REGULAR MINIMAL SETS. I(1)

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Introduction. A central problem in topological dynamics is the classification of minimal sets. In full generality, this problem appears to be quite difficult. Therefore, it seems appropriate to single out certain minimal sets which are "well behaved" in some sense, and are hopefully more amenable to classification.

It is for this purpose that we introduce the regular minimal sets in this paper They are the universal minimal sets for certain ("admissible") properties. Several characterizations are given (Theorem 3); the class of regular minimal sets is shown to coincide with the minimal right ideals of enveloping semigroups of transformation groups, and also with those minimal sets having a maximal supply of endomorphisms.

If T is a topological group, the class of regular minimal sets with phase group T forms a partially ordered set $\mathscr{R}(T)$, in which the partial ordering is defined by the existence of a homomorphism. It is shown (Theorem 5) that $\mathscr{R}(T)$ is a complete lattice.

In this paper, certain aspects of the general theory of regular minimal sets are developed. A second paper, which will be written jointly with Brindell Horelick will study in detail the regular minimal sets (X,T), with T discrete abelian, which are proximally equicontinuous (see [1, §6]). A structure theorem for these minimal sets will be given, and a number of examples constructed.

1. **Universal minimal sets.** We recall that a *minimal set* is a transformation group (X,T), with the phase space X compact Hausdorff such that for each $x \in X$, the orbit closure $\mathrm{Cl}(xT) = X$. If T is a topological group, a *universal* minimal set for T is a minimal set (M,T) such that every minimal set with phase group T is a homomorphic image of (M,T). (That is, if (X,T) is minimal, there is a continuous map $\pi: M \to X$ such that $\pi(xt) = \pi(x)t$, for $x \in X$ and $t \in T$.) For any group T, a universal minimal set exists and is unique up to isomorphism. This is proved in [6] (see also [3]). However, we prove it again here, since the ideas involved are used again in a later proof (Theorem 2), and because our proof of uniqueness differs from the proof in [6].

The existence of universal minimal sets is straightforward. Consider the class $\mathscr{M} = \{(X_\alpha, T)/\alpha \in \mathscr{A}\}$ of all minimal sets with phase group T. No logical dif-

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ficulties arise, since each X_α has a dense subspace with cardinality less than or equal to the cardinality of T. Let $X^* = \times_\alpha X_\alpha$, and let M be any minimal set in the transformation group (X^*, T). (The transformation group (X^*, T) is defined by coordinatewise action of T; that is, $(x_\alpha)t = (x_\alpha t)$ $(t \in T, x_\alpha \in X_\alpha)$.) If $\pi_\alpha : X^* \to X_\alpha$ is the αth projection, then $\pi_\alpha | M$ is a homomorphism from (M, T) to (X_α, T).

The proof of uniqueness is somewhat more difficult. It is sufficient to find a coalescent minimal set (Z, T) such that (M, T) is a homomorphic image of (Z, T). (A minimal set is said to be *coalescent* if every endomorphism is an automorphism, [1].) Indeed, suppose such a (Z, T) exists, and let (M', T) be any universal minimal set. Then there are homomorphisms $\alpha : (Z, T) \to (M, T)$, $\beta : (M, T) \to (M', T)$, and $\gamma : (M', T) \to (Z, T)$; therefore $\gamma\beta\alpha$ is an endomorphism of (Z, T). Since (Z, T), is coalescent, $\gamma\beta\alpha$ is an automorphism, and γ is an isomorphism. That is, every universal minimal set is isomorphic with (Z, T). This also shows that the universal minimal set is coalescent.

Ellis [6] observes that if I is a minimal right ideal in the enveloping semigroup of (M, T), then (I, T), (regarded as a minimal set) satisfies the above requirements. We present here another proof, which, although somewhat longer than Ellis' proof, is more elementary, since it does not use the enveloping semigroup. More generally, we prove:

THEOREM 1. *Let (X, T) be a minimal set. Then there is a cardinal number a and a minimal subset M of (X^a, T) such that (M, T) is coalescent.*

Proof. Let (Y, T) be any transformation group with Y compact Hausdorff. Recall that $y \in Y$ is an *almost periodic point* if $(\mathrm{Cl}(yT), T)$ is a minimal set. Now, let (X, T) be minimal, and let C be a subset of X. We say that C is an *almost periodic set* provided that, whenever D is a set whose cardinality is equal to that of C, and $y \in X^D$ with range $y = C$, then y is an almost periodic point of the transformation group (X^D, T). (It is clear that this definition is independent of the choice of D and y, and thus depends only on C.) Let $\mathscr{C}$ denote the family of almost periodic sets in X.

