DILATIONS ON INVERTIBLE SPACES([1])

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Abstract. This paper primarily concerns certain groups of homeomorphisms which are associated in a natural way with a variety of spaces, which satisfy a set of axiomatic conditions put forth in §1.

Let us suppose that X is a space of the type in question and that G is an appropriate group of homeomorphisms of X onto itself. In §2 we demonstrate the existence of a nonvoid subcollection $\mathscr{D}$, the "topological dilations," of G which is characterized in Theorem 1 in the following fashion: suppose $f \in \mathscr{D}$ and $g \in G$, then $g \in \mathscr{D}$ if and only if f is a G-conjugate of g, that is if and only if there exists an element h of G such that $f = hgh^{-1}$.

We proceed then to show in §3 that if f and g are nonidentity elements of G, then we may find $\delta, r \in G$ such that the product $(rgr^{-1})(\delta f \delta^{-1}) \in \mathscr{D}$. We then combine this fact with the characterization of $\mathscr{D}$ mentioned above to conclude that each element of $\mathscr{D}$ is a "universal" element of G in the sense that if $d \in \mathscr{D}$, then any element g of G may be represented as the product of two G-conjugates of d. Furthermore we conclude that if g is not the identity element of G, then g can be represented as the product of three G-conjugates of *any* nonidentity element of G.

Finally, we apply the conclusions to groups of homeomorphisms of certain spaces: for example spheres, cells, the Cantor set, etc.

1. Definitions, notation, and axioms. If X is a topological space, A and B are subsets of X, and f is a mapping of X to itself, then we write '$A \subseteq B$' for 'A is a subset of B', and '$A \subset B$' for '$A \subseteq B$ and $A \neq B$'. The mapping f is said to be supported on A if $f(x) = x$ whenever $x \in {}^{\sim}A$. A will be called a perfect subset of X if $A \neq \varnothing$ and no open subset of X contains exactly one point of A. Throughout this paper all mappings under consideration will be homeomorphisms of some space X onto itself.

Henceforth we suppose that we have given:

(a) X: a regular, first countable, Hausdorff space.

(b) X': a perfect subspace of X.

(c) $(\mathscr{U}, \mathscr{K})$: $\mathscr{U}$ is a collection of open subsets of X each having a nonvoid

Presented to the Society, November 15, 1963 under the title *Stable homeomophisms of S_r;* received by the editors March 25, 1965.

([1]) This constitutes the author's dissertation which was directed by Professor R. D. Anderson whom the author wishes to thank for his great assistance throughout the years.

intersection with X', and each point of X' has a neighborhood basis contained in $\mathscr{U}$. $\mathscr{K}$ will denote the collection of the closures of the elements of $\mathscr{U}$.

(d) $G(X,X')$: a nonvoid collection of homeomorphisms of X onto itself, each carrying X' onto X'. $G_{\mathscr{K}}$ will denote the collection of all homeomorphisms of X onto X each carrying X' onto X' and each supported on some element of $\mathscr{K}$. $G_{\mathscr{K}}^*$ will denote the group generated by $G_{\mathscr{K}}$.

If $X, X', \mathscr{K}$, and $G(X,X')$ are such as to satisfy the following axioms, then $(X,X',\mathscr{K}, G(X,X'))$ will be called an A-quadruple(2).

(1) If $K \in \mathscr{K}$, then $\mathrm{Cl}(\sim K) \in \mathscr{K}$.

(2) If $K \in \mathscr{K}$, and if $g \in G(X,X')$, then $g(K) \in \mathscr{K}$.

(3) If g_1, $g_2 \in G(X,X')$, and if $g \in G_{\mathscr{K}}$, then gg_1, $g_1g \in G(X,X')$ and $g_2g_1 \in G_{\mathscr{K}}^*$.

(4) If K_1, K_2, $K_3 \in \mathscr{K}$, with $K_1 \subset K_3^0 \supset K_2$, then there exists an element $g \in G_{\mathscr{K}}$, supported on K_3, with $(gK_1) = K_2$(3).

(5) If $g \in G_{\mathscr{K}}^*$, and if K_1, $K_2 \in \mathscr{K}$ with $K_1 \subseteq K_2^0$ and $g(K_1) = K_1$, then there exists an element $g_1 \in G_{\mathscr{K}}$, supported on K_2, such that $g_1 | K_1 = g | K_1$.

Axioms (1) and (2) will be frequently used and usually without being cited. Note that in the case where $X = X'$ Axioms (1) and (4) assert a sort of invertibility for X.

Axiom (3) implies immediately that if $g \in G_{\mathscr{K}}^*$, and if $g_1 \in G(X,X')$, then gg_1, $g_1g \in G(X,X')$. In view of this (3) also implies that if $g \in G(X,X')$, then $g^{-1} \in G(X,X')$: for $g^2 \in G_{\mathscr{K}}^*$; hence $g^{-2} \in G_{\mathscr{K}}^*$, and therefore $g^{-1} = gg^{-2} \in G(X,X)'$.

It now follows that if $g \in G_{\mathscr{K}}^*$ and $K \in \mathscr{K}$, then $g(K) \in \mathscr{K}$. For since $G(X,X') \neq \varnothing$, there exists $g_1 \in G(X,X')$. Then $gg_1^{-1} \in G(X,X')$, and $g_1(K) \in \mathscr{K}$ by Axiom (2). Hence, also by Axiom (2), $g(K) = gg_1^{-1}(g_1(K)) \in \mathscr{K}$.

Also Axiom (3) implies that if $g \in G(X,X')$ and $\delta \in G_{\mathscr{K}}^*$, then $\delta g \delta^{-1} \in G(X,X')$, i.e., the conjugate of any element of $G(X,X')$ by any element of $G_{\mathscr{K}}^*$ is an element of $G(X,X')$.

It follows also that the set $G(X,X')$ and $G_{\mathscr{K}}^*$ are either equal or disjoint; or, in other words, $G(X,X') = G_{\mathscr{K}}^*$ if and only if $e \in G(X,X')$.

Note that since each of $G(X,X')$ and $G_{\mathscr{K}}^*$ consists solely of homeomorphisms which carry X' onto X', any homeomorphism which we may subsequently construct must satisfy this condition if it is to be in either $G(X,X')$ or $G_{\mathscr{K}}^*$.

Henceforth we shall assume that $(X,X',\mathscr{K}, G(X,X'))$ is an A-quadruple.

By way of example, while reading the arguments to follow, one might think of X as the n-cell, $n > 1$, and X' as its boundary.

