HAUSDORFF MEASURES ON ABSTRACT SPACES(1)

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1. **Introduction.** Given a non-negative set function τ on a family $\mathscr{A}$ of subsets of a metric space X, an outer measure ν can be generated on X as follows: For $B \subset X$ and $\delta > 0$

$$\nu_\delta B = \inf\left\{\sum_{i\in\omega} \tau A_i \colon B \subset \bigcup_{i\in\omega} A_i \text{ and, for } i \in \omega, A_i \in \mathscr{A} \text{ and } \operatorname{diam} A_i \leqq \delta\right\}.$$

and

$$\nu B = \lim_{\delta\to 0} \nu_\delta B.$$

F. Hausdorff [3] introduced this abstract measure (a generalization of the linear measure of C. Carathéodory [2]), and proved a few basic results for it. He considered in some detail the measures obtained when various restrictions are placed on the set function τ, in particular when $\tau B = h(\operatorname{diam} B)$ for some continuous increasing function $h\colon R_+ \to R_+$, with $h(0) = 0$ and $h(t) > 0$ for $t > 0$. The measure generated using this function is called the Hausdorff h-measure, and in the case that $h(t) = t^s$, the Hausdorff s-dimensional measure. In these forms it has been studied extensively. Two recent papers by W. W. Bledsoe and A. P. Morse [1], and by C. A. Rogers and M. Sion [7], have suggested processes for defining a measure on a topological space which generalize the Hausdorff measure process in a metric space. They obtain some (in general, different) measurability and approximation results for these measures.

In this paper we introduce a process for generating a measure on an arbitrary space, which abstracts the essential idea behind all of the above Hausdorff measures and generalizations. Results are obtained which can be specialized to give many of the known results, and which throw some light on the relation between measures introduced before.

In Part I we introduce the concept of a measure generated by a gauge and a filterbase, and with any such filterbase we define an associated topology for the space, the filterbase topology. We then impose different conditions on the filterbase and deduce resulting properties of the filterbase topology and of the measure. Measurability and approximation properties of the measure are first obtained

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in terms of the filterbase. Additional conditions on the filterbase are then applied to give results, stated in terms of the filterbase topology, on measurability of closed, closed $\mathscr{G}_\delta$, and compact $\mathscr{G}_\delta$ sets, and on approximation by $\mathscr{G}_\delta, \mathscr{F}_\sigma$, open and closed sets.

In Part II we consider measures generated on a topological space. In particular, we show that the Hausdorff measure in a metric space and the measures of Bledsoe and Morse [1], and of Rogers and Sion [7] are encompassed by the general theory of Part I and that some of the measurability and approximation results can be specialized to yield existing results for these measures.

2. **Preliminaries.** In this section we collect definitions, notation, and known or elementary results in set theory, topology, and measure theory which will be needed later.

2.1. DEFINITIONS AND NOTATION.

.1. $\varnothing$ denotes the empty set.

.2. ω denotes the set of natural numbers.

.3. $A \sim B = \{x\colon x \in A \text{ and } x \notin B\}$.

Let $\mathscr{B}$ be a family of sets. Then

.4. $\pi\mathscr{B} = \bigcap_{A \in \mathscr{B}} A$;

.5. $\sigma\mathscr{B} = \bigcup_{A \in \mathscr{B}} A$;

.6. $\mathscr{B}^\sim = \{A\colon A = \sigma\mathscr{B} \sim B \text{ for some } B \in \mathscr{B}\}$;

.7. $\mathscr{B}_\sigma = \{A\colon A = \bigcup_{n \in \omega} B_n$ for some sequence B of sets in $\mathscr{B}\}$;

.8. $\mathscr{B}_\delta = \{A\colon A = \bigcap_{n \in \omega} B_n$ for some sequence B of sets in $\mathscr{B}\}$;

.9. $\mathscr{B}_{\sigma\delta} = (\mathscr{B}_\sigma)_\delta$; $\mathscr{B}_{\delta\sigma} = (\mathscr{B}_\delta)_\sigma$;

.10. $\mathscr{B}$ is a *cover* of A iff $A \subset \sigma\mathscr{B}$;

.11. $\mathscr{A}$ *refines* $\mathscr{B}$ or $\mathscr{A}$ is a *refinement* of $\mathscr{B}$ iff for each $A \in \mathscr{A}$, there exists $B \in \mathscr{B}$ such that $A \subset B$;

.12. $\mathscr{B}$ is a *σ-field* iff $\mathscr{B}^\sim \subset \mathscr{B}$ and $\mathscr{B}_\sigma \subset \mathscr{B}$;

.13. *Borel* $\mathscr{B} = \pi\{\mathscr{A}\colon \mathscr{A}$ is a σ-field and $\mathscr{B} \subset \mathscr{A}\}$ is the smallest σ-field containing $\mathscr{B}$.

.14. $\mathscr{H}$ is a *filterbase* iff $\mathscr{H}$ is a nonempty family of sets such that for every $M \in \mathscr{H}$ and $N \in \mathscr{H}$, there exists $H \in \mathscr{H}$ such that $\varnothing \neq H \subset M \cap N$.

$\mathscr{H}$ is a *filterbase in* X iff $\mathscr{H}$ is a filterbase and for every $H \in \mathscr{H}$, H is a family of subsets of X and $\varnothing \in H$.

If $\mathscr{H}$ is a filterbase in X, then $\mathscr{M}$ is a *subfilterbase* of $\mathscr{H}$ iff $\mathscr{M}$ is a filterbase in X and for some $\mathscr{A}$,

$$\mathscr{M} = \{H \cap \mathscr{A}\colon H \in \mathscr{H}\}.$$

.15. If $(X, \mathscr{G})$ is a topological space, then $\mathscr{G}$ of course denotes the family of open sets. $\mathscr{F}$ will denote the family of closed sets.

.16. $\bar{A}$ or $\operatorname{Cl} A$ denotes the closure of A.

.17. μ is an *outer measure* on X iff μ is a function on the family of subsets of X such that

(i) $\mu\emptyset = 0$, and

(ii) $0 \leqq \mu A \leqq \sum_{n\in\omega} \mu B_n$ whenever $A \subset \bigcup_{n\in\omega} B_n \subset X$.

As all measures discussed in this paper will be outer measures, we will henceforth drop the qualifying word "outer."

.18. For μ a measure on X, a set A is *μ-measurable* iff $A \subset X$ and for every $B \subset X$,

$$\mu B = \mu(B \cap A) + \mu(B \sim A).$$

.19. For μ a measure on X,

$$\mathscr{M}_\mu = \{A \subset X\colon A \text{ is } \mu\text{-measurable}\}.$$

.20. $\mu | A$, the *restriction* of μ to A is the function ν having the same domain as μ such that for every B in the domain of μ, $\nu B = \mu(B \cap A)$.

.21. ν is a *finite submeasure* of μ iff for some A with $\mu A < \infty$, $\nu = \mu | A$.

.22. If $\mathscr{B}$ is a family of sets, τ is a *gauge* on $\mathscr{B}$ iff τ is a function on $\mathscr{B} \cup \{\emptyset\}$ to the extended non-negative real line, such that $\tau\emptyset = 0$.

.23. For μ a measure on X, μ is a *regular* measure iff for every $A \subset X$, there exists $B \in \mathscr{M}_\mu$ such that $A \subset B$ and $\mu A = \mu B$.

The following theorem is well known. (See, for example, Corollary 12.1.1 in Munroe [5].)

2.2. Theorem. *If μ is a regular measure on X and A is an ascending sequence of subsets of X, then*

$$\mu\Big(\bigcup_{n\in\omega} A_n\Big) = \lim_{n\to\infty} \mu A_n.$$

The following is a form of the well-known lemma of Carathéodory.

2.3. Lemma. *Suppose μ is a measure on X, and $A \subset X$. If for every $\varepsilon > 0$ and every $T \subset X$ such that $\mu T < \infty$, there exists a sequence D of subsets of X such that*

(1) *$D_{n+1} \subset D_n$ for every $n \in \omega$;*

(2) *$\bigcap_{n\in\omega} D_n \subset A$;*

(3) *$\mu(T \cap A) \leqq \mu(T \cap D_n) + \varepsilon$ for every $n \in \omega$; and*

(4) *for every $P \subset T$ and $n \in \omega$,*

$$\mu((P \cap D_{n+1}) \cup (P \sim D_n)) = \mu(P \cap D_{n+1}) + \mu(P \sim D_n),$$

then A is μ-measurable.

Proof. Let $\varepsilon > 0$, $T \subset X$, $\mu T < \infty$, $B = \bigcap_{n\in\omega} D_n$. We show

$$\mu(T \cap A) + \mu(T \sim A) \leqq \mu T + 2\varepsilon,$$

which implies that A is μ-measurable.

We obtain first

(5) There exists $N \in \omega$ such that

$$\mu(T \sim B) \leqq \mu(T \sim D_N) + \varepsilon.$$

Setting $P = T \cap D_n$ we have

$$\begin{aligned} \mu(T \cap D_n) &= \mu P \geqq \mu((P \cap D_{n+2}) \cup (P \sim D_{n+1})) && \text{by (1)} \\ &= \mu(P \cap D_{n+2}) + \mu(P \sim D_{n+1}) && \text{by (4)} \\ &= \mu(T \cap D_{n+2}) + \mu(T \cap D_n \sim D_{n+1}) && \text{by (1).} \end{aligned}$$

Hence for any $M \in \omega$,

$$\begin{aligned} \sum_{n=0}^{M} \mu(T \cap D_n \sim D_{n+1}) &\leqq \sum_{n=0}^{M} (\mu(T \cap D_n) - \mu(T \cap D_{n+2})) \\ &= \mu(T \cap D_0) + \mu(T \cap D_1) - \mu(T \cap D_{M+1}) - \mu(T \cap D_{M+2}) \\ &\leqq 2\mu(T \cap D_0) < \infty, \end{aligned}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} \mu(T \cap D_n \sim D_{n+1}) &= \lim_{M \to \infty} \sum_{n=0}^{M} \mu(T \cap D_n \sim D_{n+1}) \\ &\leqq 2\mu(T \cap D_0) < \infty. \end{aligned}$$

Choose $N \in \omega$ so that

$$\sum_{n=N}^{\infty} \mu(T \cap D_n \sim D_{n+1}) < \varepsilon.$$

Since

$$(T \cap D_N \sim B) = \bigcup_{n \geqq N} (T \cap D_n \sim D_{n+1}) \qquad \text{by (1),}$$

we have

$$\mu(T \cap D_N \sim B) < \varepsilon.$$

But

$$\mu(T \sim B) \leqq \mu(T \sim D_N) + \mu(T \cap D_N \sim B) \leqq \mu(T \sim D_N) + \varepsilon,$$

which establishes (5). Now

$$\begin{aligned} \mu(T \cap A) + \mu(T \sim A) &\leqq \mu(T \cap A) + \mu(T \sim B) && \text{since } B \subset A, \\ &\leqq \mu(T \cap D_{N+1}) + \varepsilon + \mu(T \sim D_N) + \varepsilon && \text{by (3) and (5)} \\ &= \mu((T \cap D_{N+1}) \cup (T \sim D_N)) + 2\varepsilon && \text{by (4)} \\ &\leqq \mu T + 2\varepsilon. \end{aligned}$$

PART I. THE MEASURE GENERATED BY A GAUGE AND A FILTERBASE

In Part I we start with an abstract space X, a filterbase $\mathscr{H}$ in X (see 2.14) and a gauge τ on some family $\mathscr{A}$ of subsets of X such that $\emptyset \in \mathscr{A}$ (see 2.22). From these we generate a measure and a topology on X, and then investigate properties of the measure and of the topology. In particular we obtain conditions under which certain topological sets, such as closed, closed $\mathscr{G}_\delta$ and compact $\mathscr{G}_\delta$ sets, are measurable (§7), and also results on the approximation of a given set from above and below by measurable sets or by topological sets (§8). The topology itself is studied first (§6) and the key result, used repeatedly later, is Theorem 6.1.2, which establishes conditions under which a certain natural family forms a base for the neighborhood system of a point. From this we determine when the topology is regular (6.1.4), Hausdorff (6.1.5), or generated by a uniformity (6.2).

3. **The measure ν.** We now introduce the measure generated on X by the filterbase $\mathscr{H}$ in X and the gauge τ on $\mathscr{A}$. We may assume without any loss of generality that $\mathscr{A} \subset \sigma\mathscr{H}$.

3.1. DEFINITION. For $H \in \mathscr{H}$ and $A \subset X$, let

1. $\nu_H^{(\mathscr{H},\tau)}A = \inf\{t: t = \sum_{B \in \mathscr{B}} \tau B$ for some countable $\mathscr{B} \subset H \cap \mathscr{A}$ such that $A \subset (\sigma\mathscr{B}\}$ (note: $\inf\emptyset = \infty$).

2. $\nu^{(\mathscr{H},\tau)}A = \sup_{H \in \mathscr{H}} \nu_H^{(\mathscr{H},\tau)}A$.

If no ambiguity can arise as a result, we will drop one or both superscripts on ν.

3.2. THEOREM. *ν is an outer measure on X.*

Proof. ν_H is constructed by Method I of Munroe [5, pp. 90, 91], and so, by Theorem 11.3 in Munroe, is an outer measure. Since ν is the supremum of such measures, it is again one.