Partially order $\mathscr{C}$ by set inclusion. Since a neighborhood in a product space depends on only finitely many coordinates, $\mathscr{C}$ is of finite character, and we may apply Zorn's lemma to obtain a maximal element C of $\mathscr{C}$. Let $y \in X^C$ such that range $y = C$ and such that y is one to one, (for example $y_c = c$, for each $c \in C$).

Let $M = \mathrm{Cl}(yT)$, and let $y' \in M$. We show that $C' =$ range y' is a maximal almost periodic set and that y' is one to one. Since y' is an almost periodic point, $C' \in \mathscr{C}$. Suppose $\mathscr{C}'$ is not maximal. Then there is an $x' \in X - C'$ such that (y', x') is an almost periodic point of $(M \times X, T)$. Let $\{t_n\}$ be a net in T such that $y't_n \to y$. By choosing a subnet if necessary, we may suppose $x't_n \to x \in X$. Then (y, x) is an almost periodic point of $M \times X$, and by the maximality of $C =$ range y, $x \in C$. That is, x occurs as one of the coordinates y_c of y. Now, let $\{s_n\}$ be a net in T such that $(y,x)s_n \to (y',x')$. Then $xs_n = y_c s_n$, and therefore $x' = y'_c \in C'$. This is a contradiction, and therefore C' is maximal. The proof that y' is one to one is similar.

Now, let ϕ be an endomorphism of (M,T). Since $\mathrm{Cl}((y,\phi(y))T) = \{(y', \phi(y')) \mid y' \in M\}$, is isomorphic with $\mathrm{Cl}(yT) = Y$, it follows that $(y,\phi(y))$ is an almost periodic point of $(M \times M, T)$. But range y and range $\phi(y)$ are both maximal elements of $\mathscr{C}$, and hence range y = range $\phi(y)$.

Let γ be a permutation of C such that, if γ^* denotes the induced automorphism of (X^C, T), (that is, the map defined by $\gamma^*((y_c)) = (y_{\gamma(c)})$), then $\gamma^*(y) = \phi(y)$. Since $\phi(y) \in M$, $\gamma^*(M) \cap M \neq \Phi$, hence $\gamma^*(M) = M$. Since $\gamma^*(y) = \phi(y)$, $\phi = \gamma^* \mid M$ and ϕ is an automorphism. The proof is completed.

2. **Admissible properties and regular minimal sets.** The proofs just given may be adapted to obtain a large class of minimal sets. These are minimal sets which are universal for certain properties. To be precise, let T be a topological group, and let $\mathscr{P}$ be a property of transformation groups for which:

(i) there is at least one minimal set (X,T) satisfying $\mathscr{P}$.

(ii) if (X_α, T) $(\alpha \in \mathscr{A})$ is a collection of minimal sets, each of which satisfies $\mathscr{P}$, and if M is a minimal subset of the product transformation group $(\times_{\alpha \in \mathscr{A}} X_\alpha, T)$ then (M,T) satisfies $\mathscr{P}$.

A property $\mathscr{P}$ satisfying (i) and (ii) will be called an *admissible property for* T. Many of the properties studied in topological dynamics (for example "equicontinuous", "distal", "proximal is an equivalence relation"), are admissible in this sense. See Corollary 9 in [4], where several admissible properties are listed.

If $\mathscr{P}$ is an admissible property for T, then a $\mathscr{P}$ *universal minimal set* is a minimal set (X,T) satisfying $\mathscr{P}$, such that any minimal set for which $\mathscr{P}$ holds is a homomorphic image of (X,T).

The proof of the following theorem is identical with the proof given in the preceding section of the existence and uniqueness of universal minimal sets.

THEOREM 2. *Let $\mathscr{P}$ be an admissible property for T. Then there is a coalescent (and therefore unique up to isomorphism) $\mathscr{P}$ universal minimal set (X,T).*

The next theorem shows that the class of $\mathscr{P}$ universal minimal sets may be characterized in several ways. Some of these have to do with the enveloping semigroup of a transformation group and its minimal right ideals. If T is regarded as a subset of X^X, and E is the closure of T, then E is compact and is a semigroup under composition of functions. In particular, (E,T) is a transformation group. The minimal subsets of (E,T) coincide with the minimal right ideals of the semigroup E. For further information on the enveloping semigroup, see [5], [7], [1].

