THE LEVEL CURVES OF HARMONIC FUNCTIONS(¹)

BY

LEOPOLD FLATTO(²), DONALD J. NEWMAN(²) AND HAROLD S. SHAPIRO(³)

I. **Introduction.** In this paper we study the level curves of harmonic functions, i.e., curves Γ for which there exists a harmonic function $u(x, y)$ vanishing on Γ but not identically. A necessary condition is that Γ be an analytic curve and, if we do not permit u to have any singularities, that Γ have no closed loop. There are however much more subtle necessary conditions, given in §4 of the present paper, if we require that u be *everywhere* harmonic, which rule out many plausible-looking curves, such as $y = x^3$.

Usually in the present paper we mean by harmonic, harmonic in the whole plane. In some sections we have results valid also for harmonic functions in a less restrictive sense; these distinctions will always be pointed out explicitly. Of course, to have any nontrivial theory we must restrict attention to functions harmonic in a reasonably large neighborhood of the given curve; a purely local characterization of level curves beyond the requirement of analyticity is not possible. Indeed, if Γ is any analytic curve which divides the plane into two components, and $w = f(z)$ maps one of these components conformally on $\operatorname{Re} w > 0$, the function $u(z) = \operatorname{Re} f(z)$ is harmonic in a neighborhood of Γ and vanishes on Γ.

The results of the present paper are somewhat fragmentary, but a complete solution of the problem of characterizing even algebraic level curves seems quite difficult.

In §2 a special study is made of the level curves of harmonic polynomials. In §3 conic sections are treated in detail, the most noteworthy result being that some hyperbolas are level curves and some are not. In §4 some general necessary conditions are derived, and examples given of curves which are not level curves. An interesting feature of the problem here under study is its close connection with the study of automorphisms of analytic functions. In §5 some open questions are pointed out.

II. **The level curves of harmonic polynomials.** Let Γ denote the algebraic curve $p(x, y) = 0$ where $p(x, y)$ is a given polynomial with real coefficients. Γ will be the level curve of a harmonic polynomial if there exists a harmonic polynomial $u(x, y)$

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(²) Belfer Graduate School of Science, Yeshiva University

(³) University of Michigan and Steklov Mathematical Institute.

which vanishes on Γ. We may assume that p splits into distinct irreducible factors so that the above assertion is equivalent to $p \mid u$. The uniqueness problem reduces then to determining the divisors of harmonic polynomials. Let $p = p_n + p_{n-1} + \cdots + p_m$ where p_j $(m \leqq j \leqq n)$ is a homogeneous polynomial of degree j. We state

THEOREM 2.1. *A necessary condition for the existence of a harmonic polynomial u such that $p \mid u$ is that $p_n = \prod_{i=1}^n L_i$, $p_m = \prod_{i=1}^m K_i$ where the L_i's and K_i's are real homogeneous linear factors, the angle between any two lines $L_i = 0$ $(1 \leqq i \leqq n)$ $[K_i = 0, (1 \leqq i \leqq m)]$ being a rational multiple of π. If p is homogeneous then the above condition is also sufficient.*

Proof. Let $p \mid u$ where $u = u_t + u_{t-1} + \cdots + u_s, u_j$ $(s \leqq j \leqq t)$ denoting a homogeneous polynomial of degree j. Since u_t is homogeneous of degree t, $u_t = Cr^t \sin[t(\theta - \delta)]$ for an appropriate choice of C and δ. Thus $u_t = 0$ on the lines $\theta = \delta + k\pi/t$ $(0 \leqq k \leqq t-1)$ so that $u_t = \prod_{k=0}^{t-1} L_k$ where L_k is a real linear homogeneous polynomial, $L_k = 0$ denoting the line $\theta = \delta + k\pi/t$. $p \mid u$ implies $p_n | u_t$ so that $p_n = a \prod_{i=1}^n L_{k_i}$ where a is a constant and $0 \leqq k_1 < k_2 < \cdots < k_n \leqq t-1$. The same reasoning yields the similar result for p_m.

Conversely if $p = \prod_{i=1}^n L_{k_i}$ where $L_i = 0$ $(0 \leqq i \leqq t-1)$ denotes the line $\theta = \delta + i\pi/t$ then $p \mid \prod_{i=0}^{t-1} L_i = Cr^t \sin[t(\theta - \delta)]$ and $Cr^t \sin[t(\theta - \delta)]$ is clearly a harmonic polynomial.

III. **Conic sections.** We give a complete description of the conic sections which are level curves of harmonic functions. We may immediately rule out ellipses, for a harmonic function which vanishes on a closed curve must vanish identically. Of course, if we permit the function to have singularities this is no longer the case, e.g. the harmonic function $\log r$ vanishes on the unit circle, and has only one finite singularity. In the case of conics we are also able to obtain results for harmonic functions which are permitted to have certain singularities.