3.3. REMARK. $\mathscr{H}$ is a set directed by inclusion, so $(\nu_H A, H \in \mathscr{H})$ is a net (see Kelley [4, Chapter 2]). It is an increasing net, i.e. $H, N \in \mathscr{H}$ and $H \subset N$ implies $\nu_H A \geqq \nu_N A$, so we have

$$\nu A = \sup_{H \in \mathscr{H}} \nu_H A = \lim_{H \in \mathscr{H}} \nu_H A.$$

4. **The filterbase topology.** We now use the filterbase $\mathscr{H}$ in X to introduce a topology on X, closely related to the measure ν.

4.1. DEFINITIONS.

1. For $H \in \mathscr{H}$, $x \in X$,

$$H[x] = \{x\} \cup \sigma\{h \in H : x \in h\}.$$

2. For $H \in \mathscr{H}$, $A \subset X$,

$$H[A] = \bigcup_{x \in A} H[x] = A \cup \sigma\{h \in H : h \cap A \neq \emptyset\}.$$

3. The $\mathscr{H}$-topology, $\mathscr{G}_{\mathscr{H}} = \{G \subset X\colon$ for every $x \in G$, there exists $H \in \mathscr{H}$ such that $H[x] \subset G\}$. The subscript $\mathscr{H}$ may be dropped if no ambiguity can result.

4.2. THEOREM. *The $\mathscr{H}$-topology is a topology for X.*

Proof. Clearly $\mathscr{G}_{\mathscr{H}}$ is closed under arbitrary unions. Suppose $B, G \in \mathscr{G}_{\mathscr{H}}$ and $x \in B \cap G$. Then there exist $H, N \in \mathscr{H}$ such that $H[x] \subset B$ and $N[x] \subset G$. Since $\mathscr{H}$ is a filterbase, there exists $M \in \mathscr{H}$ such that $M \subset H \cap N$. Referring to Definition 4.1.1 we see

$$M[x] \subset (H[x] \cap N[x]) \subset B \cap G,$$

so $B \cap G \in \mathscr{G}_{\mathscr{H}}$. Finally, $\varnothing, X \in \mathscr{G}_{\mathscr{H}}$.

We note that if for a point $x \in X$ there is $H \in \mathscr{H}$ such that $x \notin \sigma H$, i.e. no element of H covers x, then $\{x\}$ is both open and closed in the $\mathscr{H}$-topology.

REMARK. Throughout the remainder of Part I all topological concepts refer to the $\mathscr{H}$-topology.

The following lemmas follow directly from the definitions.

4.3. LEMMA. *If $H, H_1, H_2 \in \mathscr{H}$; for each $i \in I$, $A_i \subset X$; and $A \subset X$, $B \subset X$, then*

.1. $H\left[\bigcup_{i \in I} A_i\right] = \bigcup_{i \in I} H[A_i]$,

.2. $H_1[H_2[A]] = \bigcup_{x \in A} H_1[H_2[x]]$, *and*

.3. $H[A] \cap B = \varnothing$ *iff* $A \cap H[B] = \varnothing$.

5. **Conditions on the filterbase in** X. We now introduce conditions on $\mathscr{H}$ which will allow us to draw conclusions about the $\mathscr{H}$-topology and about properties of the measure ν.

(5I) Given $x \in X$ and $H \in \mathscr{H}$, there exist $H_1, H_2 \in \mathscr{H}$ such that

$$H_1[H_2[x]] \subset H[x].$$

(5II) Given $H \in \mathscr{H}$, there exist $H_1, H_2 \in \mathscr{H}$ such that for every $x \in X$,

$$H_1[H_2[x]] \subset H[x].$$

(We note that by 4.1.2 an equivalent statement would be that for every $A \subset X$, $H_1[H_2[A]] \subset H[A]$.)

(5III) If A is closed, B is open and $A \subset B$, then there exists $H \in \mathscr{H}$ such that $H[A] \subset B$.

(5IV) There exists a sequence H in $\mathscr{H}$ such that for every $N \in \mathscr{H}$, there exists $n \in \omega$ such that $H_n \subset N$.

(5V) Given an open cover of X, there exists $H \in \mathscr{H}$ which refines this cover.

5.1. REMARKS.

.1. If $\mathscr{H}$ satisfies (5II), then it satisfies (5I).

.2. If $\mathscr{H}$ satisfies (5V), then it satisfies (5III).

Proof. Suppose A is closed, B is open and $A \subset B$. Then $\mathscr{E} = \{B, X \sim A\}$ is an open cover of X. By (5V), there exists $H \in \mathscr{H}$ which refines $\mathscr{E}$. Now any element of $\mathscr{E}$, and hence also of H, which intersects A is contained in B so $H[A] \subset B$.

6. **Properties of the $\mathscr{H}$-topology.** In this section we deduce properties of the $\mathscr{H}$-topology which result from imposing conditions on $\mathscr{H}$.

6.1. THEOREM. *Suppose $\mathscr{H}$ satisfies* (5I).

.1. *If $H \in \mathscr{H}$, $A \subset X$, then there exists an open G such that $A \subset G \subset H[A]$.*

.2. *For $x \in X$, $\{H[x]; H \in \mathscr{H}\}$ is a base for the neighborhood system of x. ($H[x]$ itself may not be open. See Example* 6.4.)

.3. *For $A \subset X$, the closure of A,*

$$\bar{A} = \bigcap_{H \in \mathscr{H}} H[A],$$

and if for some sequence H in $\mathscr{H}$,

$$A = \bigcap_{n \in \omega} H_n[A],$$

then A is closed.

.4. *The $\mathscr{H}$-topology is regular.*

.5. *The $\mathscr{H}$-topology is Hausdorff iff*

$$\bigcap_{H \in \mathscr{H}} H[x] = \{x\} \qquad \textit{for each } x \in X.$$

Proof of 1. Given $x \in X$ and $H \in \mathscr{H}$, we show there exists an open set G such that $x \in G \subset H[x]$. Let

$$G = \{y \in X\colon \text{for some } N \in \mathscr{H},\ N[y] \subset H[x]\}.$$

Clearly $G \subset H[x]$. Let $y \in G$. Then for some $N \in \mathscr{H}$, $N[y] \subset H[x]$. Choose $N_1, N_2 \in \mathscr{H}$ such that

$$N_1[N_2[y]] \subset N[y].$$

Then for any $z \in N_2[y]$,

$$N_1[z] \subset N_1[N_2[y]] \subset H[x],$$

so $N_2[y] \subset G$. Hence G is open.

.2 follows immediately from 1. and the definition of the $\mathscr{H}$-topology.

Proof of .3. $\bar{A} \subset \bigcap_{H \in \mathscr{H}} H[A]$: Given $H \in \mathscr{H}$, suppose $x \notin H[A]$. Then $\{x\} \cap H[A] = \varnothing$, whence by Lemma 4.3.3, $H[x] \cap A = \varnothing$. By 6.1.1 there exists a neighborhood of x free of points of A and so $x \notin \bar{A}$. We conclude that $\bar{A} \subset H[A]$ for every $H \in \mathscr{H}$.

$\bar{A} \supset \bigcap_{H \in \mathscr{H}} H[A]$: Suppose $x \notin \bar{A}$. Then since $X \sim \bar{A}$ is open, there exists $H \in \mathscr{H}$ such that $H[x] \cap \bar{A} = \varnothing$, by Definition 4.1.3. Again using Lemma 4.3.3 we have $x \notin H[\bar{A}]$. But

$$H[\bar{A}] \supset H[A] \supset \bigcap_{H \in \mathscr{H}} H[A],$$

and hence $x \notin \bigcap_{H \in \mathscr{H}} H[A]$.

If for some sequence H in $\mathscr{H}$,

$$\mathrm{A} = \bigcap_{n \in \omega} H_n[A],$$

then

$$\mathrm{A} \subset \bigcap_{H \in \mathscr{H}} H[A] \subset \bigcap_{n \in \omega} H_n[A] = A,$$

and $A = \bar{A}$.

Proof of .4. Let A be closed, $x \notin A$. By definition there exists $H \in \mathscr{H}$ such that $H[x] \cap A = \varnothing$. Choose $H_1, H_2 \in \mathscr{H}$ such that

$$H_1[H_2[x]] \subset H[x].$$

Then

$$H_1[H_2[x]] \cap A = \varnothing,$$

and so by Lemma 4.3.3,

$$H_2[x] \cap H_1[A] = \varnothing.$$

By 6.1.1 there exist disjoint open sets G_2 and G_1 such that

$$x \in G_2 \subset H_2[x] \text{ and } A \subset G_1 \subset H_1[A].$$

Proof of .5. Suppose the $\mathscr{H}$-topology is Hausdorff and $x \in X$. For any $y \neq x$, there exists $H \in \mathscr{H}$ such that $y \notin H[x]$. Hence $y \notin \bigcap_{H \in \mathscr{H}} H[x]$. (This does not use condition (5I).)

Now suppose $\bigcap_{H \in \mathscr{H}} H[x] = \{x\}$ for each $x \in X$. Then by .3 and .4, the $\mathscr{H}$-topology is T_1 and regular, and hence Hausdorff.

6.2. REMARK. (5II) is a uniform condition, and it is not hard to show, by constructing an appropriate uniformity whose topology is the $\mathscr{H}$-topology, that with condition (5II) the $\mathscr{H}$-topology is completely regular.

6.3. LEMMA. *If $\mathscr{H}$ satisfies* (5I) *and* (5IV), *and A is closed, then there exists a sequence H in $\mathscr{H}$ such that*

$$A = \bigcap_{n \in \omega} H_n[A].$$

Proof. Using (5IV) let H be a sequence in $\mathscr{H}$ such that for every $N \in \mathscr{H}$, there exists $n \in \omega$ such that $H_n \subset N$. Then since A is closed we have by 6.1.3

$$A = \bar{A} = \bigcap_{N \in \mathscr{H}} N[A] \supset \bigcap_{n \in \omega} H_n[A] \supset A.$$

6.4. EXAMPLE. Let $X = R$, $H_r = \{\{x, y\} : |x - y| \leqq r\} \cup \{\varnothing\}$, $\mathscr{H} = \{H_r : r > 0\}$. For $A \in \sigma\mathscr{H}$ let

$$\tau A = \begin{cases} \operatorname{diam} A \text{ if } A \neq \varnothing \\ 0 \text{ if } A = \varnothing. \end{cases}$$

Then $\mathscr{H}$ is a filterbase in X; the $\mathscr{H}$-topology is the usual topology; for any $x \in X$, $r > 0$,

$$H_r[x] = [x - r, x + r],$$

a closed neighborhood of x; $\mathscr{H}$ satisfies the three conditions (5I), (5II), and (5IV) but not (5III) or (5V); τ is a gauge on $\sigma\mathscr{H}$; and for $A \subset X$,

$$\nu^{(\mathscr{H}, \tau)} A = \begin{cases} 0 \text{ if } A \text{ is countable} \\ \infty \text{ if } A \text{ is uncountable.} \end{cases}$$

7. **Measurability theorems.** The following definition and lemma are taken from a paper by Bledsoe and Morse [1].

7.1. DEFINITION. For ϕ a measure on X, A is *ϕ-compact* iff $A \subset X$ and given any $\varepsilon > 0$, finite submeasure θ of ϕ, and open cover $\mathscr{B}$ of A, there is a finite subfamily $\mathscr{E}$ of $\mathscr{B}$ such that

$$\theta A \leqq \theta(A \cap \sigma\mathscr{E}) + \varepsilon.$$

7.2. LEMMA. *A closed subset of a ϕ-compact set is ϕ-compact.*

We first state two theorems and a corollary on ν-measurability of sets characterized in terms of the filterbase $\mathscr{H}$.

7.3. THEOREM. *If for some sequence B,*

$$A = \bigcap_{n \in \omega} B_n$$

where for each $n \in \omega$ there exists $M_{n+1} \in \mathscr{H}$ such that

$$M_{n+1}[B_{n+1}] \subset B_n \subset X,$$

then A is ν-measurable.

7.4. COROLLARY. *If $\mathscr{H}$ satisfies* (5II), *$A \subset X$, and for some sequence H in $\mathscr{H}$,*

$$A = \bigcap_{n \in \omega} H_n[A],$$

then A is ν-measurable.

7.5. THEOREM. *If $\mathscr{H}$ satisfies* (5I), *A is ν-compact and for some sequence H in $\mathscr{H}$,*

$$A = \bigcap_{n \in \omega} H_n[A],$$

then A is ν-measurable.

We now relate the restrictions on A in the above theorems to topological properties of A and, using additional conditions on $\mathscr{H}$, we obtain a number of theorems on the measurability of purely topological sets.