THEOREM 3. *Let (X,T) be a minimal set. Then the following are pairwise equivalent.*

(1) *(X,T) is the $\mathscr{P}$ universal minimal set for some admissible property $\mathscr{P}$.*

(2) *If a is any cardinal number, and M is a minimal subset of (X^a,T), then (M,T) and (X,T) are isomorphic.*

(3) *If I is a minimal right ideal contained in the enveloping semigroup E of (X,T), then the minimal sets (X,T) and (I,T) are isomorphic.*

(4) *(X,T) is isomorphic with (I,T), where I is a minimal right ideal in the enveloping semigroup of some transformation group (Z,T).*

(5) *If $x,y\in X$, then there is an endomorphism ϕ of (X,T) such that $\phi(x)$ and y are proximal. (That is, if α is a member of the uniformity of X, there is a $t\in T$ such that $(\phi(x)t,yt)\in\alpha$.)*

(6) *If (x,y) is an almost periodic point of $(X\times X,T)$, then there is an endomorphism ϕ of (X,T) such that $\phi(x)=y$.*

Proof. $(1)\Rightarrow(2)$ since, as observed earlier, $\mathscr{P}$ universal minimal sets are coalescent, and (X,T) is a homomorphic image of (M,T). That (3) and (5) are equivalent is proved in [1, Theorem 4]. $(3)\Rightarrow(4)$ is obvious, and, since minimal right ideals in E are minimal subsets of X^X, [5 Lemma 1], (3) follows from (2).

$(2)\Rightarrow(1)$: Let $\mathscr{P}$ be the property of being a homomorphic image of (X,T). Condition (i) in the definition of admissible property is certainly satisfied, since (X,T) is a homomorphic image of itself. Let $(Z_\beta,\ T)$ $(\beta\in\mathscr{B})$ be a family of minimal sets satisfying $\mathscr{P}$, and let M be a minimal set in the transformation group $(\times_{\beta\in\mathscr{B}}Z_\beta\ ,T)$. For each $\beta\in\mathscr{B}$, let $X_\beta=X$. Let $\lambda_\beta:X_\beta\to Z_\beta$ be homomorphisms, and let $\lambda:\times_{\beta\in\mathscr{B}}X_\beta\to\times_{\beta\in\mathscr{B}}Z_\beta$ be the induced homomorphism. Now, let M^* be a minimal set contained in $\lambda^{-1}(M)$. By assumption M^* is isomorphic with X, so M is a homomorphic image of X. The proof is completed.

$(3)\Rightarrow(2)$: Let $\eta:I\to X$ be an isomorphism, and let u be an idempotent in I. If $\eta(u)=x_0$, it follows immediately that $x_0u=x_0$. Then, if $q\in I$, $\eta(q)=\eta(uq)=x_0q$.

Since η is an isomorphism, this tells us that if q and q' are distinct elements of I, then $x_0q\neq x_0q'$. Now, let M be a minimal set in (X^a,T) and let $m\in M$ such that, for some coordinate projection $\pi:X^a\to X$, $\pi(m)=x_0$, Let m', $m''\in M$, with $m'\neq m''$. Since $mI=M$, there are $q',q''\in I$ with $q'\neq q''$ such that $m'=mq'$ and $m''=mq''$. Now $\pi(m')=\pi(mq')=\pi(m)q'=x_0q'$, and similarly $\pi(m'')=x_0q''$. By the above remark, $x_0q'\neq x_0q''$, and π (restricted to M) is one to one. Hence, (M,T) and (X,T) are isomorphic.

$(5)\Rightarrow(6)$: Let (x,y) be an almost periodic point of $(X\times X,T)$, and let ϕ be an endomorphism of (X,T) such that $y'=\phi(y)$ is proximal with y. Then there is a minimal right ideal I of $E(X)$ such that $yp=y'p$, for all $p\in I$, [5, Remark 6]. Let u be an idempotent in I such that $(x,y)u=(x,y)$. Then

$y'u = yu = y$, and $\phi(x) = \phi(xu) = \phi(x)u = y'u = y$.

$(6) \Rightarrow (5)$: Let $x, y \in X$. Let u be an idempotent in a minimal right ideal I of E such that $xu = x$. Then (x, yu) is an almost periodic point of $(X \times X, T)$, and, by (6), there is an endomorphism ϕ with $\phi(x) = yu$. Thus, $\phi(x)$ is proximal with y.

$(4) \Rightarrow (5)$: Let $p, q \in I$, and let u be the idempotent in I such that $qu = q$. Let $r \in I$ such that $rq = pu$, [5, Lemma 2]. Since left multiplication in a minimal right ideal is an endomorphism, and since pu is proximal to p, (5) is proved.

This completes the proof of Theorem 3. A minimal set which satisfies any one (and therefore all) of the properties listed in this theorem will be called *regular*.

If (Z, T) is any transformation group with Z compact, its enveloping semigroup $E(Z)$ always contains at least one minimal right ideal. This assures us of a plentiful supply of regular minimal sets.

Since regular minimal sets are coalescent, "endomorphism" may be replaced by "automorphism" in (5) and (6). Moreover, (6) tells us that regular minimal sets are characterized by possessing "as many endomorphisms as possible". For, as we observed in the course of the proof of Theorem 1, if (X, T) is any minimal set, and $x, y \in X$, then a necessary condition that $\phi(x) = y$, for some endomorphism ϕ, is that (x, y) be an almost periodic point of $(X \times X, T)$. (6) in Theorem 3 says that (X, T) is regular if and only if this is also sufficient.

