DECOMPOSABLE CHAINABLE CONTINUA([1])

BY
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1. **Introduction.** In this paper, we supply some answers to the following question. What are sufficient conditions that a compact continuum be chainable? It is well known that chainable compact metric continua must be both a-triodic and hereditarily unicoherent. If M is a compact metric continuum which is a-triodic and hereditarily unicoherent, we show in Theorem 1 that M is chainable if it is the union of two continua, each of which is chainable.

Conditions which imply chainability are of interest because of their possible application to the homogeneity problem, i.e., the problem of characterizing the homogeneous bounded plane continua. Bing [4] has shown that the only homogeneous chainable plane continuum is the pseudo-arc.

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2. **Definitions and notations.** All spaces considered are metric. A *chain* is a finite collection $\mathscr{E} = \{E_1, E_2, \cdots, E_m\}$ of open sets such that $E_i \cap E_j \neq \varnothing$ iff $|i-j| \leqq 1$. We frequently denote such a chain by $E(1,m)$; we denote $\bigcup_{i=1}^m E_i$ by $E^*(1,m)$. The individual sets of a chain are *links*; two links are *adjacent* iff they intersect. A chain $\mathscr{E}$ is an *ε-chain* iff each link of $\mathscr{E}$ has diameter $< \varepsilon$. A chain $\mathscr{E}$ is *taut* iff nonadjacent links are a positive distance apart. If $\mathscr{E}$ and $\mathscr{F}$ are chains, then $\mathscr{F}$ is a *refinement* of $\mathscr{E}$ iff each link of $\mathscr{F}$ is a subset of some link of $\mathscr{E}$. $\mathscr{F}$ is a *closed refinement* of $\mathscr{E}$ iff the closure of each link of $\mathscr{F}$ is a subset of some link of $\mathscr{E}$. If $E(1,m)$ and $F(1,n)$ are chains such that $E_i \cap F_j \neq \varnothing$ iff $i = m$ and $j = 1$, then we denote the chain $\{E_1, E_2, \cdots, E_m, F_1, F_2, \cdots, F_n\}$ by $E(1,m) \oplus F(1,n)$. If $E(1,m)$ is a chain and S is an open set intersecting the common part of each pair of adjacent links, then the chain $\{E_1 \cap S, \cdots, E_m \cap S\}$ is denoted by $E(1,m) \cap S$. A compact metric continuum M is *chainable* (*snakelike, arc-like*) iff for each $\varepsilon > 0$, there is an ε-chain $E(1,m)$ covering M.

The following is a consequence of [1, Lemma 1, p. 515].

Proposition 1. *If M is a chainable compact metric continuum, then there is a sequence $\{D_i\}$ such that for each i,*

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(1) *D_i is a taut $1/i$ chain covering M.*
(2) *D_{i+1} is a closed refinement of D_i.*
(3) *Each link of D_i intersects M.*

DEFINITION 1. A compact continuum X is a *triod* provided there exist three proper subcontinua A, B and D of X such that $X = A \cup B \cup D$ and $A \cap B = B \cap D = A \cap D = A \cap B \cap D$ is a proper subset of each of A, B and D. A compact continuum M is *a-triodic* provided M contains no triods.

DEFINITION 2. A compact continuum X is *unicoherent* provided if A, B are subcontinua of X such that $X = A \cup B$, then $A \cap B$ is a continuum. A compact continuum M is *hereditarily unicoherent* provided each subcontinuum of M is unicoherent.

PROPOSITION 2. *If M is a chainable compact metric continuum, then M is a-triodic and hereditarily unicoherent.*

One of the most useful facts about chains, and one which is trivial to establish, is

PROPOSITION 3. *If $\mathscr{E} = E(1,m)$ is a chain and A is a connected subset of $E^*(1,m)$ such that $A \cap E_1 \neq \varnothing$ and $A \cap E_m \neq \varnothing$, then A intersects each link of $E(1,m)$.*

3. **Terminal points and terminal subcontinua.** The first step in chaining $A \cup B$ when each of A and B is ε-chainable, is to chain A and B separately, in each case, chaining $A \cap B$ last. In order to do this, we introduce the concept of terminal subcontinuum, a generalization of the notion of terminal point.