Definition 1. Suppose $p \in X'$ and $\mathscr{C} = \{C_i\}_{i=t_0}^{\infty} \subseteq \mathscr{K}$. Then $\mathscr{C}$ is called a null sequence with respect to p or, more briefly, $\mathscr{C}$ is said to be null for p, if:

(a) $C_{i+1} \subset C_i^0$, $i \geq t_0$, and

(2) This definition is due to R. D. Anderson [1] except that we omit his axiom two. Without using the concept of topological dilation he obtained previously Corollaries 3 and 4 to Theorem 2.

(3) Do not forget that "$\subset$" denotes *proper* containment.

(b) $\{C_i^0\}_{i=t_0}^{\infty}$ is a neighborhood basis for p.

Of course if $\{C_i\}_{i=t_0}^{\infty}$ is null for p, then $p = \bigcap \{C_i\}_{i=t_0}^{\infty}$. Null sequences indexed from t_0 to $-\infty$ may be defined in a comparable fashion.

DEFINITION 2. Suppose p and q are distinct points of X', and suppose $\mathscr{C} = \{C_i\}_{i=-\infty}^{\infty} \subseteq \mathscr{K}$. Then $\mathscr{C}$ is called a dilation structure with respect to the ordered pair (p,q) provided:

(a) $\{C_i\}_{i=0}^{-\infty}$ is a null sequence for p, and

(b) $\{\mathrm{Cl}(\sim C_i)\}_{i=0}^{\infty}$ is a null sequence with respect to q.

DEFINITION 3. Suppose $g \in G_{\mathscr{K}}^*$ and suppose $\mathscr{C} = \{C_i\}_{i=-\infty}^{\infty}$ is a dilation structure with respect to (p,q). If $g(C_i) = C_{i+1}$, $-\infty < i < \infty$, then g is called a topological dilation with carrier $\mathscr{C}$; and $\mathscr{C}$ is said to carry g.

Note that if $\{C_i\}_{i=0}^{\infty}$ is null for p, and if $g \in G_{\mathscr{K}}^* \cup G(X,X')$, then $\{g(C_i)\}_{i=0}^{\infty}$ is null for $g(p)$. Hence if $g \in G_{\mathscr{K}}^*$ is a topological dilation with carrier $\mathscr{C} = \{C_i\}_{i=-\infty}^{\infty}$ with respect to (p,q), and if $\delta \in G_{\mathscr{K}}^*$, then $\delta g \delta^{-1}$ is a topological dilation carried by $\mathscr{C}' = \{\delta(C_i)\}_{i=-\infty}^{\infty}$ with respect to $(\delta(p), \delta(q))$.

2. **Topological dilations.** We begin this section by proving some lemmas which will be useful throughout the paper. Next we demonstrate the existence of topological dilations and, in fact, may conclude that $G_{\mathscr{K}}^*$ is richly supplied with them. We conclude §2 with Theorem 1, which asserts any two topological dilations are conjugate in $G_{\mathscr{K}}^*$.

PROPOSITION A. *Suppose p and q are distinct points of X'. Then there exists a dliation structure with respect to (p,q).*

Proof. Since X is first countable, regular, and Hausdorff, X' is perfect, and $\mathscr{U}$ is a basis, it easily follows that we may find sequences of elements of $\mathscr{K}$, $\{C_i\}_{i=0}^{-\infty}$ and $\{D_i\}_{i=1}^{\infty}$, null for p and q respectively, with $C_0 \cap D_1 = \varnothing$ and $C_0 \cup D_1 \neq X$. Setting $C_i = \mathrm{Cl}(\sim D_i)$ for $i \geqq 1$, we see that $\mathscr{C} = \{C_i\}_{i=-\infty}^{\infty}$ is the desired dilation structure.

PROPOSITION B. *Suppose K_1, $K_2 \in \mathscr{K}$ and suppose $K_1 \subset K_2^0$. Then there exists $K \in \mathscr{K}$ with $K_1 \subset K^0$ and $K \subset K_2^0$.*

Proof. Select $x \in K_1^0 \cap X'$ and $E_1 \in \mathscr{K}$ with $x \in E_1^0$ and $E_1 \subset K_1^0$. This can be done since X is regular. Now select $E_2 \in \mathscr{K}$, with $x \in E_2^0$, $E_2 \subset E_1^0$. Using Axiom (4), we obtain $g \in G_{\mathscr{K}}$, supported on K_2, with $g(K_1) = E_2$. Set $K = g^{-1}(E_1)$.

LEMMA 1. *Suppose $K \in \mathscr{K}$ and $x_0, y_0 \in (K^0 \cap X')$; suppose also that $\mathscr{C} = \{C_i\}_{i=1}^{\infty}$, $\mathscr{D} = \{D_i\}_{i=1}^{\infty} \subseteq \mathscr{K}$ are null for x_0, y_0 respectively, with $C_1 \subset K^0 \supset D_1$. Then there is a homeomorphism $h \in G_{\mathscr{K}}$, supported on K, such that $h(C_i) = D_i$ for $i \geqq 1$.*

Proof. By Axiom (4) there exists $h_1 \in G_{\mathscr{K}}$, supported on K, such that $h_1(C_1) = D_1$. For $i \geqq 2$ we define inductively, by making repeated use of Axiom (4), $h_i \in G_{\mathscr{K}}$, supported on D_{i-1}, such that $h_i h_{i-1} \cdots h_1(C_i) = D_i$.

Set $h = \prod_{i=1}^{\infty} h_i$. If $x \neq x_0$, then for some $j \geqq 1$, $x \notin C_j$. Then $\prod_{i=1}^{j} h_i(x) \notin D_j = \prod_{i=1}^{j} h_i(C_j)$. Since for $i > j$, h_i is supported on D_j, it follows that $h(x) = \prod_{i=1}^{j} h_i(x)$. From this, since $h_i \in G_{\mathscr{X}}$ and $\mathscr{C}$ and $\mathscr{D}$ are null for x_0 and y_0 respectively, it follows that h is a homeomorphism supported on K, $h(x_0) = y_0$, and h takes X' onto X'. Hence $h \in G_{\mathscr{X}}$. As a corollary we obtain Lemma 2, which resembles Anderson's telescoping Lemma [1].

LEMMA 2. *Suppose $K_0, K_1 \in \mathscr{K}$ with $K_1 \subseteq K_0^0$ and $g \in G(X, X')$ with $g(K_0) \subset K_1^0$. Suppose also that $x_0 \in K_1^0 \cap X'$, $\{D_i\}_{i=1}^{\infty}$ is null for x_0 with $D_1 \subset K_1^0$. Then there exists $h \in G_{\mathscr{X}}$, supported on K_1, such that $(hg)^i(K_0) = D_i$, for $i \geqq 1$.*

Proof. Set $C_1 = g(K_0)$ and $C_i = g(D_{i-1})$ for $i \geqq 2$. Then $\{C_i\}_{i=1}^{\infty}$ is null for $g(x_0)$. Hence Lemma 1 asserts the existence of $h \in G_{\mathscr{X}}$, supported on K_1, such that $h(C_i) = D_i$, for $i \geqq 1$. Hence $(hg)^i(K_0) = D_i$, for $i \geqq 1$.

