H^p SPACES AND EXTREMAL FUNCTIONS IN H^1

BY

T. W. GAMELIN

Let A be a uniform algebra on a compact Hausdorff space X , i.e., A is a uniformly closed separating subalgebra of $C(X)$ which contains the constant functions. Let dm be a representing measure on X for a complex-valued homomorphism ϕ of A . A_0 will denote the kernel of ϕ , and H^p and H_0^p will denote respectively the closures of A and A_0 in $L^p(dm)$, $0 < p < \infty$.

A function $f \in H^1$ is *extremal* if $f \neq 0$ and $f/\|f\|_1$ is an extreme point of the unit ball of H^1 . DeLeeuw and Rudin [1] proved that if $\Delta = \{z \mid |z| \leq 1\}$ and A is the algebra of continuous functions on Δ which are analytic on the interior of Δ , and if $f \in H^1(d\theta/2\pi)$, then Af is dense in $H^1(d\theta/2\pi)$ if and only if f is extremal. It is the purpose of this note to prove the following generalization. *Here, and throughout the paper, we assume that dm is a Szegő measure for ϕ (defined later).*

THEOREM. *Suppose $f \in H^1$. Then A_0f is dense in H_0^1 if and only if f is extremal.*

The proof of the theorem is based on the idea from [2] of projecting L^1 into H^p , where $0 < p < 1$, together with a technique of Hoffman and Wermer [9] which allows one in certain situations to modify H^p -convergence to obtain pointwise bounded convergence. Professor Forelli tells us he has used the projection of L^1 into H^p , together with some special function theory, to obtain the theorem for the H^1 spaces associated with algebras of almost periodic functions.

The Hoffman-Wermer technique is used in subsequent sections to study H^p spaces for $0 < p < \infty$. Here proofs are given of some standard results, all known for $p \geq 1$, which also cover the case $0 < p < 1$. An invariant subspace theorem is proved in the final section which shows that once the invariant subspaces of L^2 are understood, the invariant subspaces of the other L^p spaces, $0 < p < \infty$, offer no difficulty.

The author would like to acknowledge several helpful conversations with Keith Yale. We are also grateful to Professor Kenneth Hoffman for supplying us with Lemma 3, which allowed us to delete a superfluous hypothesis from the theorem.

Remarks. According to a theorem of Wermer [12], [6], the Gleason part of ϕ on H^∞ is either an analytic disc or just the one point ϕ , depending on whether or not there is an inner function F such that $FH^1 = H_0^1$. If such an inner function

exists, it is easy to see that the only extremal functions in H^1 are the outer functions. In the other case, it is not known whether there exist functions $f \in H_0^1$ such that fA is dense in H_0^1 . In [5, p. 183], this problem is raised in an equivalent form, in connection with evanescent stochastic processes.

A function f such that $A_0 f$ is dense in H_0^1 has the following property: if $g \in H_0^1$ and $|g| \leq |f|$, then $g/f \in H^\infty$. In particular, if $|g| = |f|$, then $g = Ff$ expresses g as the product of an inner function and an extremal function, and the factorization is unique, up to a constant multiple of modulus one. Consequently, the existence of extremal functions in H_0^1 would lead to a wider factorization theory than now available.

Proof of the theorem. A Szegő measure is a representing measure dm for ϕ such that Szegő's theorem is valid: for any function $h \in L^1$ such that $h \geq 0$,

$$\inf_{f \in A_0} \int |1 - f|^2 h \, dm = \exp \left\{ \int \log h \, dm \right\}.$$

Equivalently, dm is a Szegő measure if whenever dv is another representing measure for ϕ which is absolutely continuous with respect to dm , then $dv = dm$. Also, dm is a Szegő measure if and only if the algebra $H^\infty(dm)$ of bounded functions in $H^1(dm)$ is a logmodular algebra on its Šilov boundary Y [6].

We will be more interested in the logmodular algebra H^∞ on Y rather than in A . The results of [6] carry over to H^∞ . In particular, dm is the unique representing measure on Y for the extension of ϕ to H^∞ (also denoted by ϕ). Since ϕ then has a unique norm-preserving extension from H^∞ to $C(Y)$, the following approximation lemma is valid [10].