DEFINITION 3. A point p of a compact continuum M is a *terminal point* of M if and only if for each pair K, L of subcontinua of M, each containing p, either $L \subset K$ or $K \subset L$. (In [2], such points are called end points.)

DEFINITION 4. If M is a compact continuum, and A is a subcontinuum of M, then A is a *terminal subcontinuum* of M if and only if for each pair K, L of subcontinua of M, each intersecting A, either $K \subset A \cup L$ or $L \subset K \cup A$.

Note that if A is degenerate, then this definition agrees with Definition 3.

The following lemma is a generalization of [2, Theorem 10, p. 659], and its proof is similar to Bing's argument in [2].

LEMMA 1. *Suppose M is an a-triodic, hereditarily unicoherent compact metric continuum, A is a terminal subcontinuum of M and $\mathscr{E} = E(1,m)$ is a chain covering M. Then there is a chain $\mathscr{F} = F(1,n)$ covering M such that*
(1) *$\mathscr{F}$ is a refinement of $\mathscr{E}$.*
(2) *$(F_n - \mathrm{Cl}\, F_{n-1}) \cap A \neq \varnothing$.*
(3) *If $\mathscr{E}$ is taut, so is $\mathscr{F}$.*

Proof. If $A \cap E_m \neq \varnothing$, let U be an open set such that $U \cap A \neq \varnothing$ and $\mathrm{Cl}\, U \subset E_m$. If $i \neq m-1$ and $1 \leqq i \leqq m$, let $F_i = E_i$. Let $F_{m-1} = E_{m-1} - \mathrm{Cl}\, U$. Then $F(1,m)$

has properties (1), (2) and (3). If $A \cap E_1 \neq \varnothing$, we may make a similar modification on $E(m,1)$. Thus we may assume that $A \cap (E_1 \cup E_m) = \varnothing$.

Suppose the lemma fails.

If $B \subset M$, then B *has property* P if and only if B is a subcontinuum of M, $A \subset B$, and no refinement of $\mathscr{E}$ covers B and satisfies (1), (2) and (3) above. By assumption, M has property P. Clearly, A does not have property P.

We show: Property P is inductive i.e. if K is a decreasing sequence of compact sets, each having property P, then $\bigcap_{i=1}^{\infty} K_i$ has property P. For suppose there is such a sequence K such that $\bigcap_{i=1}^{\infty} K_i$ does not have property P. Clearly, $\bigcap_{i=1}^{\infty} K_i$ is a compact continuum containing A. Since $\bigcap_{i=1}^{\infty} K_i$ does not have property P, there is a chain $F(1,n)$ covering $\bigcap_{i=1}^{\infty} K_i$ satisfying (1), (2) and (3) above.

Now $F^*(1,n)$ is an open set containing $\bigcap_{i=1}^{\infty} K_i$. Since K is decreasing, there is an integer j such that $K_j \subset F^*(1,n)$. But this means that K_j does not have property P. This is impossible, hence property P is inductive. Thus there is a subcontinuum M' of M such that M' is irreducible with respect to having property P. For notational convenience, we assume $M = M'$. We may also assume that M intersects each element of $\mathscr{E}$, otherwise we may consider the subchain of $\mathscr{E}$ consisting of those elements which do intersect M.

By Lemma 3, M is irreducible from a point $p \in A$ to a point $q \in M - A$. From [7, Theorem 135, p. 58], the composant of M determined by p is dense in M. Thus there is a proper subcontinuum N of M such that $p \in N$, and N intersects each link of $\mathscr{E}$. Since $q \notin N \cup A$, $N \cup A$ is a proper subcontinuum of M intersecting each link of $\mathscr{E}$. We shall assume, without loss of generality, that $N \cup A = N$.

Since N is a proper subcontinuum of M, N does not have property P. Thus there is a chain $\mathscr{G} = G(1,b)$ covering N such that $\mathscr{G}$ is a refinement of $\mathscr{E}$, $A \cap (G_b - \mathrm{Cl}\, G_{b-1}) \neq \varnothing$, if $\mathscr{E}$ is taut, so is $\mathscr{G}$, and no chain with fewer links than $\mathscr{G}$ has these properties.