III.1. *Degenerate conic (two straight lines).* The reflection principle settles this problem. If the angle between the two lines is an irrational multiple of π, the repeated use of the reflection principle shows that a harmonic function vanishing on the two lines vanishes on a dense set of lines and hence everywhere. If the angle is a rational multiple of π, say π/nm, then the harmonic polynomial $\mathrm{Re}(cz^n)$ vanishes on both lines for an appropriate choice of c.

The case of two parallel straight lines is dealt with similarly. If we choose the lines to be $y = 0$ and $y = a$ the reflection principle shows that a harmonic function $u(x, y)$ vanishing on these lines must vanish on the lines $y = na$, $n = 0, \pm 1, \pm 2, \cdots$ and moreover $u(x, y + 2a) = u(x, y)$. For a polynomial u this implies $u \equiv 0$, whereas if u is merely assumed harmonic this does not imply $u \equiv 0$, as we see from the example $u(z) = \mathrm{Im}\exp(\pi z/a)$.

III.2. *Parabolas.* Since the class of harmonic functions remains invariant with respect to magnifications and rigid motions, there is no real loss of generality in studying the curve $\Gamma: x = y^2$. Writing $z = x + iy$, $w = u + iv$ consider the mapping $z = (w^2 + 1)/4$. It maps the half-plane $v > 1$ conformally on the region to the left of Γ, and the straight line $\Gamma^1: v = 1$ is mapped on Γ. Suppose now $U(x,y) = U(z)$ is harmonic in the z-plane and vanishes on Γ. Then $V(w) = U((w^2 + 1)/4)$ is harmonic in the w-plane and vanishes on the line Γ^1. Then, V satisfies the two relations

$$\text{(i)} \qquad V(u,v) + V(u,2 - v) = 0,$$

$$\text{(ii)} \qquad V(u,v) = V(-u, -v),$$

the first because V vanishes on Γ^1, and the reflection principle, and the second because V is a single-valued function of w^2. Conversely, it is readily seen that if V is any harmonic function satisfying (i) and (ii) then the function $U(z) = V((4z - 1)^{1/2})$ is single-valued and harmonic in the whole z-plane, and vanishes on the parabola Γ. Therefore the given problem is equivalent to the study of harmonic functions V satisfying (i) and (ii). Note that by combining (i) and (ii) we get $V(u,v) = U(u, v + 4)$ showing a periodic behaviour in v. Thus, a polynomial solution $V \not\equiv 0$ is not possible, but transcendental solutions exist, the simplest being $V(w) = \operatorname{Im} \cosh \pi w$, giving rise to the function $U(z) = \operatorname{Im} \cosh(\pi(4z - 1)^{1/2})$ which is harmonic and vanishes on the given parabola. Note that in the present case we are able to characterize all harmonic functions vanishing on Γ. Also, by further exploiting the correspondence given by the conformal mapping, we could obtain the "reflection principle" appropriate to harmonic functions which vanish on a parabola. Note the curious fact that if a harmonic function vanishes on a parabola, than it necessarily vanishes on an infinity of parabolas, and namely (in the present instance) the images of the lines $v = 2n + 1$, $n = 0 \pm 1, \pm 2, \cdots$ under the map $z = (w^2 + 1)/4$.

III. *Hyperbolas.* We may discuss hyperbolas in a similar fashion, starting from the map $z = \sin w$, or

$$x = \cosh v \sin u,$$

$$y = \sinh v \cos u.$$

The straight lines $v = v_0$ are mapped into ellipses E_{v_0} in the z-plane, and the lines $u = u_0$ map into hyperbolas H_{u_0}. Since every nondegenerate ellipse (resp. hyperbola) is equivalent (modulo magnifications and rigid motions) to one of the curves E_{v_0} (resp. H_{u_0}) there is no loss of generality in studying only the curves H_{u_0} (as regards hyperbolas) and, in the following paragraph E_{v_0} (as regards ellipses).

Suppose now $U(z) = U(x,y)$ is everywhere harmonic and vanishes on the hyperbola H_a. Then, $V(w) = U(\sin w)$ is everywhere harmonic, and vanishes on the line $u = a$. Moreover, as a function of $\sin w$ it is invariant under the transformations $w \to w + 2\pi$, and $w \to \pi - w$; and conversely since these automorphisms generate all the automorphisms of $\sin w$, any function invariant under them is a single-valued function of $\sin w$.

