QUASI-NORM SPACES⁽¹⁾

BY

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Introduction. Let S be a linear space. A function q on S is called a *quasi-norm* if it satisfies:

- (1) $0 \leq q(x) \leq +\infty$ for all $x \in S$,
- (2) $q(0) = 0$,
- (3) $|a| \leq |b|$ implies that $q(ax) \leq q(bx)$ for a, b real and $x \in S$,
- (4) $q(x + y) \leq q(x) + q(y)$ for all x and y in S .

A quasi-norm q is *proper* if $\lim_{a \rightarrow 0} q(ax) = 0$ for all $x \in S$. We say that a quasi-norm q_1 *dominates* another quasi-norm q_2 ($q_1 \succ q_2$) if, for every $\varepsilon > 0$ there is a $\delta > 0$, such that $q_1(x) < \delta$ implies $q_2(x) < \varepsilon$. A system I of quasi-norms is called an *ideal* if:

- (1) $I \ni q_1 \succ q_2$ implies $I \ni q_2$,
- (2) for any sequence $q_n \in I$ ($n = 1, 2, \dots$) there is a quasi-norm $q \in I$ such that $q \succ q_n$ for all $n = 1, 2, \dots$.

A system B of quasi-norms is a basis for an ideal I if, for each $q_1 \in I$ there is $q_2 \in B \subset I$ such that $q_1 \prec q_2$.

Given any linear space we show that there is a one-to-one correspondence between topologies on the linear space compatible with the linear operations and ideals of proper quasi-norms. Thus, study of ideals of proper quasi-norms will give us knowledge of linear topologies.

Given a set A such that $dA \subset A$ for $|d| \leq 1$ we say that A is of *finite character* c if $bA + (1 - b)A \subset cA$ for all b such that $0 < b < 1$. A set B is *bounded* by a quasi-norm q ($B \succ q$) if $\lim_{a \rightarrow 0} [\sup_{x \in B} q(ax)] = 0$. We show that if a set A is of finite character, then a quasi-norm q_2 can be constructed such that $A \succ q_2$ and $q_1 \prec q_2$ for any quasi-norm q_1 such that $A \succ q_1$. The quasi-norm constructed from A also has the following property: there are two numbers $c, d > 0$ such that $q(x) < d$ implies that $q((1/2c)x) \leq \frac{1}{2}q(x)$. Quasi-norms that have this property are said to be of *finite character* c .

We investigate completeness and quasi-completeness of a linear topological space in terms of quasi-norms. We also show that a manifold is bounded by every quasi-norm in an ideal composed of proper quasi-norms if and only if it is bounded by the corresponding linear topology.

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A system F of quasi-norms is a *filter* if $F \ni q_1 < q_2$ implies $F \ni q_2$. We consider relations between ideals and filters. Given a subspace R of the linear space S we define the relative ideal on R of a given ideal on S . Given a directed system of subspaces and ideals (filters) on these subspaces, we define an inductive limit ideal (filter) on S . We examine what properties of the individual ideals the inductive limit inherits and look at the relation between inductive limits of filters and ideals. Next, we show that, on a finite-dimensional space, every ideal composed of proper quasi-norms is contained in an ideal which has a one-element basis.

In the latter part of the paper, quasi-norms on linear lattices are investigated. We consider various properties of quasi-norms which connect them with the lattice structure. A main result shows that a universally continuous linear lattice is super-universally continuous if there is a quasi-norm q defined on S which has the following properties:

- (1) $|x| \leq |y|$ implies $q(x) \leq q(y)$ for all $x, y \in S$ (i.e. q is *monotone*).
- (2) $\text{order-lim}_{v \rightarrow \infty} a_v = 0$ implies $\lim_{v \rightarrow \infty} q(a_v) = 0$.
- (3) $q(x) = 0$ implies $x = 0$.

The next object of study is ideals of quasi-norms having special properties. We prove that the following two properties are equivalent:

- (1) I is a proper ideal which has a basis of monotone quasi-norms;
- (2) the linear topology corresponding to I has a basis $\mathfrak{B}$ of neighborhoods of 0 which satisfy $|x| \leq |y|$ and $y \in V \in \mathfrak{B}$ implies $x \in V$. The final important result says that if S is a universally continuous space and I is an ideal on S which has a basis of quasi-norms q such that $a_\lambda \uparrow_{\lambda \in \Lambda} a$ implies $q(a) = \sup_{\lambda \in \Lambda} q(a_\lambda)$ then $\{x: |x| \leq |a|\}$ is complete for all $a \in S$.

1. Quasi-norms. Let S be a linear space. A function q defined on S is called a *quasi-norm* if:

- (1) $0 \leq q(x) \leq +\infty$ for all $x \in S$,
- (2) $|a| \leq |b|$ implies $q(ax) \leq q(bx)$,
- (3) $q(x+y) \leq q(x) + q(y)$,
- (4) $q(0) = 0$.

Note that it is possible for a quasi-norm to assume infinite values. If $q(x) < +\infty$ for all $x \in S$ we say that q is a *finite quasi-norm*.

EXAMPLE 1. Define q^* on S by

$$q^*(x) = \begin{cases} 0 & \text{if } x = 0, \\ +\infty & \text{if } x \neq 0. \end{cases}$$

Then q^* is easily seen to be a quasi-norm.

EXAMPLE 2. If f is a linear functional on the linear space S then $|f|$ defined by $|f|(x) = |f(x)|$ is a finite quasi-norm.

Several properties of quasi-norms follow easily from the definition. Condition (2) shows that $q(ax) = q(|a|x)$ for all real a and $x \in S$.

By induction from condition (3) we conclude:

THEOREM 1.1. $q(nx) \leq nq(x)$ for all $n = 1, 2, \dots$.

A quasi-norm q is said to be *proper* if $\lim_{a \rightarrow 0} q(ax) = 0$ for all $x \in S$. The quasi-norm defined in Example 1 is not proper while the one in Example 2 is proper.

THEOREM 1.2. *If q is a proper quasi-norm, then q is finite.*

Proof. Since, for arbitrary $x \in S$, $\lim_{a \rightarrow 0} q(ax) = 0$ there is $\delta > 0$ such that $q(ax) \leq 1$ for $|a| \leq \delta$. Choose a positive integer n such that $1 < n\delta$. Then $q(x) \leq q((n\delta)x) \leq nq(\delta x) \leq n < +\infty$.

THEOREM 1.3. *For any quasi-norm q , if we set $q^\alpha(x) = \min\{q(x), \alpha\}$ for $\alpha > 0$, we obtain a finite quasi-norm q^α . If q is proper, then q^α is also proper.*

Proof. Clearly $0 \leq q^\alpha(x) \leq \alpha$ for all $x \in S$. If $|a| \leq |b|$ then

$$\begin{aligned} q^\alpha(ax) &= \min\{q(ax), \alpha\} \leq \min\{q(bx), \alpha\} = q^\alpha(bx), \\ q^\alpha(x+y) &= \min\{q(x+y), \alpha\} \leq \min\{q(x) + q(y), \alpha\} \\ &\leq \min\{q(x) + q(y), q(x) + \alpha, q(y) + \alpha, \alpha + \alpha\} \\ &= \min\{q(x), \alpha\} + \min\{q(y), \alpha\} = q^\alpha(x) + q^\alpha(y), \\ q^\alpha(0) &= \min\{q(0), \alpha\} = 0. \end{aligned}$$

If q is proper, then we have:

$$0 \leq \lim_{a \rightarrow 0} q^\alpha(ax) \leq \lim_{a \rightarrow 0} q(ax) = 0.$$

Theorem 1.2 shows that every proper quasi-norm is finite but the converse is false. If q^* is defined as in Example 1, then $(q^*)^1$ is a finite quasi-norm which is not proper.

THEOREM 1.4. *For any system $q_\lambda (\lambda \in \Lambda)$ of quasi-norms and for any system $a_\lambda (\lambda \in \Lambda)$ of positive real numbers the function $\sum_{\lambda \in \Lambda} a_\lambda q_\lambda$ given by: $(\sum_{\lambda \in \Lambda} a_\lambda q_\lambda)(x) = \sup_{\text{finite } H \subset \Lambda} \sum_{\lambda \in H} a_\lambda q_\lambda(x)$ is also a quasi-norm. Furthermore, if q_λ is proper for all $\lambda \in \Lambda$ and $\sum_{\lambda \in \Lambda} a_\lambda q_\lambda(x)$ is finite for all $x \in S$ then $\sum_{\lambda \in \Lambda} a_\lambda q_\lambda$ is a proper quasi-norm.*

Proof. Clearly $0 \leq \sum_{\lambda \in \Lambda} a_\lambda q_\lambda(x) \leq +\infty$. $|a| \leq |b|$ implies $\sum_{\lambda \in H} a_\lambda q_\lambda(ax) \leq \sum_{\lambda \in H} a_\lambda q_\lambda(bx)$ for every finite $H \subset \Lambda$. Thus

$$\begin{aligned}
\sum_{\lambda \in \Lambda} a_{\lambda} q_{\lambda}(ax) &\leq \sum_{\lambda \in \Lambda} a_{\lambda} q_{\lambda}(bx), \\
\sum_{\lambda \in \Lambda} a_{\lambda} q_{\lambda}(x+y) &= \sup_{\text{finite } H \subset \Lambda} \sum_{\lambda \in H} a_{\lambda} q_{\lambda}(x+y) \\
&\leq \sup_{\text{finite } H \subset \Lambda} \left\{ \sum_{\lambda \in H} a_{\lambda} q_{\lambda}(x) + \sum_{\lambda \in H} a_{\lambda} q_{\lambda}(y) \right\} \\
&\leq \sup_{\text{finite } H \subset \Lambda} \sum_{\lambda \in H} a_{\lambda} q_{\lambda}(x) + \sup_{\text{finite } H \subset \Lambda} \sum_{\lambda \in H} a_{\lambda} q_{\lambda}(y) \\
&= \sum_{\lambda \in \Lambda} a_{\lambda} q_{\lambda}(x) + \sum_{\lambda \in \Lambda} a_{\lambda} q_{\lambda}(y).
\end{aligned}$$

Clearly $\sum_{\lambda \in \Lambda} a_{\lambda} q_{\lambda}(0) = 0$.

Now suppose that q_{λ} is proper for all $\lambda \in \Lambda$ and that $\sum_{\lambda \in \Lambda} a_{\lambda} q_{\lambda}$ is finite. Given any $x \in S$ and any $\varepsilon > 0$ there is a finite set $H_0 \subset \Lambda$ such that

$$\sum_{\lambda \in H_0} a_{\lambda} q_{\lambda}(x) > \sum_{\lambda \in \Lambda} a_{\lambda} q_{\lambda}(x) - \varepsilon.$$

But if we note that:

$$\begin{aligned}
\sum_{\lambda \in \Lambda} a_{\lambda} q_{\lambda}(x) &= \sup_{\text{finite } H \subset \Lambda} \sum_{\lambda \in H} a_{\lambda} q_{\lambda}(x) \\
&= \sup_{\text{finite } H \subset \Lambda} \sum_{\lambda \in H \cup H_0} a_{\lambda} q_{\lambda}(x) \\
&= \sum_{\lambda \in H_0} a_{\lambda} q_{\lambda}(x) + \sum_{\lambda \in \Lambda - H_0} a_{\lambda} q_{\lambda}(x),
\end{aligned}$$

we see $\sum_{\lambda \in H_0} a_{\lambda} q_{\lambda}(x) > \sum_{\lambda \in H_0} a_{\lambda} q_{\lambda}(x) + \sum_{\lambda \in \Lambda - H_0} a_{\lambda} q_{\lambda}(x) - \varepsilon$ and this implies $\sum_{\lambda \in \Lambda - H_0} a_{\lambda} q_{\lambda}(x) < \varepsilon$. Therefore we have:

$$\begin{aligned}
\lim_{a \rightarrow 0} \left(\sum_{\lambda \in \Lambda} a_{\lambda} q_{\lambda}(ax) \right) &= \lim_{a \rightarrow 0} \left(\sum_{\lambda \in H_0} a_{\lambda} q_{\lambda}(ax) + \sum_{\lambda \in \Lambda - H_0} a_{\lambda} q_{\lambda}(ax) \right) \\
&\leq \lim_{a \rightarrow 0} \sum_{\lambda \in H_0} a_{\lambda} q_{\lambda}(ax) + \varepsilon = \varepsilon.
\end{aligned}$$

Since ε was arbitrary $\lim_{a \rightarrow 0} \sum_{\lambda \in \Lambda} a_{\lambda} q_{\lambda}(ax) = 0$.

For two quasi-norms q_1 and q_2 we write $q_1 < q_2$ if, for any $\varepsilon > 0$ there is $\delta > 0$ such that $q_2(x) < \delta$ implies $q_1(x) < \varepsilon$.

The next three theorems follow easily from this definition.

THEOREM 1.5. *If q_1 , q_2 and q_3 are quasi-norms such that $q_1 < q_2$ and $q_2 < q_3$, then $q_1 < q_3$.*

THEOREM 1.6. *If q_1 and q_2 are quasi-norms such that $q_1 < q_2$, then $aq_1 < bq_2$ for any $a, b > 0$.*

THEOREM 1.7. *If q_1 and q_2 are quasi-norms such that $q_1(x) \leq q_2(x)$ for all $x \in S$ then $q_1 < q_2$.*

If we denote by 0 the quasi-norm defined by $0(x) = 0$ for all $x \in S$ and let q^* be defined as in Example 1 then, for every quasi-norm q , $0(x) \leq q(x) \leq q^*(x)$ for all $x \in S$. Thus, $0 \prec q \prec q^*$ for every quasi-norm q .

THEOREM 1.8. *If q_1 and q_2 are quasi-norms such that $q_1 \prec q_2$ and q_2 is proper, then q_1 is also proper.*

Proof. Given $x \in S$ and $\varepsilon > 0$ there is $\delta_1 > 0$ such that $q_2(y) < \delta_1$ implies $q_1(y) < \varepsilon$. Since q_2 is proper there is $\delta_2 > 0$ such that $|a| < \delta_2$ implies $q_2(ax) < \delta_1$. Thus $|a| < \delta_2$ implies $q_1(ax) < \varepsilon$. Therefore q_1 is proper.

THEOREM 1.9. *If q_1 , q_2 and q_3 are quasi-norms such that $q_1 \succ q_2$ and $q_1 \succ q_3$, then $q_1 \succ q_2 + q_3$.*

Proof. For any $\varepsilon > 0$ we can find δ_1 and δ_2 , both positive, such that $q_1(x) < \delta_1$ implies $q_2(x) < \varepsilon/2$ and $q_1(x) < \delta_2$ implies $q_3(x) < \varepsilon/2$. Then for $q_1(x) < \min\{\delta_1, \delta_2\}$ we have $q_2(x) + q_3(x) < \varepsilon$.

By induction we can extend this theorem to any finite sum but a slight modification is necessary in order to extend the theorem to an infinite sum.

THEOREM 1.10. *If $q \prec q_\lambda$ for all $\lambda \in \Lambda$ and if $\sum_{\lambda \in \Lambda} (\sup_{x \in S} q_\lambda(x)) < +\infty$ then $q \succ \sum_{\lambda \in \Lambda} q_\lambda$.*

Proof. Given $\varepsilon > 0$ since $\sum_{\lambda \in \Lambda} (\sup_{x \in S} q_\lambda(x)) < +\infty$ there is a finite $H_0 \subset \Lambda$ such that:

$$\begin{aligned} \sum_{\lambda \in H_0} \left(\sup_{x \in S} q_\lambda(x) \right) &> \sum_{\lambda \in \Lambda} \sup_{x \in S} q_\lambda(x) - \varepsilon/2 \\ &= \sum_{\lambda \in H_0} \sup_{x \in S} q_\lambda(x) + \sum_{\lambda \in \Lambda - H_0} \sup_{x \in S} q_\lambda(x) - \varepsilon/2. \end{aligned}$$

Therefore, $\sum_{\lambda \in \Lambda - H_0} q_\lambda(x) < \sum_{\lambda \in \Lambda - H_0} \sup_{x \in S} q_\lambda(x) < \varepsilon/2$ for all $x \in S$. Since H_0 is finite there is $\delta > 0$ such that $q(x) < \delta$ implies $\sum_{\lambda \in H_0} q_\lambda(x) < \varepsilon/2$. Thus, if $q(x) < \delta$ we have:

$$\sum_{\lambda \in \Lambda} q_\lambda(x) = \sum_{\lambda \in H_0} q_\lambda(x) + \sum_{\lambda \in \Lambda - H_0} q_\lambda(x) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Therefore $q \succ \sum_{\lambda \in \Lambda} q_\lambda$.

