MARTINGALE THEORY AND POINTWISE CONVERGENCE OF CERTAIN ORTHOGONAL SERIES(1)

BY
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The purpose of this paper is to point out the relevance of martingale theory in the study of some classical problems concerning the pointwise convergence of orthogonal series. Particular attention is given to the Haar and the Walsh systems. The special character of their convergence properties has been the focus of some recent investigations, (see for example [13], [14], [15]) but the usefulness of ideas from martingale theory does not seem to have been recognized. We take advantage of the structure of these systems together with standard martingale techniques to obtain some new convergence theorems for a class of orthonormal systems which include the Haar and the Walsh functions as special cases.

In the first section, the connection between the Haar and Walsh systems is described, and the martingale properties of these systems are isolated. This leads to the definition of a class of orthonormal systems, called H-systems, that are generalizations of the Haar system. Our interest in H-systems stems from two facts: (a) it is shown that every complete orthonormal system of martingale differences is an H-system; and (b) optional stopping and skipping transformations are especially simple when the martingales in question come from H-systems.

The construction of the Haar system from the Rademacher functions suggests a specialization of the notion of an H-system: an H^*-system is an H-system constructed from a sequence of independent binomial functions in the manner the Rademacher functions generate the Haar system. Consequently, any H^*-system has an associated parameter sequence $\{p_k\}_{k=1}^{\infty}$, the sequence of probabilities associated with the independent binomial functions. (In this way, the Haar system is associated with the sequence $p_k = \frac{1}{2}$, $k = 1, 2, \cdots$). The influence of the parameter sequence on the convergence properties of H^*-systems is studied in the second and third sections.

In the second section of the paper, we consider H-systems in relation to the following question: Given a complete orthonormal system $\{u_k\}_{k=1}^{\infty}$ and a measurable function f, does there exist a series $\sum_{k=1}^{\infty} a_k u_k$ that converges to f

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almost everywhere (a.e.)? If such a series exists, we will say that the function f has a series representation with respect to the system $\{u_k\}_{k=1}^{\infty}$.

For the trigonometric system, this question has been answered affirmatively by Menchov [9] when f is finite a.e. It is not known whether a trigonometric series may converge to infinity on a set of positive measure. For the Haar system, N. K. Bari [7, p. 527] (see also [13]) has given an affirmative answer when f is finite a.e., and more recently, Talalyan and Arutyunyan [12] have given a negative answer for both the Haar and Walsh systems when f is infinite on a set of positive measure.

The first theorem of the section states that (a) given any complete H-system, series representations exist for every measurable function that is finite a.e., and (b) there exist complete H-systems such that every measurable function (whether finite a.e. or infinite on a set of positive measure) has a series representation. The second theorem of the section gives a more specific result related to the theorems of Bari and Talalyan-Arutyunyan. Given a complete H^*-system $\{u_k; p_k\}_{k=1}^{\infty}$ the following dichotomy holds: (1) if $\liminf_{k\to\infty} p_k > 0$, then every measurable function that is finite a.e. has a series representation, but no function that is infinite on a set of positive measure has a series representation; and (2) if $\liminf_{k\to\infty} p_k = 0$, then every measurable function has a series representation.

The proofs make use of martingale convergence theorems, and especially, a submartingale convergence theorem of Chow [2]. (Also see [3]). This theorem also gives the following result for H^*-systems where $\liminf_{k\to\infty} > p_k 0$: in such H^*-systems, it is impossible to find divergent series that oscillate boundedly a.e.. In particular, this statement is true for the Haar system, in contrast to the example given by Marcinkiewicz for the trigonometric system.

The third section of the paper presents a necessary and sufficient condition for pointwise convergence a.e. of arbitrary series from H-systems satisfying a certain regularity condition. The theorem applies to H^*-systems when $\liminf_{k\to\infty} p_k > 0$. In particular, an arbitrary Haar series $\sum_{k=1}^{\infty} a_k\chi_k$ converges a.e. on a set of positive measure E if and only if the series $\sum_{k=1}^{\infty}(a_k\chi_k)^2$ is finite a.e. on E. A version of the theorem for Walsh series follows immediately from the Haar series result.

Some of the consequences of the theorem for H^*-systems where $\liminf_{k\to\infty} p_k > 0$ are: (a) arbitrary optional skipping transformations on convergent series preserve convergence a.e.; and (b) convergence a.e. implies unconditional convergence in measure.