We therefore conclude, in a manner similar to that of III.2 that V satisfies

$$\text{(i)} \qquad V(u,v) + V(2a-u,v) = 0 \text{ (from reflection principle)},$$

$$\text{(ii)} \qquad V(u+2\pi,v) = V(u,v),$$

$$\text{(iii)} \qquad V(\pi-u,-v) = V(u,v)$$

and conversely, any harmonic V satisfying these three relations gives rise, by the transformation $U(z) = V(\arcsin z)$ to a single-valued harmonic function U vanishing on the hyperbola H_a. Now, from (i) and (iii) we deduce that $V(u + 2\pi - 4a, v) = V(u,v)$ so that V has period $2\pi - 4a$, and hence period $4a$, as well as period 2π, with repect to the variable u. This implies, if a/π is irrational, the existence of a dense set of periods, and consequently $V \equiv 0$. In view of the geometrical meaning of the number a (we recall that $2a$ is the angle between the asymptotes) we have thus proved the first part of the following(4).

THEOREM III.3. *If a harmonic function vanishes on a hyperbola, and the angle between the asymptotes is not a rational multiple of π, it vanishes identically. If the angle between the asymptotes is a rational multiple of π, there exists a nonnull harmonic polynomial vanishing on the hyperbola.*

To prove the second part, we give the following simple construction, which is motivated by the previous discussion. Suppose $a = m\pi/n$, then $V(w) = \operatorname{Im} \cos 2nw$ is harmonic and satisfies (i), (ii), (iii). $U(z) = V(\arcsin z)$ is a harmonic polynomial in z vanishing on the hyperbola H_a.

Note that in Theorem III.3 there is no need (in the uniqueness assertion) to suppose that U is everywhere harmonic; if we assume, for instance, that U is a single-valued harmonic function on a dense open connected subset of the plane, that is sufficient for the proof to work, and even much weaker hypotheses are sufficient. The same remark applies also to the following discussion concerning ellipses, where we have not attempted to formulate results of maximum generality.

III.4. *Ellipses*. Suppose $U(z)$ is harmonic in a dense open subdomain of the plane, and vanishes on the ellipse E_a (we continue to use the notation of the previous paragraph). Then $V(w) = U(\sin w)$ vanishes on the line $v = a$. The analysis is now similar to that in III.3 except that (i) is replaced by

$$\text{(i}'\text{)} \qquad V(u,v) + V(u, 2a - v) = 0.$$

(4) A somewhat simpler discussion would result from using the mapping $z = \cos w$.

This equation together with (iii) of III.3 implies

$$V(u, v + 4a) = V(u, v).$$

Thus, V has period 2π with respect to u, and period $4a$ with respect to v. In particular, V vanishes on the lines $v = (4n+1)a$, $n = 0, \pm 1, \pm 2, \cdots$. We therefore conclude that U vanishes on a certain countable set of ellipses (confocal with the given one), namely the images of these lines under the map $z = \sin w$, and must therefore have at least one singularity in the domain bounded by each pair of ellipses; this implies that U must have a sequence of a singularities z_n with $z_n \to 0$, and also a sequence of singularities z_n' with $z_n' \to \infty$. And functions U of this type are easily constructed, namely $U(z) = \operatorname{Im} F(\arcsin z)$ where $F(w)$ is an elliptic function having periods 2π and $4ai$ and is real on the line $v = a$. The limiting case $a = \infty$ corresponds to a circle; here, as the function $\log r$ shows, only one finite singularity is forced.

III.5. *An alternate method.* In the case of parabolas and hyperbolas, we may use a somewhat simpler variant of the above method which does not involve conformal mapping, but which also gives somewhat less information, and is limited to the case that U is *everywhere* harmonic (and so is the real part of an entire function). As this method will be useful to us in the sequel, we illustrate it briefly here, for the case of the parabola. The following simple lemma is fundamental:

LEMMA. *Suppose Γ is a curve given by the parametric equations $x = p(t)$, $y = q(t)$ $(a < t < b)$ where p and q are holomorphic in some region R containing the segment (a,b). If there exists a function $u(z)$ everywhere harmonic and vanishing on Γ, $u \not\equiv 0$, then there exist nonconstant entire functions f, g such that*

$$f(p(t) + iq(t)) = g(p(t) - iq(t)), \qquad t \in R. \tag{1}$$

Moreover, if u is a polynomial, f and g may be chosen to be polynomials.

Proof. The proof is immediate, since there exists an entire function $F(z)$ such that $u(z) = \operatorname{Re} F(z) = \frac{1}{2}F(z) + \frac{1}{2}\overline{F(z)} = \frac{1}{2}F(\bar z) + \frac{1}{2}F^*(\bar z)$ where $F^*(z)$ denotes the entire function $\overline{F(\bar z)}$. Writing $f(z) = F(z)/2$, $g(z) = -F^*(z)/2$ we see that (1) holds first for $a < t < b$, and therefore by analytic continuation for all $t \in R$.