2. Bounded manifolds. If A is a manifold of the linear space S and if q is a quasi-norm on S we define $q(A) = \sup_{x \in A} q(x)$. The following relations follow easily.

- (1) $q(aA) \leq q(bA)$ for $|a| \leq |b|$,
- (2) $q(-A) = q(A)$,
- (3) $q(A + B) \leq q(A) + q(B)$.

A manifold A is said to be *bounded by a quasi-norm q* (written $A| \succ q$) if

$\lim_{a \rightarrow 0} q(aA) = 0$. With this definition we see easily that if $A \succ q$, then $aA \succ q$ for any real a . Also it is clear that $A \succ q_1$ and $q_1 \succ q_2$ implies that $A \succ q_2$.

THEOREM 2.1. *If A is a manifold of S and q is a quasi-norm, then $A \succ q$ implies that $\bigcup_{|a| \leq 1} aA \succ q$.*

Proof. $q(\bigcup_{|a| \leq 1} aA) \leq \sup_{|a| \leq 1} q(aA) \leq q(A)$. Therefore, for any real b :

$$q\left(b \left\{ \bigcup_{|a| \leq 1} aA \right\}\right) \leq q(bA)$$

and the last term goes to zero as b does.

THEOREM 2.2. *If A and B are manifolds which are bounded by a quasi-norm q , then $A + B$ and $A \cup B$ are also bounded by q .*

Proof. $\lim_{a \rightarrow 0} q(a(A + B)) \leq \lim_{a \rightarrow 0} \{q(aA) + q(aB)\} = 0$.

$$\begin{aligned} \lim_{a \rightarrow 0} q(a(A \cup B)) &= \lim_{a \rightarrow 0} q(aA \cup aB) \\ &= \lim_{a \rightarrow 0} \max \{q(aA), q(aB)\} = \max \left\{ \lim_{a \rightarrow 0} q(aA), \lim_{a \rightarrow 0} q(aB) \right\} = 0. \end{aligned}$$

A manifold A is said to be *symmetric* if $A = -A$. A manifold A is said to be *star* if $A \supset aA$ for $0 \leq a \leq 1$.

It is easily seen that, for any manifold A , the manifold $\bigcup_{|a| \leq 1} aA$ is the smallest symmetric star manifold containing A .

A manifold A is said to be a *character manifold* (or A is said to be of *finite character*) if it is symmetric and star and satisfies the following: there is a positive real number c such that $aA + bA \subset cA$ for $a + b \leq 1$; $a, b \geq 0$. Such a c is called a *character of A* and A is said to be of *finite character c* .

THEOREM 2.3. *If A_λ ($\lambda \in \Lambda$) is a system of manifolds such that each one is of character c , then $A_0 = \bigcap_{\lambda \in \Lambda} A_\lambda$ is also of character c .*

Proof. Let $x, y \in A_0$. Then, for $a + b \leq 1$; $a, b \geq 0$ we have $ax + by \in cA_\lambda$ for all $\lambda \in \Lambda$ which implies $ax + by \in \bigcap_{\lambda \in \Lambda} cA_\lambda = cA_0$. Therefore, $aA_0 + bA_0 \subset cA_0$.

Since a character manifold is star by definition, we see that $0 < a < b$ implies that $aA \subset bA$ for any character manifold A . Thus, if c is a character of A , then any larger number is also a character of A .

THEOREM 2.4. *A manifold A has a character less than one if and only if A is a linear manifold.*

Proof. If A is a linear manifold $aA + bA \subset A = \frac{1}{2}A$ for $a, b \geq 0$, $a + b \leq 1$. If A has character $c < 1$, we first show that, for $a > 0$, $aA = A$. Let $x \in A$. Then $x = \frac{1}{2}x + \frac{1}{2}x = cz_1$ for $z_1 \in A$. By induction we define a sequence $z_v \in A$ ($v = 1, 2, \dots$) such that $z_v = \frac{1}{2}z_v + \frac{1}{2}z_v = cz_{v+1}$. Thus $x = c^v z_v$ (for $v = 1, 2, \dots$). Choose v_0

such that $c^{v_0}a < 1$. Then $ax = (ac^{v_0})z_{v_0} \in A$ since A is star. Thus $aA = A$ for $a > 0$. If $a < 0$, $aA = (-a)(-A) = (-a)A = A$ since A is symmetric. Given $x, y \in A$ we have $x + y = 2(\frac{1}{2}x + \frac{1}{2}y) \in 2(cA) = (2c)A = A$.

This result shows that the linear space S itself is a character manifold and that any positive number is a character of S .

Given a manifold A and a positive real number c , the character c hull of A is defined as the intersection of all manifolds of character c containing A . (This system is nonempty since S itself is of character c .) We denote the character c hull of A by $\bar{A}^c$.

Clearly $\bar{A}^c = (\bigcup_{|a| \leq 1} aA)^c$ since a manifold of character c is star and symmetric. If A is of character c , then dA is of character c for every real d since, for $a + b \leq 1$; $a, b \geq 0$, $a(dA) + b(dA) = d(aA + bA) \subset d(cA) = c(dA)$. For any manifold A consider $\overline{aA}^c$ where a is any real number $\neq 0$. B is a manifold of character c containing A if and only if aB is a manifold of character c containing aA . Therefore $\overline{aA}^c = \bigcap_{aA \subset D \text{ of char. } c} D = \bigcap_{aA \subset aB \text{ of char. } c} aB = a\bar{A}^c$.

A quasi-norm q is said to be of *finite character* if there are positive real numbers a and c such that $q(x) \leq a$ implies that $\frac{1}{2}q(x) \geq q((1/2c)x)$. If q satisfies this requirement, we say that q is of *character* c . Clearly if q is of character c , then q is of character c' for any $c' > c$.

THEOREM 2.5. *If q_1 and q_2 are quasi-norms of finite character, then $aq_1 + bq_2$ is also of finite character for $a, b > 0$.*

Proof. Let d_i, c_i ($i=1,2$) be such that $q_i(x) \leq d_i$ implies $\frac{1}{2}q_i(x) \geq q_i((1/2c_i)x)$ ($i=1,2$). Setting $d_0 = \min\{d_1, d_2\}$ and $c_0 = \max\{c_1, c_2\}$ we find that $aq_1(x) + bq_2(x) \leq \min\{ad_0, bd_0\}$ implies that $q_1(x) \leq d_0$ and $q_2(x) \leq d_0$. Therefore, $\frac{1}{2}aq_1(x) \geq aq_1((1/2c_1)x)$ and $\frac{1}{2}bq_2(x) \geq bq_2((1/2c_2)x)$. Hence,

$$\frac{1}{2}(aq_1(x) + bq_2(x)) \geq aq_1((1/2c_0)x) + bq_2((1/2c_0)x).$$

THEOREM 2.6. *A quasi-norm q is of finite character if and only if q^α is of finite character for all $\alpha > 0$.*

Proof. If q is of finite character, then there are a_0, c_0 such that $q(x) \leq a_0$ implies $\frac{1}{2}q(x) \geq q((1/2c_0)x)$. Putting $a_1 = \min\{a_0, \alpha/2\}$ and $c_1 = c_0$, we have that $q^\alpha(x) \leq a_1$ implies $q(x) = q^\alpha(x) \leq a_1 \leq a_0$. Therefore, $\frac{1}{2}q^\alpha(x) = \frac{1}{2}q(x) \geq q((1/2c_1)x) = q^\alpha((1/2c_1)x)$ since $q((1/2c_1)x) \leq \frac{1}{2}q(x) = \frac{1}{2}q^\alpha(x) \leq \frac{1}{2}\alpha$.

If q^α is of finite character for some $\alpha > 0$, then there are positive real numbers a and c such that $q^\alpha(x) \leq a$ implies $\frac{1}{2}q^\alpha(x) \geq q^\alpha((1/2c)x)$. If we let $a_0 = \min\{a, \alpha/2\}$, we have that $q(x) \leq a_0$ implies $q(x) = q^\alpha(x) \leq a$ which gives $\frac{1}{2}q(x) = \frac{1}{2}q^\alpha(x) \geq q^\alpha((1/2c)x) = q((1/2c)x)$.

THEOREM 2.7. *If $A \succ q$ and q is a quasi-norm of finite character c then $\bar{A}^{2c} \succ q$.*

Proof. Since q is of character c , we can find $a_0 > 0$ such that $q(x) \leq a_0$ implies

$\frac{1}{2}q(x) \geq q((1/2c)x)$. Hence, if $q(x) \leq a_0$ and $q(y) \leq a_0$, then for $a + b \leq 1$; $a, b \geq 0$ we have:

$$\begin{aligned} q((1/2c)(ax + by)) &\leq q((a/2c)x) + q((b/2c)y) \\ &\leq q((1/2c)x) + q((1/2c)y) \leq \frac{1}{2}q(x) + \frac{1}{2}q(y) \leq a_0. \end{aligned}$$

Therefore, if we let $V_{a_0} = \{x: q(x) \leq a_0\}$, we have: $aV_{a_0} + bV_{a_0} \subset (2c)V_{a_0}$ for $a + b \leq 1$; $a, b \geq 0$. Thus, $2c$ is a character of V_{a_0} . Now since $A \succ q$, there is $d > 0$ such that $q(dA) \leq a_0$ or, equivalently $dA \subset V_{a_0}$. The fact that V_{a_0} is of character $2c$ shows us that $d\bar{A}^{2c} = \overline{dA}^{2c} \subset V_{a_0}$. Thus, we need only show that V_{a_0} is bounded by q in order to have $d\bar{A}^{2c}$ (and therefore $\bar{A}^{2c} = ((1/d)d\bar{A}^{2c})$ also bounded by q .

Given $\varepsilon > 0$ choose v_0 such that $a_0/2^{v_0} < \varepsilon$ and let $x \in V_{a_0}$. Then

$$q((1/(2c)^{v_0})x) \leq \frac{1}{2}q((1/(2c)^{v_0-1})x) \leq \dots \leq (1/2^{v_0})q(x) < \varepsilon.$$

Hence $q((1/(2c)^{v_0})V_{a_0}) < \varepsilon$ and $V_{a_0} \succ q$.

A manifold V of the linear space S is called a *vicinity* if, for $x \in S$ we can find $b > 0$ such that $ax \in V$ for $0 \leq a \leq b$.

The following theorem is proved in [3].

THEOREM 2.8 ([3], p. 129). *If a symmetric star vicinity V is of finite character c (i.e. $aV + bV \subset cV$ for $a + b \leq 1$; $a, b \geq 0$) then there is a proper quasi-norm q on S such that*

$$\{x: q(x) < 1/2^v\} \subset (1/2c)^v V \subset \{x: q(x) \leq 1/2^v\}$$

and

$$q((1/2c)x) = \frac{1}{2}q(x) \text{ for } q(x) \leq \frac{1}{2}.$$

(Note that the definition of quasi-norm in [3] corresponds to a proper quasi-norm in our terminology.)

In the proof of the preceding theorem the fact that the set V is a vicinity is used only to establish that the quasi-norm is proper. Since every character manifold is star and symmetric, the rest of the proof gives the following result:

THEOREM 2.9. *Let A be a manifold of finite character c . Then there exists a not necessarily proper quasi-norm q of finite character c such that:*

$$\left\{x: q(x) < \frac{1}{2^v}\right\} \subset \left(\frac{1}{2c}\right)^v A \subset \left\{x: q(x) \leq \frac{1}{2^v}\right\}$$

and

$$q((1/2c)x) = \frac{1}{2}q(x) \text{ for } q(x) \leq \frac{1}{2}.$$

This quasi-norm is called the quasi-norm associated with A and denoted by q_A .

THEOREM 2.10. *If A is a manifold of finite character, then $A \succ q_A$. Furthermore, if q is any quasi-norm such that $A \succ q$ then we have $q < q_A$.*

Proof. If A is of character c , then $(1/2c)^v A \subset \{x: q_A(x) \leq 1/2^v\}$ which shows that $A \succ_{q_A}$.

If q is a quasi-norm such that $A \succ_q$ then, for any $\varepsilon > 0$ there is a $\delta > 0$ such that $q(\delta A) < \varepsilon$. Choose v_0 such that $(1/2c)^{v_0} < \delta$. Then $q_A(x) < 1/2^{v_0}$ implies that $x \in (1/2c)^{v_0} A \subset \delta A$ which implies $q(x) < \varepsilon$. Therefore $q < q_A$.

3. Ideals. A system I of quasi-norms is called an *ideal* if:

- (1) $I \ni q_1 \succ q_2$ implies $I \ni q_2$.
- (2) for any sequence $q_v \in I$ ($v = 1, 2, \dots$) there is $q \in I$ such that $q_v \prec q$ for all $v = 1, 2, \dots$.

Since $0 \prec q$ for all quasi-norms q we see that every ideal $I \ni 0$.

THEOREM 3.1. $q_v \in I$ for all $v = 1, 2, \dots$ implies that $\sum_{v=1}^{\infty} q_v^{1/2^v} \in I$.

Proof. By property (2) there is $q \in I$ such that $q_v \prec q$ for all $v = 1, 2, \dots$. But since $\sum_{v=1}^{\infty} \sup_{x \in S} q_v^{1/2^v}(x) \leq \sum_{v=1}^{\infty} (\frac{1}{2})^v < +\infty$, Theorem 1.10 shows that $\sum_{v=1}^{\infty} q_v^{1/2^v} \prec q$. Hence $\sum_{v=1}^{\infty} q_v^{1/2^v} \in I$ by (1) above.

A subsystem $B \subset I$ is called a *basis* if, for any $q_1 \in I$ there is $q_2 \in B$ such that $q_2 \succ q_1$. Clearly every basis satisfies the *basis condition*: for any sequence $q_v \in B$ ($v = 1, 2, \dots$) there is $q \in B$ such that $q_v \prec q$ for all $v = 1, 2, \dots$. Conversely we have:

THEOREM 3.2. If a system B of quasi-norms satisfies the basis condition, then there exists uniquely an ideal I such that B is a basis of I .

Proof. Let $I = \{q: q \prec q_1 \text{ for some } q_1 \in B\}$. Then if $I \ni q_1 \succ q_2$ it is clear that $I \ni q_2$. If $I \ni q_v$ ($v = 1, 2, \dots$) then $q_v \prec q'_v \in B$ ($v = 1, 2, \dots$). But then there is $q \in B$ such that $q'_v \prec q$ ($v = 1, 2, \dots$). $q \in B$ implies $q \in I$ and $q \succ q'_v \succ q_v$ shows that $q \succ q_v$ ($v = 1, 2, \dots$). Thus, I is an ideal. If any other ideal I_1 contains B it must contain I . But if B is a basis for I_1 , then $q_1 \in I_1$ implies $q_1 \prec q_2$ for some $q_2 \in B$ and hence $q_1 \in I$. Therefore, $I_1 = I$ and I is unique.

THEOREM 3.3. If B is a basis for an ideal I , then, for any $\alpha > 0$, $\{q^\alpha: q \in B\}$ is also a basis for I .

Proof. $q \geq q^\alpha$ implies $q \succ q^\alpha$ and this shows that $q^\alpha \in I$ for all $q \in B$. We need only show that $q \prec q^\alpha$ to conclude $\{q^\alpha: q \in B\}$ is a basis. For any $\varepsilon > 0$ if we choose $\delta = \min\{\varepsilon, \alpha/2\}$ we have $q^\alpha(x) < \delta$ implies $q(x) = q^\alpha(x) < \delta \leq \varepsilon$.

We say that an ideal I_1 is *stronger* than another ideal I_2 or I_2 is *weaker* than I_1 if $I_1 \supset I_2$.

THEOREM 3.4 If I_λ ($\lambda \in \Lambda$) is a system of ideals, then $I_0 = \bigcap_{\lambda \in \Lambda} I_\lambda$ is an ideal which is the strongest ideal among all those weaker than all I_λ ($\lambda \in \Lambda$).

Proof. We need only show I_0 is an ideal and the rest is clear. If $I_0 \ni q_1 \succ q_2$, then we see that $I_\lambda \ni q_1 \succ q_2$ for all $\lambda \in \Lambda$ which means that $q_2 \in I_\lambda$ for all $\lambda \in \Lambda$.