7.6. THEOREM. *If $\mathscr{H}$ satisfies* (5I), *then compact $\mathscr{G}_\delta$ sets are ν-measurable.*

7.7. THEOREM. *If $\mathscr{H}$ satisfies* (5II) *and* (5III), *then closed $\mathscr{G}_\delta$ sets are ν-measurable.*

7.8. THEOREM. *If $\mathscr{H}$ satisfies* (5II) *and* (5IV), *then closed sets are ν-measurable.*

7.9. THEOREM. *If $\mathscr{H}$ satisfies* (5I) *and* (5V), *then closed $\mathscr{G}_\delta$ sets are ν-measurable.*

7.10. THEOREM. *If $\mathscr{H}$ satisfies* (5I), (5IV), *and* (5V), *then closed sets are ν-measurable.*

7.11. REMARKS. We note that if there is any subspace $X' \subset X$ which is such that for any $x \in X'$, there is some $H \in \mathscr{H}$ such that no element of H covers x, i.e. $x \notin \sigma H$, then for every nonempty $A \subset X', \nu A = \infty$; and by the comment at the end of Theorem 4.2, $\mathscr{G}_{\mathscr{H}}$ is discrete on X'. Thus the discrete topology on X' reflects the fact that all subsets of X' are ν-measurable.

Now it may happen as a result of the nature of the domain $\mathscr{A}$ of τ that the class of measurable sets is larger than that given us by any of the Theorems 7.6 to 7.10, using the filterbase $\mathscr{H}$. (For example, if $\mathscr{A}$ is the family of singletons, then all subsets of X are ν-measurable, a result which is independent of the filterbase $\mathscr{H}$.) In this case, it may be of some advantage to consider the subfilterbase of $\mathscr{H}$,

$$\mathscr{N} = \{H \cap \mathscr{A} : H \in \mathscr{H}\}.$$

Evidently the measure $\nu^{(\mathscr{N},\tau)} = \nu^{(\mathscr{H},\tau)}$, but the $\mathscr{N}$-topology $\mathscr{G}_{\mathscr{N}}$, may be strictly larger than $\mathscr{G}_{\mathscr{H}}$. If this is the case, and if $\mathscr{N}$ satisfies the requisite conditions, we may be able to apply one of the Theorems 7.6 to 7.10 with the filterbase $\mathscr{N}$ to obtain a stronger result than that obtained using $\mathscr{H}$. (For example, if in the case above where $\mathscr{A}$ is the family of singletons, we form the filterbase $\mathscr{N}$, then trivially $\mathscr{N}$ satisfies (5II) and (5IV), and $\mathscr{G}_{\mathscr{N}}$ is the discrete topology. Then by Theorem 7.8, all subsets of X are ν-measurable.) However, $\mathscr{N}$ may not satisfy enough conditions to allow us to apply any theorems from Part I (see Example 9.6), so we cannot always automatically use $\mathscr{N}$ to get stronger measurability results.

Again, it may happen that although $\mathcal{N}$ itself does not satisfy enough conditions, another filterbase $\mathcal{M}$ can be found such that

(i) $\mathcal{N}$ is a subfilterbase of $\mathcal{M}$ so that $\nu^{(\mathcal{M})} = \nu^{(\mathcal{N})} = \nu^{(\mathcal{H})}$,

(ii) $\mathcal{G}_{\mathcal{M}}$ is strictly larger than $\mathcal{G}_{\mathcal{H}}$, and

(iii) $\mathcal{M}$ satisfies conditions allowing application of some theorem giving a stronger result than that obtained using $\mathcal{H}$ (see Example 9.6). Unfortunately, we know of no general method in such a case of choosing a filterbase in X, optimum in the sense that using it we obtain the largest possible class of measurable sets.

We note also that the nature of $\mathcal{A}$ may result in a large class of measurable sets at the same time that the $\mathcal{N}$-topology is no larger than the $\mathcal{H}$-topology, i.e. the $\mathcal{N}$-topology may not be large enough to reflect the class of measurable sets. (For instance, if in Example 6.4 H_r consisted of all sets of diameter $\leqq r$, while $\mathcal{A}$ consisted of all doubletons, the same measure would be obtained, under which all subsets of X are measurable, while both the $\mathcal{H}$ and $\mathcal{N}$-topologies would be the usual topology, and the best theorem obtainable would be 7.8, giving closed sets measurable.)

PROOFS

7.12. LEMMA. *If $A \subset X$, $B \subset X$, and there exists $M \in \mathcal{H}$ such that $M[A] \cap B = \varnothing$, then $\nu(A \cup B) = \nu A + \nu B$.*

Proof. Suppose $M \in \mathcal{H}$, $M[A] \cap B = \varnothing$, and $\nu(A \cup B) < \infty$. Let $N \in \mathcal{H}$, $N \subset M$. Then also $N[A] \cap B = \varnothing$. By Definition 4.1.2 no $h \in N$ can intersect both A and B, so any cover of $A \cup B$ by elements of $N \cap \mathcal{A}$ can be separated into disjoint covers of A and B. Checking 3.1.1 we see that

$$\nu_N(A \cup B) \geqq \nu_N A + \nu_N B.$$

Since ν_N is a measure, we have the inequality the other way also, whence

$$\nu_N(A \cup B) = \nu_N A + \nu_N B,$$

for every $N \in \mathcal{H}$ such that $N \subset M$. Hence by Remark 3.3

$$\begin{aligned} \nu(A \cup B) &= \lim_{N \in \mathcal{H}} \nu_N(A \cup B) = \lim_{N \in \mathcal{H}} (\nu_N A + \nu_N B) \\ &= \lim_{N \in \mathcal{H}} \nu_N A + \lim_{N \in \mathcal{H}} \nu_N B = \nu A + \nu B. \end{aligned}$$

Proof of 7.3. We use Lemma 2.3 with $D_n = B_n$ for each $n \in \omega$. Let $\varepsilon > 0$ and $T \subset X$ such that $\nu T < \infty$. To check (4) note

$$M_{n+1}[B_{n+1}] \cap (X \sim B_n) = \varnothing \qquad \text{for each } n \in \omega,$$

from which it follows that for each $n \in \omega$ and $P \subset T$,

$$M_{n+1}[P \cap B_{n+1}] \cap (P \sim B_n) = \varnothing.$$

Applying Lemma 7.12 we obtain

$$\nu((P \cap B_{n+1}) \cup (P \sim B_n)) = \nu(P \cap B_{n+1}) + \nu(P \sim B_n)$$

for all $n \in \omega$ and $P \subset T$.

Proof of 7.4. We show first that A can be put in the form

$$A = \bigcap_{n \in \omega} N_n[A],$$

where N is a sequence in $\mathscr{H}$ and for each $n \in \omega$

$$N_{n+1}[N_{n+1}[A]] \subset N_n[A].$$

We construct the sequence N by recursion.

Let $N_0 = H_0$ and suppose we have $N_i \in \mathscr{H}$ for $i = 1, \cdots, n$ such that

$$N_i[A] \subset H_i[A] \qquad \text{for } i = 0, \cdots, n,$$

and

$$N_{i+1}[N_{i+1}[A]] \subset N_i[A] \qquad \text{for } i = 1, \cdots, n-1.$$

We choose N_{n+1} as follows:

Using (5II) choose $M \in \mathscr{H}$ such that

$$M[M[A]] \subset N_n[A].$$

Then choose $N_{n+1} \in \mathscr{H}$ such that $N_{n+1} \subset M \cap H_{n+1}$. We have

(i) $N_{n+1}[A] \subset H_{n+1}[A]$, and

(ii) $N_{n+1}[N_{n+1}[A]] \subset N_n[A]$.

Now (i) and (ii) will be true for all $n \in \omega$. From (i) we have

$$A \subset \bigcap_{n \in \omega} N_n[A] \subset \bigcap_{n \in \omega} H_n[A] = A$$

and so

$$A = \bigcap_{n \in \omega} N_n[A].$$

Setting $B_n = N_n[A]$, the conclusion follows by application of Theorem 7.3.

Proof of 7.5. Let $T \subset X$, $\nu T < \infty$, $\varepsilon > 0$, and $\theta = \nu | T$. We employ Lemma 2.3 to show that A is ν-measurable. Sequences C, D, M, and N are constructed by recursion. To start we set $C_0 = C_1 = A$; $M_0 = M_1 = H_0$; $H_0 \supset N_0 = N_1 \in \mathscr{H}$; $D_0 = X$; and $D_1 = M_1[C_1] = H_0[A]$. Having obtained C_i, D_i, $M_i \in \mathscr{H}$ and $N_i \in \mathscr{H}$ satisfying

(a) C_i is closed for $i = 0, \cdots, n$ ($C_0 = A$ is closed by 6.1.3),

(b) $C_{i+1} \subset C_i \subset A$ for $i = 0, \cdots, n-1$;

(c) $\theta C_{i-1} \leqq \theta C_i + \varepsilon/2^{i-1}$ for $i = 1, \cdots, n$,

(d) $D_i = M_i[C_i] \subset H_{i-1}[A]$ for $i = 1, \cdots, n$, and

(e) $N_i[D_i] \subset D_{i+1}$ for $i = 1, \cdots, n$,

we construct C_{n+1}, D_{n+1}, M_{n+1}, and N_{n+1} as follows:

For each $x \in C_n$ choose, using (5I), H_{x1}, H_{x2}, H_{x3}, and $H_{x4} \in \mathscr{H}$ such that

$$H_{x1}[H_{x2}[H_{x3}[H_{x4}[x]]]] \subset M_n[x] \subset M_n[C_n] \subset D_n.$$

By (a) and Lemma 7.2, C_n is ν-compact. Since $\mathscr{H}$ satisfies (5I), for each $x \in C_n$ there is, by 6.1.1, open G_x such that

$$x \in G_x \subset H_{x4}[x].$$

Hence $\{G_x : x \in C_n\}$ is an open cover of C_n and by Definition 7.1 there is a finite subset $Q \subset C_n$ such that

$$\theta C_n \leqq \theta\Big(C_n \cap \bigcup_{x \in Q} G_x\Big) + \frac{\varepsilon}{2^n},$$

and so

$$\theta C_n \leqq \theta\Big(C_n \cap \bigcup_{x \in Q} H_{x4}[x]\Big) + \frac{\varepsilon}{2^n}.$$

Now set

$$C_{n+1} = \bigcap_{H \in \mathscr{H}} H\Big[C_n \cap \bigcup_{x \in Q} H_{x4}[x]\Big] = \mathrm{Cl}\Big(C_n \cap \bigcup_{x \in Q} H_{x4}[x]\Big),$$

and choose $M_{n+1} \in \mathscr{H}$, $N_{n+1} \in \mathscr{H}$ such that

$$M_{n+1} \subset \Big(\bigcap_{x \in Q} H_{x2}\Big) \cap H_n,$$

$$N_{n+1} \subset \bigcap_{x \in Q} H_{x1},$$

and set

$$D_{n+1} = M_{n+1}[C_{n+1}].$$

We now check:

(a) C_{n+1} is closed.

(b) $C_{n+1} \subset C_n \subset A$ since $C_{n+1} \subset \bar{C}_n = C_n \subset A$.

(c) $\theta C_n \leqq \theta C_{n+1} + \varepsilon/2^n$ since $C_{n+1} \supset (C_n \cap \bigcup_{x \in Q} H_{x4}[x])$.

(d) $D_{n+1} = M_{n+1}[C_{n+1}] \subset H_n[A]$ since $C_{n+1} \subset A$, $M_{n+1} \subset H_n$.

(e) $N_{n+1}[D_{n+1}] \subset D_n$: first,

$$C_{n+1} \subset \bigcap_{H \in \mathscr{H}} H\Big[\bigcup_{x \in Q} H_{x4}[x]\Big] = \bigcup_{x \in Q} \bigcap_{H \in \mathscr{H}} H[H_{x4}[x]] \subset \bigcup_{x \in Q} H_{x3}[H_{x4}[x]].$$

The equality is obtained using Theorem 6.1.3 and the fact that the closure of a finite union is the union of the individual closures. Now

$$\begin{aligned} N_{n+1}[D_{n+1}] &= N_{n+1}[M_{n+1}[C_{n+1}]] \\ &\subset N_{n+1}\Big[M_{n+1}\Big[\Big(\bigcup_{x \in Q} H_{x3}[H_{x4}[x]]\Big)\Big]\Big] \\ &= \bigcup_{x \in Q} N_{n+1}[M_{n+1}[H_{x3}[H_{x4}[x]]]] \\ &\subset \bigcup_{x \in Q} H_{x1}[H_{x2}[H_{x3}[H_{x4}[x]]]] \subset D_n. \end{aligned}$$

The second to last inclusion follows from the choice of M_{n+1} and N_{n+1}, and the equality from Lemma 4.3.1.

The completed sequences satisfy (a), (b), (c), (d), and (e) for each $n \in \omega$. We now check that the sequence D satisfies the hypotheses of Lemma 2.3.

(1) $D_{n+1} \subset D_n$ by (e).

(2) $\bigcap_{n \in \omega} D_n \subset \bigcap_{n \in \omega} H_n[A] = A$ follows from (d).