LEMMA 3. *Any dilation structure carries a topological dilation.*

Proof. Suppose $\mathscr{B} = \{B_i\}_{i=-\infty}^{\infty}$ is a dilation structure with respect to (p, q). By Proposition B there is an element $K_1 \in \mathscr{K}$ such that $B_0 \subset K_1^0$, $K_1 \subset B_1^0$. We now apply Lemma 1 with K_1 as K, B_{-i} as C_i, and B_{-i+1} as D_i for $i \geqq 1$, and $x_0 = p = y_0$ to obtain $h_1 \in G_{\mathscr{X}}$, supported on K_1, with $h_1(B_{-i}) = B_{-i+1}$, for $i \geqq 1$.

Since $B_0 \subset h_1(B_0^0)$, we may find $K_2 \in \mathscr{K}$ with $B_0 \subset K_2^0$, $K_2 \subset h_1(B_0^0)$. Then $\mathrm{Cl}(\sim h_1(B_0)) \subset \sim K_2$, and $\mathrm{Cl}(\sim B_i) \subset \sim K_2$, $i \geqq 1$. So we again apply Lemma 1 to obtain $h_2 \in G_{\mathscr{X}}$, supported on $\mathrm{Cl}(\sim K_2)$ with $h_2(\mathrm{Cl}(\sim h_1(B_0))) = \mathrm{Cl}(\sim B_1)$, and $h_2(\mathrm{Cl}(\sim B_i)) = \mathrm{Cl}(\sim B_{i+1})$, $i \geqq 1$.

$h = h_2 h_1 \in G_{\mathscr{X}}^*$ is the desired topological dilation.

COROLLARY. *There exists an element $g \in G_{\mathscr{X}}^*$ such that g is a topological dilation.*

Proof. Since X' is perfect, X' contains two distinct points. The corollary now follows from Proposition A and Lemma 3.

We remarked earlier that if $g \in G_{\mathscr{X}}^*$ is an arbitrary topological dilation, then any conjugate of g by an element of $G_{\mathscr{X}}^*$ is also a topological dilation. Theorem 1 asserts that the converse is true: that any topological dilation is a conjugate of g.

THEOREM 1. *Suppose g_1 and g_2 are topological dilations. Then there is an element r of $G_{\mathscr{X}}^*$ such that $g_1 = r g_2 r^{-1}$.*

Proof. Let $\mathscr{C} = \{C_i\}_{i=-\infty}^{\infty}$ be a dilation structure with respect to (p, q); and let $\mathscr{D}' = \{D_i'\}_{i=-\infty}^{\infty}$ be a dilation structure with respect to (p', q'); with $\mathscr{C}$ and $\mathscr{D}$ carrying g_2 and g_1 respectively.

We may select $a \in \sim C_0 \cap X'$ $a \neq p'$. There is an integer $i_0 \leqq 0$ such that $a \notin D_{i_0}'$. Hence we may find $K_1 \in \mathscr{K}$ with $a \in K_1^0$, $K_1 \cap (C_0 \cup D_{i_0}') = \varnothing$, and $\sim K_1 \neq (C_0 \cup D_{i_0})$. Set $D_i = D_{i+i_0}$, $-\infty < i < \infty$; then $\mathscr{D} = \{D_i\}_{i=-\infty}^{\infty}$ carries g_1, and $D_0 = D_{i_0}$.

Since $C_0 \cup D_0 \subset \sim K_1$, we may invoke Lemma 1 to obtain $\delta_1 \in G_{\mathscr{K}}$, supported on $\mathrm{Cl}(\sim K_1)$, with $\delta_1(C_i) = D_i$, $i \leqq 0$.

$\{\delta_1(\mathrm{Cl}(\sim C_i))\}_{i=1}^{\infty}$ is null for $\delta_1(q)$, and $\delta_1(\mathrm{Cl}(\sim C_0)) = \mathrm{Cl}(\sim D_0)$. Hence we may again apply Lemma 1, this time to the sequences $\{\delta_1(\mathrm{Cl}(\sim C_i))\}_{i=1}^{\infty}$ and $\{\mathrm{Cl}(\sim D_i)\}_{i=1}^{\infty}$, to obtain $\delta_2 \in G_{\mathscr{K}}$, supported on $\mathrm{Cl}(\sim D_0)$, with $\delta_2\delta_1(\mathrm{Cl}(\sim C_i)) = \mathrm{Cl}(\sim D_i)$, $i \geqq 1$.

Set $\delta = \delta_2\delta_1 \in G_{\mathscr{K}}^* \cdot \delta(C_i) = D_i$, $-\infty < i < \infty$, and $\delta(p) = p'$, $\delta(q) = q'$. Hence $\mathscr{D}$ carries the topological dilation $g = \delta g_2 \delta^{-1}$.

We may think of the construction of δ as the first step in the conjugation procedure to change g_2 into g_1, since it yields the topological dilation g which is carried by the same structure that carries g_1. Of course pointwise g need not equal g_1. To achieve pointwise equality a further conjugation in necessary, which in the event that D_0 is open and closed is quite simple. We proceed to that case:

Set $B_i = (\text{boundary of } D_i) = D_i - D_i^0$. Since $g^i(B_0) = B_i$, $-\infty < i < \infty$, $B_i = \varnothing$. if and only if $B_0 = \varnothing$. If $B_0 = \varnothing$, define α as follows:

$$\alpha(p') = p', \qquad \alpha(q') = q';$$

if $x \in D_{i+1} - D_i$, $\alpha(x) = g_1^i g^{-i}(x)$, $-\infty < i < \infty$.

Then α is a homeomorphism of X onto itself taking X' onto X'. Since α takes D_0 onto itself, and since $B_0 = \varnothing$, if we define $\alpha_1 = \alpha$ on D_0 and $\alpha_1 = e$ on $\sim D_0$; and $\alpha_2 = \alpha$ on $\sim D_0$ and $\alpha_2 = e$ on D_0, then $\alpha_1, \alpha_2 \in G_{\mathscr{K}}$ and $\alpha = \alpha_2\alpha_1 \in G_{\mathscr{K}}^*$. Since $g_1 = \alpha g \alpha^{-1}$, if we set $r = \alpha\delta$, then $g_1 = r g_2 r^{-1}$.

It is in the remaining case that Axiom (5) finds its only employment. Thus we might observe now, if the reader hasn't already done so, that in case $\mathscr{K}$ consists of sets which are open and closed in X, Axiom (5) may be omitted.