Therefore $q_2 \in I_0$. If $q_v \in I_0$ ($v = 1, 2, \dots$), then $q_v \in I_\lambda$ for all $\lambda \in \Lambda$, $v = 1, 2, \dots$. Hence, $\sum_{v=1}^{\infty} q_v^{1/2^v} \in I_\lambda$ for all $\lambda \in \Lambda$ which implies $\sum_{v=1}^{\infty} q_v^{1/2^v} \in I_0$ and clearly $\sum_{v=1}^{\infty} q_v^{1/2^v} \succ q_v$ ($v = 1, 2, \dots$).

THEOREM 3.5. *If I_λ ($\lambda \in \Lambda$) is a system of ideals, then there is a weakest ideal among all those stronger than all I_λ ($\lambda \in \Lambda$). We denote this ideal by $\bigvee_{\lambda \in \Lambda} I_\lambda$. It has a basis given by*

$$B = \left\{ \sum_{v=1}^{\infty} q_v : \sum_{v=1}^{\infty} \sup_{x \in S} q_v(x) < +\infty, q_v \in \bigcup_{\lambda \in \Lambda} I_\lambda, v = 1, 2, \dots \right\}.$$

Another basis is given by

$$B_1 = \left\{ \sum_{v=1}^{\infty} q_v^{1/2^v} : q_v \in \bigcup_{\lambda \in \Lambda} I_\lambda \text{ for } v = 1, 2, \dots \right\}.$$

Proof. If I is an ideal containing $\bigcup_{\lambda \in \Lambda} I_\lambda$ and if $\sum_{v=1}^{\infty} q_v \in B$, then there is $q \in I$ such that $q \succ q_v$ ($v = 1, 2, \dots$). But then we have, by Theorem 1.10, $q \succ \sum_{v=1}^{\infty} q_v$ which implies $\sum_{v=1}^{\infty} q_v \in I$. Thus, $I \supset B$. If we can show that B satisfies the basis condition, it will be clear that the ideal generated by B is the weakest stronger ideal. Suppose $p_v \in B$ ($v = 1, 2, \dots$). Then $p_v = \sum_{\mu=1}^{\infty} q_{v,\mu}$ where $\sum_{\mu=1}^{\infty} \sup_{x \in S} q_{v,\mu}(x) \leq \alpha_v < +\infty$ for $v = 1, 2, \dots$. Now consider $\sum_{v,\mu} (1/\alpha_v 2^v) q_{v,\mu}$. Then

$$\sum_{v,\mu} \sup_{x \in S} \frac{1}{\alpha_v 2^v} q_{v,\mu}(x) = \sum_{v=1}^{\infty} \frac{1}{\alpha_v 2^v} \sum_{\mu=1}^{\infty} \sup_{x \in S} q_{v,\mu}(x) \leq 1.$$

Therefore $\sum_{v,\mu} (1/\alpha_v 2^v) q_{v,\mu} \in B$ and clearly $\sum_{v,\mu} (1/\alpha_v 2^v) q_{v,\mu} \succ p_v$ ($v = 1, 2, \dots$) since

$$\sum_{v,\mu} \frac{1}{\alpha_v 2^v} q_{v,\mu} \succ q_{v,\mu} \quad (\mu = 1, 2, \dots) \quad (v = 1, 2, \dots).$$

Now, as for B_1 , if we can show that for any $q \in B$ there is $q_1 \in B$, such that $q < q_1$, we will have shown that B_1 is also a basis since it is clear that $B_1 \subset B$. Suppose $q = \sum_{v=1}^{\infty} q_v \in B$. Let $q_1 = \sum_{v=1}^{\infty} q_v^{1/2^v} \in B_1$. Since $\sum_{v=1}^{\infty} q_v^{1/2^v} \succ q_v^{1/2^v} \succ q_v$ for all $v = 1, 2, \dots$ we have that $q_1 = \sum_{v=1}^{\infty} q_v^{1/2^v} \succ \sum_{v=1}^{\infty} q_v = q$ by Theorem 1.10.

An ideal I is said to be *proper* if every $q \in I$ is proper. Since $q < q_1$ proper implies q is proper, we need only know that a basis of I consists entirely of proper quasi-norms to conclude I is proper.

4. Induced linear topologies. Linear topologies were defined first by Kolmogoroff [2]. After that, von Neumann gave another definition in [7]. In [3] it is proved that these two definitions are equivalent. We will use the definition of von Neumann.

A system $\mathfrak{B}$ of vicinities in the linear space S is called a *linear topology* on S if:

- (1) $V \in \mathfrak{B}$, $V \subset U$ implies $U \in \mathfrak{B}$,
- (2) $U, V \in \mathfrak{B}$ implies $U \cap V \in \mathfrak{B}$,
- (3) $V \in \mathfrak{B}$ implies $aV \in \mathfrak{B}$ for all $a \neq 0$,
- (4) for any $V \in \mathfrak{B}$ we can find $U \in \mathfrak{B}$ such that $aU \subset V$ for $0 \leq a \leq 1$,
- (5) for any $V \in \mathfrak{B}$ we can find $U \in \mathfrak{B}$ such that $U + U \subset V$.

As on p. 139 of [3] we can consider, for every vicinity V on S , the corresponding connector (connector is defined on p. 62 of [3]) V_c defined by $V_c(x) = V + x$. Then, if $\mathfrak{B}$ is a linear topology, it follows easily that $\{V_c: V \in \mathfrak{B}\}$ is a basis for a uniformity on S which is called the *induced uniformity* by $\mathfrak{B}$ and denoted by $\mathfrak{U}^{\mathfrak{B}}$. $\mathfrak{U}^{\mathfrak{B}}$, in turn, induces a topology on S which is called the induced topology by $\mathfrak{B}$ and denoted $\mathfrak{T}^{\mathfrak{B}}$. If $\mathfrak{T}$ is a topology on S induced by a linear topology $\mathfrak{B}$ we refer to S with the topology $\mathfrak{T}$ as the *linear topological space* $(S, \mathfrak{T})$.

THEOREM 4.1. *A quasi-norm q on S is uniformly continuous by the induced uniformity $\mathfrak{U}^{\mathfrak{B}}$ if and only if, for any $\varepsilon > 0$ there is $V \in \mathfrak{B}$ such that $q(V) < \varepsilon$. If q is uniformly continuous it is proper.*

Proof. We first prove the "if" part. Given $\varepsilon > 0$ by hypothesis we can choose $V \in \mathfrak{B}$ such that $q(V) < \varepsilon$. Then if $x \in V_c(y) = V + y$, we have $x - y \in V$. Then $|q(x) - q(y)| \leq q(x - y) < \varepsilon$ and we see that q is uniformly continuous. The fact that $|q(x) - q(y)| \leq q(x - y)$ shows that a quasi-norm is uniformly continuous by $\mathfrak{U}^{\mathfrak{B}}$ if it is just continuous at zero by $\mathfrak{T}^{\mathfrak{B}}$.

If q is uniformly continuous then, for any $\varepsilon > 0$, there is $V \in \mathfrak{B}$ such that $x \in V_c(0) = 0 + V$ implies $|q(x) - q(0)| = q(x) < \varepsilon$.

To show that a uniformly continuous quasi-norm is proper, suppose we are given $\varepsilon > 0$. Then there is $V \in \mathfrak{B}$ such that $q(V) < \varepsilon$. Since V is a vicinity, given any $x \in S$ there is $b > 0$ such that $ax \in V$ for $0 \leq a \leq b$. Therefore $q(ax) < \varepsilon$ for $0 \leq a \leq b$ and hence $\lim_{a \rightarrow 0} q(ax) = 0$. Since x was arbitrary q is proper.

THEOREM 4.2. *If $(S, \mathfrak{T})$ is a linear topological space, the system of all quasi-norms on S which are continuous by $\mathfrak{T}$ is a proper ideal which we denote $I(S, \mathfrak{T})$. Furthermore, if $I(S, \mathfrak{T}_1) = I(S, \mathfrak{T}_2)$ then $\mathfrak{T}_1 = \mathfrak{T}_2$.*

Proof. By Theorem 4.1 the system is composed of proper quasi-norms, and we must show that it is an ideal. If q_1 is continuous and $q_2 < q_1$ then, for any $\varepsilon > 0$ there is $\delta > 0$ such that $q_1(x) < \delta$ implies $q_2(x) < \varepsilon$. But for such a δ we can find $V \in \mathfrak{B}$ such that $q_1(V) < \delta$. Then, $q_2(V) < \varepsilon$ and hence q_2 is continuous by Theorem 4.1. If $q_1, q_2, \dots$ is a sequence of continuous quasi-norms, let $p = \sum_{v=1}^{\infty} q_v^{1/2^v}$. Then $p > q_v^{1/2^v} > q$ for all $v = 1, 2, \dots$ and we need only show that p is continuous. Given $\varepsilon > 0$ choose v_0 so that $\sum_{v=v_0+1}^{\infty} 1/2^v < \varepsilon/2$. For each q_v , $v = 1, 2, \dots, v_0$ we can find $V_v \in \mathfrak{B}$ such that $q_v(V_v) < (\varepsilon/2v_0)$. Let $V_0 = \bigcap_{v=1}^{v_0} V_v \in \mathfrak{B}$. Then $p(V_0) = \sup_{x \in V_0} p(x) \leq \sum_{v=1}^{v_0} q_v(V_0) + \varepsilon/2 \leq \sum_{v=1}^{v_0} (\varepsilon/2v_0) + \varepsilon/2 = \varepsilon$. Thus p is continuous.

Now suppose $I(S, \mathfrak{T}_1) = I(S, \mathfrak{T}_2)$ and let $\mathfrak{T}_i$ be induced by $\mathfrak{B}_i$ ($i = 1, 2$). Given $V \in \mathfrak{B}_1$ we can choose a sequence $V_v \in \mathfrak{B}_1$ ($v = 1, 2, \dots$) of star symmetric neighborhoods of zero such that $V_1 \subset V$ and $V_{v+1} + V_{v+1} \subset V_v$ ($v = 1, 2, \dots$). Then using the construction given in Theorem 3 on p. 129 of [3], we can construct a quasi-norm q such that

$$\left\{x: q(x) < \frac{1}{2^v}\right\} \subset V_v \subset \left\{x: q(x) \leq \frac{1}{2^v}\right\}.$$

Since each $V_v \in \mathfrak{B}_1$ we see that $q \in I(S, \mathfrak{T}_1)$. By assumption then, $q \in I(S, \mathfrak{T}_2)$ which implies that $\{x: q(x) < \frac{1}{2}\} \in \mathfrak{B}_2$. Since this set is contained in V we see that $V \in \mathfrak{B}_2$. Therefore $\mathfrak{T}_1 \subset \mathfrak{T}_2$ and symmetry shows that $\mathfrak{T}_1 = \mathfrak{T}_2$.

THEOREM 4.3. *If I is any proper ideal on S , then $I = I(S, \mathfrak{T})$ for some topology $\mathfrak{T}$.*

Proof. For any $q \in I$ let $V_\alpha = \{x: q(x) \leq \alpha\}$ for $\alpha > 0$. In §63 of [3] it is shown that $\mathfrak{B}_q = \{V: V \supset V_\alpha \text{ for some } \alpha > 0\}$ is a linear topology. Let $\mathfrak{B}^q$ be the topology induced by $\mathfrak{B}_q$ and consider $\mathfrak{T}_0 = \bigvee_{q \in I} \mathfrak{T}^q$. It is not difficult to show that $(S, \mathfrak{T}_0)$ is a linear topological space.

If $q \in I$ then, by the definition of $\mathfrak{T}^q$, q is continuous by $\mathfrak{T}^q$. Since $\mathfrak{T}^q \subset \mathfrak{T}_0$ we see that q is continuous by $\mathfrak{T}_0$. Therefore, $I \subset I(S, \mathfrak{T}_0)$.

Now let $p \in I(S, \mathfrak{T}_0)$. Since p is continuous, for each $v = 1, 2, \dots$ we can find $V_v \in \mathfrak{T}_0$ such that $p(V_v) < 1/v$. By the construction of $\mathfrak{T}_0 = \bigvee_{q \in I} \mathfrak{T}^q$ we see that for each V_v there is a set $q_{v,\mu} \in I$ ($\mu = 1, 2, \dots, m_v$) and $\delta_v > 0$ such that $V_v \supset \{x: q_{v,\mu}(x) < \delta_v; \mu = 1, 2, \dots, m_v\}$ ($v = 1, 2, \dots$). Since I is an ideal $\sum_{\mu=1}^{m_v} q_{v,\mu} = p_v \in I$ and clearly $V_v \supset \{x: p_v(x) < \delta_v\}$. Now let $q = \sum_{v=1}^{\infty} p_v^{1/2^v} \in I$. Given $\varepsilon > 0$ there is v_0 such that $1/v_0 < \varepsilon$. Let $\delta = \min\{1/2^{v_0}, \delta_{v_0}\}$. Then $q(x) < \delta$ implies $p_{v_0}(x) < \delta_{v_0}$ and thus, $p(x) < 1/v_0 < \varepsilon$. Hence, $p < q$ which implies $p \in I$. Therefore $I(S, \mathfrak{T}_0) \subset I$.

Theorems 4.2 and 4.3 together show that there is a one-to-one correspondence between linear topologies and proper ideals. The next theorem shows that this correspondence is order-preserving.

THEOREM 4.4. *If $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are topologies on S such that $(S, \mathfrak{T}_1)$ and $(S, \mathfrak{T}_2)$ are linear topological spaces, then $\mathfrak{T}_1 \subset \mathfrak{T}_2$ if and only if $I(S, \mathfrak{T}_1) \subset I(S, \mathfrak{T}_2)$.*

Proof. If $\mathfrak{T}_1 \subset \mathfrak{T}_2$, then we see that q continuous by $\mathfrak{T}_1$ implies q continuous by $\mathfrak{T}_2$ and therefore $I(S, \mathfrak{T}_1) \subset I(S, \mathfrak{T}_2)$.

Conversely, if $I(S, \mathfrak{T}_1) \subset I(S, \mathfrak{T}_2)$, then $q \in I(S, \mathfrak{T}_1) \subset I(S, \mathfrak{T}_2)$ implies that $\mathfrak{T}^q \subset \bigvee_{q \in I(S, \mathfrak{T}_2)} \mathfrak{T}^q$ which means that

$$\bigvee_{q \in I(S, \mathfrak{T}_1)} \mathfrak{T}^q = \mathfrak{T}_1 \subset \mathfrak{T}_2 = \bigvee_{q \in I(S, \mathfrak{T}_2)} \mathfrak{T}^q.$$

Given a proper ideal I we denote the topology associated with I by $\mathfrak{T}^I$. This topology is induced by a linear topology which we denote by $\mathfrak{B}_I$.

5. Completeness. Let $\mathfrak{B}$ be a linear topology. A manifold A of S is said to be *bounded* by $\mathfrak{B}$ if, for any $V \in \mathfrak{B}$ there is $a > 0$ such that $aA \subset V$.

THEOREM 5.1. *If $\mathfrak{B}_I$ is the linear topology associated with a proper ideal I and A is a manifold of S then A is bounded by $\mathfrak{B}_I$ if and only if $A \succ q$ for all $q \in I$.*

Proof. If A is bounded by $\mathfrak{B}_I$ then for any $q \in I$ and $\varepsilon > 0$ there is $V \in \mathfrak{B}_I$ such that $q(V) < \varepsilon$. Then there is $a > 0$ such that $aA \subset V$ and hence $q(aA) < \varepsilon$.

If $A \succ q$ for all $q \in I$, let $V \in \mathfrak{B}_I$. Then there is $q \in I$ such that $\{x: q(x) < 1\} \subset V$. But then there is $\delta > 0$ such that $q(\delta A) < 1$. Therefore $\delta A \subset V$.

THEOREM 5.2. *If A is bounded by $\mathfrak{B}$, then the closure of A in the topology $\mathfrak{T}^{\mathfrak{B}}$ is also bounded by $\mathfrak{B}$.*

Proof. For any $q \in I(S, \mathfrak{T}^{\mathfrak{B}})$ and any $\varepsilon > 0$ there is $a > 0$ such that $q(aA) < \varepsilon$. Since q is continuous $q(aA^-) = q((aA)^-) \leq \varepsilon$. Thus $A^- \succ q$ for all $q \in I(S, \mathfrak{T}^{\mathfrak{B}})$ and consequently A^- is bounded by $\mathfrak{B}$ by Theorem 5.1.