We shall show that the first link of $\mathscr{G}$ intersects $(E_1 \cup E_m) \cap N$. For suppose that $G_1 \cap (E_1 \cup E_m) \cap N = \varnothing$. Since $\mathscr{G}$ covers N and N intersects each link of $\mathscr{E}$, some link of $\mathscr{G}$ intersects $(E_1 \cup E_m) \cap N$; let G_s be the first such link. Assume, for definiteness, that $G_s \cap N \cap E_1 \neq \varnothing$. Let G_t be the first link of $\mathscr{G}$ which intersects $N \cap E_m$; clearly, $1 < s \leqq t$. There is a link $E_r \in E(2, m-1)$ such that $G_1 \subset E_r$. Now, $G(s,t)$ is a refinement of $E(1,m)$, $G_s \cap E_1 \neq \varnothing$ and $G_t \cap E_m \neq \varnothing$; it follows that E_r contains a link of $G(s,t)$. Thus E_r contains a link of $G(1,t)$ distinct from G_1.

There are links E_x and E_y of $E(1,m)$ such that $N \subset E^*(x,y)$ and N is not contained in any proper subchain of $E(x,y)$. Since N intersects each link of $\mathscr{E}$, $1 \leqq x \leqq 2 \leqq r \leqq m-1 \leqq y \leqq m$. Now $E(x,y) \cap G^*(1,t) \oplus G(t+1,b)$ is a chain covering B and having the required properties. Moreover, this chain has fewer links that $\mathscr{G}$, since each link of $E(x,y)$ contains at least one link of $G(1,t)$ and E_r

contains at least two links of $G(1,t)$. This is contrary to the choice of $\mathscr{G}$, and hence $G_1 \cap (E_1 \cup E_m) \cap N \neq \varnothing$.

Suppose $E_1 \cap G_1 \cap N \neq \varnothing$. Since $A \cap (E_1 \cup E_m) = \varnothing$, $A \cap (E_1 \cap N \cap G_1) = \varnothing$. Let V be an open set such that $V \cap N \neq \varnothing$ and $\mathrm{Cl}\, V \subset (E_1 \cap G_1) - A$. Let R be the component of $M - V$ containing A. Since each of R and N is a subcontinuum of M intersecting A, and A is a terminal subcontinuum of M, either $N \subset R \cup A$ or $R \subset N \cup A = N$. Since $V \cap N \neq \varnothing$ and $V \cap (R \cup A) = \varnothing$, $N \not\subset R \cup A$. Hence $R \subset N$. Since R is a component of $M - V$ and N is covered by $\mathscr{G}$, no continuum in $M - V$ intersects both A and $(M - V) - \mathscr{G}^* = M - \mathscr{G}^*$. Hence, by [7, Theorem 44, p. 15]. $M - V$ is the union of two disjoint closed sets, one containing A and the other containing $M - \mathscr{G}^*$. By the normality of M, there are open sets S and T such that $A \subset S$, $M - \mathscr{G}^* \subset T$, $\mathrm{Cl}\, S \cap \mathrm{Cl}\, T = \varnothing$, and $M - V \subset S \cup T$.

Define a chain $\mathscr{F}$ covering M as follows:

$$\mathscr{F} = E(m,2) \cap T \oplus (E_1 \cap T) \cup V \oplus G_1 \cap S$$
$$\oplus\ (G_2 \cap S) - \mathrm{Cl}\, V \oplus G(3,b) \cap S.$$

$\mathscr{F}$ covers M and has properties (1) and (2). Further, since $\mathscr{G}$ is taut if $\mathscr{E}$ is, $\mathrm{Cl}\, S \cap \mathrm{Cl}\, T = \varnothing$, and $V \subset G_1 \cap E_1$, $\mathscr{F}$ is taut if $\mathscr{E}$ is.

Under the additional hypothesis that M is chainable, we obtain the following converse for Lemma 1.

Proposition 4. *Suppose M is a chainable compact continuum in a metric space (X,d), and A is a subcontinuum of M with the property that for each positive number ε, there is an ε-chain $E(1,m)$ covering M such that $A \cap E_m \neq \varnothing$. Then A is a terminal subcontinuum of M.*

Proof. Suppose that A is not a terminal subcontinuum of M. Then there exist subcontinua H and K of M, each intersecting A, and neither is contained in the union of A and the other. Let $p \in H - (K \cup A)$, $q \in K - (H \cup A)$ and $\varepsilon = \frac{1}{3} \min \{d(p, K \cup A),\ d(q, H \cup A)\}$.