LEMMA 1. If u is any real-valued continuous function on Y , then

$$\int u \, dm = \inf \left\{ \int v \, dm : v \in \text{Re}(H^\infty), v \geq u \right\}.$$

The proof of the theorem begins with the main lemma used by deLeeuw and Rudin.

LEMMA 2. A nonzero function $f \in H^1$ is not extremal if and only if there exists a bounded real-valued function k such that $\int k \, dm = 0$, $kf \in H^1$, and $kf \not\equiv 0$.

COROLLARY. If the Gleason part of ϕ on H^∞ is an analytic disc, then the extremal functions in H^1 are the outer functions.

In this case f is not outer if and only if we can write $f = Fg$, where $g \in H^1$ and F is a nonconstant inner function. If $\int f \, dm \neq 0$, this follows from [6], and if $\int f \, dm = 0$, from the remarks preceding the proof. If $k = F + \bar{F}$, then the function $k - \int k \, dm$ satisfies the requirements of Lemma 2.

If f is outer, then 1 is in the L^1 -closure of Af , so any k satisfying the requirements of Lemma 2 would belong to H^1 . But H^1 contains no nonconstant real-valued functions [6]. This proves the corollary.

LEMMA 3 (HOFFMAN). *If E is a measurable subset of Y such that $0 < m(E) < 1$, there is a function $g \in H^\infty$ such that g is real on E and g is not a constant a.e. on E .*

To prove this, let h be an outer function in H^∞ such that $|h| = 1$ on E and $|h| = e$ on $Y - E$. h is invertible in H^∞ , and $g = h + 1/h$ is real on E . Also, $|\phi(h)| > 1$, since

$$\log |\phi(h)| = \int \log |h| \, dm > 0.$$

Suppose that g is a constant on E . Then h assumes at most two values on E , so h is a constant on a subset D of E of positive measure. If $h = \lambda$ on D , $h - \lambda$ vanishes on a set of positive measure, so $\phi(h - \lambda) = 0$ by Jensen's inequality. Consequently, $|\phi(h)| = |\lambda| = 1$, a contradiction.

COROLLARY. *If $f \in H^1$ vanishes on a set of positive measure, then f is not extremal.*

If $f \not\equiv 0$ vanishes on a set of positive measure, let E be the set where f does not vanish, and let g be the function of Lemma 3. If k is defined to be equal to 0 where f vanishes and equal to g where f does not, then $k - \int k \, dm$ satisfies the requirements of Lemma 2.

LEMMA 4. *$fA_0 + \overline{fA_0} + C$ is dense in L^1 if and only if f is extremal.*

Here C is the field of complex constants, and the bar denotes complex conjugation. Lemma 4 is a simple consequence of Lemma 2 and the corollary to Lemma 3. The function k appears as a linear functional on L^1 which is orthogonal to $fA_0 + \overline{fA_0} + C$. Since $H + \overline{H^1}$ is dense in L^1 , we have proved the following half of the theorem.

COROLLARY. *If fA_0 is dense in H_0^1 , then f is extremal.*

The next lemma first appears in a function algebra setting in [4]. The proof there is valid for Dirichlet algebras, and a minor adjustment using Lemma 1 covers the case at hand.

LEMMA 5. *If $0 < p < 1$, there is a constant $K(p)$ such that*

$$(1) \quad \|f\|_p \leq K(p) \|f + \bar{g}\|_1$$

for all $f \in H^\infty$ and $g \in H_0^\infty$. There is a constant $J(p)$, $0 < p < 1$, such that

$$(2) \quad \|v\|_p \leq J(p) \|u\|_1$$

for all real-valued functions u and v such that $u + iv \in H^\infty$ and $\int v \, dm = 0$.