Condition (2) in Theorem 3 indicates that regular minimal sets are "stable", in the sense that no new minimal sets are obtained by taking products of a regular minimal set with itself. It might be conjectured that minimal sets (X, T) which are isomorphic with every minimal set in $(X \times X, T)$ (and therefore, by an easy induction, with every minimal set in (X^n, T) for all positive integers n) are regular. However, this is not the case, in general. In constructing a counterexample, we make use of the following purely group theoretic lemma.

LEMMA 1. *Let A be an abelian group, let S be a subsemigroup of A, and let $\beta \in A$ such that $\beta \in S$ and $\beta^{-1} \notin S$. Then there is a subsemigroup H of A such that $S \subset H$, $\beta^{-1} \notin H$, and, if $\gamma \in A$, then at least one of γ, γ^{-1} is in H.*

Proof. Let $\mathscr{H}$ be the class of subsemigroups of A which contain S and do not contain β^{-1}. It is easy to see that there is a maximal $H \in \mathscr{H}$; we show that this semigroup H has the required properties.

Let $\gamma \in A$. We first show that some power of γ is in H. For, let H' denote the semigroup generated by H, γ, and γ^{-1}. Then (unless γ or $\gamma^{-1} \in H$, in which case we are finished), H' properly contains H. By the maximality of H, $\beta^{-1} \in H'$, so $\beta^{-1} = \gamma^r h$ $(h \in H, r \in Z)$. Then $\gamma^{-r} = \beta h \in H^2 \subset H$.

Now let $\gamma \in A$, and suppose $\gamma \notin H$ and $\gamma^{-1} \notin H$. We consider two cases. First, suppose there is an $n > 1$ such that $\gamma^n \in H$ and $\gamma^{-n} \in H$, but $\gamma^j \notin H$, $0 < |j| < n$. Let H' be the semigroup generated by H, γ, and γ^{-1}. Then, as above $\beta^{-1} = h\gamma^r$

$(r \in \mathbf{Z})$, $\gamma^{-r} = h\beta \in H$. It follows easily that $r = kn$, some $k \in \mathbf{Z}$, and therefore $\gamma^r \in H$. Then $\beta^{-1} = h\gamma^r \in H$, and this is a contradiction.

Therefore, suppose that there is an integer n such that $\gamma^n \in H$, $\gamma^{-n} \notin H$, and $\gamma^j \notin H$, for $0 < |j| < n$. We may suppose $n > 0$ (so $n > 1$). Suppose $\gamma^{-m} \in H$ where $m > 0$, and is the smallest positive integer for which this is the case. Then $m > n$, and $\gamma^{n-m} = \gamma^n \gamma^{-m} \in H$. But $0 > n - m > -m$, and $|n - m| < m$. That is $\gamma^{-(m-n)} \in H$, and this is a contradiction, since $m > m - n > 0$. Hence no negative power of γ is in H. Let H_1 denote the semigroup generated by γ and H. Since H is properly contained in $H_1, \beta^{-1} \in H_1$. Then as before, $\gamma^{-r} = \beta h \in H$ and this is a contradiction.

This completes the proof of the lemma. Now, let T be abelian, and let (X, T) be a regular minimal set with the following four properties:

(i) the automorphism group A of (X, T) is abelian,

(ii) (X, T) is not distal,

(iii) there is a $\beta \in A$ such that $\beta^m(x) \neq xt$ $(x \in X, t \in T)$ unless $m = 0$ and $t = e$.

(iv) If R is any subset of $X \times X$ which contains the diagonal Δ, and is contained in the proximal relation P, then R is closed.

An example of such a minimal set is given in [1, §7].

Let S be the semigroup generated by T and β; clearly $\beta^{-1} \notin S$. Let H be the subsemigroup of A guaranteed by Lemma 1. Now, let $(x, x') \in P$ with $x \neq x'$. If $h \in H$, identify $h(x)$ with $h(x')$, and call the resulting equivalence relation R. (That R is an equivalence relation follows from the fact that if ϕ is an endomorphism different from the identity, then $\phi(x)$ and x are not proximal, [1, Theorem 2].) Since $\Delta \subset R \subset P$, R is closed, and $Y = X/R$ is a compact Hausdorff space.

Since H is a semigroup, each $h \in H$ induces a continuous map h^* on Y such that $\pi h = h^* \pi$ (where π is the natural projection from $X \to Y$). In particular, (since $T \subset S \subset H$), T acts as a group of homeomorphisms on Y, and, for each $h \in S$, h^* is an endomorphism of the minimal set (Y, T).