(3) Using $A = C_1$, (c) and induction, we have

$$\theta A \leqq \theta C_n + \varepsilon\left(1 - \frac{1}{2^{n-1}}\right) < \theta C_n + \varepsilon \qquad \text{for every } n \in \omega,$$

or

$$\nu(T \cap A) \leqq \nu(T \cap C_n) + \varepsilon \qquad \text{for every } n \in \omega.$$

Since $C_n \subset D_n$ by (d), we have finally

$$\nu(T \cap A) \leqq \nu(T \cap D_n) + \varepsilon \qquad \text{for every } n \in \omega.$$

(4) It follows from (e) that

$$N_n[P \cap D_n] \cap (P \sim D_{n-1}) = \varnothing \quad \text{for any } P \subset T \text{ and } n \geqq 1.$$

Lemma 7.12 then gives us

$$\nu((P \cap D_n) \cup (P \sim D_{n-1})) = \nu(P \cap D_n) + \nu(P \sim D_{n-1}) \qquad \text{for every } P \subset T \text{ and } n \geqq 1.$$

Proof of 7.6. We show that for any compact $\mathscr{G}_\delta$ set A, there exists a sequence H in $\mathscr{H}$ such that

$$A = \bigcap_{n \in \omega} H_n[A],$$

where for each $n \in \omega$,

$$H_{n+1}[H_{n+1}[A]] \subset H_n[A].$$

Then setting $B_n = H_n[A]$, we apply Theorem 7.3 to obtain the conclusion.

Suppose A is compact, and

$$A = \bigcap_{n \in \omega} G_n$$

where for each $n \in \omega$, G_n is open.

We assume $G_0 = X$, set $H_0 = \{X\} \cup \sigma\mathscr{H}$ and construct H_n recursively as follows:

For each $x \in A$, using (5I) choose $H_{nx} \in \mathscr{H}$ such that

$$H_{nx}[H_{nx}[x]] \subset G_n \quad \text{and}$$

$$H_{nx}[H_{nx}[H_{nx}[x]]] \subset H_{n-1}[x].$$

Now each $H_{nx}[x]$ contains an open set containing x by Theorem 6.1.1 and since A is compact, a finite number of these open sets and hence of the sets $H_{nx}[x]$ covers A, i.e. there exists finite $Q \subset A$ such that

$$A \subset \bigcup_{x \in Q} H_{nx}[x].$$

Now choose $H_n \subset \bigcap_{x \in Q} H_{nx}$, $H_n \in \mathscr{H}$. Then

$$H_n[A] \subset H_n \left[\bigcup_{x \in Q} H_{nx}[x]\right] = \bigcup_{x \in Q} H_n[H_{nx}[x]]$$

by Lemma 4.3.1. Since $H_n \subset H_{nx}$ for each $x \in Q$,

$$H_n[A] \subset \bigcup_{x \in Q} H_{nx}[H_{nx}[x]] \subset G_n.$$

Similarly,

$$H_n[H_n[A]] \subset \bigcup_{x \in Q} H_{nx}[H_{nx}[H_{nx}[x]]] \subset H_{n-1}[A].$$

Then

$$A \subset \bigcap_{n \in \omega} H_n[A] \subset \bigcap_{n \in \omega} G_n = A,$$

and so

$$A = \bigcap_{n \in \omega} H_n[A].$$

Proof of 7.7. Condition (5III) guarantees that for any closed $\mathscr{G}_\delta$ set A, there is a sequence H in $\mathscr{H}$ such that

$$A = \bigcap_{n \in \omega} H_n[A].$$

The result follows by application of Corollary 7.4.

Proof of 7.8. We know by Lemma 6.3 that for every closed set A, there exists a sequence H in $\mathscr{H}$ such that

$$A = \bigcap_{n \in \omega} H_n[A].$$

The conclusion follows from Corollary 7.4.

Proof of 7.9. (i) By 5.1.2 $\mathscr{H}$ satisfies (5III).

(ii) As in the proof of 7.7, if A is a closed $\mathscr{G}_\delta$ set, then for some sequence H in $\mathscr{H}$

$$A = \bigcap_{n \in \omega} H_n[A].$$

(iii) X is ν-compact.

Let $T \subset X$, $\nu T < \infty$, $\theta = \nu | T$, $\varepsilon > 0$, and $\mathscr{L}$ be an open cover of X. Using (5V) choose $N \in \mathscr{H}$, N refining $\mathscr{L}$. Now choose $M \in \mathscr{H}$ such that

$$vT \leqq v_M T + \frac{\varepsilon}{2}.$$

Choose $H \in \mathscr{H}$, $H \subset N \cap M$, so by Remark 3.3,

(a) $vT \leqq v_H T + \varepsilon/2$.

Since $v_H T < \infty$, choose countable $\mathscr{B} \subset H \cap \mathscr{A}, \mathscr{B} = \{B_i\}_{i \in \omega}$, such that $T \subset \sigma\mathscr{B}$ and

$$v_H T \leqq \sum_{i \in \omega} \tau B_i < \infty.$$

Now choose $k \in \omega$ such that

$$\sum_{i=k+1}^{\infty} \tau B_i < \frac{\varepsilon}{2}.$$

Since N is a refinement of $\mathscr{L}$ and $H \subset N$, for each $i \leqq k$ choose $G_i \in \mathscr{L}$ such that $B_i \subset G_i$ and let

$$\mathscr{E} = \{G_i : i \leqq k\}.$$

Now

$$(T \sim \sigma\mathscr{E}) \subset \sigma\{B_i : i > k\}$$

and so

$$v_H(T \sim \sigma\mathscr{E}) \leqq \frac{\varepsilon}{2}.$$

Hence

$$v_H T \leqq v_H(T \cap \sigma\mathscr{E}) + v_H(T \sim \sigma\mathscr{E}) \leqq v_H(T \cap \sigma\mathscr{E}) + \frac{\varepsilon}{2} \leqq v(T \cap \sigma\mathscr{E}) + \frac{\varepsilon}{2},$$

and by (a),

$$vT \leqq v(T \cap \sigma\mathscr{E}) + \varepsilon.$$

Hence

$$\theta X \leqq \theta(X \cap \sigma\mathscr{E}) + \varepsilon$$

and by Definition 7.1, X is v-compact.

The desired conclusion now follows from (ii), (iii), Lemma 7.2 and Theorem 7.5.

Proof of 7.10. We know from the proof of Theorem 7.9 that if $\mathscr{H}$ satisfies (5V), then closed sets are v-compact, and from Lemma 6.3 that if $\mathscr{H}$ satisfies (5I) and (5IV), then for every closed set A, there is a sequence H in $\mathscr{H}$ such that

$$A = \bigcap_{n \in \omega} H_n[A].$$

The conclusion follows by application of Theorem 7.5.

8. Approximation theorems. We consider first several theorems on approximation from outside in which the only restriction on the set to be approximated is that its measure be finite. The restriction that elements of $\mathscr{A}$ be v-measurable sets is necessary in all the theorems of this section but the first.

8.1. THEOREM. *Suppose $\mathscr{H}$ satisfies (5IV) and $A \subset X$. If for every $H \in \mathscr{H}$ there is a countable subfamily of $H \cap \mathscr{A}$ which covers A (in particular if $vA < \infty$), then there exists $B \in \mathscr{A}_{\sigma\delta}$ such that $B \supset A$ and $vB = vA$.*

8.2. COROLLARY. *If* $\mathscr{H}$ *satisfies* (5IV) *and* $\mathscr{A} \subset \mathscr{M}_\nu$, *then* ν *is a regular measure.*

8.3. THEOREM. *Suppose* $\mathscr{A} \subset \mathscr{M}_\nu$, $\nu A < \infty$, *and* $E \subset A$. *Then given* $\varepsilon > 0$, *there exists* $B \in \mathscr{A}_\sigma$ *such that* $E \subset B$ *and* $\nu(A \cap B) \leqq \nu E + \varepsilon$.

8.4. COROLLARY. *Suppose* $\mathscr{A} \subset \mathscr{M}_\nu$, $\nu A < \infty$, *and* $E \subset A$. *Then there exists* $D \in \mathscr{A}_{\sigma\delta}$ *such that* $E \subset D$ *and* $\nu(A \cap D) = \nu E$.

8.5. COROLLARY. *If* $X = \bigcup_{n \in \omega} A_n$ *where for each* $n \in \omega$, $A_n \in \mathscr{M}_\nu$ *and* $\nu A_n < \infty$, *and* $\mathscr{A} \subset \mathscr{M}_\nu$, *then* ν *is a regular measure.*

By putting further restrictions on the approximated set, we can get the following results on approximation from inside.

8.6. THEOREM. *Suppose* $\mathscr{A} \subset \mathscr{B} \subset \mathscr{M}_\nu$, $A \in (\mathscr{B}_\sigma)^\sim$ (*see* 2.1.6), $\nu A < \infty$, $E \subset A$, *and* $E \in \mathscr{M}_\nu$. *Then given* $\varepsilon > 0$, *there exists* $C \in (\mathscr{B}_\sigma)^\sim$ *such that* $C \subset E$ *and* $\nu(E \sim C) < \varepsilon$.

8.7. THEOREM. *Suppose* $\mathscr{A} \subset \mathscr{B} \subset \mathscr{M}_\nu$, $A \in (\mathscr{B}_{\sigma\delta})^\sim$, $\nu A < \infty$, $E \subset A$, *and* $E \in \mathscr{M}_\nu$. *Then there exists* $C \in (\mathscr{B}_{\sigma\delta})^\sim$ *such that* $C \subset E$ *and* $\nu(E \sim C) = 0$.

8.8. THEOREM. *Suppose* $\mathscr{A} \subset \mathrm{Borel}\,\mathscr{B} \subset \mathscr{M}_\nu$, $A \in \mathrm{Borel}\,\mathscr{B}$ (*see* 2.1.13), *and* $\nu A < \infty$. *Then for each* $E \subset A$ *there exists* $B \in \mathrm{Borel}\,\mathscr{B}$ *such that* $E \subset B$ *and* $\nu E = \nu B$; *and for each* ν-*measurable* $E \subset A$, *there exists* $C \in \mathrm{Borel}\,\mathscr{B}$ *such that* $C \subset E$ *and* $\nu(E \sim C) = 0$.

If it happens that the sets of $\mathscr{A}$ have some topological properties and are ν-measurable (e.g. the open sets in the classical Hausdorff measure theory), we obtain in the above theorems approximating sets which also have topological properties. If in our hypotheses we restrict $\mathscr{A}$ to open sets, require that open sets be ν-measurable and put additional restrictions on $\mathscr{H}$ and X, we obtain some sharper results. (Recall that $\mathscr{G}$ and $\mathscr{F}$ denote respectively the families of open and closed sets.)

8.9. THEOREM. *Suppose* $\mathscr{A} \subset \mathscr{G} \subset \mathscr{M}_\nu$, $\mathscr{H}$ *satisfies* (5I) *and* (5IV), $E \subset X$, $\nu E < \infty$, *and* $E \in \mathscr{M}_\nu$. *Then there exist* $A \in \mathscr{G}_\delta$ *such that* $A \supset E$ *and* $\nu(A \sim E) = 0$, *and* $C \in \mathscr{F}_\sigma$ *such that* $C \subset E$ *and* $\nu(E \sim C) = 0$.

8.10. COROLLARY. *Suppose* $\mathscr{A} \subset \mathscr{G} \subset \mathscr{M}_\nu$, $\mathscr{H}$ *satisfies* (5I) *and* (5IV), $E \subset X$, $E \in \mathscr{M}_\nu$, *and* X *is* σ-*finite. Then the conclusions of Theorem* 8.9 *still hold.*

It is not the case that the existence of a $\mathscr{G}_\delta$ set covering $E \subset X$ and having the same measure implies that given $\varepsilon > 0$, there exists $G \in \mathscr{G}$ such that $G \supset E$ and $\nu G < \nu E + \varepsilon$. It may happen that all nonempty open sets have infinite measure (as, for example, with counting measure on R or on the rationals, and Hausdorff

$\frac{1}{2}$-dimensional measure on R). To obtain this conclusion we need an additional restriction on the space.

8.11. THEOREM. *Suppose* $\mathscr{A} \subset \mathscr{G} \subset \mathscr{M}_\nu$, $\mathscr{H}$ *satisfies* (5I) *and* (5IV), $E \subset X$, $E \in \mathscr{M}_\nu$, *and* $X = \bigcup_{n \in \omega} A_n$, *where for each* $n \in \omega$, $\nu A_n < \infty$ *and* $A_n \in \mathscr{G}$. *Then given* $\varepsilon > 0$, *there exist open* $G \supset E$ *such that* $\nu(G \sim E) < \varepsilon$ *and closed* $F \subset E$ *such that* $\nu(E \sim F) < \varepsilon$.

PROOFS

Proof of 8.1. Using (5IV), choose a sequence H in $\mathscr{H}$ such that for every $N \in \mathscr{H}$, there exists $n \in \omega$ such that $H_n \subset N$. For each $n \in \omega$ choose countable $\mathscr{B}_n \subset H_n \cap \mathscr{A}$ such that $A \subset \sigma\mathscr{B}_n$ and

$$\nu_{H_n}(\sigma\mathscr{B}_n) \leqq \sum_{D \in \mathscr{B}_n} \tau D \leqq \nu_{H_n} A + \frac{1}{n}.$$

Let

$$B = \bigcap_{n=0}^{\infty} (\sigma\mathscr{B}_n) \in \mathscr{A}_{\sigma\delta}.$$

Then $B \supset A$ and

$$\nu_{H_n} B \leqq \nu_{H_n}(\sigma\mathscr{B}_n) \leqq \nu_{H_n} A + \frac{1}{n} \qquad \text{for every } n \in \omega.$$

By Remark 3.3, taking the limit as $n \to \infty$ gives

$$\nu B \leqq \nu A.$$

Since $B \supset A$, we have $\nu B \geqq \nu A$, and so $\nu B = \nu A$.

