

3.14. THEOREM. *Let (X,T) be w.a.p. Then (X,T) is completely measurable if and only if E possesses a right invariant mean.*

Proof. Let I be a minimal right ideal of E. The existence of I is easily proved using Zorn's lemma. Now by 3.12 we have that I contains a unique idempotent, and this fact together with 3.11 and 3.13 proves that I is a group. It then follows from Theorem 2 of [6] that I is a topological group; so we have a Haar measure n on I. The natural extension of this measure to E yields a right invariant mean for E.

Conversely, let n be the measure corresponding to the right mean of E. Let Y be a closed invariant nonempty subset of X and let $y \in Y$. For any Borel set $A \subset X$ define $m(A) = n[p \mid yp \in A]$. It is easily seen that m is a T-invariant measure on X such that $m(Y) = 1$.

3.15. THEOREM. *Let (X,T) be w.a.p. If E has an invariant mean then (X,T) is simple and completely measurable.*

Proof. By the previous theorem we have that (X,T) is completely measurable. Let m be the invariant mean on E which we will treat as a measure on E. For $f \in C(X)$ and for $p \in E$ define $l(p) = fp$. By 3.9 we have that l is a weakly continuous function from E to $C(X)$. Let $g = \int_E l(p)dm(p)$. Here the weak integral of Bourbaki [2] is being used. By definition

$$(*) \qquad \int_X g(x)dn(x) = \int_E \int_X l(p)(x)dn(x)dm(p)$$

for all measures n on X. Let $x \in X$, let $t \in T$, and let n be the atomic measure concentrated at x. Then (*) becomes $g(x) = \int_E f(xp)dm(p)$. Since this holds for all $x \in X$ and since m is left invariant, we have

$$g(xt) = \int_E f(xtp)dm(p) = \int_E f(xp)dm(p) = g(x).$$

Therefore, g is T-invariant.

Now let n be a T-invariant measure on X. Then by (*) $\int_X g(x)dn(x) = \int_E \int_X f(xp)dn(x)dm(p) = \int_E \int_X f(x)dn(x)dm(p) = \int_X f(x)dn(x)$. Now by the remarks following 2.10, we see that (X,T) is simple.

3.16. LEMMA. *Let Ψ be a decomposition of X into closed T-invariant sets. Suppose that for each $A \in \Psi$, $E(A,T)$ is a group. Then $E(X,T)$ is a group.*

Proof. Let u be an idempotent of $E(X,T)$ and let $x \in X$. Now the image v of u in $E(A,T)$ under the obvious homomorphism is obtained by restricting u to A. Since $E(A,T)$ is a group, we have $xv = x$ hence $xu = x$ and, therefore, u is the identity and the conclusion follows from 3.13.

3.17. LEMMA. *Let (X,T) be w.a.p., minimal and measurable. Then $E(X,T)$ is a group.*

Proof. Let $m \in \Delta(X,T)$ and let $Y = \bigcap [Xp \mid p \in E]$. Since m is invariant, we have by 3.10 that $m(Xp) = 1$ for all $p \in E$ and, therefore, Y is not empty. Moreover, it is easily verified that Y is an invariant closed subset of X and so $Y = X$. Clearly, now if u is an idempotent of E it must be the identity map and the fact that E is a group follows from Lemma 3 of [9] which implies that a compact topological semigroup with identity is a group if the only idempotent is the identity.

3.18. LEMMA. *(X,T) is uniformly equicontinuous if and only if (X,T) is w.a.p. and E is a group.*

Proof. Let $G(X)$, the group of all homeomorphisms of X, be provided with the topology of uniform convergence. Assume (X,T) is uniformly equicontinuous and let H be the closure of T in $G(X)$. It is known [10, Chapter 11] that H is a compact topological group. Let $p \in E$ and let t_i be a net in E converging to p such that $t_i \in T$ for all i. Consider t_i as a net in H. Since H is compact, we can assume that t_i converges to q in H. It follows that $p = q$ and therefore (X,T) is w.a.p. Now consider the net t_i^{-1} in both E and H. Since E and H are compact, we may assume t_i^{-1} converges to some r in E and s in H. Again $r = s$ and since H is a topological group, gs is the identity in H and so pr is the identity of E and so E is a group.

Conversely, suppose (X, T) is w.a.p. and E is a group. Consider the map g of $X \times E$ into X defined by $g(x, p) = xp$. Since E contains only continuous maps, it follows that g is continuous in each variable separately. Now by a result of Ellis [6], we have that g is continuous in both variables simultaneously. This implies that the topology on E is that of uniform convergence and since E is compact the result follows.

3.19. Lemma. *(X, T) is uniformly equicontinuous if and only if (X, T) is w.a.p., completely measurable and Φ is a partition of X.*

Proof. Assume (X, T) is uniformly equicontinuous. Then by the previous lemma, (X, T) is w.a.p. It is proved in [10, p. 18] that Φ is a partition of X. Complete measurability follows from that fact that the closure of T in the space of all homeomorphisms (with the uniform convergence topology) is a compact topological group, say G. Now G has a Haar measure μ. Let Y be a closed invariant subset of X and let $y \in Y$. For $f \in C(X)$ define $J(f) = \int f(xg)d\mu(g)$. It is easily seen that J induces an invariant measure m with $m(Y) = 1$.

3.20. Theorem. *Let (X, T) be w.a.p. and let E possess an invariant mean. Let $A = \bigcup \Phi$ and $B = \bigcap [Xp \mid p \in E]$. Then $A = B = kX$. Moreover, (kX, T) is uniformly equicontinuous.*

Proof. Let $x \notin A$. Then the closure, D, of xT is not minimal. Since D is compact and invariant, it follows that D contains a minimal set M. Let $J = [p \mid p \in E$ and $xp \in M]$. It is easily seen that J is not empty. Moreover, J is a right ideal for if $p \in J$ and $q \in E$, we let t_i be a net in T converging, in E, to q. Then xpt_i converges to xpq. However, $xpt_i \in M$ and so $xpq \in M$. Now there exists an idempotent $u \in J$ and $x \notin Xu$ for if $x = yu$ then $x = xu$ but $xu \in M$ and $x \notin M$. Therefore, $B \subset A$. We also know by 3.15 that (X, T) is simple and completely measurable. Now if m is an invariant measure on X, $m(Xp) = 1$ for all $p \in E$ and so $m(B) = 1$. Therefore, $B \supset kX$. We also have by 2.30 that $kX \supset A$ and so $A = B = kX$. To prove the last statement notice that by 3.17 $E(M, T)$ is a group for all minimal sets M and that by 3.16 $E(kX, T)$ is a group. Now apply 3.18.

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