It is easy to show that inequalities of the forms (1) and (2) are equivalent (cf. [7], [13]). To prove (2), one first assumes that $u \in \text{Re}(H^\infty)$ is positive, and proceeds as in [4], or as in [13, p. 254]. If $w \in \text{Re}(H^\infty)$ is arbitrary, Lemma 1 produces a $u \in \text{Re}(H^\infty)$ such that $u \geq \max(0, w)$ and $\|u\|_1 \leq \|w\|_1$. Then w is expressed as the difference of the positive functions u and $u - w$, and the inequality is extended to w .

Now suppose $f \in H^1$, and let $h = \max(|f|, 1)$. h and $\log h$ are integrable, so there is an outer function $G \in H^1$ such that $|G| = h$ [6]. In particular, $|f/G| \leq 1$. If g_n is a sequence in A such that $g_n G \rightarrow 1$ in H^1 , then

$$\int |f g_n - f/G| dm \leq \int |g_n G - 1| dm \rightarrow 0,$$

so $f/G \in H^\infty$.

LEMMA 6. f is extremal if and only if f/G is extremal. fA_0 is dense in H_0^1 if and only if $(f/G)A_0$ is dense in H_0^1 .

The proof of this lemma is straightforward, and will be omitted.

We now complete the proof of the theorem. Let f be an extremal function in H^1 . Replacing f by f/G as in Lemma 6, we can assume that f is bounded. Let $g \in A_0$. By Lemma 4, there are sequences $p_n, q_n \in A_0$ and complex numbers λ_n such that $p_n f + q_n f + \lambda_n \rightarrow g$ in L^1 . Integrating both sides of this limit relation, we see that $\lambda_n \rightarrow 0$. So we can assume that $\lambda_n = 0$.

By Lemma 5, $q_n f \rightarrow g$ in H^p , $0 < p < 1$. Passing to a subsequence, if necessary, we can also assume that $q_n f \rightarrow g$ a.e. The remainder of the proof involves reproducing a technique due to Hoffman and Wermer [9] for modifying the sequence to obtain pointwise convergence.

We can assume that $\|f\|_\infty < 1$ and $\|g\|_\infty < 1$. Let $w_n = \log_+ |q_n f|$, then $w_n \geq 0$ and $w_n \rightarrow 0$ a.e. Let $E_n = \{x: |q_n(x)f(x)| > 1\}$. Since $p \log_+ s \leq s^p$ for $s \geq 0$,

$$\begin{aligned} p \int w_n dm &= p \int_{E_n} w_n dm \leq \int_{E_n} |q_n f|^p dm \\ &\leq \int_{E_n} |q_n f - g|^p dm + \int_{E_n} |g|^p dm \\ &\leq \|q_n f - g\|_p^p + m(E_n). \end{aligned}$$

Since $\|g\|_\infty < 1$ and $q_n f \rightarrow g$ in L^p , $m(E_n) \rightarrow 0$. Consequently, $\int w_n dm \rightarrow 0$.

By Lemma 1, we can find $u_n \in \text{Re}(H^\infty)$ such that $u_n \geq w_n$ and $\int u_n dm \rightarrow 0$. Choose v_n real such that $u_n + iv_n \in H^\infty$ and $\int v_n dm = 0$. If $g_n = \exp(-u_n - iv_n)$, then $g_n \in H^\infty$ and $\|g_n q_n f\|_\infty \leq 1$.

Now $\|g_n\|_\infty \leq 1$ and $\int g_n dm = \exp\{\int u_n dm\} \rightarrow 1$. Passing to a subsequence, we can assume that $g_n \rightarrow 1$ a.e. Hence $g_n q_n f \rightarrow g$ a.e. In particular, $g_n q_n f \rightarrow g$ in

$H^1(dm)$, so every function $g \in A_0$ is in the L^1 -closure of A_0f . This proves that A_0f is dense in H_0^1 .

COROLLARY. *If $f \in H^2$, then A_0f is dense in H_0^1 if and only if A_0f is dense in H_0 .*

Density in H_0^2 trivially implies density in H_0^1 . So suppose A_0f is dense in H_0^1 . The technique used in Lemma 6 allows us to assume that f is bounded. The Hoffman-Wermer argument then shows that every $g \in A_0$ is a bounded pointwise limit of functions in $H_0^\infty f$. In particular, A_0f is dense in H_0^2 .