THEOREM 5.3. *$S \ni x_\delta \rightarrow_{\delta \in \Delta} x$ by $\mathfrak{B}_I$ (that is, for any $V \in \mathfrak{B}_I$ there is δ_0 belonging to the directed system Δ such that $\delta \leq \delta_0$ implies $x_\delta \in x + V$) if and only if $q(x_\delta - x) \rightarrow_{\delta \in \Delta} 0$ for all $q \in \mathcal{B}$ where $\mathcal{B}$ is a basis of I .*

Proof. If $x_\delta \rightarrow_{\delta \in \Delta} x$ by $\mathfrak{B}_I$, then given any $q \in \mathcal{B}$ and any $\varepsilon > 0$ there is $V \in \mathfrak{B}_I$ such that $q(V) < \varepsilon$. But $x_\delta \rightarrow_{\delta \in \Delta} x$ means there is $\delta_0 \in \Delta$ such that $x_\delta - x \in A$ for all $\delta \leq \delta_0$. Hence, $q(x_\delta - x) < \varepsilon$ for all $\delta \leq \delta_0$.

As for the converse, if we are given any $V \in \mathfrak{B}_I$, there is a quasi-norm $q_V \in I$ such that $\{x: q_V(x) < 1\} \subset V$. Since $\mathcal{B}$ is a basis of I , there is $q \in \mathcal{B}$ such that $q_V < q$. Hence, there is $a > 0$ such that $q(x) < a$ implies $q_V(x) < 1$. Since $q(x_\delta - x) \rightarrow_{\delta \in \Delta} 0$, there is $\delta_0 \in \Delta$ such that $q(x_\delta - x) < a$ for $\delta \leq \delta_0$. Hence, $\delta \leq \delta_0$ implies $q(x_\delta - x) < a$ which means $q_V(x_\delta - x) < 1$ and hence $x_\delta - x \in V$. Therefore $x_\delta \rightarrow_{\delta \in \Delta} x$ by $\mathfrak{B}_I$.

A system $S \ni x_\delta (\delta \in \Delta)$ is a *Cauchy system* by $\mathfrak{B}$ if, for any $V \in \mathfrak{B}$ there is $\delta_0 \in \Delta$ such that $x_{\delta_1} - x_{\delta_2} \in V$ for $\delta_1, \delta_2 \leq \delta_0$.

THEOREM 5.4. *A directed system $S \ni x_\delta (\delta \in \Delta)$ is a Cauchy system by $\mathfrak{B}_I$ if and only if $q(x_{\delta_1} - x_{\delta_2}) \rightarrow_{\delta_1, \delta_2 \in \Delta} 0$ for all $q \in \mathcal{B}$ where $\mathcal{B}$ is a basis of I .*

Proof. Follows the same pattern as the proof of Theorem 5.3.

We say that a linear space S with linear topology $\mathfrak{B}$ is *complete* if, for every Cauchy system $x_\delta (\delta \in \Delta)$ there is $x \in S$ such that $x_\delta \rightarrow_{\delta \in \Delta} x$.

THEOREM 5.5. *S is complete by $\mathfrak{B}_I$ if and only if $q(x_{\delta_1} - x_{\delta_2}) \rightarrow_{\delta_1, \delta_2 \in \Delta} 0$ for all $q \in \mathcal{B}$ (a basis of I) implies that $q(x_\delta - x) \rightarrow_{\delta \in \Delta} 0$ for some $x \in S$ and all $q \in \mathcal{B}$.*

Proof. This is an immediate consequence of the two preceding theorems.

On p. 155 of [3] it is shown that, with every linear space S with linear topology $\mathfrak{V}$, we can associate another linear space $\bar{S}$ with linear topology $\bar{\mathfrak{V}}$ having the following properties:

- (1) $\bar{S}$ is complete by $\bar{\mathfrak{V}}$,
- (2) S is a linear manifold of $\bar{S}$,
- (3) $\mathfrak{V}$ is the relative linear topology of $\bar{\mathfrak{V}}$,
- (4) S is dense in $\bar{S}$ by $\bar{\mathfrak{V}}$,
- (5) $\{0\}^{\mathfrak{V}-} \subset S$.

$\bar{S}$ is called the *completion* of S and is unique up to isomorphism. Using Theorem 6 on p. 90 of [3], we see that every continuous quasi-norm q on S can be extended to a function $\bar{q}$ on $\bar{S}$ which is continuous by $\bar{\mathfrak{V}}$. Since the real numbers form a Hausdorff space $\bar{q}$ is uniquely determined. It can be easily shown that $\bar{q}$ is actually a quasi-norm on $\bar{S}$. Thus, if I is an ideal of quasi-norms on S , we can correspond the ideal $\bar{I} = \{\bar{q}: q \in I\}$ on $\bar{S}$. ($\bar{I}$ can easily be shown to be an ideal if we note that $q_1 < q_2$ implies $\bar{q}_1 < \bar{q}_2$. This follows from the relation:

$$\{x \in \bar{S}: \bar{q}_2(x) < \delta\} \subset \{x \in S: q_2(x) < \delta\}^- \subset \{x \in S: q_1(x) < \varepsilon\}^- \subset \{x \in \bar{S}: \bar{q}_1(x) \leq \varepsilon\}.$$

If q is a quasi-norm on S , we can consider the quasi-norm q^S on S defined by $q^S(x) = q(x)$ for $x \in S$. Then, if I' is an ideal on S , we correspond the ideal $I = \{q^S: q \in I'\}$. Then we see that $\bar{I} = I'$ and we have a one-to-one correspondence between ideals on S and ideals on $\bar{S}$.

A linear space S with linear topology $\mathfrak{V}$ is said to be *conditionally complete* if the closure of every set which is bounded by $\mathfrak{V}$ is complete by $\mathfrak{U}^{\mathfrak{V}}$.

THEOREM 5.6. *A linear space S with linear topology $\mathfrak{V}_I$ is conditionally complete if and only if $q(x_{\delta_1} - x_{\delta_2}) \rightarrow_{\delta_1, \delta_2 \in \Delta} 0$ and $\{x_{\delta}: \delta \in \Delta\} \succ q$ for all $q \in B$, a basis of I implies there is $x \in S$ such that $q(x_{\delta} - x) \rightarrow_{\delta \in \Delta} 0$ for all $q \in B$.*

Proof. If S is conditionally complete the result is clear. Conversely, if A is bounded by $\mathfrak{V}_I$ and $A \ni x_{\delta}$ ($\delta \in \Delta$) then $\{x_{\delta}: \delta \in \Delta\} \subset A$ implies $\{x_{\delta}: \delta \in \Delta\} \succ q$ for all $q \in B$. Theorems 5.3 and 5.4 then show that there is $x \in A^-$ such that $x_{\delta} \rightarrow_{\delta \in \Delta} x$.

6. Filters. A system F of quasi-norms is called a *filter* if $F \ni q < q_1$ implies $F \ni q_1$.

Given any system B of quasi-norms if we set $F = \{q: q \succ q_1 \text{ for all } q_1 \in B\}$ then F is a filter. We denote this filter by $B^{\mathfrak{V}}$ and call it the *associated filter of B* . If we set $I = \{q: q < q_1 \text{ for all } q_1 \in B\}$ then we obtain an ideal I . We denote this ideal $B^{\mathfrak{V}}$ and call it the *associated ideal of B* . (The fact that $B^{\mathfrak{V}}$ is an ideal is a consequence of the fact that $q_v < q$ for all $v = 1, 2, \dots$ implies that $\sum_{v=1}^{\infty} q_v^{1/2^v} < q$.)

A filter F (ideal I) is called *reflexive* if $F^{\mathfrak{F}\mathfrak{F}} = F$ ($I^{\mathfrak{F}\mathfrak{F}} = I$.)

THEOREM 6.1. For any system B , $B^{\mathfrak{F}}$ and $B^{\mathfrak{F}\mathfrak{F}}$ are reflexive.

Proof. Note first that, for any system C , $C \subset C^{\mathfrak{F}\mathfrak{F}}$ and $C \subset C^{\mathfrak{F}\mathfrak{F}\mathfrak{F}}$. Thus, $B^{\mathfrak{F}} \subset (B^{\mathfrak{F}})^{\mathfrak{F}\mathfrak{F}}$. But on the other hand, $B \subset B^{\mathfrak{F}\mathfrak{F}}$ implies that $B^{\mathfrak{F}} \supset B^{\mathfrak{F}\mathfrak{F}\mathfrak{F}}$. Thus $B^{\mathfrak{F}} = B^{\mathfrak{F}\mathfrak{F}\mathfrak{F}}$ and a similar argument shows that $B^{\mathfrak{F}} = B^{\mathfrak{F}\mathfrak{F}\mathfrak{F}}$.

A manifold A is said to be *bounded by an ideal I* (we write $A \succ I$) if $A \succ q$ for all $q \in I$.

THEOREM 6.2. If A is a character manifold and $A \succ I$ then there is $q_1 \in I^{\mathfrak{F}}$ such that $A \succ q_1$.

Proof. If we let $q_1 = q_A$, the quasi-norm defined in Theorem 2.9, then Theorem 2.10 shows that $q_1 \in I^{\mathfrak{F}}$ and $A \succ q_1$.

Note that the converse holds even when A is not a character manifold. In fact, since $A \succ q_1 \succ q_2$ implies $A \succ q_2$ we have $A \succ q \in I^{\mathfrak{F}}$ implies $A \succ I$.

An ideal I is said to be of *finite character* if $A \succ I$ implies there exists a character manifold B such that $A \subset B \succ I$.

An ideal I is said to be of *bounded character* if there is a basis B of I and $c > 0$ such that every $q \in B$ is of character c .

THEOREM 6.3. If I is of bounded character, then I is of finite character.

Proof. If $A \succ I$ and I is of bounded character, then there is a basis B of I and $c > 0$ such that every $q \in B$ is of character c . Consider the character $(2c)$ hull of A , $\bar{A}^{2c}$. Then, by Theorem 2.7, $A \succ q$ and q of character c implies that $\bar{A}^{2c} \succ q$. But for any $q_1 \in I$ there is $q_2 \in B$ such that $q_1 \prec q_2$. Then $\bar{A}^{2c} \succ q_2 \succ q_1$ implies $\bar{A}^{2c} \succ q_1$. Therefore, since q_1 was arbitrary $\bar{A}^{2c} \succ I$.

If I_1 and I_2 are ideals, we say that I_1 is *equivalent* to I_2 and write $I_1 \sim I_2$ if, for every character manifold A , $A \succ I_1$ if and only if $A \succ I_2$.

THEOREM 6.4. If I_1 and I_2 are ideals, then $I_1 \sim I_2$ if and only if; for every finite character quasi-norm q , $q \in I_1^{\mathfrak{F}}$ if and only if $q \in I_2^{\mathfrak{F}}$.

Proof. If $I_1 \sim I_2$ and q is of finite character and such that $q \in I_1^{\mathfrak{F}}$ then there is $a > 0$ such that $A = \{x: q(x) < a\}$ is of finite character and furthermore $A \succ q$. Now since $q \in I_1^{\mathfrak{F}}$ we see that $A \succ I_1$ and since $I_1 \sim I_2$ we have $A \succ I_2$. Given any $q_1 \in I_2$ and $\varepsilon > 0$ there is a positive integer n such that $q_1((1/n)A) < \varepsilon$. Let $\delta = a/n$. Then $q(x) < \delta$ implies $q(nx) \leq nq(x) \leq n\delta = a$. Thus, $q(x) < \delta$ implies $nx \in A$ which means $x \in (1/n)A$ and therefore $q_1(x) < \varepsilon$. Thus, $q_1 \prec q$ and since q_1 was arbitrary $q \in I_2^{\mathfrak{F}}$.

For the converse, let A be of finite character and let q_A be the quasi-norm associated with A . We see that $q_A \in I_1^{\mathfrak{F}}$ and q_A is of finite character. Therefore, by assumption $q_A \in I_2^{\mathfrak{F}}$. But then $A \succ q_A$ gives $A \succ I_2$.

Given any ideal I_0 consider $\bar{I}_0 = \bigvee_{I \sim I_0} I$. $\bar{I}_0$ is called the *equivalent hull* of I_0 .

THEOREM 6.5. *If $M = \{q: q \text{ is of finite character and } q \in I_0^{\mathfrak{F}}\}$ then $\bar{I}_0 = M^{\mathfrak{F}}$.*

Proof. Clearly M is never empty since q^* defined in Example 1.1 is of finite character. Theorem 6.4 shows that if $I \sim I_0$, then $M \subset I^{\mathfrak{F}}$. Therefore, $M^{\mathfrak{F}} \supset I^{\mathfrak{F}\mathfrak{F}} \supset I$. Hence, $I \sim I_0$ implies $I \subset M^{\mathfrak{F}}$.

Now suppose I_1 is such that $I \sim I_0$ implies that $I \subset I_1$. For any $q \in M^{\mathfrak{F}}$ consider $(\{q\} \cup I_0)^{\mathfrak{F}\mathfrak{F}}$. Then, if q_1 is of finite character and $q_1 \in I_0^{\mathfrak{F}}$ we see that $q_1 \succ q$ since $q \in M^{\mathfrak{F}}$. Therefore, $q_1 \in (\{q\} \cup I_0)^{\mathfrak{F}} = (\{q\} \cup I_0)^{\mathfrak{F}\mathfrak{F}\mathfrak{F}}$. Conversely, if q_2 is of finite character and $q_2 \in (\{q\} \cup I_0)^{\mathfrak{F}\mathfrak{F}\mathfrak{F}}$, then $(\{q\} \cup I_0)^{\mathfrak{F}\mathfrak{F}\mathfrak{F}} = (\{q\} \cup I_0)^{\mathfrak{F}} \subset I_0^{\mathfrak{F}}$ and we have $q_2 \in I_0^{\mathfrak{F}}$. Thus, we see that $(\{q\} \cup I_0)^{\mathfrak{F}\mathfrak{F}} \sim I_0$. Therefore, $I_1 \supset (\{q\} \cup I_0)^{\mathfrak{F}\mathfrak{F}} \ni q$. Hence, $q \in M^{\mathfrak{F}}$ implies $q \in I_1$ and we have $M^{\mathfrak{F}} \subset I_1$. Therefore $M^{\mathfrak{F}} = \bigvee_{I \sim I_0} I$.

COROLLARY. $\bar{I}_0$ is reflexive.

Proof. $\bar{I}_0^{\mathfrak{F}\mathfrak{F}} = M^{\mathfrak{F}\mathfrak{F}\mathfrak{F}} = M^{\mathfrak{F}} = \bar{I}_0$.

An ideal I is *simple* if $I \ni q_0 \succ q$ for all $q \in I$.

THEOREM 6.6. *An ideal I is simple if and only if $I \cap I^{\mathfrak{F}} \neq \emptyset$.*

Proof. If I is simple $I \ni q_0 \succ q$ for all $q \in I$ implies that $q_0 \in I \cap I^{\mathfrak{F}}$.

If $q_0 \in I \cap I^{\mathfrak{F}}$ then $I \ni q_0 \succ q$ for all $q \in I$ and I is simple.

7. Relative ideals. Let R be a linear manifold contained in S . Given a quasi-norm q on S we define a function q^R on R by setting $q^R(x) = q(x)$ for $x \in R$. It can easily be shown that q^R is a quasi-norm on the linear space R . It is called the *relative quasi-norm of q on R* . For any system B of quasi-norms on S , we let $B^R = \{q^R: q \in B\}$. The following properties are immediate consequences of the definition.

- (1) If q is proper, then q^R is proper.
- (2) If q is of finite character χ then q^R is of character χ .
- (3) $q_1 < q_2$ implies $q_1^R < q_2^R$.

Let p be a quasi-norm on the linear space R . If we set

$$p^\infty(x) = \begin{cases} p(x) & \text{for } x \in R, \\ +\infty & \text{for } x \notin R, \end{cases}$$

we obtain a quasi-norm p^∞ on S which will be called the *maximum extension* of p over S . For any system B of quasi-norms on R we let $B^\infty = \{p^\infty: p \in B\}$.

THEOREM 7.1. *Let q be a quasi-norm on S and let p be a quasi-norm on R . Then $q^R < p$ implies $q < p^\infty$.*

Proof. For any $\varepsilon > 0$ there is $\delta > 0$ such that $p(x) < \delta$ implies $q^R(x) < \varepsilon$ for $x \in R$. But $p^\infty(x) < \delta$ implies $x \in R$ and hence $p(x) = p^\infty(x) < \delta$ gives us $q(x) = q^R(x) < \varepsilon$.