Let (y_1, y_2) be an almost periodic point of $(Y \times Y, T)$, and let (x_1, x_2) be an almost periodic point of $(X \times X, T)$ for which $\pi(x_i) = y_i$ $(i = 1, 2)$. Since (X, T) is regular, there is an $h \in A$ such that $h(x_1) = x_2$. Either h or h^{-1} is in H; suppose $g = h^{-1} \in H$. Then $g^*(y_2) = g^*(\pi(x_2)) = \pi(g(x_2)) = \pi(x_1) = y_1$. Then $(y_1, y_2) = (g^*(y_2), y_2)$, and $\mathrm{Cl}((y_1, y_2)T) = \mathrm{Cl}((g^*(y_2), y_2)T)$ is isomorphic with $\mathrm{Cl}(y_2 T) = Y$. Thus every minimal set in $(Y \times Y, T)$ is isomorphic with (Y, T). However, (Y, T) is not coalescent, since $\beta^{-1} \notin H$, and therefore β^* is not one to one. Hence (Y, T) is not regular.

The choice of a noncoalescent example was not fortuitous, as the next theorem shows.

Theorem 4. *Let (X, T) be a coalescent minimal set and suppose that (X, T)*

is isomorphic with every minimal set (M,T) contained in $(X \times X, T)$. Then (X,T) is regular.

Proof. Let I be a minimal right ideal in $E(X)$. We show that (X,T) and (I,T) are isomorphic. Let $x \in X$. By [1, Lemma 3], it is sufficient to show that $xu = x$, for a unique $u \in I$. Suppose $xu = xr = x$, where $u, r \in I$, and u is an idempotent. Let $z \in X$, and let $y = zu$. Then $(x,y)u = (x, zu^2) = (x, zu) = (x,y)$, so (x,y) is an almost periodic point of $(X \times X, T)$. Let $M = \mathrm{Cl}((x,y)T)$, and let $\pi : M \to X$ be the first coordinate projection. Since M is isomorphic with X and (X,T) is coalescent, π is an isomorphism. Now $\pi((x,y)r) = \pi(xr, yr) = xr = x = \pi(x,y)$. Then $(x,y)r = (x,y)$, and $yr = y$. Therefore $zr = zur = yr = y = zu$. Since z is arbitrary $r = u$. The proof is completed.

3. **The lattice $\mathscr{R}(T)$.** Let T be a fixed topological group. We let $\mathscr{R}(T)$ denote the class of regular minimal sets with phase group T. $\mathscr{R}(T)$ may be partially ordered by defining $(X,T) \geqq (Y,T)$ if there is a homomorphism $\pi : (X,T) \to (Y,T)$. Since regular minimal sets are coalescent, $\geqq$ is indeed a partial ordering.

If (M,T) is the universal minimal set for T, and $(1,T)$ is the trivial minimal set (that is, the minimal set whose phase space consists of a single point) then $M \geqq X \geqq 1$, for all $X \in \mathscr{R}(T)$.

One might attempt to partially order the class of all minimal sets with phase group T in this manner. However, it seems likely (although no example is known) that there can exist two non isomorphic minimal sets, each of which is a homomorphic image of the other.

We now proceed to investigate the structure of $\mathscr{R}(T)$.

THEOREM 5. *$\mathscr{R}(T)$ is a complete lattice.*

Proof. It is sufficient to show that every nonempty family in $\mathscr{R}(T)$ has a greatest lower bound, [2, Theorem 2, Ch. IV]. Let $(X_\alpha, T) \in \mathscr{R}(T)$, $(\alpha \in \mathscr{A})$, and let $\mathscr{P}$ denote the property of being a homomorphic image of each (X_α, T). As in the proof of (2) $\Rightarrow$ (1) in Theorem 3, one verifies that $\mathscr{P}$ is an admissible property. Let (X,T) be the $\mathscr{P}$ universal minimal set. Then $(X_\alpha, T) \geqq (X,T)$ for all $\alpha \in \mathscr{A}$, so certainly (X,T) is a lower bound for the family $[(X_\alpha, T) \mid \alpha \in \mathscr{A}]$. If $(Y,T) \in \mathscr{R}(T)$ is another lower bound, then, by the defining property of "$\mathscr{P}$ universal", $(X,T) \geqq (Y,T)$, and (X,T) is a greatest lower bound.

We employ the usual lattice theoretic notations $\bigvee_i X_i$ and $\bigwedge_i X_i$ for the least upper bound and greatest lower bound of the subfamily $\{(X_i, T) \mid i \in \mathscr{I}\}$ of $\mathscr{R}(T)$. Now, we show how $\bigvee_i X_i$ and $\bigwedge_i X_i$ may be explicitly constructed.