In applying this lemma, we exploit the fact that the function $h(t) = f(p(t) + iq(t)) = g(p(t) - iq(t))$ is holomorphic in R, and admits all the automorphisms, i.e., makes all the identifications, of the two functions $p(t) + iq(t)$ and $p(t) - iq(t)$. For example, let us use this approach to prove that a harmonic polynomial which vanishes on the parabola $x = y^2$ vanishes identically (less trivial applications will be given in the following sections). Here $p(t) = t^2$, $q(t) = t$. It is enough to show that the relations

$$h(t) = f(t^2 + it) = g(t^2 - it)$$

where f, g are polynomials, implies $h =$ constant. Now, $t^2 + it$ is invariant under the substitution $t \to -i - t$ and $t^2 - it$ is invariant under the substitution $t \to i - t$. Since h is invariant with respect to both of these, it is invariant with respect to the substitution $t \to t + 2i$ which they generate; i.e., $h(t)$ has period $2i$, and being a polynomial it is constant.

REMARK. Actually, we do not have to assume u is a polynomial, it is enough to assume for instance that

$$|u(z)| < Ae^{B|z|^p}$$

where $p < \frac{1}{2}$, and it is easy to formulate sharper theorems of this kind.

IV. **Various examples of curves which are not level lines of harmonic functions.** We consider the curves $y = x^n$ $(1 \leqq n < \infty)$ and show that these are not level lines of a harmonic function except for $n = 1, 2$. We give two different proofs, the second one being applicable to a larger class of curves described later on. The first proof gives a local version of the result.

THEOREM IV.1. *Let n be an integer > 2. If $u(x,y)$ is harmonic in the disc $x^2 + y^2 < 5$ and vanishes on $y - x^n = 0$, then u vanishes throughout the disc.*

Proof. The reasoning employed to establish the lemma of §III.5 shows that we may write $u(x,y) = F(y + ix) - G(y - ix)$ where $F(z)$ and $G(z)$ are analytic in $|z| < 5^{1/2}$. Note then, that $F(x^n + ix) = G(x^n + ix)$ for all x, $-1 \leqq x \leqq 1$. Since $F(z^n + iz)$ and $G(z^n - iz)$ are both analytic for $|z| \leqq 1$ we conclude that (1) $H(z) = F(z^n + iz) = G(z^n - iz)$ for $|z| \leqq 1$.

The crux of our proof lies in the notion of a local automorphism, (l.a.). Namely, we say that a map $a + bt \to c + dt$ is an l.a. for the function $f(z)$ if $f(z)$ is analytic at a and at c and if the expansion of $f(a + bt)$ and $f(c + dt)$ about $t = 0$ agree in the constant and the first nonvanishing terms, (the convention being that the map is an l.a. if $f(z)$ is a constant). Let us take note of certain trivial properties of these l.a.'s. We have

(2) If $a + bt \to c + dt$ and $c + dt \to e + ft$ are l.a.'s for $f(z)$ then so is $a + bt \to e + ft$.

(3) If $a + bt \to c + dt$ is an l.a. for $f(z)$ and $g(w)$ is analytic at $w = f(a)(= f(c))$ then $a + bt \to c + dt$ is an l.a. for $g(f(z))$.

(4) If $a + t \to a + \lambda t$ is an l.a. for $f(z)$, and if λ is not a root of unity, then $f(z)$ is a constant. Now set

$$p = \begin{cases} \text{any prime divisor of } n - 1 \text{ if } n \text{ is even,} \\ 2^k \text{ if } n - 1 = 2^k m,\ k \geqq 1,\ m \text{ odd} \end{cases}$$

and

$$\omega = e^{\pi i/p}, \quad a = (\tan \pi/2p)^{1/n-1}, \quad \zeta = \frac{(n+1)\omega - (n-1)}{(n+1) - (n-1)\omega}.$$

Thus

$$\text{(5)}\qquad \omega^{n-1} = -1; \quad a^{n-1} = \tan \pi/2p = \frac{\omega - 1}{i(\omega + 1)}.$$

Using (5) one can easily show that

$$\text{6(a)}\qquad (a\omega^j)^n + (-1)^{j-1} ia\omega^j = (a\omega^{j+1})^n + (-1)^{j-1} ia\omega^{j+1},$$

$$\text{6(b)}\qquad [n(a\omega^j)^{n-1} + (-1)^{j-1} i]\zeta^j = [n(a\omega^{j+1})^{n-1} + (-1)^{j-1} i]\zeta^{j+1}.$$

It follows from 6(a) and 6(b) that

(7) For j odd, $a\omega^j + \zeta^j t \to a\omega^{j+1} + \zeta^{j+1} t$ is an l.a. for $z^n + iz$.