THEOREM 7.2. *If I is an ideal on S , then $I^R = \{q^R: q \in I\}$ is the basis of an ideal $I^{(R)}$ on R . This ideal is called the relative ideal of I on R . If F is a filter on S , then F^R is a filter on R called the relative filter of F on R .*

Proof. For a sequence $q_v^R \in I^R$ ($v = 1, 2, \dots$) we see that $I \ni q_v$ ($v = 1, 2, \dots$) and therefore there is $q_0 \in I$ such that $q_0 > q_v$ ($v = 1, 2, \dots$). Thus, $I^R \ni q_0^R > q_v^R$ ($v = 1, 2, \dots$). This shows that I^R is a basis for an ideal on R .

If F is a filter on S and p is a quasi-norm on R such that $p > q^R$ for some $q \in F$, then $p^\infty > q$ which implies $p^\infty \in F$. But then $p = (p^\infty)^R \in F^R$.

Given a system B of quasi-norms remember that $B^{\mathfrak{J}}(B^{\mathfrak{F}})$ is the ideal (filter) associated with this system.

THEOREM 7.3. *For a system B of quasi-norms on S we have $(B^R)^{\mathfrak{F}} = (B^{\mathfrak{F}})^R$ and $(B^{\mathfrak{J}})^{(R)} \subset (B^R)^{\mathfrak{J}}$.*

Proof. If $p \in (B^{\mathfrak{F}})^R$, then there is $q \in B^{\mathfrak{F}}$ such that $p = q^R$. But $q \in B^{\mathfrak{F}}$ implies that $q > q_1$ for all $q_1 \in B$ and hence $p = q^R > q_1^R$ for all $q_1 \in B$ which is the same as saying $p \in (B^R)^{\mathfrak{F}}$.

Conversely, if $p \in (B^R)^{\mathfrak{F}}$ then $p > q^R$ for all $q \in B$ and consequently $p^\infty > q$ for all $q \in B$. Thus, $p^\infty \in B^{\mathfrak{F}}$ implies $p = (p^\infty)^R \in (B^{\mathfrak{F}})^R$.

If $p \in (B^{\mathfrak{J}})^{(R)}$, then there is $q \in B^{\mathfrak{J}}$ such that $p < q^R$. Since $q < q_1$ for all $q_1 \in B$ we have $p < q^R < q_1^R$ for all $q_1 \in B$ which implies $p \in (B^R)^{\mathfrak{J}}$.

THEOREM 7.4. *If a filter F is reflexive, then the relative filter F^R is also reflexive.*

Proof. If $F = F^{\mathfrak{J}\mathfrak{F}}$ then we have $F^R = (F^{\mathfrak{J}\mathfrak{F}})^R = [(F^{\mathfrak{J}})^R]^{\mathfrak{F}}$. Since $[(F^{\mathfrak{J}})^R]^{\mathfrak{F}}$ is reflexive by Theorem 6.1, we see that F^R is reflexive.

We now consider a generalization of the inductive limit defined in [1].

If $R_\lambda (\lambda \in \Lambda)$ is a system of linear manifolds contained in S such that $S = \bigcup_{\lambda \in \Lambda} R_\lambda$ and such that, for any $\lambda_1, \lambda_2 \in \Lambda$ there is $\lambda_3 \in \Lambda$ such that $R_{\lambda_1} \cup R_{\lambda_2} \subset R_{\lambda_3}$, then we write $R_\lambda \uparrow_{\lambda \in \Lambda} S$ and say that the system $R_\lambda (\lambda \in \Lambda)$ increases to S .

THEOREM 7.5. *If $R_\lambda \uparrow_{\lambda \in \Lambda} S$ and I_λ is a reflexive ideal on R_λ for all $\lambda \in \Lambda$ then $\{q: q^{R_\lambda} \in I_\lambda \text{ for all } \lambda \in \Lambda\} = (\bigcup_{\lambda \in \Lambda} (I_\lambda^{\mathfrak{F}})^\infty)^{\mathfrak{J}}$.*

Proof. Let $I_0 = \{q: q^{R_\lambda} \in I_\lambda \text{ for all } \lambda \in \Lambda\}$. Then, $q \in I_0$ implies $q^{R_\lambda} \in I_\lambda$ for all $\lambda \in \Lambda$. Hence $q^{R_\lambda} < p$ for all $p \in I_\lambda^{\mathfrak{F}}$ and therefore $q < p^\infty$ for all $p \in I_\lambda^{\mathfrak{F}}$. Since this is true for all $\lambda \in \Lambda$ we have $q \in (\bigcup_{\lambda \in \Lambda} (I_\lambda^{\mathfrak{F}})^\infty)^{\mathfrak{J}}$. Conversely, if $q \in (\bigcup_{\lambda \in \Lambda} (I_\lambda^{\mathfrak{F}})^\infty)^{\mathfrak{J}}$, then $q^{R_\lambda} < p^{R_\lambda}$ for all $p \in (I_\lambda^{\mathfrak{F}})^\infty$ and all $\lambda \in \Lambda$. But $p \in I_\lambda^{\mathfrak{F}}$ if and only if $p = (p^\infty)^{R_\lambda}$ where $p^\infty \in (I_\lambda^{\mathfrak{F}})^\infty$. Therefore $I_\lambda^{\mathfrak{F}} = ((I_\lambda^{\mathfrak{F}})^\infty)^{R_\lambda}$ for all $\lambda \in \Lambda$ and hence $q^{R_\lambda} \in I_\lambda^{\mathfrak{F}\mathfrak{J}} = I_\lambda$. Thus, $q \in I_0$ and we have $I_0 = (\bigcup_{\lambda \in \Lambda} (I_\lambda^{\mathfrak{F}})^\infty)^{\mathfrak{J}}$.

Now suppose $R_\lambda \uparrow_{\lambda \in \Lambda} S$ and I_λ is an ideal on R_λ for all $\lambda \in \Lambda$. The *upper limit* of $I_\lambda (\lambda \in \Lambda)$ (written $\limsup_{\lambda \in \Lambda} I_\lambda$) is defined by: $\limsup_{\lambda \in \Lambda} I_\lambda = \bigvee \{I: \forall \lambda_0 \in \Lambda \exists \lambda \in \Lambda \text{ such that } R_{\lambda_0} \subset R_\lambda \text{ and } I^{R_\lambda} \subset I_\lambda\}$. The *lower limit* of $I_\lambda (\lambda \in \Lambda)$ is defined by $\liminf_{\lambda \in \Lambda} I_\lambda = \bigvee \{I: \text{there is } \lambda_0 \in \Lambda \text{ such that } R_\lambda \supset R_{\lambda_0} \text{ implies } I^{R_\lambda} \subset I_\lambda\}$. Since $R_\lambda (\lambda \in \Lambda)$ is an increasing system it is clear that $\liminf_{\lambda \in \Lambda} I_\lambda \subset \limsup_{\lambda \in \Lambda} I_\lambda$. If $\liminf_{\lambda \in \Lambda} I_\lambda = \limsup_{\lambda \in \Lambda} I_\lambda$ then we say that the system $I_\lambda (\lambda \in \Lambda)$ is *convergent* to $\lim_{\lambda \in \Lambda} I_\lambda = \limsup_{\lambda \in \Lambda} I_\lambda = \liminf_{\lambda \in \Lambda} I_\lambda$.

THEOREM 7.6. *If $R_\lambda \supset R_\rho$ implies that $I_\lambda^{R_\rho} \subset I_\rho$ (which implies $I_\lambda^{(R_\rho)} \subset I_\rho$) then $I_\lambda (\lambda \in \Lambda)$ is convergent and $\lim_{\lambda \in \Lambda} I_\lambda = \{q: q^{R_\lambda} \in I_\lambda \text{ for all } \lambda \in \Lambda\}$. Furthermore, if for any $\lambda_0 \in \Lambda$ there is $\lambda \in \Lambda$ such that $R_{\lambda_0} \subset R_\lambda$ and I_λ is reflexive, then $\lim_{\lambda \in \Lambda} I_\lambda$ is also reflexive.*

Proof. Let I be an ideal such that for any $\lambda_0 \in \Lambda$ there is $\lambda \in \Lambda$ such that $R_{\lambda_0} \subset R_\lambda$ and $I^{R_\lambda} \subset I_\lambda$. Then, if $q \in I$ for any $\lambda_0 \in \Lambda$ there is $\lambda \in \Lambda$ such that $q^{R_\lambda} \in I_\lambda$ and $R_\lambda \supset R_{\lambda_0}$. By hypothesis $q^{R_{\lambda_0}} = (q^{R_\lambda})^{R_{\lambda_0}} \in I_\lambda^{R_{\lambda_0}} \subset I_{\lambda_0}$. Therefore, $q \in \{q: q^{R_\lambda} \in I_\lambda \text{ for all } \lambda \in \Lambda\}$. (Call this last set I_0 .) Then $I \subset I_0$ and this shows that $\limsup_{\lambda \in \Lambda} I_\lambda \subset I_0$.

Now suppose $q \in I_0$. Then, for any $\lambda_0 \in \Lambda$ we have $I_0^{R_{\lambda_0}} \subset I_\lambda$ for $R_\lambda \supset R_{\lambda_0}$. Therefore $I_0 \subset \liminf_{\lambda \in \Lambda} I_\lambda$. Thus, we see that $I_\lambda (\lambda \in \Lambda)$ is convergent and $\lim_{\lambda \in \Lambda} I_\lambda = I_0$.

The second part of the theorem corresponds to the assumption that there is a system $\lambda_\rho (\rho \in P)$ such that, for any $\lambda \in \Lambda$, there is $\rho \in P$ such that $R_\lambda \subset R_{\lambda_\rho}$ and I_{λ_ρ} is reflexive. Since $R_\lambda \subset R_{\lambda_\rho}$ implies $q^{R_\lambda} = (q^{R_{\lambda_\rho}})^{R_\lambda}$ and since $I_{\lambda_\rho}^{R_\lambda} \subset I_\lambda$ we see that $\lim_{\lambda \in \Lambda} I_\lambda = \{q: q^{R_\lambda} \in I_\lambda \text{ for all } \lambda \in \Lambda\} = \{q: q^{R_{\lambda_\rho}} \in I_{\lambda_\rho} \text{ for all } \rho \in P\}$. Then, by Theorem 7.5 we have that $\lim_{\lambda \in \Lambda} I_\lambda = (\bigcup_{\rho \in P} (I_{\lambda_\rho}^{\mathfrak{F}})^\alpha)^{\mathfrak{F}}$ which is reflexive.

Now suppose $R_\lambda \uparrow_{\lambda \in \Lambda} S$ and F_λ is a filter on R_λ for all $\lambda \in \Lambda$. The *upper limit* of $F_\lambda (\lambda \in \Lambda)$ is $\limsup_{\lambda \in \Lambda} F_\lambda = \{p: \text{for all } \lambda_0 \in \Lambda \text{ there is } \lambda \in \Lambda \text{ such that } R_{\lambda_0} \subset R_\lambda \text{ and } p^{R_\lambda} \in F_\lambda\}$. The *lower limit* is defined to be: $\liminf_{\lambda \in \Lambda} F_\lambda = \{p: \text{there is } \lambda_0 \in \Lambda \text{ such that } p^{R_\lambda} \in F_\lambda \text{ for all } \lambda \in \Lambda \text{ such that } R_\lambda \supset R_{\lambda_0}\}$. It is clear that both the upper limit and lower limit are filters such that $\liminf_{\lambda \in \Lambda} F_\lambda \subset \limsup_{\lambda \in \Lambda} F_\lambda$. If $\limsup_{\lambda \in \Lambda} F_\lambda = \liminf_{\lambda \in \Lambda} F_\lambda$, then $F_\lambda (\lambda \in \Lambda)$ is said to be *convergent* to $\lim_{\lambda \in \Lambda} F_\lambda$.

THEOREM 7.7. *If $R_\lambda \supset R_\rho$ implies $F_\lambda^{R_\rho} \subset F_\rho$, then $F_\lambda (\lambda \in \Lambda)$ is convergent and $\lim_{\lambda \in \Lambda} F_\lambda = \{p: p^{R_\lambda} \in F_\lambda \text{ for all } \lambda \in \Lambda\}$.*

Proof. In any case it is clear that

$$\{p: p^{R_\lambda} \in F_\lambda \text{ for all } \lambda \in \Lambda\} \subset \liminf_{\lambda \in \Lambda} F_\lambda \subset \limsup_{\lambda \in \Lambda} F_\lambda.$$

Suppose $p \in \limsup_{\lambda \in \Lambda} F_\lambda$. Then, for any $\lambda_0 \in \Lambda$ there is $\lambda \in \Lambda$ such that $R_{\lambda_0} \subset R_\lambda$ and $p^{R_\lambda} \in F_\lambda$. By assumption $F_\lambda^{R_{\lambda_0}} \subset F_{\lambda_0}$. Therefore $p^{R_{\lambda_0}} = (p^{R_\lambda})^{R_{\lambda_0}} \in F_\lambda^{R_{\lambda_0}} \subset F_{\lambda_0}$. This gives us $\limsup_{\lambda \in \Lambda} F_\lambda \subset \{p: p^{R_\lambda} \in F_\lambda \text{ for all } \lambda \in \Lambda\}$.

THEOREM 7.8. *If $F_\lambda^{R_\rho} = F_\rho$ for $R_\rho \subset R_\lambda$, then, setting $\lim_{\lambda \in \Lambda} F_\lambda = F_0$, we have $F_0^{R_\lambda} = F_\lambda$ for $\lambda \in \Lambda$.*

Proof. From the preceding theorem we know that $\lim_{\lambda \in \Lambda} F_\lambda = \{p: p^{R_\lambda} \in F_\lambda \text{ for all } \lambda \in \Lambda\}$. From this it is clear that $F_0^{R_\lambda} \subset F_\lambda$ for all $\lambda \in \Lambda$. If $p \in F_\lambda$, then $p = (p^\infty)^{R_\lambda}$ and we need only show that $p^\infty \in F_0$. For any $\lambda_1 \in \Lambda$ there is $\lambda_2 \in \Lambda$ such that $R_{\lambda_1} \cup R_{\lambda_1} \subset R_{\lambda_2}$. Then by assumption, there is $q \in F_{\lambda_2}$ such that $q^{R_{\lambda_2}} = p$. Then $(p^\infty)^{R_{\lambda_2}} \supset q$ implies that $(p^\infty)^{R_{\lambda_2}} \in F_{\lambda_2}$. But then we see that

$$(p^\infty)^{R_{\lambda_1}} = ((p^\infty)^{R_{\lambda_2}})^{R_{\lambda_1}} \in F_{\lambda_2}^{R_{\lambda_1}} = F_{\lambda_1}.$$

Thus, $p^\infty \in F_0$ and we have $F_0^{R_\lambda} = F_\lambda$.

An ideal I on S is said to be *compatible* to $R_\lambda \uparrow_{\lambda \in \Lambda} S$ if $\lim_{\lambda \in \Lambda} I^{(R_\lambda)} = I$. Note that the required limit always exists by Theorem 7.6 since $(I^{(R_\lambda)})^{R_\rho} = I^{(R_\rho)}$ for $R_\rho \subset R_\lambda$ (which gives $(I^{(R_\lambda)})^{(R_\rho)} = I^{(R_\rho)}$). Also we have $\lim_{\lambda \in \Lambda} I^{(R_\lambda)} = \{q: q^{R_\lambda} \in I^{(R_\lambda)}\} \supset I$ in any case.

THEOREM 7.9. *If $I_\lambda^{R_\rho} \subset I_\rho$ for $R_\rho \subset R_\lambda$, then $\lim_{\lambda \in \Lambda} I_\lambda$ is an ideal which is compatible to $R_\lambda \uparrow_{\lambda \in \Lambda} S$.*

Proof. Let $I_0 = \lim_{\lambda \in \Lambda} I_\lambda = \{q: q^{R_\lambda} \in I_\lambda \text{ for all } \lambda \in \Lambda\}$. Then clearly $I_0^{R_\lambda} \subset I_\lambda$ for all $\lambda \in \Lambda$. Then $\lim_{\lambda \in \Lambda} I_0^{(R_\lambda)} = \{q: q^{R_\lambda} \in I_0^{(R_\lambda)} \text{ for all } \lambda \in \Lambda\} \subset \{q: q^{R_\lambda} \in I_\lambda \text{ for all } \lambda \in \Lambda\} = I_0$. Therefore $\lim_{\lambda \in \Lambda} I_0^{(R_\lambda)} \subset I_0$ and the reverse inclusion is always true.

Given an ideal I on S and a system $R_\lambda \uparrow_{\lambda \in \Lambda} S$ suppose there is an ideal $\hat{I}$ on S such that: (1) $\hat{I} \supset I$; (2) $\hat{I}$ is compatible to $R_\lambda \uparrow_{\lambda \in \Lambda} S$; and (3) $I_1 \supset I$ and I_1 compatible to $R_\lambda \uparrow_{\lambda \in \Lambda} S$ implies $\hat{I} \subset I_1$. Such an ideal is called the *compatible hull* of I .