The same proof could be used to study extremal functions in H_0^1 , i.e., functions $f \in H_0^1$ such that $f/\|f\|_1$ is an extreme point of ball H_0^1 . The analogous result is the following.

THEOREM. *Let $f \in H_0^1$. fA is dense in H_0^1 if and only if f is extremal in H_0^1 .*

The altered form of Lemma 4 needed to prove this theorem is that $fA + \overline{fA} + C$ is dense in L^1 if and only if f is extremal in H_0^1 .

COROLLARY. *Suppose that the Gleason part of ϕ on H^∞ is the one point $\{\phi\}$. Then the extreme points of ball H^1 are the outer functions in H^1 of norm 1, together with the extreme points of ball H_0^1 .*

The problem here is to show that every extreme point f of ball H_0^1 is extremal in H^1 . Now Af is dense in H_0^1 . If f were not extremal in H^1 , then A_0f would not be dense in H_0^1 . Consequently, H_0^1 would be a simply invariant subspace of H^1 (cf. [11], or Theorem 7), and $H_0^1 = FH^1$ for some inner function F . As remarked earlier, this would imply that ϕ is the center of an analytic disc.

H^p spaces. For $p = 1$ and 2, the results of this section are found in [6] for logmodular algebras. The reduction of the general case of a Szegő measure to the logmodular case is in [8]. Here it is shown that H^∞ is logmodular, and that A is weak* dense in H^∞ , so that the H^p spaces associated with H^∞ are the same as those associated with A .

Not all results about logmodular algebras transfer to A . In fact, the Hoffman-Wermer argument establishes the following theorem, which is not valid for arbitrary Szegő measures.

THEOREM 1. *Suppose that dm is a unique representing measure on X for ϕ , considered as a homomorphism of A . If $0 < p < \infty$, and f is a bounded function in H^p , then there is a sequence $f_n \in A$ such that $\|f_n\|_X \leq \|f\|_\infty$ and $f_n \rightarrow f$ a.e.*

Applied to the H^p space of an arbitrary Szegő measure, this yields the following corollary.

COROLLARY. *If $0 < p < \infty$, and f is a bounded function in H^p , then $f \in H^\infty$.*

We will need first some facts about H^p spaces for $p \geq 1$. Recall that a function $g \in H^1$ is outer if

$$\log \left| \int g \, dm \right| = \int \log |g| \, dm > -\infty,$$

and $g \in H^1$ is inner if $|g| = 1$ a.e.

LEMMA 7. Suppose $p \geq 1$. A non-negative function h is the modulus of an outer function f in H^p if and only if $h \in L^p$ and $\log h \in L^1$.

For $p = 1$ or 2 , this is proved in [6]. The general case $1 < p < \infty$ is a consequence of the direct sum decomposition $L^p = H^p \oplus \bar{H}_0^p$. The boundedness of the projection of L^p onto H^p is due, in the classical case, to M. Riesz (cf. [7], [13]). His proof carries over, with some minor adjustments as in Lemma 4, to the general case. An immediate corollary of this direct sum decomposition is that H^p is the orthogonal complement in L^p of A_0 . Consequently, $H^p = H^1 \cap L^p$, and the H^p theorem now follows from the H^1 theorem.

LEMMA 8. Suppose $1 \leq p < \infty$. $f \in H^p$ is an outer function if and only if Af is dense in H^p .

Again the theorem is known for H^1 [6], so we assume $1 < p < \infty$. If Af is dense in H^p , then Af is dense in H^1 , so f is outer.

Suppose that f is outer. Then Af is dense in H^1 . Let q be the conjugate index to p , and let $g \in L^q$ be orthogonal to Af and also to $\bar{H}_0^p$. Then the L^1 function gf is orthogonal both to A and to $\bar{A}_0$. Since $A + \bar{A}_0$ is weak* dense in L^∞ [6], $gf \equiv 0$. f cannot vanish on a set of positive measure, so $g \equiv 0$, and Af must be dense in H^p .