THEOREM 6. *Let $(X_i, T) \in \mathscr{R}(T)$ $(i \in \mathscr{I})$ and let M be any minimal subset of the product transformation group $(\bigtimes_i X_i, T)$. Then $M = \bigvee_i X_i$.*

Proof. We first show that $(M,T) \in \mathscr{R}(T)$. Let $x = (x_i)$ and $y = (y_i)$ in M such that (x,y) is an almost periodic point of $(M \times M, T)$. Then, for each $i \in \mathscr{I}$,

(x_i, y_i) is an almost periodic point of $(X_i \times X_i, T)$, and, since (X_i, T) is regular, there is an automorphism ϕ_i of (X_i, T) with $\phi_i(x_i) = y_i$. Let ϕ denote the automorphism of $(\times_i X_i, T)$ defined by applying ϕ_i to each coordinate. Then, if $\lambda = \phi \mid M$, $\lambda(x) = y$, so $\lambda(M) = M$, and λ is an automorphism of (M, T). Thus (6) of Theorem 3 is satisfied, and (M, T) is regular. Suppose $(M', T) \in \mathscr{R}(T)$ with $M' \geqq X_i$ $(i \in \mathscr{I})$. Putting $M_i' = M'$, for $i \in \mathscr{I}$, one shows, again as in $(2) \Rightarrow (1)$ of Theorem 3, that there is a homomorphism $\phi' : M' \to M$. Therefore $M' \geqq M$, and $M = \bigvee_i X_i$.

In order to discuss the construction of the greatest lower bound of a collection of regular minimal sets, we require a lemma, which is of independent interest. It asserts that if $(X, T) \geqq (Y, T)$ there is "essentially" only one homomorphism from X to Y.

Lemma 2. *Let (X, T), (Y, T) be minimal, with $(Y, T) \in \mathscr{R}(T)$. Let ϕ_1 and ϕ_2 be homomorphisms from (X, T) to (Y, T). Then there is a unique automorphism α of (Y, T) such that $\phi_2 = \alpha\phi_1$.*

Proof. Consider the homomorphism ϕ from $(X \times X, T)$ to $(Y \times Y, T)$ defined by $\phi(x_1, x_2) = (\phi_1(x_1), \phi_2(x_2))$. Let $x \in X$ and let $y_i = \phi_i(x)$ $(i = 1, 2)$. Then $(y_1, y_2) = \phi(x, x)$ is an almost periodic point of $(Y \times Y, T)$.

Hence there is an automorphism α of (Y, T) such that $\alpha(y_1) = y_2$. Then $\alpha\phi_1(x) = \alpha(y_1) = y_2 = \phi_2(x)$. Since $\alpha\phi_1$ and ϕ_2 agree at x, they are identical. That α is unique is clear.

Now we proceed to discuss the construction of $\bigwedge_i X_i$, where $(X_i, T) \in \mathscr{R}(T)$. Let $X^{\#}$ denote the disjoint union of the X_i and let T act on $X^{\#}$ in the obvious manner. Let $x = (x_i)$ be an almost periodic point in $(\times_i X_i, T)$ and let $M = \mathrm{Cl}(xT)$. (So $M = \bigvee_i X_i$). Now, in $X^{\#}$ identify x_i with x_j, for all $i, j \in \mathscr{I}$, and let R denote the closed, T invariant equivalence relation generated by this identification. (That is, R is the smallest closed T invariant equivalence relation in $X^{\#}$ containing all (x_i, x_j).) Let $X^* = X^{\#}/R$. X^* is compact, and since R is T invariant, it is meaningful to speak of the transformation group (X^*, T).

Theorem 7. $X^* = \bigwedge_i X_i$.

Proof. It is easy to see that (X^*, T) is minimal, and is a homomorphic image of each (X_i, T). Let $(Y, T) \in \mathscr{R}(T)$ such that all $X_i \geqq Y$ and consider homomorphisms $\psi_i : X_i \to Y$ $(i \in \mathscr{I})$. Let $\pi_i : M \to X_i$ be the restriction to M of the projection from $\times_j X_j \to X_i$. Let $\phi_i = \psi_i \pi_i : M \to Y$. Fix $j \in \mathscr{I}$, and, for each $i \in \mathscr{I}$, let α_i be an automorphism of (Y, T) such that, as in Lemma 2, $a_i\phi_i = \phi_j$. Then, if we write ψ_i' for $\alpha_i\psi_i$, all $\psi_i'\pi_i : M \to Y$ are identical. Now, if $i, k \in \mathscr{I}$, $\psi_i'(x_i) = \psi_i'\pi_i(x) = \psi_k'\pi_k(x) = \psi_k(x_k)$. Let $\psi' : X^{\#} \to Y$ be the homomorphism defined by $\psi' \mid X_i = \psi_i'$ $(i \in \mathscr{I})$. Since $\psi_i'(x_i) = \psi_k'(x_k)$, for all $i, k \in \mathscr{I}$, ψ' induces a homomorphism $\psi^* : X^* \to Y$.