1. **Complete orthonormal systems and martingales.** Let $\{r_n\}_{n=0}^{\infty}$ be the system of Rademacher functions:

$$r_0(x) = \operatorname{sgn}\sin 2\pi x,$$
$$r_n(x) = r_0(2^n x).$$

Both the Haar system and the Walsh system may be constructed from the Rademacher system. The Haar system is defined as follows:

$$\chi_1(x) \equiv \chi_0^{(0)}(x) = 1,$$

$$\chi_2(x) \equiv \chi_0^{(1)}(x) = r_0(x).$$

If $N = 2^n + k$ with $1 \leqq k \leqq 2^n$

$$\chi_N(x) \equiv \chi_n^{(k)}(x) = 2^{n/2} r_n(x) \text{ if } (k-1)/2^n \leqq x < k/2^n,$$
$$= 0 \text{ otherwise.}$$

The Walsh functions are defined as follows:

$$\psi_0(x) = 1,$$

$$\psi_N(x) = r_{n_1}(x) \cdot r_{n_2}(x) \cdots r_{n_k}(x),$$

if $N = 2^{n_1} + \cdots + 2^{n_k}$ where $n_1 > n_2 > \cdots > n_k$.

Both the Haar and Walsh systems are known to be complete orthonormal in $L^2[0,1]$ [11].

Let $\{H_n\}_{n=1}^{\infty} = \{ \sum_{k=1}^{n} a_k \chi_k\}_{n=1}^{\infty}$ designate the sequence of partial sums of the Haar Fourier series of a function f, and $\{W_n\}_{n=1}^{\infty} = \{ \sum_{k=0}^{n-1} b_k \psi_k\}_{n=1}^{\infty}$ the sequence of partial sums of the Walsh Fourier series for the same function. It is pointed out in [11] that $H_{2^n} = W_{2^n}$ for all $n = 0, 1, \cdots$. It is clear from the verification of this relation for Fourier series that the relation holds for general Walsh and Haar series. That is, given the sequence of partial sums H_n of any Haar series, there is a unique Walsh series such that $H_{2^n} = W_{2^n}$ for all n, and conversely. One of the consequences of this identity is that the sequence W_{2^n} is a martingale. This fact has also been pointed out by Burkholder [1](2). The relation also means that, in many cases, theorems on Haar series may be applied to Walsh series.

The relation between Haar series and Rademacher series should also be noted. Since any series of Rademacher functions may be considered as a Walsh lacunary series (the coefficients are zero except for integers $N = 2^n$; $n = 0, 1, \cdots$), the above remarks show that any Rademacher series may also be considered as a Haar series.

From the standpoint of harmonic analysis on groups, the interest in the Walsh functions steme from the fact that they are the continuous characters of a compact abelian group, the so-called dyadic group. (For the definitions and an extensive study of harmonic analysis on this group, see Fine [5].)

The Haar system is interesting, from the standpoint of martingale theory, in that the sequence of partial sums of an arbitrary Haar series forms a martingale. Verification of this fact is a straightforward application of the definition of a

(2) Also, see M. Jerison and G. Robinson, *Convergence theorems obtained from inducted homomorphisms of a group algebra*, Ann. of Math. **63** (1956), 176–190.

martingale. (For the definitions and an extensive study of martingales, see Doob [4, Chapter 7].)

The martingale property of Haar series suggests the following definition.

Definition 1.1. An orthonormal system (o.n.s.) $\{u_k\}_{k=1}^{\infty}$ defined on an arbitrary probability space is called an H-system if:

(1) Each u_k assumes at most two nonzero values with positive probability.

(2) The σ-field generated by $\{u_k\}_{k=1}^{N}$ denoted by $\sigma(u_1,\cdots,u_N)$, consists of exactly N atoms.

(3) $E(u_{k+1} \| u_1,\cdots,u_k) = 0$; $k \geqq 1$. (That is, the functions u_k are martingale differences.)

The following alternative definition is sometimes useful.

Definition 1.2. The o.n.s. $\{u_k\}_{k=1}^{\infty}$ is called an H-system if and only if for any $f \in L^2[0,1]$

$$E(f \| u_1,\cdots,u_n) = \sum_{k=1}^{n} a_k u_k,$$

where $\{a_k\}_{k=1}^{\infty}$ are the Fourier coefficients of f with respect to $\{u_k\}_{k=1}^{\infty}$.