Let $E(1,m)$ be an ε-chain covering M such that $E_m \cap A \neq \varnothing$. Let E_j be the first link of $E(1,m)$ which intersects $H \cup K$. Suppose $p \in E_r$ and $q \in E_s$. Clearly, $j \leqq r$, $j \leqq s$, $E_r \cap (K \cup A) = \varnothing$ and $E_s \cap (H \cup A) = \varnothing$, by choice of ε. We may suppose, without loss of generality, that $E_j \cap H \neq \varnothing$. By Proposition 3, $H \cup A$ intersects each link of $E(j,m)$. In particular, $(H \cup A) \cap E_s \neq \varnothing$. This contradiction establishes the proposition.

Definition 5. If M is a compact continuum, A is a subcontinuum of M, $\mathscr{E} = E(1,m)$ is a chain covering M, and $1 \leqq j \leqq k \leqq m$, then A *is contained exactly in the subchain* $E(j,k)$ (in symbols, $A \subset^e E^*(j,k)$) if and only if

(1) $A \subset E^*(j,k)$.

(2) A is not contained in any proper subchain of $E(j,k)$.

(3) $A \cap [\mathrm{Cl}\,E^*(1,j-1) \cup \mathrm{Cl}\,E^*(j+1,n)] = \varnothing$.

The concept of exact containment will prove useful in the succeeding section. Hence we obtain the following, slightly strengthened, version of Lemma 1.

LEMMA 2. *Suppose M is an a-triodic, hereditarily unicoherent compact metric continuum, A is a terminal subcontinuum of M and $\mathscr{E} = E(1,m)$ is a chain covering M. Then there is a chain $\mathscr{G} = G(1,n)$ covering M and an integer s, $1 \leqq s \leqq n$, such that,*

(1) *$\mathscr{G}$ is a refinement of $\mathscr{E}$.*

(2) *$A \subset^{e} G^*(s,n)$.*

(3) *If $\mathscr{E}$ is taut, so is $\mathscr{G}$.*

Proof. By Lemma 1, there exists a chain $\mathscr{F} = F(1,n)$ covering M such that $\mathscr{F}$ is a refinement of $\mathscr{E}$, $(F_n - \mathrm{Cl}\,F_{n-1}) \cap A \neq \varnothing$, and if $\mathscr{E}$ is taut, so is $\mathscr{F}$. There is an integer s such that $A \subset F^*(s,n)$ and A is not contained in any proper subchain of $F(s,n)$. Since $F(s,n)$ is a chain, $F^*(1,s-2) \cap F^*(s,n) = \varnothing$. Thus, $(\mathrm{Cl}\,F^*(1,s-2)) \cap F^*(s,n) = \varnothing$, and therefore $(\mathrm{Cl}\,F^*(1,s-2)) \cap A = \varnothing$. If $\mathrm{Cl}\,F_{s-1} \cap A = \varnothing$, we may take $\mathscr{G} = \mathscr{F}$.

Suppose $(\mathrm{Cl}\,F_{s-1}) \cap A \neq \varnothing$. Since $A \subset F^*(s,n)$, A and $M - F^*(s,n)$ are disjoint closed subsets of a normal space. There exist disjoint open sets Y and W such that $A \subset Y$ and $M - F^*(s,n) \subset W$. For each positive integer i, $1 \leqq i \leqq s-1$, let $G_i = F_i \cap W$. For each positive integer i, $s \leqq i \leqq n$, let $G_i = F_i$. Clearly $G(1,n)$ is a chain which refines $\mathscr{F}$, hence $G(1,n)$ is a refinement of $\mathscr{E}$. Furthermore, $A \subset G^*(s,n)$ and A is not contained in any proper subchain of $G(s,n) = F(s,n)$. Also, $A \cap \mathrm{Cl}\,G^*(1,s-1) \subset A \cap \mathrm{Cl}\,W = \varnothing$. Thus $A \subset^{e} G^*(s,n)$. Finally, since nonadjacent links of $\mathscr{G}$ are contained in nonadjacent links of $\mathscr{F}$, $\mathscr{G}$ is taut if $\mathscr{F}$ is; hence $\mathscr{G}$ is taut if $\mathscr{E}$ is.