Now suppose $B_0 \neq \varnothing$.

By Proposition B there exists $K_1 \in \mathscr{K}$ with $D_0 \subset K_1^0$, $K_1 \subset D_1^0$. Set $\alpha_1 = g_1 g^{-1} \in G_{\mathscr{K}}^*$; α_1 takes D_i onto D_i, $-\infty < i < \infty$. By Axiom (5) there exists $\bar{\alpha}_1 \in G_{\mathscr{K}}$, supported on $\mathrm{Cl}(\sim K_1)$ such that $\bar{\alpha}_1 \mid \mathrm{Cl}(\sim D_1) = \alpha_1$. In particular, $\bar{\alpha}_1$ takes D_i onto D_i and $\bar{\alpha}_1 \mid B_1 = \alpha_1$.

Proceeding inductively, for $i \geqq 2$ we may select $K_i \in \mathscr{K}$ with $D_{i-1} \subset K_i^0$, $K_i \subset D_i^0$. Set

$$\alpha_i = g_1^i g^{-i} (\bar{\alpha}_{i-1} \cdot \bar{\alpha}_{i-2} \cdots \bar{\alpha}_1)^{-1}.$$

$\alpha_i \in G_{\mathscr{K}}^*$ and α_i takes D_j onto D_j, $-\infty < j < \infty$. By Axiom (5) there is $\bar{\alpha}_i \in G_{\mathscr{K}}$, supported on $\mathrm{Cl}(\sim K_i)$, such that $\bar{\alpha}_i \mid \mathrm{Cl}(\sim D_i) = \alpha_i$. In particular, $\bar{\alpha}_i$ takes D_j onto D_j and $\bar{\alpha}_i \mid B_i = \alpha_i$.

Set $\alpha' = \prod_{i=1}^{\infty} \bar{\alpha}_i$. α' is a homeomorphism, supported on $\mathrm{Cl}(\sim K_1)$, and taking X' onto X' and X onto X; hence $\alpha' \in G_{\mathscr{K}}$.

Observe that $\alpha' \mid B_i = g_1^i g^{-i}$, $i \geqq 0$; and α' takes D_i onto D_i, $-\infty < i < \infty$.

Now take $K_{-1} \in \mathscr{K}$ with $D_{-1} \subset K_{-1}^0$, $K_{-1} \subset D_0^0$. Set $\alpha_{-1} = g_1^{-1} g$. $\alpha_{-1} \in G_{\mathscr{K}}^*$, and α_{-1} takes D_i onto D_i, $-\infty < i < \infty$. As before, from Axiom (5) we obtain $\bar{\alpha}_{-1} \in G_{\mathscr{K}}$, supported on K_{-1}, with $\bar{\alpha}_{-1} \big| D_{-1} = \alpha_{-1}$. In particular, $\bar{\alpha}_{-1}$ takes D_i onto D_i, and $\bar{\alpha}_{-1} \big| B_{-1} = \alpha_{-1}$.

Again proceeding inductively for $i \geqq 2$, select $K_{-i} \in \mathscr{K}$ with $D_{-i} \subset K^0_{-i}$, $K_{-i} \subset D^0_{-i+1}$. Set

$$\alpha_{-i} = g_1^{-i} g^i (\bar{\alpha}_{-i+1} \cdot \bar{\alpha}_{-i+2} \cdots \bar{\alpha}_{-1})^{-1}.$$

Then α_{-i} takes D_j onto D_j, $-\infty < j < \infty$, and $\alpha_{-i} \in G^*_{\mathscr{K}}$. We use Axiom (5) to obtain $\bar{\alpha}_{-i} \in G_{\mathscr{K}}$, supported on K_{-i}, with $\bar{\alpha}_{-i} \big| D_{-i} = \alpha_{-i}$. And $\bar{\alpha}_{-i}$ takes D_j onto D_j, and $\bar{\alpha}_{-i} \big| B_{-i} = \alpha_{-i}$.

Set $\bar{\alpha} = \prod_{i=1}^{\infty} \bar{\alpha}_{-i}$. $\bar{\alpha} \in G_{\mathscr{K}}$, is supported on K_{-1}, takes D_i onto D_i, $-\infty < i < \infty$, and $\bar{\alpha} \big| B_{-i} = g_1^{-i} g^i$, $i \geqq 0$.

Now set $\alpha = \bar{\alpha} \cdot \alpha' \in G^*_{\mathscr{K}}$. Since $K_{-1} \cap \mathrm{Cl}(\sim K_1) = \varnothing$, α takes D_i onto D_i and $\alpha \big| B_i = g_1^i g^{-i}$, $-\infty < i < \infty$.

Setting $\bar{g} = \alpha g \alpha^{-1}$, we observe that $\bar{g} \big| B_i = g_1$, $-\infty < i < \infty$.

Finally, we define β as follows:

$$\beta(p') = p', \qquad \beta(q') = q';$$

if $x \in D_{i+1} - D_i$, $\beta(x) = g_1^i \bar{g}^{-i}(x)$, $-\infty < i < \infty$.

β is a homeomorphism of X onto itself, carrying X' onto X'. Also $\beta \big| B_i = e$, $-\infty < i < \infty$. And since, in addition, β takes D_0 onto D_0, if we define $\beta_1 = \beta$ on D_0 and $\beta_1 = e$ on $\sim D_0$; and $\beta_2 = \beta$ on $\sim D_0$ and $\beta_2 = e$ on D_0, then $\beta_1, \beta_2 \in G_{\mathscr{K}}$; hence $\beta = \beta_2 \beta_1 \in G^*_{\mathscr{K}}$. And since $g_1 = \beta \bar{g} \beta^{-1}$, if we set $r = \beta \alpha \delta \in G^*_{\mathscr{K}}$, we have $g_1 = r g_2 r^{-1}$.

3. **The principal theorem; its corollaries.** In this section we prove the principal theorem of the paper and deduce some corollaries.

THEOREM 2. *Suppose f and g are elements of $G(X, X')$ neither of which is the identity on X'. Then there exists elements h_1 and h_2 of $G^*_{\mathscr{K}}$ such that $(h_2 g h_2^{-1})(h_1 f h_1^{-1})$ is a topological dilation. Furthermore h_1 and h_2 may be chosen as topological dilations.*

Proof. We distinguish two cases: *Case* 1. At least one of f and g, say f, is not of period two on X'. *Case* 2. Both f and g are of period two on X'.

Case 1. Since f is not of period two on X', neither f nor g is the identity on X', and X' is perfect, we may find elements C_1, D_1, A_0, and B_0 of $\mathscr{K}$ such that the sets $\{C_1, D_1, A_0, B_0, f(A_0), f^{-1}(A_0), g(B_0)\}$ are mutually disjoint and such that the sets $\{C_1, D_1, A_0, B_0, f(A_0), f^{-1}(A_0), g^{-1}(B_0)\}$ are mutually disjoint.