THEOREM 2. Suppose that $f \in H^p$ and $\log |f|$ is integrable. For some integer n such that $np \geq 1$, there is an outer function $g \in H^{np}$ such that $f = Fg^n$, where F is an inner function.

To prove this, choose the integer k such that $n = 2^k \geq 1/p$. By Lemma 7, $|f|^{1/n}$ is the modulus of an outer function in H^{np} . Choose a sequence $g_j \in A$ such that $g_j \rightarrow g$ in L^{np} . Then

$$\int |g_j^2 - g^2|^{p/2} \, dm \leq \left\{ \int |g_n - g|^{np} \, dm \right\}^{1/2} \left\{ \int |g_n + g|^{np} \, dm \right\}^{1/2},$$

and the right-hand side tends to zero. Then $g_j^2 \rightarrow g^2$ in $L^{np/2}$. By induction, $g_j^n \rightarrow g^n$ in L^p , so that $g^n \in H^p$.

By Lemma 8, there is a sequence $h_j \in A$ such that $h_j g \rightarrow 1$ in L^{np} . By the same estimate as above, we see that $\int |h_j^2 g^2 - 1|^{np/2} \, dm \rightarrow 0$. Proceeding by induction, we see that $h_j^n g^n \rightarrow 1$ in L^p . Consequently,

$$\int |f/g^n - f h_j^n| \, dm = \int |1 - g^n h_j^n| \, dm \rightarrow 0,$$

and $f/g^n = F$ belongs to H^p . Also, $|F| = 1$ a.e., so F is an inner function in H^∞ .

THEOREM 3. The functional $\phi(f) = \int f \, dm$ has a continuous extension to H^p , $0 < p < 1$, which will also be denoted by ϕ . Jensen's inequality is valid for functions $f \in H^p$:

$$\log |\phi(f)| \leq \int \log |f| \, dm.$$

Suppose $f \in A$ and $\int f \, dm \neq 0$. Then $\int \log |f| \, dm \neq 0$, and by Theorem 2 we can write $f = Fg^n$, where $np \geq 1$, F is inner, and g is outer. Then

$$\begin{aligned} |\phi(f)| &= |\phi(F)\phi(g)^n| \leq |\phi(g)|^n \\ &= \left| \int g \, dm \right|^n \leq \left\{ \int |g|^{np} \, dm \right\}^{1/p} \\ &= \left\{ \int |f|^p \, dm \right\}^{1/p} = \|f\|_p. \end{aligned}$$

This estimate shows that ϕ extends continuously to H^p . Now Jensen's inequality holds for functions in A , and the same proof which extends it to functions in H^1 also extends it to functions in H^p , $0 < p < 1$ (cf. [6]).

THEOREM 4. Let $f \in H^p$. Then Af is dense in H^p if and only if $\log |\phi(f)| = \int \log |f| \, dm > -\infty$.

If Af is dense in H^p , then $\phi(f) \neq 0$, so $\int \log |f| \, dm > -\infty$. By Theorem 2, $f = Fg^n$, where F is inner and $g \in H^{np}$ is outer. Since 1 is in the closure of Af , $\bar{F}$ is in the closure of Ag , and $\bar{F} \in H^\infty$. Consequently, F is a constant of modulus 1. Now the equality in Theorem 4 for f follows from the corresponding equality for g , and the fact that $|\phi(f)| = |\phi(g)|^n$.

Conversely, if $\log |\phi(f)| = \int \log |f| \, dm > -\infty$, and $f = Fg^n$ as above, then the inner factor F must be a constant, so we can assume $f = g^n$. As in the proof of Theorem 2, we can approximate the function 1 in H^p by functions in Ag^n . Thus, Ag^n is dense in H^p .

Another consequence of Lemma 7 and Theorem 2 is the following.

THEOREM 5. If $f \in H^p$ and $|f| \in L^q$, where $0 < p < q \leq \infty$, then $f \in H^q$.

Adjusting f by a constant, if necessary, we can assume that $\phi(f) \neq 0$. Then $\int \log |f| \, dm > -\infty$, and we can write $f = Fg^n$ as in Theorem 2. But now $g \in H^{np}$ and $|g| \in L^{nq}$, so by Lemma 7, $g \in H^{nq}$. It follows that $g^n \in H^q$, and so $f \in H^q$.