In order to complete the proof, it is only necessary to show that $(X^*, T) \in \mathscr{R}(T)$. Let $\mathscr{P}$ denote the property of being a homomorphic image of each X_i. $\mathscr{P}$ is an admissible property, and we show that (X^*, T) is the $\mathscr{P}$ universal minimal set. Let (Z, T) satisfy $\mathscr{P}$. Then there are homomorphisms $\eta_i : X_i \to Z$. Let I be a minimal right ideal in $E(Z)$, and let $\theta_i : E(X_i) \to E(Z)$ be the induced semigroup homomorphism, [7, Lemma 2]. Let I_i be a minimal right ideal in $E(X_i)$ such that $\theta_i(I_i) = I$. Since $(X_i, T) \in \mathscr{R}(T)$, X_i and I_i are isomorphic, so there are homomorphisms $\lambda_i : X_i \to I$. Now, $(I, T) \in \mathscr{R}(T)$, so, by the first part of this proof, there is a homomorphism $\lambda : X^* \to I$. Since there is always a homomorphism from I to Z, (Z, T) is a homomorphic image of (X^*, T), and the proof is completed.

Let (X, T) be any minimal set, and let I be a minimal right ideal in $E(X)$. Then the minimal set (I, T), which is in $\mathscr{R}(T)$, can be mapped homomorphically onto (X, T). Moreover, the method used in the last paragraph of the preceding proof shows that (I, T) is the "least" (with respect to the partial ordering $\geqq$) regular minimal set with this property. Using the method of proof of $(2) \Rightarrow (1)$ in Theorem 3, as well as Theorem 6, it is also easy to prove the existence of a "largest" regular minimal set (Y, T) which is a homomorphic image of (X, T). However, an explicit description of (Y, T) is not readily available.

The admissible property $\mathscr{P}$ is said to be *divisible* if every homomorphic image of a minimal set satisfying $\mathscr{P}$ also satisfies $\mathscr{P}$.

THEOREM 8. *Let $\mathscr{P}_i$ be divisible admissible properties $(i \in \mathscr{I})$, and let $\mathscr{P}$ denote the conjunction of the $\mathscr{P}_i$ (i.e., $\mathscr{P}$ holds if and only if each $\mathscr{P}_i$ holds). Then*

(i) *$\mathscr{P}$ is an admissible divisible property.*

(ii) *If X_i is the $\mathscr{P}_i$ universal minimal set, and X is the $\mathscr{P}$ universal minimal set, then $X = \bigwedge_i X_i$.*

Proof. The proof of (i) is immediate. To prove (ii), observe that $X_i \geqq X$, since (X, T) satisfies each $\mathscr{P}_i$. Therefore $\bigwedge_i X_i \geqq X$. Since each $\mathscr{P}_j$ is divisible, $\bigwedge_i X_i$ satisfies $\mathscr{P}_j$ $(j \in \mathscr{I})$. That is, $\bigwedge_i X_i$ satisfies $\mathscr{P}$, and $X = \bigwedge_i X_i$.

Let $\mathscr{P}$ be an admissible property, and let (X, T) be any minimal set. In [7], Gottschalk and Ellis prove that there is a smallest, closed T invariant equivalence relation S such that the quotient minimal set $(X/S, T)$ satisfies $\mathscr{P}$. S is called the $\mathscr{P}$ structure relation for (X, T). Using this result and Theorem 8, we easily obtain:

COROLLARY 1. *Let $\mathscr{P}$ be a divisible admissible property. Let $(X, T) \in \mathscr{R}(T)$, let S denote the $\mathscr{P}$ structure relation for (X, T), and let (Y, T) be the $\mathscr{P}$ universal minimal set. Then $X/S = X \wedge Y$.*

Proof. Let $\mathscr{P}'$ denote the property of being a homomorphic image of (X, T) and satisfying $\mathscr{P}$. Apply Theorem 8.

With every topological group T, we have associated the lattice $\mathscr{R}(T)$ of regular minimal sets with phase group T. We conclude by showing that the association is "functorial".

To be precise, let T and T' be topological groups, and let $\alpha : T \to T'$ be a (continuous) epimorphism. We define a map $i_\alpha = i : \mathscr{R}(T') \to \mathscr{R}(T)$ by $i(X', T') = (X', T)$, where the action of T on X' is defined by $x't = x'\alpha(t)$. It is immediate that this defines (X', T) as a transformation group, and, since α is onto, (X', T) is a minimal set. That (X', T) is regular follows from the trivial facts that x' and y' are proximal in (X', T) if and only if they are proximal in (X', T'), and that a map $\phi : X' \to X'$ is an endomorphism of (X', T) if and only if it is an endomorphism of (X', T').