(8) For j even, $a\omega^j + \zeta^j t \to a\omega^{j+1} + \zeta^{j+1} t$ is an l.a. for $z^n - iz$.

We conclude then from (1) and (3) that for every j, $a\omega^j + \zeta^j t \to a\omega^{j+1} + \zeta^{j+1} t$ is an l.a. for $H(z)$. Repeated application of (2) now insures that $a + t \to a + \zeta^{2(n-1)} t$ is an l.a. for $H(z)$. And so, by (4), the following lemma is all that is required to prove our theorem.

Lemma. *ζ is not a root of unity.*

Proof. We will produce the irreducible polynomial for $\zeta + 1$ and thereby show that $\zeta + 1$ is not even an algebraic integer by dint of the fact that the leading coefficient of this polynomial does not divide its constant term.

Case I. n odd; p a power of 2 and the irreducible polynomial for ω is $x^p + 1$. The irreducible polynomial for $\zeta + 1$ is $[(n + 1)x - 2]^p + [(n - 1)x + 2]^p$. Here the leading coefficient, $(n + 1)^p + (n - 1)^p$ is $> 2^p + 2^p$, which is the constant term.

Case II. n even; p is an odd prime and the irreducible polynomial for ω is $(x^p + 1)/(x + 1)$. Hence the irreducible polynomial for $\zeta + 1$ is $[(n + 1)x - 2]^p/x + [(n - 1)x + 2]^p/x$. The leading coefficient is $(n + 1)^p + (n - 1)^p$ and the constant term is $np2^p$. This time we have

$$\begin{aligned}(n + 1)^p + (n - 1)^p &= 2n[(n + 1)^{p-1} + (n + 1)^{p-2}(n - 1) + \cdots] \\ &> 2n[2^{p-1} + 2^{p-1} + \cdots] = 2np2^{p-1} = np2^p.\end{aligned}$$

The proof of the lemma, and so of the theorem, is now complete.

Remark. It is clear that harmonicity throughout the entire disc $x^2 + y^2 < 5$ was not necessary to the proof. Of course 5 could be replaced by any number larger than 4 but even further reduction of the region is possible. We leave these details to the reader.

In the next two Theorems IV. 2 and IV. 3, $p(z)$ and $q(z)$ denote meromorphic functions defined in a region R_0 of the extended z plane. $p(z)$ and $q(z)$ are assumed to be real for $a < z < b$ where $(a, b) \subset R_0$. It is furthermore assumed that the finite images of R_0 under $p + iq$, $p - iq$ are contained respectively in R and $\bar{R}$

where R is a region of the z plane and $\bar{R} = \{z \mid \bar{z} \in R\}$. Γ denotes the curve $x = p(t)$, $y = q(t)$ $(a < t < b)$ so that Γ is contained in R.

THEOREM IV.2. (a) *Suppose there exist a point $z_0 \in R_0$ such that $p(z)$ and $q(z)$ are both analytic at z_0 and $m \nmid 2n$, $n \nmid 2m$, m and n denoting the respective orders of the zeros of $p(z) - p(z_0)$ and $q(z) - q(z_0)$ at z_0. If $u(x,y)$ is harmonic in R and vanishes on Γ, then u vanishes throughout R.*

(b) *Let p and q have a common pole at $z_0 \in R_0$, m and n denoting the respective orders of these poles. Suppose that $m \neq n/2$, n, $2n$. If $u(x,y)$ is harmonic in R and vanishes on Γ, then u vanishes throughout R.*

COROLLARY. *Let p and q be nonconstant polynomials with real coefficients and assume that $m \neq n/2$, $n, 2n, m$ and n denoting the respective degrees of p and q. In this case R_0 is the extended z-plane; p and q have a common pole at ∞, the orders of these poles being respectively m and n. We conclude from Theorem* IV. 2(b) *that if $u(x,y)$ is harmonic in the entire (x,y) plane and vanishes on the curve $x = p(t)$, $y = q(t)$ $(-\infty < t < \infty)$, then u vanishes for all (x,y).*