Select $c \in C_1^0 \cap X'$, $d \in D_1^0 \cap X'$, $a \in A_0^0 \cap X'$, and $b \in B_0^0 \cap X'$. We now take sequences of elements of $\mathscr{K}$ — $\{C_i\}_{i=1}^{\infty}$, $\{D_i\}_{i=1}^{\infty}$, $\{A_i\}_{i=0}^{-\infty}$, and $\{B_i\}_{i=0}^{-\infty}$-null for c, d, a, and b respectively.

Set $A_i = \mathrm{Cl}(\sim C_i)$, $B_i = \mathrm{Cl}(\sim D_i)$, $i \geqq 1$. Then $\mathscr{A} = \{A_i\}_{i=-\infty}^{\infty}$ and $\mathscr{B} = \{B_i\}_{i=-\infty}^{\infty}$

are dilation structures with respect to (a,c) and (b,d) respectively. By Lemma 3 $\mathscr{A}$ and $\mathscr{B}$ respectively carry topological dilations δ and r.

Set $\beta=(rgr^{-1})(\delta f\delta^{-1})$. $\beta\in G^*_{\mathscr{X}}$ by Axiom (3).

Now a remark about what is to follow: It is true that $\beta(A_1)\subset A_1^0$ and $\beta^{-1}(C_1)\subset C_1^0$ ($A_1=\mathrm{Cl}(\sim C_1)$). Hence $\beta^i(A_1)\subset\beta^{i-1}(A_1^0)$ for $i\geqq 1$, and $\beta^{-i}(C_1)\subset\beta^{-i+1}(C_1^0)$, $i\geqq 1$. Therefore if $\{\beta^i(A_1)\}_{i=0}^\infty$ were null for a point of X' and if $\{\beta^{-i}(C_1)\}_{i=0}^\infty$ were null for a point of X', then it is apparent that β would then be a topological dilation. Therefore, in order to achieve this we are going to multiply r and δ by appropriate elements of $G^*_{\mathscr{X}}$.

Consider $\beta(A_1)$:

$$\begin{aligned}\delta^{-1}(A_1) &= A_0,\\ f(A_0) &\subset A_1^0-A_0,\\ \delta(A_1^0-A_0) &= A_2^0-A_1=C_1^0-C_2.\end{aligned}$$

Hence

$$\begin{aligned}\delta f\delta^{-1}(A_1) &\subset C_1^0-C_2,\\ r^{-1}(C_1) &\subset B_0^0-B_{-1},\\ g(B_0) &\subset B_1^0-B_0,\\ r(B_1^0-B_0) &= B_2^0-B_1=D_1^0-D_2.\end{aligned}$$

Hence $\beta(A_1)\subset D_1^0-D_2$. Set $K_1=rg(B_0)$. $\beta(A_1)\subset K_1^0$, $K_1\subset D_1^0-D_2$. Select $x_1\in K_1^0\cap X'$ and a sequence of elements of $\mathscr{K}$, $\{T_i\}_{i=1}^\infty$, null for x_1, with $T_1\subset K_1^0$. We now apply Lemma 2, with A_1 as K_0 and with β as g, to obtain $\alpha_1\in G_{\mathscr{X}}$, supported on K_1, with $(\alpha_1\beta)^i(A_1)=T_i$, $i\geqq 1$.

Now set $\bar\beta=(\delta f^{-1}\delta^{-1})(\alpha_1 rg^{-1}r^{-1}\alpha_1^{-1})$ and consider $\bar\beta(C_1)$.

Since $K_1\cap C_1=\varnothing$, $rg^{-1}r^{-1}\alpha_1^{-1}(C_1)=rg^{-1}r^{-1}(C_1)$,

$$\begin{aligned}r^{-1}(C_1) &\subset B_0^0,\\ g^{-1}(B_0) &\subset B_1^0-B_0,\\ r(B_1^0-B_0) &= B_2^0-B_1=D_1^0-D_2.\\ \text{So } rg^{-1}r^{-1}(C_1) &\subset D_1^0-D_2. \text{ Since } K_1\subset D_1^0-D_2,\\ \alpha_1 rg^{-1}r^{-1}(C_1) &\subset D_1^0-D_2\subset D_1^0,\\ \delta^{-1}(D_1) &\subset A_0^0; \text{ and } f^{-1}(A_0)\subset A_1^0-A_0,\\ \delta(A_1^0-A_0) &= A_2^0-A_1=C_1^0-C_2.\end{aligned}$$

Hence $\bar\beta(C_1)\subset C_1^0-C_2$.

Set $K_2=\delta f^{-1}(A_0)$. Then $\bar\beta(C_1)\subset K_2^0$, $K_2\subset C_1^0-C_2$. Note that $K_2\cap\delta f(A_0)=\varnothing$.

Select $x_2 \in K_2^0 \cap X'$ and a sequence of elements of $\mathscr{K}$, $\{T_i\}_{i=-1}^{-\infty}$ null for x_2, with $T_{-1} \cap K_2^0$. We again apply Lemma 2, this time with C_1 as K_0 and with $\bar{\beta}$ as g, to obtain $\alpha_2 \in G_{\mathscr{X}}$, supported on K_2, with $(\alpha_2\bar{\beta})_i(C_1) = T_{-i}$, $i \geqq 1$.

Set $d = (\alpha_1 rgr^{-1}\alpha_1^{-1})(\alpha_2\delta f\delta^{-1}\alpha_2^{-1})$ and observe that if $d^i(A_1) = T_i$ and $d^{-i}(C_1) = T_{-i}$ for $i \geqq 1$, then d is a topological dilation carried by the dilation structure $\{T_i'\}_{i=-\infty}^{\infty}$ with respect to (x_2, x_1), where $T_i' = T_i$, $i \leqq -1$, $T_0' = C_1$, $T_i' = \mathrm{Cl}(\sim T_i)$, $i \geqq 1$.

Consider $d(A_1)$:

Since $K_2 \cap A_1 = \varnothing$, $\delta f\delta^{-1}\alpha_2^{-1}(A_1) = \delta f\delta^{-1}(A_1) = \delta f(A_0)$. Since $K_2 \cap \delta f(A_0) = \varnothing$, $\alpha_2\delta f(A_0) = \delta f(A_0) \subset C_1$. But $K_1 \cap C_1 = \varnothing$; so $\alpha_1^{-1}\delta f(A_0) = \delta f(A_0)$. Hence $d(A_1) = \alpha_1 rgr^{-1}\delta f\delta^{-1}(A_1) = \alpha_1\beta(A_1) = T_1$. Since for $i \geqq 1$, $T \subset A_1$, $d(T_i) = \alpha_1\beta(T_i) = T_{i+1}$. Therefore $d^{-i}(A_1) = T_{-i}$, $i \geqq 1$.