The reader may compare the following lemma with [2, Theorem 12, p. 661].

LEMMA 3. *If M is an a-triodic hereditarily unicoherent compact metric continuum and A is a subcontinuum of M, then A is a terminal subcontinuum of M if and only if for each subcontinuum B of M which intersects A, $A \cup B$ is irreducible between some pair of points, one of which belongs to A.*

Proof. Suppose A is a terminal subcontinuum of M and B is a subcontinuum of M intersecting A. Since $A \cup B$ is a a-tridic and hereditarily unicoherent, there are points p and q in $A \cup B$ such that $A \cup B$ is irreducible from p to q.

Assume that the lemma fails. Then $\{p,q\} \subset B - A$. Let $r \in A$. $A \cup B$ is not irreducible from p to r, nor from q to r. Hence, there are proper subcontinua L and K of $A \cup B$ such that $\{p,r\} \subset L, \{q,r\} \subset K$. Since A is a terminal subcontinuum of M, one of L and K is contained in the union of A and the other. Suppose the notation has been chosen so that $L \subset A \cup K$. Thus $p \in A \cup K$. Since $p \notin A, p \in K$; thus K is a proper subcontinuum of $A \cup B$, containing p and q. This is impossible, hence either $p \in A$ or $q \in A$.

Conversely, suppose that A is not a terminal subcontinuum of M. We shall show that there is a subcontinuum of M which fails to satisfy the conditions of the lemma.

There exist subcontinua D and E of M, each intersecting A, such that $D \not\subset A \cup E$, $E \not\subset A \cup D$. If $p \in A$ and $q \in A \cup D \cup E$, then $A \cup D \cup E$ is clearly not irreducible from p to q.

4. **Continua which are the union of chainable continua.** Once we know that we can chain each of A and B and chain $A \cap B$ last, we still must "fit the chains together properly." The next lemma describes this fitting process.

LEMMA 4. *Suppose each of A and B is a chainable compact continuum in a metric space (X,d), $A \cap B \neq \varnothing$, and $A \cap B$ is a terminal subcontinuum of A and B. Then, for each $\varepsilon > 0$, there exist ε-chains $\mathscr{E} = E(1,m)$ and $\mathscr{F} = F(1,n)$ such that*

1. *$\mathscr{E}$ and $\mathscr{F}$ are taut ε-chains covering A and B respectively.*
2. *$A \cap B \subset^{e} E^*(j,m)$ and $A \cap B \subset^{e} F^*(k,n)$.*
3. *$F(k,n)$ is a closed refinement of $E(j,m)$.*
4. *$\mathrm{Cl}\,E^*(1,j-1) \cap \mathscr{F}^* = \varnothing$.*
5. *$\mathrm{Cl}\,F^*(1,k-2) \cap \mathscr{E}^* = \varnothing$.*
6. *$\mathrm{Cl}\,F_{k-1} \cap A = \varnothing$ and there is a positive integer t, $j \leqq t \leqq m$, such that $\mathrm{Cl}\,F_{k-1} \cap \mathscr{E}^* \subset E_t$ and $\mathrm{Cl}\,F_{k-1}$ intersects no other link of $\mathscr{E}$.*

Proof. By Proposition 1, there is a taut ε-chain $\mathscr{G}'$ covering A. By Lemma 2, there is a refinement $\mathscr{G}$ of $\mathscr{G}'$, $\mathscr{G} = G(1,m)$ such that $\mathscr{G}$ is a taut ε-chain covering A and $A \cap B \subset^{e} G^*(j,m)$. Let $b > 0$ denote the minimum distance between nonadjacent links of $\mathscr{G}$.