LEMMA 3. *Let* $(X', T') \in \mathscr{R}(T')$, *and let* $(X', T) = i_\alpha(X', T')$. *Suppose* $(Y', T) \in \mathscr{R}(T)$ *such that* $(X', T') \geqq (Y', T)$. *Then, there is an action of* T' *on* Y' *such that* $(Y', T') \in \mathscr{R}(T')$, *and* $(Y', T) = i_\alpha(Y', T')$.

Proof. Let $y' \in Y'$, and $s, t \in T$ such that $\alpha(s) = \alpha(t)$. We show $y's = y't$. Let $\sigma : (X', T) \to (Y', T)$ be a homomorphism, and let $x' \in X'$ such that $\sigma(x') = y'$. Now $x's = x'\alpha(s) = x'\alpha(t) = x't$, and it follows that $y's = y't$.

Now, we define the action of T' on Y' by $y't' = y't$, where $t \in T$ is such that $\alpha(t) = t'$. Then $y'\alpha(t) = y't$, and $\sigma(x't') = \sigma(x')\, t' (x' \in X', t' \in T')$. The remarks in the preceding paragraph guarantee that this action is well defined.

Let $\{y_n'\}$, $\{t_n'\}$ be nets in Y' and T' respectively, with $y_n \to y' \in Y'$, $t_n' \to t' \in T'$. Let $x_n', x' \in X'$ with $x_n' \to x'$ and $\sigma(x_n') = y_n'$, (so that $\sigma(x') = y'$). Then $y_n't_n' = \sigma(x_n')t_n' = \sigma(x_n't_n') \to \sigma(x't') = \sigma(x')t' = y't'$. That $(y's')t' = y'(s't')$ $(y' \in Y', s', t' \in T')$, and that (Y', T') is regular minimal are immediate.

It is clear that the map i_α is one to one into. Using Lemma 3, we show that i_α imbeds $\mathscr{R}(T')$ as a sublattice of $\mathscr{R}(T)$.

THEOREM 9. *If* $X_j' \in \mathscr{R}(T')$ $(j \in \mathscr{I})$, *then* $i(\bigvee_j X_j') = \bigvee_j i(X_j')$ *and* $i(\bigwedge_j X_j') = \bigwedge_j i(X_j')$.

Proof. The proof is entirely lattice theoretic. Observe first that $X' \geqq Y'$ in $\mathscr{R}(T')$ if and only if $i(X') \geqq i(Y')$ in $\mathscr{R}(T)$. Now $\bigvee_j X_j' \geqq X_k'$ $(k \in \mathscr{I})$, so $i(\bigvee_j X_j') \geqq \bigvee_j i(X_j')$. By Lemma 3, there is a $Z' \in \mathscr{R}(T')$ such that $i(Z') = \bigvee_j i(X_j')$. Then $i(Z') \geqq i(X_j')$ $(j \in \mathscr{I})$, so $Z' \geqq X_j'$. But $i(\bigvee_j X_j') \geqq \bigvee_j i(X_j') = i(Z')$, so $\bigvee_j X_j' = Z'$. That is, $i(\bigvee_j X_j') = \bigvee_j i(X_j')$. The proof of the other equality is similar.

This discussion may be conveniently summarized by using the language of categories. Let $\mathscr{C}$ be the category of topological groups and epimorphisms, and let $\mathscr{D}$ be the category of lattices and imbeddings. Then the association $T \to \mathscr{R}(T)$ and $\alpha \to i_\alpha$ defines a contravariant functor from $\mathscr{C}$ to $\mathscr{D}$.

References

1. J. Auslander, *Endomorphisms of minimal sets*, Duke Math. J. **30** (1963), 605–614.

2. G. Birkhoff, *Lattice theory*, rev. ed., Amer. Math. Soc. Colloq. Publ. Vol. 25, Amer. Math. Soc., Providence, R. I., 1961.

3. H. Chu, *On universal transformation groups*, Illinois J. Math. **6** (1962), 317–326.

4. J. P. Clay, *Variations on equicontinuity*, Duke Math. J. **30** (1963), 423–432.

5. R. Ellis, *A semigroup associated with a transformation group*, Trans. Amer. Math. Soc. **94** (1960), 272–281.

6. ———, *Universal minimal sets*, Proc. Amer. Math. Soc. **11** (1960), 540–543.

7. R. Ellis and W. H. Gottschalk, *Homomorphisms of transformation groups*, Trans. Amer. Math. Soc. **94** (1960), 258–271.

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