Proof. We first treat part (a). Without loss of generality, we may assume $z_0 = 0$, $m \leqq n$, $p(0) = q(0) = 0$. Hence $p(z) = a_m z^m + \cdots, q(z) = b_n z^n + \cdots$, $(a_m b_n \neq 0)$ in a neighborhood of 0. Let $\Gamma_1 = \Gamma_1(c)$, $\Gamma_2 = \Gamma_2(c)$ denote respectively the curves $|p + iq| = c$, $|p - iq| = c$ where $c > 0$. For c sufficiently small Γ_1 and Γ_2 will be closed Jordan curves contained in R_0 and surrounding 0. Furthermore for c_1, c_2 sufficiently small and $c_1 < c_2, \Gamma_i(c_1)$ will be in the interior of $\Gamma_i(c_2)$ $(i = 1,2)$. (For a discussion of the facts concerning the Γ_i's see [1, p. 108 and the relevant Figure 21].) Let R_1 and R_2 denote the closed regions enclosed respectively by Γ_1 and Γ_2. The reasoning employed to establish the lemma of §III.5 shows that if $u(x,y)$ is harmonic in R then $u(x,y) = F(x + iy) - G(x - iy)$ with $F(z)$ analytic in R and $G(z)$ analytic in $\bar{R}$. Since $u = 0$ on Γ we have $H(z) = F(p(z) + iq(z)) = G(p(z) - iq(z))$ for $a < z < b$. Hence, by analytic continuation,

$$H(z) = F(p(z) + iq(z)) = G(p(z) - iq(z))$$

at all z in R_0 where p and q are both regular.

By the maximum modulus theorem the maximum modulus M_1 of $|H|$ in R_1 is attained at some point z_1 on Γ_1. $p(z) + iq(z)$ has a zero of order m at 0. It follows that there exist $m - 1$ points on Γ_2 distinct from z_1 (we call them $z_2, \cdots, z_m$) such that

$$p(z_1) + iq(z_1) = p(z_2) + iq(z_2) = \cdots = p(z_m) + iq(z_m) \qquad [1, \text{p. } 108].$$

Hence $|H|$ attains its maximum M_1 in R_1 at $z_1, \cdots, z_m$. A similar argument shows that $|H|$ attains its maximum M_2 in R_2 at m points $z_1', \cdots, z_m'$ on Γ_2.

Now $|p + iq| > |p - iq| \Leftrightarrow |p + iq|^2 > |p - iq|^2 \Leftrightarrow (p + iq)(\bar{p} + i\bar{q}) >$

$(p - iq)(\bar{p} + i\bar{q}) \Leftrightarrow \operatorname{Im} q\bar{p} < 0$. Similarly $|p + iq| < |p - iq| \Leftrightarrow \operatorname{Im} q\bar{p} > 0$. For sufficiently small z we have

$$q(z)\overline{p(z)} = b_n\overline{a_m} z^n \bar{z}^m + \cdots$$

so that

$$q\bar{p} = b_n\overline{a_m} r^{m+n} \exp[i(n-m)\theta +]O(r^{m+n+1})$$

where $z = re^{i\theta}$. Let $\theta_1 = \arg z_1 = \delta + 2\pi/m$. Then $\theta_j = \arg z_j = \delta + 2\pi j/m + \varepsilon_j$ $(\varepsilon_1 = 0)$ where $\varepsilon_j \to 0$ as $c \to 0$ $(1 \leqq j \leqq m)$ [1, p. 108]. Hence

$$q(z_j)\overline{p(z_j)} = b_n\bar{a}_m r^{m+n} \exp[i(n - m)(\delta + \varepsilon_j)] \exp[2\pi inj/m] + O(r^{m+n+1}).$$

Let $(n,m) = d$, $n = n_1 d$, $m = m_1 d$. The set of numbers $\exp[2\pi inj/m]$ $(1 \leqq j \leqq m)$ is identical with the set of numbers $\exp[2\pi ij/m]$ $(1 \leqq j \leqq m_1)$. It follows that for $m_1 \geqq 3$, $\operatorname{Im}(q(z_j)\overline{p(z_j)}) > 0$ for some j, $1 \leqq j \leqq m$. Thus

$$|p(z_j) - iq(z_j)| < |p(z_j) + iq(z_j)| = c$$

so that z_j is inside Γ_2. Hence $M_1 \leqq M_2$. Repeating the argument we find a z'_k $(1 \leqq k \leqq m)$ for which $\operatorname{Im}(q(z'_k)\overline{p(z'_k)}) < 0$ so that z'_k lies inside Γ_1. Hence $M_2 \leqq M_1$ and therefore $M_1 = M_2$. Since $|H|$ assumes its maximum inside Γ_1, it follows that H is constant in R_1 and hence constant in R_0. Thus $u \equiv 0$ in R. One can readily check that $m_1 = 1$ or $2 \Leftrightarrow m \mid 2n$. Hence $m_1 \geqq 3 \Leftrightarrow m \nmid 2n$ and we have proven part (a) of Theorem IV.2).