THEOREM 7.10. *For any ideal I the compatible hull $\hat{I}$ exists and is given by $\hat{I} = \lim_{\lambda \in \Lambda} I^{(R_\lambda)}$.*

Proof. The remark before Theorem 7.9 shows that $I \subset \lim_{\lambda \in \Lambda} I^{(R_\lambda)}$ and Theorem 7.9 shows that $\lim_{\lambda \in \Lambda} I^{(R_\lambda)}$ is compatible. If $I \subset I_1$ is compatible, then we have $I^{R_\lambda} \subset I_1^{R_\lambda}$ and therefore:

$$\begin{aligned} \lim_{\lambda \in \Lambda} I^{(R_\lambda)} &= \{q: q^{R_\lambda} \in I^{R_\lambda} \forall \lambda \in \Lambda\} \subset \{q: q^{R_\lambda} \in I_1^{R_\lambda} \text{ for all } \lambda \in \Lambda\} \\ &= \lim_{\lambda \in \Lambda} I_1^{(R_\lambda)} = I_1. \end{aligned}$$

A filter F is said to be *compatible* to $R_\lambda \uparrow_{\lambda \in \Lambda} S$ if $\lim_{\lambda \in \Lambda} F^{R_\lambda} = F$. As in the case of ideals, this limit always exists and we have $\lim_{\lambda \in \Lambda} F^{R_\lambda} = \{p: p^{R_\lambda} \in F^{R_\lambda} \text{ for all } \lambda \in \Lambda\} \supset F$.

THEOREM 7.11. *If $F_\lambda^{R_\rho} \subset F_\rho$ for $R_\rho \subset R_\lambda$, then $\lim_{\lambda \in \Lambda} F_\lambda$ is compatible to $R_\lambda \uparrow_{\lambda \in \Lambda} S$.*

Proof. Let $F_0 = \lim_{\lambda \in \Lambda} F_\lambda = \{p: p^{R_\lambda} \in F_\lambda \text{ for all } \lambda \in \Lambda\}$. Then $F_0^{R_\lambda} \subset F_\lambda$ for all $\lambda \in \Lambda$ and we have: $\lim_{\lambda \in \Lambda} F_0^{R_\lambda} = \{p: p^{R_\lambda} \in F_0^{R_\lambda} \text{ for all } \lambda \in \Lambda\} \subset \{p: p^{R_\lambda} \in F_\lambda \text{ for all } \lambda \in \Lambda\} = F_0$. Therefore $\lim_{\lambda \in \Lambda} F_0^{R_\lambda} = F_0$.

Given a filter F on S and a system $R_\lambda \uparrow_{\lambda \in \Lambda} S$ suppose there is a filter $\hat{F}$ such that (1) $\hat{F} \supset F$; (2) $\hat{F}$ is compatible to $R_\lambda \uparrow_{\lambda \in \Lambda} S$; and (3) $F \subset F_1$ compatible implies $\hat{F} \subset F_1$. Such a filter $\hat{F}$ is called the *compatible hull* of F .

THEOREM 7.12. *For any filter F the compatible hull $\hat{F}$ exists and is given by $\hat{F} = \lim_{\lambda \in \Lambda} F^{R_\lambda}$.*

Proof. Previous results show that (1) and (2) are satisfied and we need only prove (3). If $F \subset F_1$ compatible, then we have $F^{R_\lambda} \subset F_1^{R_\lambda}$ for all $\lambda \in \Lambda$.

$$\begin{aligned} \lim_{\lambda \in \Lambda} F^{R_\lambda} &= \{q: q^{R_\lambda} \in F^{R_\lambda} \text{ for all } \lambda \in \Lambda\} \supset \{q: q^{R_\lambda} \in F_1^{R_\lambda} \text{ for all } \lambda \in \Lambda\} \\ &= \lim_{\lambda \in \Lambda} F_1^{R_\lambda} = F_1. \end{aligned}$$

THEOREM 7.13. *If $R_\rho \subset R_\lambda$ implies $F_\lambda^{R_\rho} = F_\rho$, then*

$$\left(\bigcup_{\lambda \in \Lambda} F_\lambda^\infty \right)^\wedge = \{p: p^{R_\lambda} \in F_\lambda \text{ for all } \lambda \in \Lambda\} = \lim_{\lambda \in \Lambda} F_\lambda.$$

Proof. We first show that $(\bigcup_{\lambda \in \Lambda} F_\lambda^\infty)^{R_\lambda} = F_\lambda$ for all $\lambda \in \Lambda$. If $p \in \bigcup_{\lambda \in \Lambda} F_\lambda^\infty$, then there is $\lambda_0 \in \Lambda$ such that $p = q^\infty$ for some $q \in F_{\lambda_0}$. For any other $\lambda_1 \in \Lambda$ there is $\lambda_2 \in \Lambda$ such that $R_{\lambda_0} \cup R_{\lambda_1} \subset R_{\lambda_2}$. Then, since $F_{\lambda_2}^{R_{\lambda_0}} = F_{\lambda_0}$, $p^{R_{\lambda_2}} \in F_{\lambda_2}$ and therefore $p^{R_{\lambda_1}} = (p^{R_{\lambda_2}})^{R_{\lambda_1}} \in F_{\lambda_2}^{R_{\lambda_1}} = F_{\lambda_1}$. Thus, we see that $(\bigcup_{\lambda \in \Lambda} F_\lambda^\infty)^{R_\lambda} \subset F_\lambda$. The opposite inclusion follows from the fact that $F_\lambda = (F_\lambda^\infty)^{R_\lambda}$. By the previous theorem we have $(\bigcup_{\lambda \in \Lambda} F_\lambda^\infty)^\wedge = \lim_{\lambda \in \Lambda} (\bigcup_{\lambda \in \Lambda} F_\lambda^\infty)^{R_\lambda} = \lim_{\lambda \in \Lambda} F_\lambda = \{p: p^{R_\lambda} \in F_\lambda \text{ for all } \lambda \in \Lambda\}$.

THEOREM 7.14. *For any ideal I on S and any system $R_\lambda \uparrow_{\lambda \in \Lambda} S$ we have $\hat{I}^\mathfrak{F} \subset (\hat{I}^\mathfrak{F})^\wedge$.*

Proof. $q \in \hat{I}^\mathfrak{F}$ implies $q \in I^\mathfrak{F}$ since $\hat{I} \supset I$. Consequently $q^{R_\lambda} \in I^{\mathfrak{F}R_\lambda}$ for all $\lambda \in \Lambda$. Then $q \in \{p: p^{R_\lambda} \in I^{\mathfrak{F}R_\lambda} \text{ for all } \lambda \in \Lambda\} = (I^\mathfrak{F})^\wedge$. Thus $\hat{I}^\mathfrak{F} \subset (I^\mathfrak{F})^\wedge$.

THEOREM 7.15. *For any system $R_\lambda \uparrow_{\lambda \in \Lambda} S$ we have $\hat{F}^\mathfrak{F} \subset \lim_{\lambda \in \Lambda} F^{R_\lambda \mathfrak{F}}$.*

Proof. $q \in \hat{F}^\mathfrak{F}$ implies $q^{R_\lambda} \in \hat{F}^{\mathfrak{F}R_\lambda} \subset \hat{F}^{R_\lambda \mathfrak{F}} = F^{R_\lambda \mathfrak{F}}$ for all $\lambda \in \Lambda$. (The last relation follows from the fact that: $F^{R_\lambda} \subset \hat{F}^{R_\lambda} = \{p: p^{R_\lambda} \in F^{R_\lambda} \text{ for all } \lambda \in \Lambda\}^{R_\lambda} \subset F^{R_\lambda}$.) $F^{R_\lambda \mathfrak{F}}$ ($\lambda \in \Lambda$) is a convergent system of ideals since $R_\lambda \subset R_\mu$ implies

$$(F^{R_\mu \mathfrak{F}})^{R_\lambda} \subset (F^{R_\mu})^{R_\lambda \mathfrak{F}} = F^{R_\lambda \mathfrak{F}} \text{ for all } \lambda \in \Lambda.$$

$q^{R_\lambda} \in F^{R_\lambda \mathfrak{F}}$ for all $\lambda \in \Lambda$ gives $q \in \{p: p^{R_\lambda} \in F^{R_\lambda \mathfrak{F}} \text{ for all } \lambda \in \Lambda\} = \lim_{\lambda \in \Lambda} F^{R_\lambda \mathfrak{F}}$. Therefore $\hat{F}^\mathfrak{F} \subset \lim_{\lambda \in \Lambda} F^{R_\lambda \mathfrak{F}}$.

8. Finite-dimensional spaces.

THEOREM 8.1. *If S is a finite-dimensional space and q and p are quasi-norms such that q is pure and proper and p is proper, then $q \succ p$.*

Proof. Choose a basis $a_1, a_2, \dots, a_n$ for S and let $m(\sum_{v=1}^n \alpha_v a_v) = \max_{v=1,2,\dots,n} |\alpha_v|$. If we can show that q proper implies $m \succ q$ and that q proper and pure implies $q \succ m$, we will have proved the desired result.

Let q be proper. Then, given $\varepsilon > 0$, for each a_v ($v = 1, 2, \dots, n$) there is $\delta_v > 0$ such that $q(\xi a_v) < \varepsilon/n$ for $|\xi| \leq \delta_v$. Let $\delta = \min_{v=1,2,\dots,n} \delta_v$. Then, when $m(\sum_{v=1}^n \alpha_v a_v) < \delta$, we have $q(\sum_{v=1}^n \alpha_v a_v) \leq \sum_{v=1}^n q(\alpha_v a_v) \leq \sum_{v=1}^n \varepsilon/n = \varepsilon$. Thus $m \succ q$.

Now we want to show that if q is proper and q does not dominate m , then q is not pure. If q is proper and q does not dominate m , then there is $\varepsilon > 0$ such that, for each $\delta > 0$ there is $x \in \{x: q(x) < \delta\}$ satisfying $m(x) \geq \varepsilon$. For $v = 1, 2, \dots$ we can find $x_v \in \{x: q(x) < 1/v\}$ such that $m(x_v) \geq \varepsilon$. By multiplying x_v by a scalar of absolute value less than or equal to one, we can assume that $\varepsilon \leq m(x_v) \leq 2\varepsilon$ for $v = 1, 2, \dots$. If $x_v = \sum_{\mu=1}^n \alpha_{v,\mu} a_\mu$ this implies $|\alpha_{v,\mu}| \leq 2\varepsilon$ for all $v = 1, 2, \dots$ and all $\mu = 1, 2, \dots, n$. We then have n bounded sequences of real numbers and we can choose a subsequence v_ρ ($\rho = 1, 2, \dots$) such that they all converge: $\lim_{\rho \rightarrow \infty} \alpha_{v_\rho, \mu} = \beta_\mu$ ($\mu = 1, 2, \dots, n$). Let $x_0 = \sum_{\mu=1}^n \beta_\mu a_\mu$. Then we see that $m(x_{v_\rho} - x_0) \rightarrow 0$ since $m(x_{v_\rho} - x_0) = \max_{\mu=1,2,\dots,n} |\alpha_{v_\rho, \mu} - \beta_\mu| \rightarrow 0$. The fact that $m(x_0) \geq m(x_{v_\rho}) - m(x_{v_\rho} - x_0) \geq \varepsilon - m(x_{v_\rho} - x_0)$ for all $\rho = 1, 2, \dots$ shows that $m(x_0) \geq \varepsilon > 0$.

Since q is proper we know from the first part of the proof that $m \succ q$. Therefore $m(x_{v_\rho} - x_0) \rightarrow 0$ implies that $q(x_{v_\rho} - x_0) \rightarrow 0$. But then $q(x_0) \leq q(x_{v_\rho}) + q(x_0 - x_{v_\rho})$ for all $\rho = 1, 2, \dots$. Since both of the last terms tend to zero, we see that $q(x_0) = 0$ which shows that q is not pure.

THEOREM 8.2. *If S is a linear space such that every pure and proper quasi-norm on S dominates every proper quasi-norm on S , then S is finite-dimensional.*

Proof. In light of Theorem 8.1 we need only show that if S is infinite-dimensional there is a pure proper quasi-norm on S which does not dominate every proper quasi-norm. Let a_α ($\alpha \in A$) be a Hamel basis for S . Define quasi-norms q and p by $q(\sum_{\alpha \in A} v_\alpha a_\alpha) = \max_{\alpha \in A} |\beta_\alpha|$ and $p(\sum_{\alpha \in A} \beta_\alpha a_\alpha) = \sum_{\alpha \in A} |\beta_\alpha|$. Then q is proper and pure but q does not dominate p . To see this choose $\varepsilon = 1$ and for any $\delta > 0$ choose n such that $n(\delta/2) > 1$. Then, let $\sum_{\alpha \in A} v_\alpha a_\alpha$ be such that $v_\alpha = \delta/2$ for n different values of α and such that $v_\alpha = 0$ for all other $\alpha \in A$. Then we have $q(\sum_{\alpha \in A} v_\alpha a_\alpha) = \delta/2 < \delta$ but $p(\sum_{\alpha \in A} v_\alpha a_\alpha) = n(\delta/2) > 1$. Hence, q does not dominate p .

An ideal I is said to be *pure* if $x \neq 0$ implies that there is $q \in I$ such that $q(x) \neq 0$.

THEOREM 8.3. *If S is finite-dimensional and I is pure, then I contains a quasi-norm q_0 which is pure.*

Proof. If S is one-dimensional and I is pure, then $x \neq 0$ implies there is $q_0 \in I$ such that $q_0(x) \neq 0$. But then $(1/v)q_0(x) \leq q_0((1/v)x)$ shows that $q_0(\alpha x) \neq 0$ for all $\alpha \neq 0$.

To complete the induction argument we assume the result holds when S is of dimension less than or equal to n . If S is $(n+1)$ -dimensional, choose $0 \neq q_1 \in I$ and let $R = \{x: q_1(x) = 0\}$. Since $q_1 \neq 0$ (dimension R) $\leq n$ and the fact that $I^{(R)}$

is pure on R shows that there is $p \in I^{(R)}$ such that p is pure on R by the induction hypothesis. By definition of $I^{(R)}$ there is $q_2 \in I$ such that $q_2^R \succ p$ which implies q_2^R is pure on R . Now let $q_0 = (q_1 + q_2) \in I$. For any $0 \neq z \in S$ we have either $z \in R$ which implies

$$q_0(z) = q_2(z) = q_2^R(z) \neq 0$$

or $z \notin R$ which gives

$$q_0(z) \geq q_1(z) \neq 0.$$

Therefore q_0 is pure.

THEOREM 8.4. *A linear space S is finite-dimensional if and only if there is exactly one pure, proper ideal on the space.*

Proof. If S is finite-dimensional and I is a pure, proper ideal on S then, by the previous theorem I contains a pure, proper quasi-norm q . By Theorem 8.1, if p is any proper quasi-norm on S we have $q \succ p$. Thus $\{q\}$ is a basis for I and I is uniquely defined as the strongest proper ideal on S .

If S is not finite-dimensional, then by Theorem 8.2 there is a pure, proper quasi-norm q and a proper quasi-norm p such that $p \notin \{q\}^3$. But then $p \in \{p + q\}^3$ and we see that $\{q\}^3 \neq \{p + q\}^3$ even though both are pure, proper ideals.

THEOREM 8.5. *On a finite-dimensional space the unique pure, proper ideal has a basis $\{q\}$ where q is a quasi-norm of character 1.*

Proof. Let a_v ($v = 1, 2, \dots, n$) be a basis for the space S and define q by:

$$q(x) = q\left(\sum_{v=1}^n \alpha_v a_v\right) = \max_{v=1, \dots, n} |\alpha_v|.$$

Then q is clearly a pure, proper quasi-norm and therefore $\{q\}^3$ is the unique, pure, proper ideal on S . Since $q(\frac{1}{2}x) = \max_{v=1, 2, \dots, n} |\frac{1}{2}\alpha_v| = \frac{1}{2}q(x)$ we see that q is of character 1.

Now let S be a linear space and let S_λ ($\lambda \in \Lambda$) be a system of finite-dimensional submanifolds of S such that $S_\lambda \uparrow_{\lambda \in \Lambda} S$.

THEOREM 8.6. *If I_1 and I_2 are pure, proper ideals on S then $I_1^{(S_\lambda)} = I_2^{(S_\lambda)}$ for all $\lambda \in \Lambda$.*

Proof. This follows from the previous theorem since $I_1^{(S_\lambda)}$ and $I_2^{(S_\lambda)}$ are pure, proper ideals.