Definition 1.1 is equivalent to Definition 1.2. To prove this, suppose $\{u_k\}_{k=1}^{\infty}$ is an H-system according to Definition 1.1 and let ϕ $[= E(f \| u_1,\cdots,u_n)]$ be a function measurable on $\sigma(u_1,\cdots,u_n)$. Definition 1.1 implies that ϕ may be written as the sum of n linearly independent functions, $\{I_k\}_{k=1}^{n}$, the n indicator functions of the atoms of $\sigma(u_1,\cdots,u_n)$. From this fact, and the definition of conditional expectation, it follows that the range of the orthogonal projection $E(\cdot \| u_1,\cdots,u_n)$ is n-dimensional. On the other hand, the linear manifold generated by $\{u_k\}_{k=1}^{n}$ is an n-dimensional manifold of functions measurable with respect to $\sigma(u_1,\cdots,u_n)$. Therefore, $E(\cdot \| u_1,\cdots,u_n)$ must coincide with the orthogonal projection onto the span of $\{u_k\}_{k=1}^{n}$. This is exactly the condition required in Definition 2.2.

Now suppose $\{u_k\}_{k=1}^{\infty}$ is an H-system relative to the Definition 1.2. We show by induction that the system $\{u_k\}_{k=1}^{\infty}$ satisfies the conditions (1), (2), and (3) of Definition 1.1.

Consider the first member, u_1, of the given system. Let 1 designate the constant function identically equal to 1. The condition of Definition 1.2 and the properties of conditional expectation together imply that

$$1 = E(1 \| u_1) = a_1 u_1.$$

In other words, u_1 is equal almost everywhere to the constant $1/a_1$. Therefore, conditions (1) and (2) are satisfied for the function u_1. Condition (3) imposes no restriction on the function u_1.

Suppose that we have shown that the partial collection $\{u_k\}_{k=1}^{N}$ satisfies conditions (1), (2), and (3). Consider the augmented collection $\{u_k\}_{k=1}^{N+1}$. Let $\sum_{k=1}^{N} b_k u_k$ be the Nth partial sum of the Fourier series of the function u_{N+1} with respect to the system $\{u_k\}_{k=1}^{\infty}$. Since u_{N+1} is orthogonal to the other members of the family, the coefficients b_k, $k = 1,2,\cdots,N$ are all equal to zero. If we apply this fact in conjunction with the condition of Definition 1.2, we obtain

$$E(u_{N+1} \| u_1,\cdots,u_N) = \sum_{k=1}^{N} b_k u_k = 0,$$

so that condition (3) of Definition 1.1 is satisfied for the collection $\{u_k\}_{k=1}^{N+1}$. The function u_{N+1} cannot be constant almost everywhere since it is normalized and orthogonal to the constant function u_1. Therefore, u_{N+1} must assume at least two nonzero values with positive probability. In fact, u_{N+1} assumes exactly two nonzero values with positive probability, and it does this by splitting one of the N atoms of $\sigma(u_1,\cdots,u_N)$. The argument for this is quite similar to one used previously. Notice that the linear manifold generated by the functions $\{u_k\}_{k=1}^{N+1}$ is an $N+1$-dimensional manifold of functions measurable with respect to $\sigma(u_1,\cdots,u_{N+1})$. By Definition 1.2, the orthogonal projection $E(\cdot \| u_1,\cdots,u_{N+1})$ must coincide with the orthogonal projection onto the span of $\{u_k\}_{k=1}^{N+1}$. Therefore, the range of $E(\cdot \| u_1,\cdots,u_{N+1})$ must also be $N+1$-dimensional. In other words, $\sigma(u_1,\cdots,u_{N+1})$ consists of exactly $N+1$ atoms of positive measure. Since $\sigma(u_1,\cdots,u_N)$ consists of exactly N atoms, the function u_{N+1} augments $\sigma(u_1,\cdots,u_N)$ by splitting a single atom. Therefore, conditions (1), (2), and (3) of Definition 1.1 are satisfied for the partial collection $\{u_k\}_{k=1}^{N+1}$ and the induction is complete.

The Haar system is the most immediate example of a complete o.n.s. that is an H-system. In fact, the following proposition is also true:

PROPOSITION 1.1. *Any complete orthonormal system of martingale differences is an H-system.*

Proof of Proposition 1.1. Let $\{u_k\}_{k=1}^{\infty}$ be the system in question and $\sum_{k=1}^{\infty} a_k u_k$ be the Fourier series of an arbitrary function f belonging to L^2. Fix n and define the function

$$d_n = E(f \| u_1,\cdots,u_n) - \sum_{k=1}^{n} a_k u_k.$$

Because the functions $\{u_k\}_{k=1}^{\infty}$ are martingale differences, the Fourier coefficients of d_n vanish identically. Since the system $\{u_k\}_{k=1}^{\infty}$ is complete, this implies that the function d_n vanishes almost everywhere, or

$$E(f \| u_1,\cdots,u_n) = \sum_{1}^{n} a_k u_k$$

almost everywhere. Since n is arbitrary, the condition of Definition 1.2 is satisfied, so that $\{u_k\}_{k=1}^{\infty}$ is an H-system.