We sketch the proof of part (b) as it is very similar to that of part (a). We assume again without loss of generality that $z_0 = 0$ and $m \leqq n$ so that $p(z) = a_m/z^m + \cdots$, $q(z) = b_n/z^n + \cdots$ $(a_m b_n \neq 0)$ in a neighborhood of 0. We choose a fixed circle $K: |z| = r$ so that $p(z)$ and $q(z)$ are analytic for $0 < |z| \leqq r$. $\Gamma_i = \Gamma_i(c)$ $(i = 1,2)$ is defined as before and R_i is now defined as the closed annular region bounded by K and Γ_i. Γ_2 lies inside K for c sufficiently large and for large c_1, c_2 $(c_1 > c_2)$ $\Gamma_i(c_1)$ lies inside $\Gamma_i(c_2)$. Since $p + iq$, $p - iq$ have poles of order n at 0, H assumes its maximum in R_i at n points $z_{i1}, \cdots, z_{in}$ on Γ_i, provided c is sufficiently large. We have

$$\theta_{ij} = \arg z_{ij} = \delta_i + \frac{2\pi j}{n} + \varepsilon_{ij} \qquad (i = 1,2;\ 1 \leqq j \leqq n)$$

where $\theta_{i1} = \delta_i + 2\pi/n$ (so that $\varepsilon_{11} = \varepsilon_{21} = 0$) and $\varepsilon_{ij} \to 0$ as $c \to \infty$.

$$\alpha(z_{ij})\overline{p(z_{ij})} = \frac{b_n a_m}{r^{m+n}} \exp[i(m-n)(\delta_i + \varepsilon)] + \exp[2\pi im\ /n] + O\left(\frac{1}{r^{m+n-1}}\right).$$

The set of numbers $\exp[2\pi imj/n]$ $(1 \leqq j \leqq n)$ is identical with the set of numbers

$\exp[2\pi ij/n_1]$ $(1 \leqq j \leqq n_1)$. If $n_1 \geqq 3$, then we conclude as before that $u \equiv 0$ in R. It is easily verified that $n_1 = 1 \Leftrightarrow n = m$ and $n_2 = 2 \Leftrightarrow n = 2m$. Hence $n_1 \geqq 3 \Leftrightarrow n \neq m$, $2m$ and we have proven part (b) of Theorem 4.2([5]).

We have not been able to settle the exceptional cases mentioned in Theorem IV.2. Some restriction on m and n is of course necessary as $x = t$, $y = t$ and $x = t$, $y = t^2$ $(-\infty < t < \infty)$ are level curves of harmonic functions. We do however obtain the following partial results for the exceptional cases.

THEOREM IV.3. *Let $p(z)$ and $q(z)$ have a common pole at ∞ of multiplicity n so that $p(z) = A_n z^n + \cdots + A_0 + \sum_{k=1}^{\infty} a_k/z^k$, $q(z) = B_n z^n + \cdots + B_0 + \sum_{k=1}^{\infty} b_k/z^k$ $(A_n B_n \neq 0)$ for large z. Assume the A_i's and B_i's real $(0 \leqq i \leqq n)$ and $B_n A_{n-s} - B_{n-s} A_n \neq 0$ for some s, $0 < s < n/2$. If $u(x, y)$ is harmonic in R and vanishes on Γ, then u vanishes throughout R.*

Proof. The reasoning is similar to that of Theorem 4.2b. We choose $z_{i1}, \cdots, z_{in}$ $(i = 1, 2)$ as in the proof of Theorem 4.2b. It suffices to show that for c sufficiently large there exist points z_{1j}, z_{2k} $(1 \leqq j \leqq n)$ such that $\operatorname{Im}(q(z_{1j})\overline{p(z_{1j})}) > 0$, $\operatorname{Im}(q(z_{2k})\overline{p(z_{2k})}) < 0$. Let $B_n A_{n-s} - B_{n-s} A_n \neq 0$, $B_n A_{n-j} - B_{n-j} A_n = 0$ for $0 < j < s$. The vectors $(B_{n-j}, A_{n-j})(1 \leqq j < s)$ are all multiples of (B_n, A_n) so that any two are linearly dependent. Thus $B_{n-j} A_{n-k} - B_{n-j} A_{n-k} = 0$ for $0 \leqq j < s$, $0 \leqq k < s$, a direct computation yields

$$\operatorname{Im}[q\bar{p}] = (B_n A_{n-s} - B_{n-s} A_n) r^{2n-s} \sin s\theta + O(r^{2n-s-1})$$

for large z. Let $\theta_{ij} = \arg z_{ij}$ $(i = 1, 2; 1 \leqq j \leqq n)$. The distance between two successive zeros of $\sin s\theta$ is π/s and $\theta_{ij} - \theta_{i,j-1} \to 2\pi/n$ as $c \to \infty$ $(i = 1, 2; 2 \leqq j \leqq n)$. Since $2\pi/n < \pi/s$, it follows that for large c there exists a z_{ij} for which $\operatorname{Im}(q(z_{1j})\overline{p(z_{1j})}) > 0$. Similarly, for c sufficiently large there exists a z_{2k} for which $\operatorname{Im}(q(z_{2k})\overline{p(z_{2k})}) < 0$. The rest of the proof is identical with that of Theorem IV.2b.