Now consider $d^{-1}(C_1) = \alpha_2\delta f^{-1}\delta^{-1}\alpha_2^{-1}\alpha_1 rg^{-1}r^{-1}\alpha_1^{-1}(C_1)$:

As seen earlier, $\alpha_1 rg^{-1}r^{-1}\alpha_1^{-1}(C_1) \subset D_1^0$. Since $K_2 \cap D_1 = \varnothing$,

$$\alpha_2^{-1}\alpha_1 rg^{-1}r^{-1}\alpha_1^{-1}(C_1) = \alpha_1 rg^{-1}r^{-1}\alpha_1^{-1}(C_1).$$

Hence $d^{-1}(C_1) = \alpha_2\delta f^{-1}\delta^{-1}\alpha_1 rg^{-1}r^{-1}\alpha_1^{-1}(C_1) = \alpha_2\bar{\beta}(C_1) = T_{-1}$. And since for $i \geqq 1$, $T_{-i} \subset C_1$, $d^{-1}(T_{-i}) = \alpha_2\bar{\beta}(T_{-i}) = T_{-i-1}$. Hence $d^{-i}(C_1) = T_{-i}$, and hence d is a topological dilation.

Furthermore, since α_1 is supported on $K_1 \subset D_1^0 - D_2$ and α_2 is supported on $K_2 \cap C_1^0 - C_2$, $h_1 = \alpha_2\delta$ and $h_2 = \alpha_1 r$ are topological dilations carried by $\mathscr{A}$ and $\mathscr{B}$ respectively; and $d = (h_2 g h_2^{-1})(h_1 f h_1^{-1})$.

In the above we had set $K_2 = \delta f^{-1}(A_0)$ and had then remarked that $K_2 \subset \delta f(A_0) = \varnothing$. It was this that enabled us to construct α_1 independently of α_2. Of course the construction of α_2 was dependent on α_1; however we defined $K_1 = rg(B_0)$ and had we been able to assert that $K_1 \cap rg^{-1}(B_0) = \varnothing$ we could have constructed α_1 and α_2 independently of each other. But, since in the event that g was of period two on X', $K_1 \cap rg^{-1}(B_0) \neq \varnothing$, no such assertion was possible. Had f also been of period two on X' then neither α_1 nor α_2 could have been constructed independently of the other. For this reason Case 2 requires a slightly modified procedure.

Case 2. Since neither f nor g is the identity on X', we may select elements of $\mathscr{K}$, C_0, D_0, A_0, and $^*B_0$, such that the sets $\{C_0, D_0, A_0, {}^*B_0, f(A_0), g({}^*B_0)\}$ are mutually disjoint and the sets $\{C_0, D_0, A_0, {}^*B_0, f^{-1}(A_0), g^{-1}({}^*B_0)\}$ are mutually disjoint.

Now select $c \in C_0^0 \cap X'$, $d \in D_0^0 \cap X'$, $a \in A_0^0 \cap X'$, and $b \in {}^*B_0^0 \cap X'$. As before, we may select null sequences $\{C_i\}_{i=0}^{\infty}$, $\{D_i\}_{i=0}^{\infty}$, $\{A_i\}_{i=0}^{-\infty}$, and $\{B_i\}_{i=0}^{-\infty}$ for c, d, a, and b respectively with $B_0 \subset {}^*B_0^0$.

We now set $A_i = \mathrm{Cl}(\sim C_i)$, $B_i = \mathrm{Cl}(\sim D_i)$ for $i \geqq 1$ and observe that $\mathscr{A} = \{A_i\}_{i=-\infty}^{\infty}$ and $\mathscr{B} = \{B_i\}_{i=-\infty}^{\infty}$ are dilation structures with respect to (a, c) and (b, d) respectively and hence, by Lemma 3, carry topological dilations δ_1 and r_1 respectively.

$r_1^{-1}(C_0) \subset B_0^0 - B_1$ and, since $gr_1^{-1}(c) = g^{-1}r_1^{-1}(c) \in g(B_0^0)$, we may select $N \in \mathscr{K}$ with $r_1^{-1}(c) \in N \subset r_1^{-1}(C_0^0)$ and such that $g^{-1}(N) \subset g(B_0^0)$. Since $C_1 \subset C_0^0 \supset r_1(N)$, we may invoke Axiom (4) to obtain $\alpha_1 \in G_{\mathscr{X}}$, supported on C_0, with $\alpha_1 r_1(N) = C_1$.

Set $r = \alpha_1 r_1$ and observe that r is a topological dilation carried by $\mathscr{B}$ and $r(N) = C_1$.

Employing an identical procedure, we can obtain $K_1 \in \mathscr{K}$, and $\alpha_2 \in G_{\mathscr{X}}$ such that α_2 is supported on D_0, $K_1 \subset A_0^0 - A_{-1}$, $f^{-1}(K_1) \subset f(A_0^0)$, and $\alpha_2\delta_1(K_1) = D_1$.

Set $\delta = \alpha_2\delta_1$ and note that δ is a topological dilation carried by $\mathscr{A}$ and $\delta(K_1) = D_1$.

Now $rg(b) \in rg({}^*B_0) \subset D_1^0 - D_2$. Hence $\delta^{-1}rg(b) \in K_1^0$. Set $x_0 = \delta^{-1}rg(b)$. Since $g^{-1}r^{-1}\delta f^{-1}(x_0) = gr^{-1}\delta f(x_0)$, we may select $\{K_i\}_{i=2}^{\infty}$, null for x_0, with $K_2 \subset K_1^0$, such that $g^{-1}r^{-1}\delta f^{-1}(K_i) \subset gr^{-1}\delta f(K_{i-1}^0)$ for $i \geqq 2$ and with $\delta(K_2) \subset rg({}^*B_0^0)$.

Set $E_i = r^{-1}\delta f^{-1}(K_i)$, $i \geqq 2$. Then $r(E_i) = \delta f^{-1}(K_i)$ and $g^{-1}(E_i) \subset gr^{-1}\delta f(K_{i-1}^0)$, $i \geqq 2$.

Now define $\pi \in G_{\mathscr{X}}^*$ as: $\pi = rgr^{-1}\delta f\delta^{-1}$.

Consider $\pi(A_1)$: $\delta^{-1}(A_1) = A_0$, so $\delta f\delta^{-1}(A_1) = \delta f(A_0)$. Since $r^{-1}\delta f(A_0) \subset B_0^0$, $\pi(A_1) = rgr^{-1}\delta f\delta^{-1}(A_1) = rgr^{-1}\delta f(A_0) \subset rg(B_0^0)$.