Optional stopping and skipping transformations [4, p. 310] are especially simple when the martingales under consideration arise from H-systems.

PROPOSITION 1.2. *Let the sequence of partial sums* $S_n = \sum_{k=1}^n a_k u_k$ *be a martingale formed from the H-system* $\{u_k\}_{k=1}^\infty$. *Then, every system of optional stopping and or skipping on* S_n *is generated by a "multiplier" transformation of the form* $\sum_{k=1}^n \delta_k a_k u_k$ *where* $\delta_k = 1$ *or* 0.

Proof of Proposition 1.2. The martingale from any optional stopping or skipping scheme on S_n may be written as $\sum_{k=1}^n I_k a_k u_k$ where I_k is the characteristic function of a set measurable with respect to $\sigma(u_1, \cdots, u_{k-1})$. In order to prove Proposition 1.2, it must be shown that the functions I_k may be replaced by constants δ_k so that

$$\sum_{k=1}^n I_k a_k u_k = \sum_{k=1}^n \delta_k a_k u_k$$

holds almost everywhere for each n. This can be done as follows. If u_k is a member of an H-system, then $|u_k| > 0$ on a single atom of $\sigma(u_1, \cdots, u_{k-1})$. Since the function I_k is measurable on this σ-field, I_k is either identically one or identically zero almost everywhere on the set where $|u_k| > 0$. Therefore, $\sum_{k=1}^n I_k a_k u_k = \sum_1^n \delta_k a_k u_k$ almost everywhere, where δ_k is constant, equal to the value assumed by I_k on the support of u_k. The proposition is proved.

The Haar system is constructed from a sequence of independent two-valued functions, the Rademacher functions. This suggests that we may achieve a sharper comparison of the Haar system with other H-systems if we limit the comparison to systems generated by a sequence of independent two-valued functions in the same way the Haar system is defined in terms of the Rademacher system.

DEFINITION 1.3. An H^*-system is a H-system $\{u_k\}_{k=1}^\infty$ generated by a sequence of two-valued independent orthogonal functions $\{\rho_k\}_{k=1}^\infty$ as follows:

$$u_1 = 1,$$

$$u_2 = O(\rho_1),$$

$$u_3 = \begin{cases} O(\rho_2) & \text{on the set where } u_2 > 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$u_4 = \begin{cases} O(\rho_2) & \text{on the set where } u_2 < 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $O(\rho_k)$ indicates multiplication of the function ρ_k by a constant defined so that $\| u_k \|_2 = 1$. The remaining functions are defined recursively as follows: Having defined $u_1, u_2, \cdots, u_{2^n}$, we define

$$u_{2^n+(2k-1)} = \begin{cases} O(\rho_{n+1}) & \text{on the set where } u_{2^{n-1}+k} > 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$u_{2^n+(2k)} = \begin{cases} O(\rho_{n+1}) & \text{on the set where } u_{2^{n-1}+k} < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Associated with each H^*-system, there is a sequence of probabilities $\{p_k\}_{k=1}^{\infty}$, $0 \leqq p_k \leqq \frac{1}{2}$, the parameters of the independent functions generating H^*-system. That is,

$$p_k = \min(P\{\rho_k > 0\},\ P\{\rho_k < 0\})$$

where $P\{\ \}$ is the probability of the set in brackets.

The influence of the parameter sequence $\{p_k\}_{k=1}^{\infty}$ on pointwise convergence of H^*-series will be studied in the next two sections.

2. **H-systems and the representation of functions by series.** Let $\{u_k\}_{k=1}^{\infty}$ be a complete H-system defined on the unit interval. The theorems of this section relate to the following question. Given a measurable function f defined on the unit interval, does there exist a series $\sum_{k=1}^{\infty} a_k u_k$ that converges to f a.e.? If such a series exists, we say that f has a series representation with respect to the system in question.

THEOREM 2.1. (a) *Every measurable function that is finite a.e. has a series representation with respect to any complete H-system.*

(b) *There exist complete H-systems such that every measurable function has a series representation.*

A more specific result holds for complete H^*-systems.

THEOREM 2.2. *Given a complete H^*-system with parameter sequence $\{p_k\}_{k=1}^{\infty}$, the following dichotomy holds:*

(a) *If* $\liminf_{k\to\infty} p_k = 0$, *then every measurable function, whether finite a.e. or infinite on a set of positve measure, has a series representation with respect to the system in question.*

(b) *If* $\liminf_{k\to\infty} p_k > 0$, *then a measurable function has a series representation if and only if it is finite a.e.*

REMARK. It can be shown that there are complete H^*-systems such that $\lim_{k\to\infty} p_k = 0$. In fact, part of the following result will be needed for the proof of Theorems 2.1 and 2.2 but may also be of independent interest in this context.