Now $\{G_j - \mathrm{Cl}\,G^*(1,j-1), G_{j+1}, \cdots, G_m\}$ is an open cover of the compact set $A \cap B$. Let $\varepsilon' > 0$ be a Lebesgue number for this cover of $A \cap B$. Let $\varepsilon'' = \frac{1}{2}\min\{\varepsilon, \varepsilon', b, d(A \cap B, \mathrm{Cl}\,G^*(1,j-1))\}$. By Proposition 1, there is a taut ε'' chain $\mathscr{H}_1$ covering B. Since $\varepsilon'' \leqq \frac{1}{2}\varepsilon'$, the closure of each link of $\mathscr{H}_1$ intersecting $A \cap B$ is contained in a link of $G(j,m)$ and, since $\varepsilon'' \leqq \frac{1}{2} d(A \cap B, \mathrm{Cl}G^*(1,j-1))$, does not intersect $\mathrm{Cl}\,G^*(1,j-1)$. By Lemma 2, there is a taut ε''-chain $\mathscr{H} = H(1,n)$ covering B such that $\mathscr{H}$ is a refinement of $\mathscr{H}_1$ and $A \cap B \subset^{e} H^*(k,n)$. Thus the closure of each link of $H(k,n)$ is a subset of some link of $G(j,m)$ and does not intersect $\mathrm{Cl}\,G^*(1,j-1)$. Let $t = \mathrm{glb}\{i : \mathrm{Cl}\,H_k \subset G_i,\ j \leqq i \leqq m\}$. Since $B - A$ and $A - B$ are separated and metric spaces are completely normal, there are disjoint open sets S and T such that $A - B \subset S$ and $B - A \subset T$. Define a chain $\mathscr{F} = F(1,n)$ as follows:

$$F_i = H_i \cap T, \text{ if } 1 \leqq i \leqq k-1, \qquad F_i = H_i, \text{ if } k \leqq i \leqq n.$$

Define a chain $\mathscr{E} = E(1,m)$ as follows: if $1 \leqq i \leqq j-1$, $E_i = G_i \cap S$; if $j \leqq i \leqq m$ and $i \neq t$, $E_i = G_i - \mathrm{Cl}\,F^*(1,k-1)$. Let $E_t = G_t - \mathrm{Cl}\,F^*(1,k-2)$. Properties (1) and (2) for $\mathscr{E}$ and $\mathscr{F}$ follow from the corresponding properties for $\mathscr{G}$ and $\mathscr{H}$. Since $F(k,n) = H(k, n)$, $F(k, n)$ is a closed refinement of $G(j, m)$ and $\mathrm{Cl}(F^*(k,n)) \cap \mathrm{Cl}(G^*(1,j-1)) = \varnothing$. Since $\mathscr{F}$ is taut, $\mathrm{Cl}(F^*(1,k-2)) \cap \mathrm{Cl}(F^*(k,n)) = \varnothing$; it follows from this and the definition of $E(j,m)$ that $F(k,n)$ is a closed refinement of $E(j,m)$, and (3) holds. Now $\mathrm{Cl}\,E^*(1,j-1) \subset \mathrm{Cl}\,S$ and $F^*(1,k-1) \subset T$, hence $\mathrm{Cl}(E^*(1,j-1)) \cap F^*(1,k-1) = \varnothing$. Since the closure of each link of $H(k,n)$ is disjoint from $\mathrm{Cl}\,G^*(1,j-1)$, $\mathrm{Cl}(F^*(k,n)) \cap \mathrm{Cl}(E^*(1,j-1)) = \varnothing$. The conclusions of the two preceding statements yield: $\mathrm{Cl}(E^*(1,j-1)) \cap \mathscr{F}^* = \varnothing$, and (4) is satisfied. Property (5) follows directly from the definition of $\mathscr{E}$. Since $A \cap B \subset^{e} F^*(k,n)$, $\mathrm{Cl}\,F_{k-1} \cap A \cap B = \varnothing$. Now $F_{k-1} = H_{k-1} \cap T$ and $\mathrm{Cl}\,T \cap (A-B) = \varnothing$, thus $\mathrm{Cl}\,F_{k-1} \cap A = \varnothing$. The remaining part of (6) follows from the definition of $E(j,m)$.

THEOREM 1. *Suppose that each of A and B is a chainable compact continuum in a metric space (X,d) and $A \cap B \neq \varnothing$. Then $A \cup B$ is chainable iff $A \cup B$ is a-triodic and unicoherent.*

Proof. Suppose $A \cup B$ is chainable. Then $A \cup B$ is certainly a-triodic and unicoherent; in fact, $A \cup B$ is hereditarily unicoherent.