Since $\delta(K_1) \subset A_1^0$, the sequence:

$$rg(B_0) \supset \pi(A_1) \supset \pi(\delta(K_1)) \supset \dots \supset \pi(\delta(K_i)) \supset \dots$$

is null for $\pi\delta(x_0)$. Also the sequence $\{\delta(K_i)\}_{i=2}^{\infty}$ is null for $\delta(x_0)$. Finally we have $\delta(K_2) \cup rg(B_0) \subset rg({}^*B_0^0) \subset D_1^0 - D_2$.

Hence by Lemma 1 there exists $\alpha_3 \in G_{\mathscr{X}}$, supported on $rg({}^*B_0)$, such that $\alpha_3 rg(B_0) = \delta(K_2)$, $\alpha_3\pi(A_1) = \delta(K_3)$, and $\alpha_3\pi(\delta(K_i)) = \delta(K_{i+3})$, $i \geqq 1$.

$\alpha_3 r$ is a topological dilation carried by $\mathscr{B}$.

We shall now show that $d = \alpha_3 rgr^{-1}\alpha_3^{-1}\delta f\delta^{-1}$ is a topological dilation.

Consider $d(A_1)$:

Since $\delta f\delta^{-1}(A_1) \subset C_1$, and α_3 is supported on D_1, $\alpha_3^{-1}\delta f\delta^{-1}(A_1) = \delta f\delta^{-1}(A_1)$. Hence $d(A_1) = \alpha_3 rgr^{-1}\delta f\delta^{-1}(A_1) = \alpha_3\pi(A_1) = \delta(K_3)$. And since $\delta(K_i) \subset A_1^0$, $i \geqq 1$, $\alpha_3^{-1}\delta f\delta^{-1}(\delta(K_i)) = \delta f\delta^{-1}(\delta(K_i))$. Hence $d(\delta(K_i)) = \alpha_3\pi(\delta(K_i)) = \delta(K_{i+3})$, $i \geqq 1$. Therefore $\{d^i(A_1)\}_{i=0}^{\infty}$ is null.

We now consider $d^{-1}(C_1) = \delta f^{-1}\delta^{-1}\alpha_3 rg^{-1}r^{-1}\alpha_3^{-1}(C_1)$: Since α_3 is the identity on C_1, $rg^{-1}r^{-1}\alpha_3^{-1}(C_1) = rg^{-1}r^{-1}(C_1)$. But $g^{-1}r^{-1}(C_1) \subset g(B_0)$, so $rg^{-1}r^{-1}(C_1) \subset rg(B_0)$. Hence $\alpha_3 rg^{-1}r^{-1}(C_1) \subset \alpha_3 rg(B_0) = \delta(K_2)$. Therefore $d^{-1}(C_1) \subset \delta f^{-1}\delta^{-1}(\delta(K_2)) = \delta f^{-1}(K_2) \subset C_1^0$.

Now consider $d^{-1}(\delta f^{-1}(K_i))$, $i \geqq 2$: Since α_3 is the identity on $\delta f^{-1}(K_i) \subset C_1$, $r^{-1}\alpha_3^{-1}\delta f^{-1}(K_i) = r^{-1}\delta f^{-1}(K_i) = E_i$. Since $g^{-1}(E_i) \subset gr^{-1}\delta f(K_{i-1})$, $rg^{-1}(E_i) \subset rgr^{-1}\delta f(K_{i-1}) = rgr^{-1}\delta f\delta^{-1}\delta(K_{i-1}) = \pi(\delta(K_{i-1}))$. Hence $d^{-1}(\delta f^{-1}(K_i)) \subset \delta f^{-1}\delta^{-1}\alpha_3\pi(\delta(K_{i-1})) = \delta f^{-1}\delta^{-1}(\delta(K_{i+2})) = \delta f^{-1}(K_{i+2})$, $i \geqq 2$.

So we see that $d^{-1}(C_1) \subset \delta f^{-1}(K_2)$, $d^{-2}(C_1) \subset d^{-1}(\delta f^{-1}(K_2)) \subset \delta f^{-1}(K_4)$, etc. In general, $d^{-i}(C_1) \subset \delta f^{-1}(K_{2i})$, $i \geqq 1$.

From this, together with the fact that $d^{-i}(C_1) \subset d^{-i+1}(C_1^0)$, $i \geqq 1$, we conclude that $\{d^{-i}(C_1)\}_{i=0}^{\infty}$ is null for $\delta f^{-1}(x_0)$. If we set $h_1 = \delta$ and $h_2 = \alpha_3 r$, then $d = (h_2 g h_2^{-1})(h_1 f h_1^{-1})$.

Let $d \in G_{\mathscr{X}}^*$ be an arbitrary but fixed topological dilation:

COROLLARY 1. *If $f, g \in G(X, X')$, neither f nor g being the identity on X', then d is the product of a conjugate of f by a conjugate of g, the conjugating homeomorphisms being elements, of $G_{\mathscr{X}}^*$.*

Proof. By Theorem 2 there exist $h_1, h_2 \in G_{\mathscr{X}}^*$ such that $\bar{d} = (h_2 g h_2^{-1})(h_1 f h_1^{-1})$, where $\bar{d}$ is a topological dilation. But by Theorem 1 there exists $h_3 \in G_{\mathscr{X}}^*$ such that $h_3 d h_3^{-1} = \bar{d}$. Hence $h_3 d h_3^{-1} = (h_2 g h_2^{-1})(h_1 f h_1^{-1})$. Therefore

$$d = h_3^{-1}(h_2 g h_2^{-1})(h_1 f h_1^{-1})h_3 = (h_3^{-1} h_2 g h_2^{-1} h_3)(h_3^{-1} h_1 f h_1^{-1} h_3).$$

COROLLARY 2. *If $f \in G(X, X')$ and either $f \mid X' \neq e$ or $f = e$ and if $d \in G(X, X')$, then f is a product of two conjugates of d, the conjugating homeomorphisms being elements of $G_{\mathscr{X}}^*$.*

Proof. If $f \mid X' \neq e$, then by Corollary 1, with d^{-1} as g, we have

$$d = (h_2 d^{-1} h_2^{-1})(h_1 f h_1^{-1})$$

for some h_2, $h_1 \in G_{\mathscr{X}}^*$. Hence $(h_2 d h_2^{-1})\ d = h_1 f h_1^{-1}$; or $h_1^{-1}(h_2 d h_2^{-1})\ d h_1 = f$. Therefore $(h_1^{-1} h_2 d h_2^{-1} h_1)(h_1^{-1} d h_1) = f$.