Proof of 8.2. 8.2 is a direct consequence of 8.1.

Proof of 8.3. Choose $H \in \mathscr{H}$ such that

$$\nu A \leqq \nu_H A + \frac{\varepsilon}{2}.$$

Suppose $B \subset X$ is ν-measurable. Then

$$\nu(A \cap B) + \nu(A \sim B) = \nu A \leqq \nu_H A + \frac{\varepsilon}{2} \leqq \nu_H(A \cap B) + \nu_H(A \sim B) + \frac{\varepsilon}{2}$$

$$\leqq \nu_H(A \cap B) + \nu(A \sim B) + \frac{\varepsilon}{2}$$

Cancelling $\nu(A \sim B)$ in the first and last expressions gives

(a) $\nu(A \cap B) \leqq \nu_H(A \cap B) + \varepsilon/2$ for ν-measurable B. Now given $E \subset A$, choose countable $\mathscr{B} \subset H \cap \mathscr{A}$ such that $E \subset \sigma\mathscr{B} = B$ and

$$\sum_{D \in \mathscr{B}} \tau D < \nu_H E + \frac{\varepsilon}{2}.$$

But $B \in \mathscr{A}_\sigma$ and so is ν-measurable, whence by (a),

$$\begin{aligned}\nu(A \cap B) &\leqq \nu_H(A \cap B) + \frac{\varepsilon}{2} \leqq \nu_H B + \frac{\varepsilon}{2} \\ &\leqq \sum_{D \in \mathscr{B}} \tau D + \frac{\varepsilon}{2} \text{ (since } \mathscr{B} \text{ is a cover of } B) \\ &\leqq \nu_H E + \varepsilon \leqq \nu E + \varepsilon.\end{aligned}$$

8.4 follows immediately from 8.3, and 8.5 directly from 8.4.

Proof of 8.6. By Theorem 8.3 there exists $B \in \mathscr{A}_\sigma$ such that $A \sim E \subset B$ and

$$\nu(A \cap B) \leqq \nu(A \sim E) + \varepsilon.$$

Now $A \sim E$ is ν-measurable so

$$\nu(E \cap B) = \nu((A \cap B) \sim (A \sim E)) = \nu(A \cap B) - \nu(A \sim E) < \varepsilon.$$

Setting $C = A \sim B$ we have by Definition 2.1.6, $C \in (\mathscr{B}_\sigma)^\sim$ and since $E \cap B = E \sim C$,

$$\nu(E \sim C) < \varepsilon.$$

Proof of 8.7. The proof is identical to that of 8.6 except that the result of Corollary 8.4 is used instead of that of 8.3.

Proof of 8.8. We obtain B from Corollary 8.4 and C from Theorem 8.7.

Proof of 8.9. (i) Use Theorem 8.1 to choose $A \in \mathscr{A}_{\sigma\delta} \subset \mathscr{G}_\delta$ such that $A \supset E$ and $\nu A = \nu E$. Since E is ν-measurable and $\nu E < \infty$, we have $\nu(A \sim E) = 0$.

(ii) We show now that if $B \in \mathscr{G}_\delta$ and $\nu B < \infty$, there exists $D \in \mathscr{F}_\sigma$ such that $D \subset B$ and $\nu(B \sim D) = 0$.

By Lemma 6.3 and Theorem 6.1.1 we have $\mathscr{F} \subset \mathscr{G}_\delta$, so $\mathscr{G} \subset \mathscr{F}_\sigma$ and we may set, for $B \in \mathscr{G}_\delta$,

$$B = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} F(n,i),$$

where for each $n \in \omega$ and $i \in \omega$, $F(n,i) \in \mathscr{F}$. We may assume that $F(n,i+1) \supset F(n,i)$ for each $i \in \omega$.

By Corollary 8.2, ν is a regular measure, so by Theorem 2.2 we have for each $n \in \omega$,

$$\nu B = \nu\left(B \cap \bigcup_{i=1}^{\infty} F(n,i)\right) = \lim_i \nu(B \cap F(n,i)).$$

Hence for each $n \in \omega$ there exists a sequence i_n such that for each $k \in \omega$,

$$\nu(B \sim [B \cap F(n, i_{n_k})]) = \nu B - \nu(B \cap F(n, i_{n_k})) \leqq \frac{1}{k2^n}$$

since $\nu B < \infty$ and $B \cap F(n, i_{n_k})$ is ν-measurable.

Let

$$F(k) = \bigcap_{n=1}^{\infty} F(n, i_{n_k}) \qquad \text{for each } k \in \omega.$$

Then $F(k) \in \mathscr{F}$ and

$$F(k) \subset \bigcup_{i=1}^{\infty} F(n, i) \qquad \text{for every } n \in \omega,$$

whence $F(k) \subset B$ and

$$\begin{aligned} \nu B - \nu F(k) &= \nu(B \sim F(k)) = \nu\left(B \sim \bigcap_{n=1}^{\infty} F(n, i_{n_k})\right) \\ &= \nu\left(\bigcup_{i=1}^{\infty} [B \sim (B \cap F(n, i_{n_k}))]\right) \text{ by de Morgan's law,} \\ &\leqq \sum_{n=1}^{\infty} \frac{1}{k2^n} = \frac{1}{k}. \end{aligned}$$

Set

$$D = \bigcup_{k=1}^{\infty} F(k).$$

Then $D \in \mathscr{F}_\sigma$, $D \subset B$, and

$$\nu(B \sim D) = \nu B - \nu D \leqq \frac{1}{k} \qquad \text{for every } k \in \omega.$$

Hence

$$\nu(B \sim D) = 0.$$

Now let $E \in \mathscr{M}_\nu$, $\nu E < \infty$. Using (i), choose $B \in \mathscr{G}_\delta$ such that $B \supset E$ and $\nu B = \nu E$. Since $\nu(B \sim E) = 0$, we choose $Q \in \mathscr{G}_\delta$ such that $Q \supset B \sim E$ and $\nu Q = 0$. Using (ii), choose $D \in \mathscr{F}_\sigma$ such that $D \subset B$ and $\nu(B \sim D) = 0$. Now

$$\nu E = \nu B = \nu D = \nu D - \nu Q = \nu(D \sim Q).$$

Set $C = D \sim Q$. Then $C \in \mathscr{F}_\sigma$, $C \subset E$, $\nu C = \nu E$ and so

$$\nu(E \sim C) = 0.$$

Proof of 8.10. Let $X = \bigcup_{n \in \omega} A_n$ where for each $n \in \omega$, $\nu A_n < \infty$. By Theorem 8.1, choose for each $n \in \omega$, $B_n \in \mathscr{G}_\delta$ such that $B_n \supset A_n$ and $\nu B_n = \nu A_n$. Let $E_n = E \cap B_n$, so $E_n \in \mathscr{M}_\nu$ and $\nu E_n < \infty$. By 8.9 choose for each $n \in \omega$, $C_n \in \mathscr{F}_\sigma$ such that $C_n \subset E_n$ and $\nu(E_n \sim C_n) = 0$. Set

$$C = \bigcup_{n \in \omega} C_n.$$

Then $C \in \mathscr{F}_\sigma$, $C \subset E$ and

$$E \sim C \subset \bigcup_{n \in \omega} (E_n \sim C_n),$$

whence

$$v(E \sim C) \leqq \sum_{n=0}^{\infty} v(E_n \sim C_n) = 0.$$

We now apply this result to $X \sim E$ to obtain $A \in \mathscr{G}_\delta$ such that $A \supset E$ and $v(A \sim E) = 0$.

Proof of 8.11. For each $n \in \omega$ let $E_n = E \cap A_n$ and using 8.9 choose $B_n \in \mathscr{G}_\delta$ such that $B_n \supset E_n$ and $v(B_n \sim E_n) = 0$. Let

$$B_n = \bigcap_{i \in \omega} B_{n_i},$$

where for each $i \in \omega$, $B_{n_i} \in \mathscr{G}$ and $B_{n_i} \supset B_{n_{i+1}}$. Then

$$\infty > vE_n = vB_n \geqq v(A_n \cap B_n) = \lim_{i \to \infty} v(A_n \cap B_{n_i}).$$

Choose $i \in \omega$ such that

$$v(A_n \cap B_{n_i}) < vE_n + \frac{\varepsilon}{2^{n+1}}$$

and set

$$G_n = A_n \cap B_{n_i} \in \mathscr{G}.$$

Since E_n is v-measurable and $vE_n < \infty$,

$$v(G_n \sim E_n) < \frac{\varepsilon}{2^{n+1}}.$$

Set

$$G = \bigcup_{n \in \omega} G_n \in \mathscr{G}.$$

Then $E \subset G$,

$$(G \sim E) \subset \bigcup_{n \in \omega} (G_n \sim E_n),$$

and

$$v(G \sim E) \leqq \sum_{n \in \omega} v(G_n \sim E_n) < \varepsilon.$$

To obtain $F \in \mathscr{F}$, $F \subset E$, $v(E \sim F) < \varepsilon$, apply the above result to $X \sim E$.

PART I . MEASURES ON TOPOLOGICAL SPACES

In Part II we start with a topological space $(X,\mathscr{G})$ and a gauge τ on some family $\mathscr{A}$ of subsets of X such that $\varnothing \in \mathscr{A}$. Our aim is to study measures on X generated by τ and $\mathscr{G}$ through processes which are generalizations of the well-known Hausdorff process in a metric space (see Method II of Munroe [5, p. 105]).

We first consider the Hausdorff process itself, showing that the standard results can be obtained by application of the general theory developed in Part I. Generalizations of the process were introduced by Bledsoe and Morse [1] and by Rogers and Sion [7]. We show that each of these cases can be obtained as an application of the theory in Part I. More specifically, in each case we consider a filterbase $\mathscr{H}$ in X and see that the known measure is $\nu^{(\mathscr{H},\tau)}$. Since properties of $\nu^{(\mathscr{H},\tau)}$ are stated in terms of the $\mathscr{H}$-topology, it is important to study the relation between the given topology $\mathscr{G}$ and the $\mathscr{H}$-topology. In particular we determine how conditions on $\mathscr{G}$ affect the $\mathscr{H}$-topology and its relation to $\mathscr{G}$, thereby throwing some light on the role played by such conditions.

Finally, we investigate other approaches to generating a measure on a topological space using a quasi-uniformity, and relate the measures obtained thereby to those considered previously.

9. The measure ζ in a metric space. In this section we suppose that the topology $\mathscr{G}$ is induced by some metric ρ on X. All metric concepts refer to ρ.

The standard metric measure ζ generated by τ (Method II of Munroe [5, p. 105]), is given by

9.1. DEFINITION. For $A \subset X$ and $\delta > 0$

$$\zeta_\delta A = \inf \left\{ \sum_{i\in\omega} \tau B_i : A \subset \bigcup_{i\in\omega} B_i, \text{ for each } i \in \omega,\, B_i \in \mathscr{A} \text{ and diam } B_i \leqq \delta \right\}.$$

$$\zeta A = \lim_{\delta\to 0} \zeta_\delta A.$$

To see that the theory of Part I applies to ζ, let

9.2. DEFINITIONS.

$H_r = \{A \subset X : \operatorname{diam} A \leqq r\}$,

$\mathscr{H} = \{H_r : r > 0\}$,

$\mathscr{N} = \{H \cap \mathscr{A} ; H \in \mathscr{H}\}$.

Then $\mathscr{H}$ and $\mathscr{N}$ are filterbases in X, $\mathscr{N}$ is a subfilterbase of $\mathscr{H}$, and $\zeta = \nu^{(\mathscr{N},\tau)} = \nu^{(\mathscr{H},\tau)}$. The well-known properties of ζ will follow from the results of Part I and the following easily verified lemmas.

9.3. LEMMAS.

1. *The $\mathscr{H}$-topology is the metric topology, i.e.* $\mathscr{G}_{\mathscr{H}} = \mathscr{G}$.
2. $\mathscr{H}$ *satisfies* (5II) and (5IV).

Specifically, we have the following theorems.

9.4. THEOREM. *If A is closed in $\mathscr{G}$, then A is ζ-measurable.*

We note that stronger measurability results may be available (see Remarks 7.11 and Example 9.6).

9.5. THEOREMS. *Suppose $\mathscr{A} \subset \mathscr{G}$. Then*

.1. *ζ is a regular measure.*

.2. *If $\zeta E < \infty$ and $E \in \mathscr{M}_\zeta$, then there exist $D \in \mathscr{G}_\delta$ such that $D \supset E$ and $\zeta(D \sim E) = 0$, and $C \in \mathscr{F}_\sigma$ such that $C \subset E$ and $\zeta(E \sim C) = 0$*

.3. *If X is ζ-σ-finite, $E \in \mathscr{M}_\zeta$, then there exist $D \in \mathscr{G}_\delta$ such that $D \supset E$ and $\zeta(D \sim E) = 0$, and $C \in \mathscr{F}_\sigma$ such that $C \subset E$ and $\zeta(E \sim C) = 0$.*

.4. *If $X = \bigcup_{n \in \omega} G_n$, where for each $n \in \omega$, $G_n \in \mathscr{G}$ and $\zeta G_n < \infty$, $E \in \mathscr{M}_\zeta$ and $\varepsilon > 0$, then there exist open $G \supset E$ such that $\zeta(G \sim E) < \varepsilon$ and closed $F \subset E$ such that $\zeta(E \sim F) < \varepsilon$.*

PROOFS

Proof of 9.4. Use 9.3 and Theorem 7.8.