A function $f \in H^p$, $0 < p < 1$, is said to be *outer* if

$$\log |\phi(f)| = \int \log |f| \, dm > -\infty.$$

The characterization of the moduli of outer functions can now be carried over from [6]. The proof is straightforward.

THEOREM 6. If $h \geq 0$, and $0 < p < \infty$, the following assertions are equivalent:

- (i) $h \in L^p$ and $\log h \in L^1$.
- (ii) $h = |g|$ for some outer function $g \in H^p$.
- (iii) $h = |f|$ for some function $f \in H^p$ such that $\phi(f) \neq 0$.

Invariant subspaces. A (closed) subspace $\mathcal{M}$ of L^p is *invariant* if $A\mathcal{M} \subseteq \mathcal{M}$. $\mathcal{M}$ is *simply invariant* if $\mathcal{M}$ is invariant and $A_0\mathcal{M}$ is not dense in $\mathcal{M}$. In order to carry over the invariant subspace theorems in the form given them by Srinivasan [11], we first state a strengthened form of Theorem 1, established by the same argument.

THEOREM 7. Suppose that $f_n \in L^\infty$, $f \in L^\infty$, and $f_n \rightarrow f$ in the L^p metric for some p , $0 < p < \infty$. Then there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and functions $g_k \in H^\infty$ such that $\|g_k f_{n_k}\|_\infty \leq \|f\|_\infty$ and $g_k f_{n_k} \rightarrow f$ a.e. If dm is a unique representing measure on X for ϕ , and if the f_n are continuous, we can choose the g_k to belong to A .

LEMMA 9. If $0 < p \leq \infty$, and $\mathcal{M}$ is a (weak*) closed subspace of L^p such that $A\mathcal{M} \subseteq \mathcal{M}$, then $H^\infty \mathcal{M} \subseteq \mathcal{M}$.

If dm were a unique representing measure, Lemma 8 would be a consequence of Theorem 7. In case dm is only a Szegő measure for ϕ we use the fact [8] that A is weak* dense in H^∞ . Hence Lemma 8 is true if $p = \infty$.

Suppose $1 \leq p < \infty$. Given $f \in H^\infty$ and $g \in \mathcal{M}$, choose $f_n \in A$ such that $f_n \rightarrow f$ weak*. If $h \in (L^p)^*$ and $h \perp \mathcal{M}$, then $0 = \int f_n g h \, dm \rightarrow \int f g h \, dm$, so $h \perp fg$. Hence $fg \in (\mathcal{M}^\perp)^\perp = \mathcal{M}$, and $H^\infty \mathcal{M} \subseteq \mathcal{M}$.

Now suppose $0 < p < 1$. By induction we can assume the theorem is true for L^{2p} . Let $f \in H^\infty$ and $g \in \mathcal{M}$. Write $g = g_0 g_1$, where g_0 and g_1 belong to L^{2p} . Then fg_0 is in the L^{2p} -closure of Ag_0 , so there is a sequence $f_n \in A$ such that $f_n g \rightarrow fg$ in L^{2p} . From

$$\int |f_n g - fg|^p \, dm \leq \left\{ \int |f_n g_0 - fg_0|^{2p} \, dm \right\}^{1/2} \left\{ \int |g_1|^{2p} \, dm \right\}^{1/2},$$

we see that $f_n g \rightarrow fg$ in L^p . Again $fg \in \mathcal{M}$, and $H^\infty \mathcal{M} \subseteq \mathcal{M}$.

LEMMA 10. If $0 < p < \infty$, and $\mathcal{M}$ is an invariant subspace of L^p , then $\mathcal{M} \cap L^\infty$ is dense in $\mathcal{M}$.