THEOREM 8.7. *There is a unique, pure, proper ideal which is compatible to $S_\lambda \uparrow_{\lambda \in \Lambda} S$. We denote this ideal by I_0 .*

Proof. If I_1 and I_2 are pure, proper and compatible to $S_\lambda \uparrow_{\lambda \in \Lambda} S$ then $I_1 = \lim_{\lambda \in \Lambda} I_1^{S_\lambda} = \lim_{\lambda \in \Lambda} I_2^{S_\lambda} = I_2$. This shows that any such ideal will be unique.

But there does exist such an ideal since, if I_λ is the unique, pure, proper ideal on S_λ , it is clear that $\lim_{\lambda \in \Lambda} I_\lambda$ is pure and proper and compatible with $S_\lambda \uparrow_{\lambda \in \Lambda} S$.

THEOREM 8.8. *For any pure, proper ideal I on S the compatible hull $\hat{I}$ of I equals I_0 .*

Proof. We know $\hat{I} = \lim_{\lambda \in \Lambda} I^{S_\lambda} = \lim_{\lambda \in \Lambda} I_\lambda = I_0$. (I_λ being the unique, pure, proper ideal on S_λ .)

Thus, we see that I_0 is the strongest pure, proper ideal. Since we can always find a system of finite-dimensional submanifolds S_λ ($\lambda \in \Lambda$) such that $S_\lambda \uparrow_{\lambda \in \Lambda} S$ we see that every linear space S has a strongest pure, proper ideal I_0 . Let I_1 be the ideal of all proper quasi-norms on S . Then, since $I_1 \supset I_0$ we see that I_1 is pure and therefore $I_1 = I_0$.

9. Monotone quasi-norms. In the rest of the paper we will always assume that S is a linear lattice. We will make use of the notation $B_x = \{y \in S: |y| \leq |x|\}$ for $x \in S$.

A quasi-norm q on S is said to be *semimonotone* if $B_x \succ q$ for all $x \in S$.

THEOREM 9.1. *If q is semimonotone, then q is proper.*

Proof. $x \in B_x$ implies $0 \leq \lim_{a \rightarrow 0} q(ax) \leq \lim_{a \rightarrow 0} q(aB_x) = 0$.

A quasi-norm q on S is said to be *monotone* if $|x| \leq |y|$ implies $q(x) \leq q(y)$.

THEOREM 9.2. *Let q be a quasi-norm on S . If we define a quasi-norm $\bar{q}$ by $\bar{q}(x) = q(B_x)$, we obtain a monotone quasi-norm. $\bar{q}$ is proper if and only if q is semimonotone.*

Proof. Clearly $\bar{q}(0) = 0$ and $0 \leq \bar{q}(x)$ for all $x \in S$. If $|a| \leq |b|$, then

$$\bar{q}(ax) = q(B_{ax}) \leq q(B_{bx}) = \bar{q}(bx).$$

To prove the triangle inequality, we need the following lemma:

LEMMA. $|z| \leq |x| + |y|$ if and only if we can find z_1 and z_2 such that $z = z_1 + z_2$ while $|z_1| \leq |x|$ and $|z_2| \leq |y|$.

Proof. $|z| \leq |x| + |y|$ implies $0 \leq |z| = z^+ + z^- \leq |x| + |y|$ and hence $0 \leq z^+ \leq |x| + |y|$. We can then write $z^+ = a + b$ where $0 \leq a \leq |x|$ and $0 \leq b \leq |y|$. Similarly $z^- = c + d$ where $0 \leq c \leq |x|$ and $0 \leq d \leq |y|$. Then $z = z^+ - z^- = (a + b) - (c + d) = (a - c) + (b - d)$ where $-|x| \leq a - c \leq |x|$ and $-|y| \leq b - d \leq |y|$. Thus, $|a - c| \leq |x|$ and $|b - d| \leq |y|$ and the lemma is established.

$$\begin{aligned} \bar{q}(x + y) &= q(B_{(x+y)}) = \sup_{|z| \leq |x+y|} q(z) \leq \sup_{|z| \leq |x| + |y|} q(z) \\ &= \sup_{|z_1| \leq |x|; |z_2| \leq |y|} q(z_1 + z_2) \leq \sup_{|z_1| \leq |x|; |z_2| \leq |y|} \{q(z_1) + q(z_2)\} = \bar{q}(x) + \bar{q}(y) \end{aligned}$$

Thus, we see that $\bar{q}$ is a quasi-norm. It is clear that $\bar{q}$ is monotone since if $|x| \leq |y|$ then $B_x \subset B_y$ implies $\bar{q}(x) = q(B_x) \leq q(B_y) = \bar{q}(y)$. It also follows easily that $\bar{q}$ is proper if and only if q is semimonotone.

THEOREM 9.3. *If q_1, q are quasi-norms such that q_1 is monotone and $q < q_1$, then $\bar{q} < q_1$.*

Proof. For any $\varepsilon > 0$ there is $\delta > 0$ such that $q_1(x) < \delta$ implies $q(x) < \varepsilon$. If $q_1(x) < \delta$ and $|y| \leq |x|$ then $q_1(y) \leq q_1(x) < \delta$ implies $q(y) < \varepsilon$. Therefore $\bar{q}(x) = q(B_x) = \sup_{|y| \leq |x|} q(y) \leq \varepsilon$.

THEOREM 9.4. *For a quasi-norm q on S we have $\bar{q} < q$ if and only if for any $\varepsilon > 0$ there is $\delta > 0$ such that $q(x) < \delta$ implies $q(B_x) < \varepsilon$.*

Proof. This is clear by definition of $\bar{q}$.

A quasi-norm q is called *sequentially continuous* if $\text{order-lim}_{v \rightarrow \infty} x_v = 0$ implies $\lim_{v \rightarrow \infty} q(x_v) = 0$.

THEOREM 9.5. *If q, q_1 are quasi-norms such that q_1 is sequentially continuous and $q < q_1$, then q is sequentially continuous.*

Proof. Suppose $\text{order-lim}_{v \rightarrow \infty} x_v = 0$. For $\varepsilon > 0$ there is $\delta > 0$ such that $q_1(x) < \delta$ implies $q(x) < \varepsilon$. For this δ there is v_0 such that $q_1(x) < \delta$ for $v \geq v_0$. But then $q(x_v) < \varepsilon$ for $v \geq v_0$ and therefore $\lim_{v \rightarrow \infty} q(x_v) = 0$.

THEOREM 9.6. *If q is a sequentially continuous quasi-norm and S is Archimedean, then q is semimonotone.*

Proof. Suppose q is not semimonotone. Then there is $x \in S$ and $\varepsilon > 0$ such that $q((1/v)B_x) \geq \varepsilon$ for all $v = 1, 2, \dots$. Therefore, there is a sequence $x_v \in B_x$ ($v = 1, 2, \dots$) such that $q((1/v)x_v) > \varepsilon/2$ for all $v = 1, 2, \dots$. But $x_v \in B_x$ implies that $|x_v| \leq |x|$. Hence $|(1/v)x_v| \leq |(1/v)x|$. This means $\text{order-lim}_{v \rightarrow \infty} (1/v)x_v = 0$ since S is Archimedean. Therefore, we have $\lim_{v \rightarrow \infty} q((1/v)x_v) = 0$ which contradicts $q((1/v)x_v) > \varepsilon/2$ for all $v = 1, 2, \dots$.

Since semimonotone implies proper, we see that every sequentially continuous quasi-norm on an Archimedean space is proper.

A quasi-norm q is *semicontinuous* if $|x_\lambda| \uparrow_{\lambda \in \Lambda} |x|$ implies that $q(x) = \sup_{\lambda \in \Lambda} q(x_\lambda)$. If we let $\Lambda = \{1, 2\}$ and consider $|x_1| \leq |x_2|$, then $q(x_2) = \sup_{\lambda \in \{1, 2\}} q(x_\lambda) \geq q(x_1)$. Therefore, if a quasi-norm is semicontinuous, it is monotone.

A quasi-norm q is *monotone continuous* if it is monotone and $|x_\lambda| \downarrow_{\lambda \in \Lambda} 0$ implies $\inf_{\lambda \in \Lambda} q(x_\lambda) = 0$.

THEOREM 9.7. *If q is monotone continuous, then it is semicontinuous.*

Proof. Suppose $|x_\lambda| \uparrow_{\lambda \in \Lambda} |x|$. Then

$$|x| - |x_\lambda| = ||x| - |x_\lambda|| \downarrow_{\lambda \in \Lambda} 0.$$

Since q is monotone continuous $\inf_{\lambda \in \Lambda} q(|x| - |x_\lambda|) = 0$. But

$$0 \leq q(|x|) - q(|x_\lambda|) = q(x) - q(x_\lambda)$$

since q is monotone. Also, for any quasi-norm it is true that

$$|q(y) - q(z)| \leq q(y - z).$$

Therefore $0 \leq \inf_{\lambda \in \Lambda} \{q(x) - q(x_\lambda)\} \leq \inf_{\lambda \in \Lambda} q(|x| - |x_\lambda|) = 0$. Hence $0 = \inf_{\lambda \in \Lambda} \{q(x) - q(x_\lambda)\} = q(x) - \sup_{\lambda \in \Lambda} q(x_\lambda)$.

THEOREM 9.8. *If a quasi-norm q is monotone continuous, then it is sequentially continuous.*

Proof. If $\text{order-lim}_{v \rightarrow \infty} x_v = 0$, then, by definition, there is a sequence l_v ($v = 1, 2, \dots$) such that $|x_v| \leq l_v \downarrow_{(v=1,2,\dots)} 0$. But then we have $0 \leq q(x_v) \leq q(l_v) \downarrow_{(v=1,2,\dots)} 0$. Hence, $\lim_{v \rightarrow \infty} q(x_v) = 0$.

We obtain a generalization of a theorem proved in [4].

THEOREM 9.9. *If S is a sequentially continuous linear lattice and if there exists a monotone, sequentially continuous pure quasi-norm defined on S , then S is super-universally continuous. (A linear lattice S is super-universally continuous if, for every family $x_\lambda \in S$ ($\lambda \in \Lambda$) satisfying*

- (1) $x_\lambda \geq 0$ for all $\lambda \in \Lambda$,
- (2) $\lambda_1, \lambda_2 \in \Lambda$ then there is $\lambda_3 \in \Lambda$ such that $x_{\lambda_1} \wedge x_{\lambda_2} \geq x_{\lambda_3}$, there exists a sequence $x_{\lambda_v} \in S$ ($v = 1, 2, \dots$) such that $\bigwedge_{\lambda \in \Lambda} x_\lambda$ exists and equals $\bigwedge_{v=1}^\infty x_{\lambda_v}$.

Proof. Let x_λ ($\lambda \in \Lambda$) be such a family of positive elements. Let

$$A = \inf_{\lambda \in \Lambda} \left\{ \sup_{\{\beta: x_\lambda \geq x_\beta\}} q(x_\lambda - x_\beta) \right\}.$$

To prove $A = 0$ we suppose there is $\varepsilon > 0$ such that $A > \varepsilon$ and derive a contradiction. If $A > \varepsilon$ there is a sequence $b_1 \geq b_2 \geq \dots \geq b_v \geq \dots$ with $b_v \in \{x_\lambda: \lambda \in \Lambda\}$ for $v = 1, 2, \dots$ such that $q(b_v - b_{v+1}) > \varepsilon$ for $v = 1, 2, \dots$.

Now let $b_0 = \bigwedge_{v=1}^\infty b_v$ which exists since S is sequentially continuous. Since q is monotone $b_v - b_{v+1} \leq b_v - b_0$ implies $q(b_v - b_{v+1}) \leq q(b_v - b_0)$. Therefore $q(b_v - b_0) > \varepsilon$ for $v = 1, 2, \dots$.

But this contradicts the fact that q is sequentially continuous since $b_v - b_0 \downarrow_{(v=1,2,\dots)} 0$ but $q(b_v - b_0) > \varepsilon$ for $v = 1, 2, \dots$.

Thus, we must have $A = 0$. This allows us to select a sequence of elements $x'_1, x'_2, \dots$ with $x'_v \in \{x_\lambda: \lambda \in \Lambda\}$ for $v = 1, 2, \dots$ such that

$$\sup_{\{\beta: x'_v \geq x_\beta\}} q(x'_v - x_\beta) < \frac{1}{2^v} \quad (v = 1, 2, \dots).$$

Since $\{x_\lambda: \lambda \in \Lambda\}$ is a decreasing system, there exists a sequence $x_1 \geq x_2 \geq \dots \geq x_v \geq \dots$ with $x_v \in \{x_\lambda: \lambda \in \Lambda\}$ for all $v = 1, 2, \dots$ such that

$x'_v \geq x_v$ ($v = 1, 2, \dots$). (To get such a sequence we need only let $x_1 = x'_1$ and choose $x_{v+1} \leq x_1 \wedge x_2 \wedge \dots \wedge x_v \wedge x'_{v+1}$. Then since q is monotone we have:

$$(1) \quad \sup_{\{\beta: x_v \geq x_\beta\}} q(x_v - x_\beta) \leq \sup_{\{\beta: x_v \geq x_\beta\}} q(x'_v - x_\beta) < \frac{1}{2^v}$$

Now let $x_0 = \bigwedge_{v=1}^{\infty} x_v$. Since q is sequentially continuous $q(x_v - x_0) \leq (1/2^v)$ ($v = 1, 2, \dots$).

Given any $\lambda \in \Lambda$ we have:

$$(2) \quad q((x_v \wedge x_\lambda) - (x_0 \wedge x_\lambda)) \leq q(x_v - x_0) \leq \frac{1}{2^v}.$$

Since $\{x_\lambda: \lambda \in \Lambda\}$ is a decreasing system, there is $\beta \in \Lambda$ such that $x_v \wedge x_\lambda \geq x_\beta$. Then, according to (1)

$$q(x_v - (x_v \wedge x_\lambda)) \leq q(x_v - x_\beta) < \frac{1}{2^v}.$$

Hence (2) gives us

$$q(x_v - (x_0 \wedge x_v)) \leq q(x_v - (x_v \wedge x_\lambda)) + q((x_v \wedge x_\lambda) - (x_0 \wedge x_\lambda)) \leq \frac{1}{2^{v-1}}.$$

Therefore, since q is sequentially continuous $q(x_0 - (x_0 \wedge x_\lambda)) = 0$. But then the fact that q is pure implies that $x_0 = x_0 \wedge x_\lambda$ or $x_0 \leq x_\lambda$. Since λ was arbitrary we have $x_0 \leq x_\lambda$ for all $\lambda \in \Lambda$ which gives, us since

$$x_0 = \bigwedge_{v=1}^{\infty} x_v \quad (x_v \in \{x_\lambda: \lambda \in \Lambda\} \text{ for all } v = 1, 2, \dots), \quad x_0 = \bigwedge_{\lambda \in \Lambda} x_\lambda.$$

10. Monotone ideals. An ideal I is said to be *semimonotone* if $B_a \succ I$ for all $a \in S$. The following result follows easily from this definition.

THEOREM 10.1. *I is semimonotone if and only if every quasi-norm in I is semimonotone.*

An ideal I is said to be *monotone* if I has a basis consisting of monotone quasi-norms.

THEOREM 10.2. *If I is monotone and proper, then it is semimonotone.*

Proof. Let $q \in I$. Then there exists $q_1 \in I$ such that q_1 is monotone and $q < q_1$. Then, for $a \in S$, B_a is bounded by q_1 since $q_1(B_a) = q_1(a)$ and q_1 is proper. $B_a \succ q_1 \succ q$ shows that $B_a \succ q$.

For any manifold $A \subset S$ we let $B_A = \bigcup_{a \in A} B_a = \{x: |x| \leq |a| \text{ for some } a \in A\}$.