Proof of 9.5. 1. Use 9.3, 9.4 and Corollary 8.2.

Proof of 9.5.2, 9.5.3, and 9.5.4. Use 9.3 and then respectively Theorems 8.9, 8.10, and 8.11.

9.6. EXAMPLE. On obtaining a stronger measurability result by choice of an appropriate filterbase.

Let $X = R^2$,

$$\mathscr{A} = \Big\{A\text{: for some } x_0 \in R,\ y \in R, \text{ and } k \in \omega,$$
$$A = \Big\{(x,y): x = x_0 \text{ or for some } n > k, x = x_0 \pm \frac{1}{2^n}\Big\}\Big\}.$$

Let $\mathscr{H}, \mathscr{N}$ be defined as in 9.2. Then by 7.8, closed sets in the usual topology are ζ-measurable. Now $\mathscr{N}$ does not satisfy (5I), so we cannot get any measurability results using $\mathscr{N}$. Let

$$\mathscr{B} = \{A\text{: for some } x_0 \in R \text{ and } y \in R,\ A \subset \{(x,y)\text{: for some } s > 0,\ |x - x_0| < s\}\},$$

and

$$\mathscr{M} = \{H \cap \mathscr{B} : H \in \mathscr{H}\}.$$

Then $\mathscr{N}$ is a subfilterbase of $\mathscr{M}$, $\zeta = \nu^{(\mathscr{M})}$, $\mathscr{M}$ satisfies (5II) and (5IV), and so closed sets in $\mathscr{G}_{\mathscr{M}}$ are ζ-measurable. Clearly $\mathscr{G}_{\mathscr{M}}$ strictly contains $\mathscr{G}_{\mathscr{H}}$.

10. **The measures ϕ, ϕ_1, ϕ_2, in a topological space.** The measures ϕ and ϕ_2 below were introduced and studied by Bledsoe and Morse [1] and by C. A. Rogers and M. Sion (unpublished) respectively.

10.1 DEFINITIONS.

1. Families of open covers

$$\mathfrak{G} = \{\mathscr{B}:\mathscr{B} \subset \mathscr{G},\ \sigma\mathscr{B} = X, \text{ and } \emptyset \in \mathscr{B}\}.$$

$$\mathfrak{G}_1 = \{\mathscr{B}:\mathscr{B} \subset \mathscr{G},\ \mathscr{B} \text{ is countable, } \sigma\mathscr{B} = X, \text{ and } \emptyset \in \mathscr{B}\}.$$

$$\mathfrak{G}_2 = \{\mathscr{B}:\mathscr{B} \subset \mathscr{G},\ \mathscr{B} \text{ is finite, } \sigma\mathscr{B} = X, \text{ and } \emptyset \in \mathscr{B}\}.$$

.2. For $A \subset X$, $\mathscr{B}$ a cover of X,

$$\phi_{\mathscr{B}} A = \inf\left\{\sum_{B \in \mathscr{E}} \tau B : \mathscr{E} \text{ is a countable refinement of } \mathscr{B},\ \mathscr{E} \subset \mathscr{A}, \text{ and } A \subset \sigma\mathscr{E}\right\}.$$

.3. For $A \subset X$,

$$\phi A = \sup_{\mathscr{B} \in \mathfrak{G}} \phi_{\mathscr{B}} A.$$

$$\phi_1 A = \sup_{\mathscr{B} \in \mathfrak{G}_1} \phi_{\mathscr{B}} A.$$

$$\phi_2 A = \sup_{\mathscr{B} \in \mathfrak{G}_2} \phi_{\mathscr{B}} A.$$

To apply the theory of Part I we set

10.2. DEFINITIONS.

$$H_{\mathscr{B}} = \{A : A \subset B \text{ for some } B \in \mathscr{B}\}.$$

$$\mathscr{H} = \{H_{\mathscr{B}} : \mathscr{B} \in \mathfrak{G}\}.$$

$$\mathscr{H}_1 = \{H_{\mathscr{B}} : \mathscr{B} \in \mathfrak{G}_1\}.$$

$$\mathscr{H}_2 = \{H_{\mathscr{B}} : \mathscr{B} \in \mathfrak{G}_2\}.$$

$$\mathscr{G}^0 = \mathscr{H}\text{-topology}.$$

$$\mathscr{G}^1 = \mathscr{H}_1\text{-topology}.$$

$$\mathscr{G}^2 = \mathscr{H}_2\text{-topology}.$$

Then $\mathscr{H}$, $\mathscr{H}_1$, $\mathscr{H}_2$ are filterbases in X, and $\phi = \nu^{(\mathscr{H},\tau)}$, $\phi_1 = \nu^{(\mathscr{H}_1,\tau)}$, $\phi_2 = \nu^{(\mathscr{H}_2,\tau)}$.

The relations between the given topology $\mathscr{G}$ and the induced topologies $\mathscr{G}^0$, $\mathscr{G}^1$, $\mathscr{G}^2$ and properties of the filterbases $\mathscr{H}$, $\mathscr{H}_1$, $\mathscr{H}_2$ are indicated in the following theorem.

10.3. THEOREM.

.1. $\mathscr{G}^2 \subset \mathscr{G}^1 \subset \mathscr{G}^0 \subset \mathscr{G}$.

If $\mathscr{G}$ is regular, then

.2. $\mathscr{G}^2 = \mathscr{G}^1 = \mathscr{G}^0 = \mathscr{G}$,

.3. $\mathscr{H}$, $\mathscr{H}_1$, $\mathscr{H}_2$ *satisfy condition* (5I), *and*

.4. $\mathscr{H}$ *satisfies condition* (5V).

In general, $\mathscr{G} \neq \mathscr{G}^0$ as we show in 10.6. On the other hand, regularity of $\mathscr{G}$ is not needed for $\mathscr{G} = \mathscr{G}^2$, as we show in the proof. In view of 6.1.4, $\mathscr{G} = \mathscr{G}^0$ and $\mathscr{H}$ satisfies (5I) iff $\mathscr{G}$ is regular.

Applying the results of Part I we then get the following measurability theorems (already known for ϕ and ϕ_2).

10.4. THEOREMS.

.1. *If $\mathscr{G}$ is regular, then closed $\mathscr{G}_\delta$ sets are ϕ-measurable and compact $\mathscr{G}_\delta$ sets are ϕ_1, ϕ_2-measurable.*

.2. *If $\mathscr{G}$ is normal, then closed $\mathscr{G}_\delta$ sets in $\mathscr{G}$ are ϕ, ϕ_1, ϕ_2-measurable.*

(Since singletons are not assumed closed, normality does not imply regularity.)

Again, stronger results may be available, as indicated in the discussion in Remarks 7.11.

To obtain approximation results we require that $\mathscr{A} \subset \mathscr{M}_\phi$ or $\mathscr{A} \subset \mathscr{M}_{\phi_1}$ or $\mathscr{A} \subset \mathscr{M}_{\phi_2}$. In any of these cases we can apply directly Theorems 8.3 to 8.8.

In general the three measures ϕ, ϕ_1, ϕ_2 are distinct, as is shown in 10.7. It follows immediately from the definitions, however, that we always have

10.5. THEOREM. $\phi_2 \leqq \phi_1 \leqq \phi$.

PROOFS AND EXAMPLES

Proof of 10.3.1. Let $G \in \mathscr{G}^0$ and let $x \in G$. Then for some $H \in \mathscr{H}$, $H[x] \subset G$. But for some $\mathscr{B} \in \mathfrak{G}$, $H = H_{\mathscr{B}}$ and

$$H[x] = \sigma\{G \in \mathscr{B} : x \in G\} \in \mathscr{G}.$$

Hence $G \in \mathscr{G}$, and $\mathscr{G}^0 \subset \mathscr{G}$.

Clearly $\mathscr{H}_2 \subset \mathscr{H}_1 \subset \mathscr{H}$ and so $\mathscr{G}^2 \subset \mathscr{G}^1 \subset \mathscr{G}^0$.

Proof of 10.3.2. We need only show that $\mathscr{G} \subset \mathscr{G}^2$.

Let $G \in \mathscr{G}$ and $x \in G$. By regularity choose closed C such that $x \in C \subset G$, and set

$$\mathscr{B} = \{G, X \sim C\} \in \mathfrak{G}_2.$$

Then letting $H = H_{\mathscr{B}}$, we have $H[x] = G$. Thus, for each $x \in G$, there exists $H \in \mathscr{H}_2$ such that $H[x] \subset G$, i.e. G is open in the $\mathscr{H}_2$-topology.

Note that in this proof we need only that $\mathrm{Cl}\{x\} \subset G$. Thus, if $\mathscr{G}$ is a T_1-topology, then $\mathscr{G} = \mathscr{G}^2$.

Proof of 10.3.3. Suppose $x \in X$ and $H \in \mathscr{H}$. Then for some $\mathscr{B} \in \mathfrak{G}$, $H = H_{\mathscr{B}}$. Now $x \in G_0$ for some $G_0 \in \mathscr{B}$, and

$$G_0 \subset H[x] = \sigma\{G \in \mathscr{B} : x \in G\}.$$

By regularity choose G_1, $G_2 \in \mathscr{G}$ such that

$$x \in G_2 \subset \bar{G}_2 \subset G_1 \subset \bar{G}_1 \subset G_0$$

and let

$$\mathscr{B}_1 = \{G_0, X \sim \bar{G}_1\} \in \mathfrak{G}_2,$$
$$\mathscr{B}_2 = \{G_1, X \sim \bar{G}_2\} \in \mathfrak{G}_2,$$
$$H_1 = H_{\mathscr{B}_1} \in \mathscr{H}_2, \text{ and}$$
$$H_2 = H_{\mathscr{B}_2} \in \mathscr{H}_2.$$

Then

$$H_2[x] = G_1,$$
$$H_1[G_1] = G_0,$$

and so

$$H_2[H_1[x]] \subset H[x].$$

Proof of 10.3.4. By .2, any cover $\mathscr{B}$ consisting of sets open in the $\mathscr{H}$-topology is a cover of sets open in $\mathscr{G}$, i.e. $\mathscr{B} \in \mathfrak{G}$, and so $H_{\mathscr{B}} \in \mathscr{H}$ and $H_{\mathscr{B}}$ refines $\mathscr{B}$.

Proof of 10.4.1. Use 10.3. and Theorems 7.6 and 7.9.

Proof of 10.4.2. The result follows from Theorem 7.3 after it has been shown that if A is a closed $\mathscr{G}_\delta$ set in $\mathscr{G}$, then there exists a sequence B of subsets of X such that

$$A = \bigcap_{n \in \omega} B_n,$$

and for each $n \in \omega$, there exists $N_{n+1} \in \mathscr{H}_2$ such that

$$N_{n+1}[B_{n+1}] \subset B_n.$$

Suppose A is closed in $\mathscr{G}$ and

$$A = \bigcap_{n \in \omega} G_n,$$

where for each $n \in \omega$, $G_n \in \mathscr{G}$. The sequences N in $\mathscr{H}_2$ and B are defined recursively. To start, set $\mathscr{B}_0 = \{G_0, X \sim A\}$, $N_0 = H_{\mathscr{B}_0}$, and $B_0 = G_0$. Having obtained B_i and N_i such that

(a) $N_i[B_i] \subset B_{i-1}$ for $i = 0, \cdots, n$ (take $B_{-1} = X$),
(b) B_i is open for $i = 0, \cdots, n$, and
(c) $A \subset B_i \subset G_i$ for $i = 0, \cdots, n$,

we construct B_{n+1} and N_{n+1} as follows: Let

$$D_{n+1} = G_{n+1} \cap B_n \in \mathscr{G}.$$

Using normality choose open B_{n+1} such that

$$A \subset B_{n+1} \subset \bar{B}_{n+1} \subset D_{n+1}.$$

Let

$$\mathscr{B}_{n+1} = \{D_{n+1}, X \sim \bar{B}_{n+1}\} \in \mathscr{G}_2,$$

and $N_{n+1} = H_{\mathscr{B}_{n+1}} \in \mathscr{H}_2$. Then the only element of N_{n+1} which intersects B_{n+1} is D_{n+1}, so

$$N_{n+1}[B_{n+1}] = D_{n+1} \subset B_n.$$

For the sequences B and N, (a) and (c) hold for every $n \in \omega$. (c) assures us that

$$A = \bigcap_{n \in \omega} B_i.$$

10.6. EXAMPLE. $\mathscr{G}^0$ does not always coincide with $\mathscr{G}$. Let $X = R_+$, $\mathscr{G} = \{[0,a) : a > 0\}$. Then for any open cover $\mathscr{B}$ of X, and $x \in X$, $H_{\mathscr{B}}[x] = X$, so $\mathscr{G}^0$ is the trivial topology.

10.7. EXAMPLES. We can have $\phi_2 \neq \phi_1$, $\phi_1 \neq \phi$.

.1. A case where $\phi_2 \neq \phi_1$.