Conversely, suppose $A \cup B$ is a-triodic and unicoherent. By [6, Theorem 1.5, p. 180], if $A \cup B$ contains a nonunicoherent subcontinuum D, then $D \subset A$ or $D \subset B$. Since each of A and B is chainable, this is impossible, hence $A \cup B$ is hereditarily unicoherent.

CLAIM 1. $A \cap B$ is a terminal subcontinuum of A and of B. Assume, to the contrary, that $A \cap B$ is not a terminal subcontinuum of A. Then there exist subcontinua H and K of A such that $H \cap A \cap B \neq \varnothing$, $K \cap A \cap B \neq \varnothing$, $H \not\subset K \cup (A \cap B)$, and $K \not\subset H \cup (A \cap B)$. Thus $K \cup (A \cap B)$, $H \cup (A \cap B)$ and B are three continua which intersect, no one of which is contained in the union of the other two. By [8, p. 440], the union of these three continua is a triod. This is impossible, hence $A \cap B$ is a terminal subcontinuum of A. Similarly, $A \cap B$ is a terminal subcontinuum of B, and Claim 1 is established.

Let $\mathscr{E} = E(1, m)$ and $\mathscr{F} = F(1, n)$ be ε-chains having properties (1)–(6) of Lemma 4.

Case i. $\mathrm{Cl}(F_{k-1}) \cap \mathscr{E}^* \subset E_m$, i.e., $t = m$.

Since $\mathrm{Cl}\,F_{k-1}$ intersects no link of $\mathscr{E}$ except E_t, an ε-chain covering $A \cup B$ is $\mathscr{E} \oplus F(k-1,1)$.

Case ii. $\mathrm{Cl}(F_{k-1}) \cap \mathscr{E}^* \not\subset E_m$.

Thus, $\mathrm{Cl}(F_{k-1}) \cap E_m = \varnothing$, by (6).

Let $r = \mathrm{glb}\{i : \mathrm{Cl}\,F_i \not\subset E^*(j,m-1),\ k \leqq i \leqq n\}$. Since $A \cap B \subset^{e} E^*(k, m)$, r exists; clearly $\mathrm{Cl}\,F_r \subset E_m$. Since $\mathrm{Cl}\,F_{k-1} \cap E_m = \varnothing$, $r \geqq k+1$; thus $r-2 \geqq k-1$. Let $E'_m = E_m - \mathrm{Cl}\,F^*(k, r-2)$, if $r-2 \geqq k$; let $E_m = E'_m$, if $r-2 = k-1$. Let $\mathscr{E}' = E(1, m-1) \oplus E'_m$. $\mathscr{E}'$ and $\mathscr{F}$ have properties (1)–(6). For notational convenience, we will assume that $\mathscr{E}' = \mathscr{E}$ i.e., $\mathscr{E}$ and $\mathscr{F}$ have properties (1)–(6) and in addition, $E_m \cap \mathrm{Cl}\,F^*(k-1, r-2) = \varnothing$ and $F^*(k-1,k) \subset F^*(k-1,r-1)$.

Let $D = (A \cup B) \cap (\mathrm{Cl}(E^*(j,m-1)) - E_m)$.

CLAIM 2. $D \cap \mathrm{Cl}\, F_{k-1}$ and $D - F^*(k-1,r-1)$ are disjoint.

Assume to the contrary, that the claim fails, and there is a point $p \in (D \cap \mathrm{Cl}\, F_{k-1}) \cap (D - F^*(k-1,r-1))$. From (6), $\mathrm{Cl}\, F_{k-1} \cap A = \emptyset$, thus $p \in B$. Since $\mathscr{F}$ is taut, $B \cap \mathrm{Cl}\, F_{k-1} \subset F^*(k-2,k)$. Now $F^*(k-1,k) \subset F^*(k-1,r-1)$, thus $p \notin F^*(k-1,k)$, and therefore $p \in F_{k-2}$. But $F_{k-2} \cap D = \emptyset$, from (5). This is impossible, hence Claim 2 is established.

CLAIM 3. No continuum in D intersects both $D \cap \mathrm{Cl}\, F_{k-1}$ and $D - F^*(k-1,r-1)$.

Suppose, to the contrary, that there is such a continuum, M. Since $M \cap \mathrm{Cl}\, F_{k-1} \neq \emptyset$, $M \cap (B - A) \neq \emptyset$.