V. **Concluding remarks.** A number of questions suggest theselves for further investigation; we wish here to point out only a few of these.

(a) Analogous problems for harmonic functions of more than two variables, and for partial differential equations other than the Laplace equation. Insofar as the present paper uses essentially the methods of classical function theory, it cannot cope with these questions.

(b) Considering uniqueness curves for more general classes of harmonic functions; for example, can the theorems of the last section be strengthened so as to permit isolated singularities?

(c) We have shown in §III.2 that the parabola is an algebraic level curve of a har-

([5]) A technique very similar to that used here was employed for a different purpose by A. and C. Rényi in their paper *Some remarks on periodic entire functions*, J. Analyse Math. **14** (1965), 303–310.

monic function without being the level curve of a harmonic polynomial. Are there any other algebraic curves for which this is true?

(d) Given, say two polynomials $p(z)$ and $q(z)$, when do there exist nonconstant entire (or meromorphic) functions f, g with $f(p(z)) = g(q(z))$? (Certain necessary conditions follow from the analysis in §4.)(6)

(e) If Γ is a level curve, i.e. such a curve that the values of $u(z)$ on Γ do not uniquely determine u, what additional data will suffice to determine u uniquely? Knowledge of the normal derivative of u along Γ is enough, but in general "over-determines" the problem; i.e. there is no corresponding existence theorem in general. Another type of additional condition is a restriction on the growth at infinity, as we observed in the case of the parabola.

(f) The whole question of existence of a harmonic function with prescribed values on a given curve seems an interesting one. More precisely, the curve Γ will be said to have the *existence property* if, given any entire function $F(x, y)$ of two complex variables, which is real when both x and y are real, there exists an everywhere harmonic function $u(x, y)$ such that $u(x, y) = F(x, y)$ for $(x, y) \in \Gamma$.

To gain some insight into the problem, let us consider two simple examples: (1) Γ is a straight line. Here we can prescribe both u and its normal derivative as arbitrary real-entire functions; the solution is then unique. (These statements are readily proved, e.g. by use of the Cauchy-Kowalewski procedure.) (2) Γ is a parabola. We now show: Γ *has the existence property.*

Indeed, as is readily seen from the discussion in §III.2, it is enough to show, if $\phi(u)$ is any entire function of u we can construct an everywhere harmonic function $V(w)$, $w = u + iv$, such that

(i) $V(-w) = V(w)$,

(ii) $V(u, 1) = \phi(u)$.

Moreover, considering the decomposition $\phi(u) = \phi_1(u) + \phi_2(u)$ where ϕ_1 is even and ϕ_2 is odd, it is enough to solve the problem when $\phi(u)$ is an even function and when ϕ is an odd function of z. Consider first the case of an even function. It is known from the classical theory of difference equations that there exists an entire function $f(u)$ such that the equation

$$\left(\cos\frac{d}{du}\right) f(u) = f(u) - \frac{f''(u)}{2!} + \frac{f^{(iv)}(u)}{4!} - \cdots = \phi(u)$$

is satisfied. If $\phi(u)$ is even, and real for real u, then we may find a solution f which is even and real for real u. The harmonic function

$$V(u, v) = f(u) - \frac{f''(u)}{2!}v^2 + \frac{f^{(iv)}(u)}{4!}v^4 - \cdots$$

(6) For some results on this question, see a forthcoming paper by H. S. Shapiro, *On the functional equation* $f(P(z)) = g(Q(z))$, in the Publications of the Hungarian Academy of Sciences.

then satisfies (i) and (ii). The case ϕ odd is treated similarly, here

$$V(u,v) = \frac{g(u)}{1!}v - \frac{g''(u)}{3!}v^3 + \cdots$$

where g satisfies the difference equation $(\sin d/du)g(u) = \phi(u)$.

Added in proof. An example of a set without the existence property is the set $A: xy = 0$. There is no harmonic function $u(x, y)$ equal to y^2 on A, since vanishing on $y = 0$ implies $u(x, -y) = -u(x, y)$.

Bibliography

1. L. V. Ahlfors, *Complex Analysis*, McGraw-Hill, New York, 1953.

Yeshiva University,
New York, New York

University of Michigan,
Ann Arbor, Michigan

Steklov Mathematical Institute,
Moscow, Russia