Let $X = R_+$; $\mathscr{G} = \{[0,a) : a > 0\}$, for $E \subset X$,

$$\tau E = \begin{cases} 0 & \text{if } E = X \text{ or } E = \varnothing \\ 1 & \text{otherwise.} \end{cases}$$

Then for any $A \subset X$, $\mathscr{B} \in \mathfrak{G}_2$, we have $\phi_{\mathscr{B}} A = 0$, since any finite open cover of X must include X as an element. Hence ϕ_2 is just the zero measure.

On the other hand, for any unbounded $A \subset X$, and $\mathscr{B} \in \mathfrak{G}_1$ such that $X \notin \mathscr{B}$, we have $\phi_{\mathscr{B}} A = \infty$ and so $\phi_1 A = \infty$.

.2. A case where $\phi_1 \neq \phi$.

Let X be any uncountable space with the discrete topology and $\tau E = 0$ for any $E \subset X$. Then for any $A \subset X$ and $\mathscr{B} \in \mathfrak{G}_1$, A can be covered by a countable refinement of $\mathscr{B}$ and so $\phi_{\mathscr{B}} A = 0$ and ϕ_1 is the zero measure. (By 10.5, ϕ_2 is also the zero measure.)

On the other hand, if $A \subset X$ is uncountable and $\mathscr{B}$ is the open cover consisting of singletons, then $\phi_{\mathscr{B}} A = \infty$ and so too $\phi A = \infty$.

11. **The measure λ in a topological space.** The measure λ of this section was studied by Rogers and Sion [7].

11.1 DEFINITIONS.

$\mathfrak{D} = \{\mathscr{B} : \mathscr{B}$ is a finite disjoint cover of X consisting of differences of open sets$\}$.

For $A \subset X$, $\mathscr{B}$ a cover of X,

$$\lambda_{\mathscr{B}} A = \inf \left\{ \sum_{B \in \mathscr{E}} \tau B : \mathscr{E} \text{ is a countable refinement of } \mathscr{B}, \mathscr{E} \subset \mathscr{A}, \text{and } A \subset \sigma\mathscr{E} \right\}.$$

$$\lambda A = \sup_{\mathscr{B} \in \mathfrak{D}} \lambda_{\mathscr{B}} A.$$

It can be shown that the same measure is obtained if the covers of differences of open sets are taken to be countable rather than finite. The process breaks down, however, if we attempt to use arbitrary covers. If the topology is T_1, then a cover consisting of singletons is of the required kind and the resulting measure would be infinite on any uncountable set, regardless of what gauge τ was used.

To apply the theory of Part I we set

11.2. DEFINITIONS.

$$H_{\mathscr{B}} = \{A : A \subset B \text{ for some } B \in \mathscr{B}\}.$$

$$\mathscr{H} = \{H_{\mathscr{B}} : \mathscr{B} \in \mathfrak{D}\}. \quad \mathscr{G}_{\mathscr{H}} = \mathscr{H}\text{-topology}.$$

Then $\mathscr{H}$ is a filterbase in X (the intersection of two sets which are differences of open sets is again such a set) and $\lambda = \nu^{(\mathscr{H},\tau)}$.

11.3. THEOREMS.

.1. *For any* $H \in \mathscr{H}$, *and* $x \in X$, $H[H[x]] = H[x]$, *and so* $\mathscr{H}$ *satisfies* (5II).

.2. $\mathscr{G} \subset \mathscr{G}_{\mathscr{H}}$.

.3. $\mathscr{G}_{\mathscr{H}}$ *is completely regular.*

.4. *If A is closed in $\mathscr{G}$, then A is both open and closed in $\mathscr{G}_{\mathscr{H}}$.*

.5. *If $\mathscr{G}$ is T_1, then $\mathscr{G}_{\mathscr{H}}$ is the discrete topology.*

Theorem 11.4.2 following was obtained by Rogers and Sion [7].

11.4. THEOREMS.

.1. *Compact $(\mathscr{G}_{\mathscr{H}})_\delta$ sets are λ-measurable.*

.2. *If $G \in \mathscr{G}$, then G is λ-measurable.*

Again, as discussed in 7.11, stronger results may be available.

To obtain results on approximation, we require that $\mathscr{A} \subset \mathscr{M}_\lambda$. (For example, suppose $\mathscr{A}$ consists of differences of open sets. Such sets will be λ-measurable by 11.4.2.) In this case we can apply Theorems 8.3 to 8.8.

PROOFS

Proof of 11.3.1. Let $x \in X$ and $H \in \mathscr{H}$. For some $\mathscr{B} \in \mathfrak{D}$, $H = H_{\mathscr{B}}$, and for some $B \in \mathscr{B}$, $x \in B$. Now $B \in \mathscr{B}$ implies $B \in H_{\mathscr{B}}$ so by Definition 4.1.1

$$B \subset H[x].$$

But any element of H containing x must be contained in B, since H refines $\mathscr{B}$ and the elements of $\mathscr{B}$ are disjoint, so we have $H[x] \subset B$. Hence $H[x] = B$ and

$$H[H[x]] = H[B].$$

But $H[B] = B$ by Definition 4.1.2 and an argument similar to the one above. Hence

$$H[H[x]] = H[x] \qquad \text{for each } x \in X.$$

Proof of 11.3.2. Let $G \in \mathscr{G}$ and set $\mathscr{B} = \{G, X \sim G\}$. Then $\mathscr{B} \in \mathfrak{D}$ and $H_{\mathscr{B}} \in \mathscr{H}$. Further, for every $x \in G$, $H_{\mathscr{B}}[x] = G$. Hence $G \in \mathscr{G}_{\mathscr{H}}$.

Proof of 11.3.3. Use 11.3.1 and Theorem 6.2.

Proof of 11.3.4. A is closed in $\mathcal{G}_{\mathcal{H}}$ because $\mathcal{G} \subset \mathcal{G}_{\mathcal{H}}$. Let $\mathcal{B} = \{A, X \sim A\}$. Then $\mathcal{B} \in \mathfrak{D}$ and $H_{\mathcal{B}} \in \mathcal{H}$. For every $x \in A$, $H_{\mathcal{B}}[x] = A$. Hence $A \in \mathcal{G}_{\mathcal{H}}$.

Proof of 11.3.5. If $\mathcal{G}$ is T_1, then points are closed in $\mathcal{G}$ and hence by .4 above, open and closed in $\mathcal{G}_{\mathcal{H}}$.

Proof of 11.4.1. Use 11.3.1 and Theorem 7.6.

Proof of 11.4.2. Suppose $G \in \mathcal{G}$ and let

$$\mathcal{B} = \{G, X \sim G\}.$$

Then $\mathcal{B} \in \mathfrak{D}$ and $H_{\mathcal{B}} \in \mathcal{H}$. As in the argument in the proof of 11.3.1, we show $H_{\mathcal{B}}[G] = G$. The conclusion follows from Theorem 7.3, setting $B_n = G$ for every $n \in \omega$.

11.5. EXAMPLE. $\mathcal{G}_{\mathcal{H}}$ may be strictly larger than $\mathcal{G}$. Let $X = R_+$, $\mathcal{G} = \{[0,a) : a > 0\}$. Then $\mathcal{G}_{\mathcal{H}}$ is the half-open interval topology, which is not only larger than $\mathcal{G}$, but larger than the usual topology on R_+ as well. (Compare 10.6.)

11.6. EXAMPLE. Theorem 11.4.1 cannot be strengthened to closed $(\mathcal{G}_{\mathcal{H}})_\delta$ sets. It was shown by Rogers and Sion [7, Theorem 8], that the measure λ defined on the real line, with the gauge τ on the subsets of R defined by $\tau A = \operatorname{diam} A$ is just the measure ζ, which in this case is known to be the same as Lebesgue measure. But the $\mathcal{H}$-topology in this case is discrete by 11.3.5 and so all subsets of R are closed $(\mathcal{G}_{\mathcal{H}})_\delta$ sets.

12. **Relations between measures.** In this section we establish some relations among some of the measures we have studied.

The following result was obtained by Bledsoe and Morse [1].

12.1. THEOREM. *If $(X, \mathcal{G})$ is a metric space, then $\phi = \zeta$.*

12.2. REMARK. It was shown by Rogers and Sion ([7, Theorem 8], and in some unpublished work) that if $(X, \mathcal{G})$ is a separable metric space, and τ is well behaved in a certain sense, then $\lambda = \zeta = \phi_2$. In this case then (which includes Lebesgue measure and the classical Hausdorff measures) $\zeta = \lambda = \phi_2 = \phi_1 = \phi$ by Theorems 10.5 and 11.2.

12.3. REMARK. In the example in 10.7.2, the space is metric and it is easy to see that $\phi = \zeta$, as is assured by Theorem 12.1. On the other hand, by 10.5, ϕ_2 is the zero measure, and since $\mathfrak{D} = \mathfrak{G}_2$, we have also that $\lambda = \phi_2$. We have then

$$\zeta = \phi \neq \lambda = \phi_1 = \phi_2.$$

In this sense, ϕ is the most successful of these measures in generalizing from the metric case.

12.4. REMARK. We note that in Example 10.7.1, λ is counting measure, different from both ϕ_1 and ϕ_2.

13. **An approach using quasi-uniformities.** A natural extension of the idea of using covers of sets of smaller and smaller diameter in a metric space is the idea of using covers of sets defined in terms of the elements of a uniformity in a uniform space (Kelley, [4, Chapter 6]). Now the idea of "coming down" through elements of a uniformity is independent of the symmetry requirement for a uniformity and applies as well to a quasi-uniformity (see Definition 13.1.2 below). Since for any topological space there is a quasi-uniformity which induces the topology (see Pervin [6]), in this section we generate a measure on the given topological space $(X, \mathscr{G})$ using a quasi-uniformity. There may be different quasi-uniformities which induce the same topology on a space and it turns out that the measure generated depends in general both on the particular quasi-uniformity chosen and on the way of "coming down" defined in terms of the quasi-uniformity. The measures ζ and λ of §§9 and 11 can always be obtained through this approach, whereas an additional condition on the topology is needed to obtain the measure ϕ of §10.

We first establish some results on quasi-uniformities (see Pervin [6]).

13.1. DEFINITIONS.

.1. $A \circ B = \{(x,z)\colon \text{for some } y, (x,y) \in B \text{ and } (y,z) \in A\}$.

$\mathscr{U}$ is a quasi-uniformity for X iff

.2. $\mathscr{U}$ *is* a family of subsets of $X \times X$ such that for every $U \in \mathscr{U}$ and $V \in \mathscr{U}$,

(a) $U \supset \Delta = \{(x,x)\colon x \in X\}$,

(b) $W \supset U$ and $W \subset X \times X \Rightarrow W \in \mathscr{U}$,

(c) $U \cap V \in \mathscr{U}$, and

(d) there exists $W \in \mathscr{U}$ such that $W \circ W \subset U$.

.3. If U is an element of a quasi-uniformity,

$$U[A] = \{y\colon (x,y) \in U \text{ for some } x \in A\},$$

$$U[x] = U[\{x\}].$$

13.2. REMARK. A quasi-uniformity $\mathscr{U}$ for X generates a topology $\mathscr{T}_{\mathscr{U}}$ on X consisting of all subsets G of X such that for each $x \in G$, there exists $U \in \mathscr{U}$ such that $U[x] \subset G$. For $x \in X$, $\{U[x]\colon U \in \mathscr{U}\}$ is a neighborhood system for x (see Pervin [6]).

13.3. THEOREM. *For any topological space, there is a quasi-uniformity which induces the topology.*

Proof. (see (Pervin [6]). Let $(X, \mathscr{G})$ be a topological space. For each $G \in \mathscr{G}$ let

$$S_G = (G \times G) \cup ((X \sim G) \times X),$$

and let $\mathscr{A} = \{S_G; G \in \mathscr{G}\}$. Pervin shows that $\mathscr{A}$ is a subbase for a quasi-uniformity $\mathscr{U}$ for X (hereafter referred to as *Pervin's quasi-uniformity*), and that $\mathscr{T}_{\mathscr{U}} = \mathscr{G}$.

13.4. REMARK. For a given topological space $(X,\mathscr{G})$ there is a largest quasi-uniformity $\mathscr{U}$ such that $\mathscr{T}_{\mathscr{U}}=\mathscr{G}$. We take as a subbase for $\mathscr{U}$ the union of all quasi-uniformities $\mathscr{V}$ such that $\mathscr{T}_{\mathscr{V}}=\mathscr{G}$. Then $\mathscr{U}$ is a quasi-uniformity and $\mathscr{T}_{\mathscr{U}}=\mathscr{G}$.

We now assume that $\mathscr{U}$ is a quasi-uniformity for X which induces the topology $\mathscr{G}$, and introduce the measures μ and $\mu^\dagger$.

13.5. DEFINITIONS.

For $U\subset X\times X$,

.1. $U^* = \{A\subset X\colon A\times A\subset U\}$.

$U^\dagger = \{A\subset X\colon \text{for some } x\in X,\ A\subset U[x]\}$.

For $A\subset X$,

.2. $\mu_U A = \inf\left\{\sum_{B\in\mathscr{B}}\tau B\colon \mathscr{B} \text{ is countable, } \mathscr{B}\subset U^*\cap\mathscr{A}, \text{ and } A\subset\sigma\mathscr{B}\right\}$.