THEOREM 10.3. *An ideal I on a linear lattice S is monotone if and only if for each $q_1 \in I$ and $\delta_1 > 0$ there is $q_2 \in I$ and $\delta_2 > 0$ such that $B_{\{x: q_2(x) < \delta_2\}} \subset \{x: q_1(x) < \delta_1\}$. (In the case when I is proper this is equivalent to the requirement that, in the corresponding linear topology, there is a basis $\mathfrak{B}$ of neighborhoods of zero such that $V \in \mathfrak{B}$ implies $B_V = V$.)*

Proof. If I is monotone, consider any $q_1 \in I$. Then, by assumption, there is a monotone $q_2 \in I$ such that $q_1 < q_2$. For any $\delta_1 > 0$ there is a $\delta_2 > 0$ such that $q_2(x) < \delta_2$ implies $q_1(x) < \delta_1$. Let $A = \{x: q_2(x) < \delta_2\}$. Then $B_A = A$ since, if $x \in A$ and $|y| \leq |x|$, then $q_2(y) \leq q_2(x) < \delta_2$. Hence

$$B_{\{x: q_2(x) < \delta_2\}} = \{x: q_2(x) < \delta_2\} \subset \{x: q_1(x) < \delta_1\}.$$

For the converse we assume the condition is satisfied and show that I is monotone. For $q \in I$ let $\bar{q}$ be defined as before by $\bar{q}(x) = q(B_x)$. In Theorem 8.2 it was shown that $\bar{q}$ is a monotone quasi-norm. It is clear that $\bar{q}(x) \geq q(x)$ for all $x \in S$. If we can show that $\bar{q} \in I$ for all $q \in I$, then $\{\bar{q}: q \in I\}$ will be a basis of monotone quasi-norms.

For $q \in I$ and $\varepsilon_n = 1/n$ there is a $q_n \in I$ and $\delta_n > 0$ such that $q_n(x) < \delta_n$ and $|y| \leq |x|$ imply $q(y) < 1/n$.

Therefore $q_n(x) < \delta_n$ implies $\bar{q}(x) = q(B_x) \leq 1/n$. Since I is an ideal, there is $p \in I$ such that $p > q_n$ for all $n = 1, 2, \dots$. Then, for any $\varepsilon > 0$, there is a positive integer n_0 such that $1/n_0 < \varepsilon$. We know $q_{n_0}(x) < \delta_{n_0}$ implies $\bar{q}(x) \leq 1/n_0$. For this δ_{n_0} there is $\delta > 0$ such that $p(x) < \delta$ implies $q_{n_0}(x) < \delta_{n_0}$. Therefore $p(x) < \delta$ implies $\bar{q}(x) < \varepsilon$. Hence $\bar{q} < p \in I$ and this implies $\bar{q} \in I$.

If q_1 and q_2 are quasi-norms such that $q_1 < q_2$, then $\bar{q}_1 < \bar{q}_2$. This follows by Theorem 8.3 since $\bar{q}_2 > q_2 > q_1$ and $\bar{q}_2$ is monotone. Given an ideal I consider $\{\bar{q}: q \in I\}$. This system of quasi-norms satisfies the basis condition since, if $\bar{q}_v$ ($v = 1, 2, \dots$) is a sequence from $\{\bar{q}: q \in I\}$, then there is a $q \in I$ such that $q > q_v$ ($v = 1, 2, \dots$) which implies $\bar{q} > \bar{q}_v$ ($v = 1, 2, \dots$). The ideal $\bar{I}$ which has $\{\bar{q}: q \in I\}$ for a basis is called the *monotone hull* of I .

THEOREM 10.4. *If I is an ideal contained in a monotone ideal I_1 , then $\bar{I} \subset I_1$. (It follows from this that $\bar{I} = \bigwedge_{I \in I_1 \text{ monotone}} I_1$.)*

Proof. If $q \in I$, then there exists a monotone $q_1 \in I$ such that $q < q_1$. Thus, we have $\bar{q} < q_1$ which implies $\bar{q} \in I_1$. Therefore $\bar{I} \subset I_1$.

For any $a \in S$, B_a is of character 1. Obviously B_a is star and symmetric and, for $x, y \in B_a$, $\lambda + \mu \leq 1$, $\lambda, \mu \geq 0$, we have

$$|\lambda x + \mu y| \leq |\lambda x| + |\mu y| \leq |\lambda + \mu| |a| \leq |a|.$$

If I is any semimonotone ideal, we know that $B_a > I$ for all $a \in S$. In other words, $B_a > q$ for all $a \in S$ and $q \in I$. But if q_{B_a} is the quasi-norm associated with the finite character set B_a , Theorem 2.10 shows that $q_B > q$ for all $q \in I$

and all $a \in S$. In other words, $q_{B_a} \in I^{\mathfrak{F}}$ for all $a \in S$. Now let $F = \{q: q \succ q_{B_a} \text{ for some } a \in S\}$. Then we see that $F \subset I^{\mathfrak{F}}$ implies $F^{\mathfrak{F}} \supset I^{\mathfrak{F}\mathfrak{F}} \supset I$ for any semimonotone ideal I . If $q \in F^{\mathfrak{F}}$, then $q < q_{B_a}$ for all $a \in S$ which implies $B_a \succ q$ for all $a \in S$ and this means q is semimonotone. Therefore $F^{\mathfrak{F}}$ is semimonotone itself and it is clearly the strongest semimonotone ideal. Since B_a is of finite character for every $a \in S$ we see that $I_1 \sim I_2$ and I_1 semimonotone implies I_2 is semimonotone. Then we conclude that $I \sim F^{\mathfrak{F}}$ implies that I is semimonotone which means $I \subset F^{\mathfrak{F}}$. Thus, $F^{\mathfrak{F}}$ is equivalently strongest (i.e. $F^{\mathfrak{F}} = \bigvee_{I \sim F^{\mathfrak{F}}} I$). Also $F^{\mathfrak{F}}$ is clearly reflexive.

THEOREM 10.5. *If I is a semimonotone ideal, then the equivalent hull of I and the reflexive extension of I are also semimonotone.*

Proof. If $I_1 \sim I$, then I_1 is semimonotone which implies $I_1 \subset F^{\mathfrak{F}}$. Therefore $\bigvee_{I_1 \sim I} I_1 \subset F^{\mathfrak{F}}$ which shows that the equivalent hull is semimonotone. Similarly $I \subset F^{\mathfrak{F}}$ gives $I^{\mathfrak{F}\mathfrak{F}} \subset F^{\mathfrak{F}\mathfrak{F}\mathfrak{F}} = F^{\mathfrak{F}}$ and this shows that $I^{\mathfrak{F}\mathfrak{F}}$ is semimonotone.

An ideal I is said to be *sequentially continuous* if I contains a basis consisting of sequentially continuous quasi-norms. Theorem 9.5 then shows that in a sequentially continuous ideal every quasi-norm is sequentially continuous.

THEOREM 10.6. *If I is sequentially continuous and S is Archimedean, then I is semimonotone.*

Proof. This follows easily from Theorem 9.6.

We say that an ideal is *semicontinuous* if it contains a basis consisting of semicontinuous quasi-norms.

Using the terminology in [5] we have:

THEOREM 10.7. *If q is a semicontinuous quasi-norm on a universally continuous linear lattice S , there is a unique normal manifold $N \subset S$ such that q is pure in N and $q(x) = 0$ for all $x \in N^{\perp}$.*

Proof. Let $M = \{x: q(x) = 0\}$. Then it is easily seen that M is a linear manifold. Since q is monotone, $|x| \leq |y|$ and $y \in M$ implies $x \in M$. Therefore M is a seminormal manifold. If $x_{\lambda} \uparrow_{\lambda \in \Lambda} x$ and $x_{\lambda} \in M$ for all $\lambda \in \Lambda$, then $q(x) = \sup_{\lambda \in \Lambda} q(x_{\lambda}) = 0$ by assumption and $x \in M$. Hence, by a theorem in linear lattices M is normal. Clearly $q(x) = 0$ for all $x \in M$. Also $q(x) = 0$ implies that $x \in M$ which means $[M^{\perp}]x = 0$. Therefore q is pure in $M^{\perp}$. To show that M is unique, we suppose q is pure in a normal manifold P and $q(x) = 0$ for all $x \in P^{\perp}$. Then clearly $P^{\perp} = M$.

THEOREM 10.8. *Given any system of semicontinuous quasi-norms q_{λ} ($\lambda \in \Lambda$) on a universally continuous linear lattice S there is a unique normal manifold $N \subset S$ such that q_{λ} ($\lambda \in \Lambda$) satisfies $q_{\lambda}(x) = 0$ for all $\lambda \in \Lambda$ if and only if $x \in N$.*

Proof. Let $N_\lambda = \{x: q_\lambda(x) = 0\}$ and let $N = \bigcap_{\lambda \in \Lambda} N_\lambda$. Then N is the intersection of normal manifolds and is therefore normal itself. If $x \neq 0$ and $x \in N^\perp$, then by definition of N there is $\lambda \in \Lambda$ such that $q_\lambda(x) \neq 0$.

THEOREM 10.9. *Let S be a sequentially continuous linear lattice. Suppose there is a pure, proper, semicontinuous quasi-norm q defined on S . Then, for all $a \in S$, B_a is complete by the uniformity induced by $\{q\}^3$.*

Proof. Let $B_a \ni x_\delta$ ($\delta \in \Delta$) be a Cauchy system. Then, by Theorem 5.3, for every positive integer v there is $\delta'_v \in \Delta$ such that $q(x_{\delta_1} - x_{\delta_2}) \leq 1/2^v$ for all $\delta_1, \delta_2 \leq \delta'_v$. Since Δ is a directed system we can define by induction a sequence $\delta_v \in \Delta$ ($v = 1, 2, \dots$) such that $\delta_v \leq \delta_v, \delta'_{v+1}$. For ease of notation, we write $a_v = x_{\delta_v}$ ($v = 1, 2, \dots$). Now consider

$$q\left(\sum_{v=\mu}^{\sigma} |a_{v+1} - a_v|\right) \leq \sum_{v=\mu}^{\sigma} q(a_{v+1} - a_v) \leq \sum_{v=\mu}^{\sigma} \frac{1}{2^v},$$

$$0 \leq \bigvee_{v=\mu}^{\sigma} a_v - a_\mu = \bigvee_{v=\mu}^{\sigma} (a_v - a_\mu) \leq \sum_{v=\mu}^{\sigma} |a_{v+1} - a_v|.$$

This last follows since, for $\mu \leq v \leq \sigma$, we have

$$a_v - a_\mu \leq |a_v - a_\mu| = |a_v - a_{v-1} + a_{v-1} - a_{v-2} + \dots + a_{\mu+1} - a_\mu|$$

$$\leq \sum_{v=\mu}^{\sigma} |a_{v+1} - a_v|.$$

Then, since q is monotone,

$$q\left(\bigvee_{v=\mu}^{\sigma} a_v - a_\mu\right) \leq q\left(\sum_{v=\mu}^{\sigma} |a_{v+1} - a_v|\right) \leq \sum_{v=\mu}^{\sigma} \frac{1}{2^v} \leq \frac{1}{2^{\mu-1}}.$$

Therefore, since S is sequentially continuous, $|x_\delta| \leq |a|$ ($\delta \in \Delta$) and q is semi-continuous;

$$q\left(\bigvee_{v=\mu}^{\infty} a_v - a_\mu\right) = \sup_{\sigma=\mu, \mu+1, \dots} q\left(\bigvee_{v=\mu}^{\sigma} a_v - a_\mu\right) \leq \frac{1}{2^{\mu-1}}.$$

Similarly we find $q(a_\mu - \bigwedge_{v=\mu}^{\infty} a_v) \leq 1/2^{\mu-1}$.

Thus $q(\bigvee_{v=\mu}^{\infty} a_v - \bigwedge_{v=\mu}^{\infty} a_v) \leq 1/2^{\mu-2}$.

Now let $l_\mu = \bigvee_{v=\mu}^{\infty} a_v - \bigwedge_{v=\mu}^{\infty} a_v$ and set $l = \bigwedge_{\mu=1}^{\infty} l_\mu$.

Then $0 \leq l \leq l_\mu$ implies $q(l) \leq q(l_\mu) \leq 1/2^{\mu-2}$. Since this is true for all $\mu = 1, 2, \dots$ we see that $q(l) = 0$ which means, since q is pure, that $l = 0$. Therefore there is $a_0 \in S$ such that $\text{order-lim}_{v \rightarrow \infty} a_v = a_0$.

Given any $\varepsilon > 0$ since $\bigvee_{v=\mu}^{\infty} a_v \geq a_0 \geq \bigwedge_{v=\mu}^{\infty} a_v$ we have:

$$|a_0 - a_\mu| \leq \left(\bigvee_{\nu=\mu}^{\infty} a_\nu - a_\mu \right) \vee \left(a_\mu - \bigwedge_{\nu=\mu}^{\infty} a_\nu \right) \leq l_\mu$$

which implies $q(a_0 - a_\mu) \leq 1/2^{\mu-2}$. Now choose μ_0 such that $5/2^{\mu_0} < \varepsilon$. Then, for $\delta \leq \delta_{\mu_0}$, we have

$$\begin{aligned} q(a_0 - x_\delta) &\leq q(a_0 - x_{\delta_{\mu_0}}) + q(x_{\delta_{\mu_0}} - x_\delta) \\ &= q(a_0 - a_{\mu_0}) + q(x_{\delta_{\mu_0}} - x_\delta) \leq 1/2^{\mu_0-2} + 1/2^{\mu_0} = 5/2^{\mu_0} < \varepsilon. \end{aligned}$$

Therefore $x_\delta \rightarrow_{\delta \in \Delta} a_0$.

As a generalization of Theorem 3.4 in [6], we have:

THEOREM 10.10. *Let S be a universally continuous linear lattice. Let I be a proper semicontinuous ideal defined on S . Then, for any $a \in S$, B_a is complete by $\mathfrak{U}^3$.*

Proof. Suppose x_δ ($\delta \in \Delta$) is a Cauchy system in B_a . Let L be a basis for I composed of semicontinuous quasi-norms and choose $q \in L$. Then, by Theorem 9.6, there is a unique normal manifold N_q of S such that q^{N_q} is pure in N_q and $q(x) = 0$ for all $x \in N_q^\perp$. Then q^{N_q} is a semicontinuous, pure quasi-norm on N_q .

$$\begin{aligned} q([N_q](x_{\delta_1} - x_{\delta_2})) &= q(|[N_q](x_{\delta_1} - x_{\delta_2})|) \leq q(|x_{\delta_1} - x_{\delta_2}|) \\ &= q(x_{\delta_1} - x_{\delta_2}) \rightarrow_{\delta_1, \delta_2 \in \Delta} 0. \end{aligned}$$

This shows that $[N_q]x_\delta$ ($\delta \in \Delta$) is a Cauchy system in N_q by the uniformity induced by q^{N_q} . Thus, by the previous theorem there is an $x_q \in N_q$ such that $[N_q]x_\delta \rightarrow_{\delta \in \Delta} x_q$.

Now let $q_1, q_2 \in L$.

$$\begin{aligned} [N_{q_1}][N_{q_2}]x_{q_1} &= [N_{q_1}][N_{q_2}] \lim_{\delta \in \Delta} [N_{q_1}]x_\delta \\ &= [N_{q_1}] \lim_{\delta \in \Delta} [N_{q_2}][N_{q_1}]x_\delta \\ &= [N_{q_1}][N_{q_1}] \lim_{\delta \in \Delta} [N_{q_2}]x_\delta \\ &= [N_{q_1}]x_{q_2} = [N_{q_1}][N_{q_2}]x_{q_2}. \end{aligned}$$

Since S is universally continuous and since $|x_q| \leq a$ for all $q \in L$ we see that $x_1 = \bigvee_{q \in L} x_q^+$ and $x_2 = \bigvee_{q \in L} x_q^-$ both exist.

$$\begin{aligned} [N_p]x_1 &= [N_p] \bigvee_{q \in L} x_q^+ = [N_p] \bigvee_{q \in L} [N_q]x_q^+ = \bigvee_{q \in L} [N_p][N_q]x_q^+ \\ &= \bigvee_{q \in L} [N_p][N_q]x_p^+ = [N_p]x_p^+ = x_p^+. \end{aligned}$$

Similarly $[N_p]x_2 = x_p^-$ and therefore, if $x_0 = x_1 - x_2$ we have $[N_p]x_0 = x_p^+ - x_p^- = x_p$.

Then $x_\delta \rightarrow_{\delta \in \Delta} x_0$ since, for any $\varepsilon > 0$, and for any $q \in L$ there is $\delta_0 \in \Delta$ such that

$$q([N_q]x_\delta - x_0) = q([N_q](x_\delta - x_0)) < \varepsilon \text{ for } \delta \leq \delta_0.$$

But $x_\delta - x_0 = [N_q](x - x_0) + [N_q^\perp](x_\delta - x_0)$ and therefore

$$q(x_\delta - x_0) \leq q([N_q](x_\delta - x_0)) + q([N_q^\perp](x_\delta - x_0)) \leq \varepsilon + 0 = \varepsilon$$

for all $\delta \leq \delta_0$.

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