If $f = e$, then notice that d^{-1} is a topological dilation, and hence, for some $h_1 \in G_{\mathscr{X}}^*$, $h_1 d h_1^{-1} = d^{-1}$. Then $d(h_1 d h_1^{-1}) = d d^{-1} = e = f$.

COROLLARY 3. *Suppose f, g, h, and k are elements of $G(X, X')$, none of which are the identity on X'. Then f is the product of a conjugate of g by a conjugate of h by a conjugate of k, the conjugating elements being in $G_{\mathscr{X}}^*$.*

Proof. By Corollary 1 $(h_1 f h_1^{-1})(h_2 g^{-1} h_2^{-1}) = d = (h_3 k h_3^{-1})(h_4 h h_4^{-1})$ for some $h_1, h_2, h_3, h_4 \in G_{\mathscr{X}}^*$. Then $h_1 f h_1^{-1} = (h_3 k h_3^{-1})(h_4 h h_4^{-1})(h_2 g h_2^{-1})$. Hence $f = (h_1^{-1} h_3 k h_3^{-1} h_1)(h_1^{-1} h_4 h h_4^{-1} h_1)(h_1^{-1} h_2 g h_2^{-1} h_1)$.

COROLLARY 4. *Suppose $f, g \in G(X, X')$ and $f \mid X' \neq e \neq g \mid X'$. Then f is the product of three conjugates of g, the conjugating elements being in $G_{\mathscr{X}}^*$.*

Proof. Corollary 3 with $h = k = g$.

4. **Examples.** We now make mention of a few examples of A-quadruples. In each instance, of course, the existence of topological dilations is guaranteed by the corollary to Lemma 3. However in the examples to follow a canonical topological dilation d, suggests itself and in some instances will be given. Then, by Theorem

1, the collection of topological dilations coincides with the collection of $G^*_{\mathscr{X}}$-conjugates of d.

(1) Let $X = X' = S_n$, the n-sphere for $n \geqq 1$. Let $G(X, X') = G^*_{\mathscr{X}} =$ the group of stable(4) homeomorphisms of S_n onto itself. $\mathscr{X}$ is the collection of images of a geometric n-cell under the elements of $G^*_{\mathscr{X}}$. That Axioms (1), (2), and (3) are satisfied in this case is evident. The proof that Axioms (4) and (5) obtain may be found in [3]. Let $p \in S_n$ and let q be the point of S_n antipodal to p. We may take as d an ordinary radial expansion homeomorphism of S_n onto itself which fixes only p and q. (We think of d as the natural extension to S_n of $d': E_n \to E_n$ given by $d'(x) = r_0 x$ for some fixed, positive $r_0 \neq 1$.)

(2) Take $X, X', \mathscr{X}$, $G^*_{\mathscr{X}}$, and d as above. Let $G(X, X')$ be the set of all homeomorphisms which are the product of some geometric orientation-reversing involution, h, and an element of $G^*_{\mathscr{X}}$.

Since $d \notin G(X, X')$, Corollary 2 is not applicable here. However $dh \in G(X, X')$ so we may substitute for Corollary 2 the following: if $f \in G(X, X')$, then f is the product of a conjugate of d by a conjugate of dh, the conjugating homeomorphisms, of course, being in $G^*_{\mathscr{X}}$(5).

(3) Let $X = S_n$, $n \geqq 1$. Let X' be a countable dense subset of X, and let $G^*_{\mathscr{X}} = G(X, X')$ be those stable homeomorphisms which take X' onto X'. $\mathscr{X}$ is the collection of images of a geometric n-cell under elements of $G^*_{\mathscr{X}}$.

(4) Let $X = D_n =$ the n-cell, $n \geqq 2$. Call X' the boundary of D_n, and $G_{\mathscr{X}}$ is the set of all homeomorphisms of D_n onto itself each of which is the identity on some neighborhood of some point of X'. $G(X, X') = G^*_{\mathscr{X}}$, and $\mathscr{X}$ is the collection of images of a geometric half-cell under the elements of $G^*_{\mathscr{X}}$. Axioms (1), (2), and (3) are obviously satisfied, and Axioms (4) and (5) follow from arguments like those given in [3].

(5) Let $X = X'$ be the rationals, the irrationals, or the Cantor set. Let $G^*_{\mathscr{X}} = G(X, X')$ be the collection of all homeomorphisms of X onto itself. $\mathscr{X}$ is the collection of all nonvoid, proper, open and closed subsets of X.

If X is the Cantor set, realized as the 'middle third' set in $[0,1]$, define $\{C_i\}_{i=-\infty}^{\infty}$ as follows: for $i \leqq 0$,

$$C_i = \left[\frac{2}{3^{-i+2}}, \frac{1}{3^{-i+1}}\right] \cap X, \text{ and for } i \geqq 1,\ C_i = \left[\frac{3^i - 1}{3^i}, \frac{3^{i+1} - 2}{3^{i+1}}\right] \cap X.$$

Then $\bigcup_{i=-\infty}^{\infty}\{C_i\} \cup \{0\} \cup \{1\} = X$. Take d as any homeomorphism of X onto itself such that $d(0) = 0$, $d(1) = 1$, and $d(C_i) = C_{i+1}$, $-\infty < i < \infty$.

(4) $G^*_{\mathscr{X}}$ is the group generated by the collection of all those homeomorphisms of S_n onto itself each of which is the identity on some nonvoid open set. The term 'stable' used to denote this group appears to be due to Brown and Gluck [3].

(5) If the annulus problem is solved affirmatively, then in (1) we may take $G^*_{\mathscr{X}} = G(X, X')$ to be the collection of all orientation-preserving homeomorphisms, and in (2) we may take $G(X, 'X)$ to be all the orientation-reversing homeomorphisms.

A few words about possible extensions of the preceding conclusions. Let G denote the group of stable homeomorphisms of E_n. Set $d(x) = r_0 x$, $x \in E_n$, $r_0 > 1$. By arguments similiar in spirit to those of Theorems 1 and 2, one can show that if $f \in G_n$, then there exist stable homeomorphisms δ and r such that $\delta d \delta^{-1} f = r d r^{-1}$. Therefore f is the product of a stable conjugate of d by a stable conjugate of d^{-1}.

It is impossible to obtain Corollary 4 for the group G_n, since it is not simple. However the following question might well be asked: suppose f, $g \in G_n$ with f not the identity and g not supported on a cell. Is f the product of three conjugates of g? Or if three won't work, what is the smallest number which will? Three appears to suffice, if g moves a tame ray off of itself.

Finally we remark that the number of conjugates of g employed in Corollary 4 cannot, in general, be reduced: In each of the examples (1)–(5) above it is easy to find nonidentity elements f and g of $G(X, X')$ such that f is not the product of two conjugates of g.

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