$\mu A = \sup_{U\in\mathscr{U}} \mu_U A$.

.3. $\mu_U^\dagger A = \inf\left\{\sum_{B\in\mathscr{B}}\tau B\colon \mathscr{B} \text{ is countable, } \mathscr{B}\subset U^\dagger\cap\mathscr{A}, \text{ and } A\subset\sigma\mathscr{B}\right\}$.

$\mu^\dagger A = \sup_{U\in\mathscr{U}} \mu_U^\dagger A$.

It is clear that the same measures are obtained if the supremum is taken over any base for $\mathscr{U}$ (see Kelley [4, p. 177].)

13.6. REMARK. Pervin [6] points out that two noncomparable quasi-uniformities for X may give identical topologies for X. They may at the same time yield different measures. In the case in 13.7 below, note that $\mathscr{T}_{\mathscr{U}}$ is just the metric topology. In the case in 13.8, applied in a metric space, $\mathscr{T}_{\mathscr{U}}$ is by construction again the metric topology but we have seen that ζ and λ do not always agree on a metric space (see Remark 12.3).

13.7. REMARK. μ is a direct generalization of ζ. Let X be a metric space with metric d. If we set

$$U_r = \{(x,y)\colon d(x,y)\leqq r\}.$$

and

$$\mathscr{U} = \{U_r\colon r>0\},$$

then $\mathscr{U}$ is clearly a base for a quasi-uniformity for X. Since $A\times A\subset U_r$ iff $\operatorname{diam} A\leqq r$, we have $\zeta_r=\mu_{U_r}$ for $r>0$.

Using Remark 3.3, we conclude $\zeta=\mu$.

13.8. THEOREM. *If $\mathscr{U}$ is Pervin's quasi-uniformity, then $\mu=\lambda$ (§11).*

13.9. THEOREM. *If $\mathscr{U}$ is Pervin's quasi-uniformity, then $\mu^\dagger=\phi_2$ (§10).*

The following property of a topological space is needed for a comparison of μ and ϕ (§10).

13.10. DEFINITION. A topological space has *property Q* iff for any open cover $\mathscr{A}$ of X, there exists an open cover $\mathscr{B}$ refining $\mathscr{A}$ and such that for every $x \in X$,

$$\pi\{G \in \mathscr{B} : x \in G\} \text{ is open.}$$

13.11. THEOREM. *$\mu^\dagger \leqq \phi$. If $\mathscr{U}$ is the largest quasi-uniformity inducing $\mathscr{G}$ on X and $(X, \mathscr{G})$ has property Q, then $\mu^\dagger = \phi$.*

13.12. REMARK. We can have $\mu^\dagger \neq \phi$ in a space having property Q. If $\mu^\dagger$ is obtained using Pervin's quasi-uniformity for the space in Example 10.7.1, then $\mu^\dagger = \phi_2 \neq \phi$.

PROOFS

Proof of 13.8. $\mu \geqq \lambda$: Suppose $\mathscr{B} \in \mathfrak{D}$,

$$\mathscr{B} = \{G_i \sim G_{i+n} : i = 1, \cdots, n\}.$$

Let

$$U = \bigcap_{j=1}^{2n} S_{G_j},$$

where

$$S_{G_j} = (G_j \times G_j) \cup ((X \sim G_j) \times X) \qquad \text{(see 13.3).}$$

Then $U \in \mathscr{U}$ and

$$\begin{aligned} A \in U^* \quad &\text{iff } A \times A \subset U \\ &\text{iff } A \times A \subset S_{G_j} \text{ for } j = 1, \cdots, 2n \\ &\text{iff } A \subset G_j \text{ or } A \cap G_j = \varnothing \text{ for } j = 1, \cdots, 2n \\ &\text{iff } A \subset G_i \sim G_{i+n} \text{ for some } i,\ 1 \leqq i \leqq n. \end{aligned}$$

Hence any family of sets in U^* is a refinement of $\mathscr{B}$ and so

$$\mu_U \geqq \lambda_{\mathscr{B}}$$

and

$$\mu \geqq \lambda.$$

$\lambda \geqq \mu$: Let $U \in \mathscr{U}$. Then there exists $V \in \mathscr{U}$, $V \subset U$ such that

$$V = \bigcap_{j=1}^{m} S_{G_j}$$

where $G_j \in \mathscr{G}$ for $j = 1, \cdots, m$. Let

$$\mathscr{E} = \{G_j\}_{j=1, \cdots, m},$$

and

$$\mathscr{B} = \{A \sim B : A = \pi\mathscr{L} \text{ for some } \mathscr{L} \subset \mathscr{E},\ \mathscr{L} \neq \varnothing, \text{ and } B = \sigma(\mathscr{E} \sim \mathscr{L})\}.$$

Then $\mathscr{B} \in \mathfrak{D}$ and if $D \subset A \sim B$ for some $(A \sim B) \in \mathscr{B}$, then for $j = 1, \cdots, m$, $D \subset G_j$ or $D \cap G_j = \varnothing$, whence $D \times D \subset V$, or $D \in V^*$. Hence any refinement of $\mathscr{B}$ is contained in V^* and so

$$\lambda_{\mathscr{B}} \geqq \mu_V \geqq \mu_U$$

and

$$\lambda \geqq \mu.$$

Proof of 13.9. $\mu^\dagger \geqq \phi_2$: Let $\mathscr{B} \in \mathfrak{G}_2$, $\mathscr{B} = \{G_1, \cdots, G_n\}$, where $G_i \in \mathscr{G}$ for $i = 1, \cdots, n$. Let

$$U = \bigcap_{i=1}^{n} S_{G_i} \in \mathscr{U}.$$

Suppose $A \in U^\dagger$. Then for some $x \in X$, $A \subset U[x]$. Now $x \in G_j$ for some $G_j \in \mathscr{B}$, and since $U \subset S_{G_j}$, we have

$$U[x] \subset S_{G_j}[x] = G_j.$$

Hence $A \subset G_j$ and so $\mathscr{E} \subset U^\dagger$ implies $\mathscr{E}$ refines $\mathscr{B}$. Therefore

$$\mu_U^\dagger \geqq \phi_{\mathscr{B}}$$

and

$$\mu^\dagger \geqq \phi_2.$$

$\mu^\dagger \leqq \phi_2$: Suppose $U \in \mathscr{U}$. Choose $V \in \mathscr{U}$, $V \subset U$,

$$V = \bigcap_{j=1}^{m} S_{G_j} \qquad \text{where } G_j \in \mathscr{G} \text{ for } j = 1, \cdots, m.$$

Let

$$\mathscr{B} = \{V[x]: x \in X\}.$$

Now for $x \in X$, either $V[x] = X$, or

$$V[x] = \bigcap_{i=1}^{k} G_{j_i}$$

for some k, $1 \leqq k \leqq m$, and some function j on $\{1, \cdots, m\}$ onto $\{1, \cdots, m\}$, for: Suppose $x \notin G_i$ for $i = 1, \cdots, m$. Then

$$\begin{aligned} V[x] &= \left(\bigcap_{i=1}^{m} S_{G_i}\right)[x] \\ &= \{y : (x, y) \in (G_i \times G_i) \cup ((X \sim G_i) \times X) \text{ for } i = 1, \cdots, m\} \\ &= X. \end{aligned}$$

On the other hand, suppose $x \in G_{j_i}$ for $i = 1, \cdots, k$, $1 \leqq k \leqq m$, and $x \notin G_{j_i}$ for $i = k + 1, \cdots, m$, for some function j on $\{1, \cdots, m\}$ onto $\{1, \cdots, m\}$. Then

$$\begin{aligned}(x,y)\in V \ \text{ iff } & (x,y)\in G_{j_i}\times G_{j_i} \text{ for } i=1,\cdots,k\\ \text{iff } & y\in G_{j_i} \text{ for } i=1,\cdots,k\\ \text{iff } & y\in \bigcap_{i=1}^{k} G_{j_i}\end{aligned}$$

i.e. $V[x]=\bigcap_{i=1}^{k} G_{j_i}$.

We conclude that each element of $\mathscr{B}$ is open and $\mathscr{B}$ is finite. Clearly $\mathscr{B}$ is a cover of X, so $\mathscr{B}\in\mathfrak{G}_2$. Trivially, if $A\subset B$ for some $B\in\mathscr{B}$, then $A\subset V[x]$ for some $x\in X$, so $\mathscr{E}$ refines $\mathscr{B}$ implies $\mathscr{E}\subset U^{\dagger}$ and hence

$$\phi_{\mathscr{B}}\geqq \mu_V^{\dagger}\geqq \mu_U^{\dagger} \text{ and } \phi_2\geqq \mu^{\dagger}.$$

Proof of 13.11. Let $U\in\mathscr{U}$. For each $x\in X$, $U[x]$ is a neighborhood of x so there exists open G_x such that

$$x\in G_x\subset U[x].$$

Let

$$\mathscr{B}=\{G_x\colon x\in X\}.$$

Then $\mathscr{B}\in\mathfrak{G}$ and if $\mathscr{E}$ refines $\mathscr{B}$, $\mathscr{E}\subset U^{\dagger}$, so

$$\phi_{\mathscr{B}}\geqq \mu_U^{\dagger}$$

and

$$\phi\geqq\mu^{\dagger}.$$

Suppose now $\mathscr{U}$ is the maximal quasi-uniformity inducing $\mathscr{G}$ on X, and $(X,\mathscr{G})$ has property Q. Let $\mathscr{E}\in\mathfrak{G}$ and let $\mathscr{B}\in\mathfrak{G}$, $\mathscr{B}$ refining $\mathscr{E}$ and such that

$$\pi\{G\in\mathscr{B}\colon x\in G\} \qquad \text{is open for every } x\in X.$$

Set

$$U=\bigcap_{G\in\mathscr{B}} S_G$$

where again $S_G=(G\times G)\cup((X\sim G)\times X)$. Then

(1) for every $x\in X$, $U[x]$ is a neighborhood of x, and

(2) $U\circ U=U$.

(1): We show $U[x]=\pi\{G\in\mathscr{B}\colon x\in G\}$.

$$\begin{aligned}y\in U[x] \ \text{ iff } & (x,y)\in U\\ \text{iff } & (x,y)\in S_G \text{ for every } G\in\mathscr{B}\\ \text{iff } & (x,y)\in G\times G \text{ or } (x,y)\in(X\sim G)\times X \text{ for every } G\in\mathscr{B}\\ \text{iff } & x,y\in G \text{ or } x\notin G \text{ for every } G\in\mathscr{B}\\ \text{iff } & y\in G \text{ for each } G\in\mathscr{B} \text{ such that } x\in G\\ \text{iff } & y\in\pi\{G\in\mathscr{B}\colon x\in G\}.\end{aligned}$$

(2): By definition,

$$U\circ U=\{(x,y)\colon \text{for some } z,(x,z)\in U \text{ and } (z,y)\in U\}.$$

Now if $(x, y) \in U$, then since $(x, x) \in U$, we have $(x, y) \in U \circ U$, and so $U \subset U \circ U$.

Suppose now $(x, y) \in U \circ U$. Then for some $z, (x, z) \in U$ and $(z, y) \in U$. Hence $(x, z) \in S_G$ and $(z, y) \in S_G$ for every $G \in \mathscr{B}$. Let $G \in \mathscr{B}$.,

(a) If $(x, z) \in G \times G$ and $(z, y) \in G \times G$, then $(x, y) \in G \times G \subset S_G$.

(b) If $(x, z) \in (X \sim G) \times X$ and $(z, y) \in (X \sim G) \times X$, then $(x, y) \in (X \sim G) \times X \subset S_G$.

(c) If $(x, z) \in (X \sim G) \times X$, $(z, y) \in G \times G$, then $(x, y) \in (X \sim G) \times X \subset S_G$

(d) $(x, z) \in G \times G$ and $(z, y) \in (X \sim G) \times X$ is impossible.

Hence $(x, y) \in S_G$ for every $G \in \mathscr{B}$ and so $(x, y) \in U$ and we have $U \circ U \subset U$.

Now (1) implies that $\Delta \subset U$, and this with (2) implies that $\{U\}$ is the base for a quasi-uniformity for X. Hence by Theorem 6.3 of Kelley [4], $\mathscr{U} \cup \{U\}$ is the subbase for a quasi-uniformity $\mathscr{V}$ for X. But $\mathscr{T}_{\mathscr{V}} = \mathscr{G}$ and so since $\mathscr{U}$ is the maximal quasi-uniformity inducing $\mathscr{G}$ on X, we have $\mathscr{V} = \mathscr{U}$ and $U \in \mathscr{U}$.

Now if $A \subset U[x]$ for some $x \in X$ (i.e. $A \in U^{\dagger}$), then by the proof of (1), $A \subset G$ for each $G \in \mathscr{B}$ such that $x \in G$, and hence $\mathscr{L} \subset U^{\dagger}$ implies $\mathscr{L}$ is a refinement of $\mathscr{B}$. Therefore

$$\mu_U^{\dagger} \geqq \phi_{\mathscr{B}} \geqq \phi_{\mathscr{E}}$$

and we have

$$\mu^{\dagger} \geqq \phi.$$

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