Let $f \in \mathcal{M}$, and let G_n be the outer function in H^p whose modulus is $|G_n| = \max(1, |f|/n)$. Then $|f/G_n| = \min(n, |f|)$, and f/G_n is bounded. If $h_k \in A$ is a sequence such that $\lim_{n \rightarrow \infty} h_k G_n = 1$ in L^p , then

$$\int |f/G_n - fh_k|^p \, dm \leq n^p \int |1 - h_k G_n|^p \, dm \rightarrow 0,$$

as $k \rightarrow \infty$. So $f/G_n \in \mathcal{M} \cap L^\infty$.

Now $\{|G_n|\}_{n=1}^\infty$ is a monotone decreasing sequence of real functions in L^p , and $|G_n| \rightarrow 1$ a.e. Consequently $\int \log |G_n| dm \rightarrow 0$, and $|\phi(G_n)| \rightarrow 1$. Adjusting by a complex constant of modulus 1, we can assume $\phi(G_n) \geq 1$, so that $\phi(G_n) \rightarrow 1$.

Let $G_n = g_n^k$, when k is a fixed integer as in Theorem 1 with $kp > 1$, g_n is outer in H^{kp} , and $\phi(g_n) \rightarrow 1$. Passing to a subsequence, we can assume that $g_n \rightarrow g$ weakly in L^{kp} . Since $|g_n| \rightarrow 1$ by decreasing, $\|g_n\|_{kp} \rightarrow 1$, and $\|g\| \leq 1$. However, $\|g\| \geq |\int g dm| = \lim_{n \rightarrow \infty} \int g_n dm = 1$, and we see that $\|g\|_{kp} = 1 = \lim \|g_n\|_{kp}$. It follows that $g_n \rightarrow g$ strongly in L^{kp} . Assuming that $g_n \rightarrow g$ a.e., we see that $|g| = 1$ a.e. Since $\int g dm = 1$, g is identically 1. Then $G_n \rightarrow 1$ a.e. also.

Now $|f - f/G_n|^p$ is integrable, $|f - f/G_n|^p \rightarrow 0$ a.e., and

$$|f - f/G_n|^p \leq |f|^p + |f/G_n|^p \leq 2|f|^p, \quad 0 < p \leq 1,$$

with a similar inequality holding if $p > 1$.

It follows from the dominated convergence theorem that $f/G_n \rightarrow f$ in L^p .

THEOREM 8. *If $0 < p < q < \infty$, there is a one-to-one correspondence between invariant subspaces $\mathcal{M}_p$ of L^p and $\mathcal{M}_q$ of L^q , such that $\mathcal{M}_q = L^q \cap \mathcal{M}_p$, and $\mathcal{M}_p$ is the closure in L^p of $\mathcal{M}_q$. There is a one-to-one correspondence between invariant subspaces $\mathcal{M}_p$ of L^p and weak* closed invariant subspaces $\mathcal{M}_\infty$ of L^∞ , such that $\mathcal{M}_\infty = L^\infty \cap \mathcal{M}_p$ and $\mathcal{M}_p$ is the closure of $\mathcal{M}_\infty$ in L^p .*

To prove this, it clearly suffices to prove the part dealing with the correspondence between $\mathcal{M}_p$ and $\mathcal{M}_\infty$.

Let $\mathcal{M}$ be an invariant subspace of L^p , and let $\mathcal{M}_\infty = \mathcal{M} \cap L^\infty$. By Lemma 10, the closure of $\mathcal{M}_\infty$ in L^p is $\mathcal{M}$. Also, $\mathcal{M}_\infty$ is weak* closed. In fact, a consequence of the Krein-Schmullian theorem [14] is that the space of bounded functions in any closed subspace of $L^p(dm)$, dm a finite measure, is weak* closed in $L^\infty(dm)$.

To complete the argument, we must show that if $\mathcal{M}_\infty$ is a weak* closed invariant subspace of L^∞ , and $\mathcal{M}_p$ is the closure of $\mathcal{M}_\infty$ in L^p , then $\mathcal{M}_p \cap L^\infty = \mathcal{M}_\infty$. By Theorem 7, one can modify any sequence $f_n \in \mathcal{M}_\infty$ converging in L^p to a function $f \in L^\infty$, to obtain a sequence $g_n \in \mathcal{M}_\infty$ converging pointwise boundedly to f . Then g_n converges weak* to f , so $f \in \mathcal{M}_\infty$, and $\mathcal{M}_p \cap L^\infty \subset \mathcal{M}_\infty$. The reverse inclusion is immediate.