We shall show that $M \subset B$. If $M \cap (A - B) \neq \emptyset$, then each of $A \cap M$ and $B \cap M$ is a continuum, since M is hereditarily unicoherent. Furthermore, since M is a continuum and $A - B$ and $B - A$ are separated, $A \cap M$ and $B \cap M$ each intersect $A \cap B$. Thus $A \cap B$, $A \cap M$ and $B \cap M$ are three continua which intersect, and no one is contained in the union of the other two. By [8, p. 440], their union is a triod. This is impossible, hence $M \cap (A - B) = \emptyset$ and $M \subset B$.

Now $M \cap F^*(1,k-2) = \emptyset$, since $M \subset D \subset \mathrm{Cl}\,\mathscr{E}^*$, and $\mathrm{Cl}\,\mathscr{E}^* \cap F^*(1,k-2) = \emptyset$, from (5). Since $M \not\subset F^*(k-1,r-1)$ and $M \subset B \subset \mathscr{F}^*$, there is an $i, i \geqq r$, such that $M \cap F_i \neq \emptyset$. It follows that $M \cap F_r \neq \emptyset$. This is impossible, since $F_r \subset E_m$, $M \subset D$ and $D \cap E_m = \emptyset$. Thus Claim 3 is established.

Since $D \cap \mathrm{Cl}\, F_{k-1}$ and $D - F^*(k-1,r-1)$ are closed, from [7, Theorem 44, p. 15] it follows that D is the union of two disjoint closed sets, one containing $D \cap \mathrm{Cl}\, F_{k-1}$ and the other containing $D - F^*(k-1,r-1)$. Normality guarantees the existence of open sets S and T such that

$$(\#) \quad \mathrm{Cl}\, S \cap \mathrm{Cl}\, T = \emptyset,\ D \subset S \cup T,\ D - F^*(k-1,r-1) \subset S, \text{ and } D \cap \mathrm{Cl}\, F_{k-1} \subset T.$$

Define a chain $\mathscr{G}$ by

$$\mathscr{G} = E(1,j-1) \oplus E(j,m-1) \cap S \oplus E_m \oplus F(r-1,k) \cap T \oplus (F_{k-1} \cap T)$$
$$\cup (F_{k-1} - \mathrm{Cl}\,\mathscr{E}^*) \oplus F(k-2,1).$$

Clearly, each link of $\mathscr{G}$ is an open set of diameter $< \varepsilon$. From the tautness of $\mathscr{E}$ and $\mathscr{F}$, properties (4) and (5), and the disjointness of $\mathrm{Cl}\, S$ and $\mathrm{Cl}\, T$, it follows that links of $\mathscr{G}$ intersect only if their indices differ by at most 1. The definition of $\mathscr{G}$ and (#) show that $\mathscr{G}$ covers $A \cup B$, and hence $\mathscr{G}$ is a chain. Thus $A \cup B$ is chainable.

COROLLARY 1. *Suppose (X,d) is a metric space and for each positive integer i, $1 \leqq i \leqq n$, A_i is chainable compact continuum in X such that $A_i \cap A_{i+1} \neq \emptyset$. Then $\bigcup_{i=1}^{n} A_i$ is chainable if and only if $\bigcup_{i=1}^{n} A_i$ is a-triodic and unicoherent.*

COROLLARY 2. *Suppose M is a decomposable unicoherent compact continuum in a metric space (X,d) and each proper subcontinuum of M is chainable. Then M is chainable.*

Proof. First, M is not a triod. For if, to the contrary, there exist three proper subcontinua A, B and D of M such that $M = A \cup B \cup D$ and $A \cap B = A \cap D = B \cap D = A \cap B \cap D$ is a proper subset of each of A, B and D, then $A \cup B$ is a proper subcontinuum of M. Let $p \in A \cap B$, $q \in D - (A \cup B)$. There exists a subcontinuum D' of D such that D' is irreducible from p to q. Since the composant of D' determined by p is dense in D', there exists a proper subcontinuum D'' of D' such that $D'' \cap (D - (A \cup B)) \neq \varnothing$. Then $D'' \cup (A \cap B)$, A and B form a triod. This is impossible. Hence M is not a triod. Since each proper subcontinuum of M is chainable, M is a-triodic. Apply Theorem 1.

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