THEOREM 9. *Let $0 < p < \infty$, and let $\mathcal{M}$ be a simply invariant subspace of L^p . There exists a function $F \in \mathcal{M}$ such that $|F| = 1$ a.e. and $\mathcal{M} = FH^p$.*

The generalized form of Beurling's theorem is due in this form to Srinivasan. The general case $0 < p < \infty$ is now a consequence of Theorem 7 and the case $p = 2$, since the invariant subspaces $\mathcal{M}_p$ of Theorem 7 are simultaneously simply invariant or not simply invariant.

The proof for $p = 2$ is so beautiful that we cannot resist setting it down again. In this case, let F be a function of norm 1 in $\mathcal{M}$ which is orthogonal to $A_0\mathcal{M}$. In particular, $F \perp A_0F$, so $\int g|F|^2 dm = 0$, all $g \in A_0$. Since $\int |F|^2 dm = 1$,

$|F|^2 dm$ is a representing measure for ϕ . Consequently, $|F|^2 dm = dm$ and $|F| = 1$ a.e. If $g \in \mathcal{M}$ and $g \perp FH^2$, then $\int fF\bar{g} dm = 0$, all $f \in A$. Since $F \perp A_0g$, $\int fF\bar{g} dm = 0$, all $f \in A_0$. So $F\bar{g} \perp A + \bar{A}$. Since $A + \bar{A}$ is weak* dense in L^∞ , $F\bar{g} \equiv 0$, and $g \equiv 0$. Thus $\mathcal{M} = FH^2$.

Remark added in proof. Let dm be a representing measure for a homomorphism ϕ of a uniform algebra A such that

$$\log |\phi(f)| \leq \int \log |f| dm, \quad f \in A.$$

Then ϕ extends continuously to all $H^p(dm)$, $p > 0$. Otherwise there would be a sequence $f_n \in A$ such that $\|f_n\|_p \rightarrow 0$ while $|\phi(f_n)| \rightarrow \infty$. But this is impossible in view of the inequality $\log |\phi(f)| \leq (\int |f|^p dm)/p$, $f \in A$, derived from $\log s \leq s^p/p$, $p > 0$, $s > 0$.

REFERENCES

1. K. deLeeuw and W. Rudin, *Extreme points and extremum problems in H^1* , Pacific J. Math. **8** (1958) 467–485.
2. F. Forelli, *Extreme points in $H^1(R)$* , (to appear).
3. T. Gamelin, *Extreme points in spaces of analytic functions*, (to appear).
4. H. Helson, *Conjugate series and a theorem of Paley*, Pacific J. Math. **8** (1958), 437–446.
5. H. Helson and D. Lowdenslager, *Prediction theory and Fourier series in several variables*, II, Acta Math. **106** (1961), 175–213.
6. K. Hoffman, *Analytic functions and logmodular Banach algebras*, Acta Math. **108** (1962), 271–317.
7. ———, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, N. J., 1962.
8. K. Hoffman and H. Rossi, *Function theory from a multiplicative linear functional*, Pacific J. Math. (to appear).
9. K. Hoffman and J. Wermer, *H^∞ for logmodular Banach algebras*, Pacific J. Math. (to appear).
10. G. Lumer, *Analytic functions and Dirichlet problem*, Bull. Amer. Math. Soc. **70** (1964), 98–104.
11. T. Srinivasan, *Simply invariant subspaces*, Bull. Amer. Math. Soc. **69** (1963), 706–709.
12. J. Wermer, *Dirichlet algebras*, Duke Math. J. **27** (1960), 373–382.
13. A. Zygmund, *Trigonometric series*, 2nd ed., Cambridge Univ. Press, New York, 1959.
14. N. Dunford and J. T. Schwartz, *Linear operators*, Part I, Interscience, New York, 1958.

UNIVERSIDAD NACIONAL DE LA PLATA,
LA PLATA, ARGENTINA