THEOREM 2.3. *If $\{p_k\}_{k=1}^{\infty}$ is the parameter sequence of a complete H^*-system, then $\sum_{k=1}^{\infty} p_k = \infty$. Conversely, given any sequence $\{p_k\}_{k=1}^{\infty}$ such that $0 \leqq p_k \leqq \frac{1}{2}$ and $\sum_{k=1}^{\infty} p_k = \infty$ there is a complete H^*-system with the prescribed parameter sequence $\{p_k\}_{k=1}^{\infty}$.*

This theorem is a direct consequence of Theorem 2 of [6]. In fact, it is not difficult to see that a given H^*-system is complete if and only if the associated orthogonal system, composed of the constant function and all finite products $\prod_{i=1}^{m} \rho_{n_i}$, $m = 1, 2, \cdots$ of the independent functions generating the H^*-system, is complete. In other words the statement of Theorem 2.3 is equivalent to the assertion of Theorem 2 of [6].

The proof of the first part of Theorem 2.1 depends on a series of lemmas.

Lemma 2.1. *Let $\{u_k\}_{k=1}^{\infty}$ be an H-system, and f a function measurable on $\sigma(u_1, u_2, \cdots)$. Then there is a sequence of functions f_k such that*
(i) *f_k is measurable with respect to $\sigma(u_1, u_2, \cdots, u_k)$,*
(ii) *$\lim_{k\to\infty} f_k = f$ a.e.,*

The proof of this lemma is an application of standard approximation arguments, and will be omitted.

The problem posed by part (a) of Theorem 2.1 is to show that the sequence f_k, whose existence is assured by the preceding lemma, may be chosen so that f_n is the nth partial sum of fixed series $\sum_{k=1}^{\infty} a_k u_k$. When f is integrable, the Fourier series of f with respect to the system in question converges to f a.e. by the martingale convergence theorem [4]. Recall that in this case, the coefficients are defined as

$$a_k = \int f(x)\, u_k(x)\, dx.$$

However, when f is not integrable, it can happen that none of the above integrals exist.

When the H-system in question is the Haar system, the assertion of part (a) of Theorem 2.1 is due to N. K. Bari [7, p. 527], as mentioned in the introduction. Her proof makes use of a theorem of Luzin [7, p. 77] to the effect that every function f which is finite a.e. has a continuous primitive F; that is, $F'(x) = f(x)$ a.e. The following lemma is a martingale generalization of Bari's theorem, and is proved along the lines suggested by Luzin's theorem, but without topological considerations.

Lemma 2.2. *Let $\{u_k\}_{k=1}^{\infty}$ be an H-system such that $\sigma(u_1, u_2, \cdots)$ is nonatomic. If f is finite a.e. and measurable with respect to $\sigma(u_1, u_2, \cdots)$, then f has a series representation with respect to $\{u_k\}_{k=1}^{\infty}$.*

Proof. We construct a sequence of mutually singular measures $\{\mu_n\}_{n=1}^{\infty}$ such that (a) each μ_n is the sum of an absolutely continuous and a purely discrete measure, and (b) $\sum_{n=1}^{\infty} d\mu_n/dx = f$ a.e. The coefficients of the proposed series representation of f are then defined by the (convergent) series

$$a_k = \sum_{n=1}^{\infty} \int u_k(x)\, d\mu_n(x).$$

We may assume, without loss of generality, that $f \geqq 0$. By Lemma 2.1, there is a sequence of functions $\{f_k\}_{k=1}^{\infty}$; $f_k \geqq 0$ converging to f a.e., such that each f_k is measurable on $\sigma(u_1, u_2, \cdots, u_k)$. Let

$$A_n = \{x : \sup_k f_k(x) > n\}; \quad n = 0, 1, \cdots$$

and

$$B_n = A_{n-1} - A_n, \quad n = 1, 2, \cdots.$$

Then the sets B_n are disjoint, on each B_n we have $f < n$, and $\sum_{n=1}^{\infty} P(B_n) = 1$. (Here, as before, $P(E)$ means the measure of E.) Furthermore, from the construction of the sets B_n, it is not difficult to see that each set may be covered by a countable collection of atoms or null sets $G_r^{(n)}$, $r = 1, 2, \cdots$, so that, if $D^{(n)} = \bigcup_{r=1}^{\infty} G_r^{(n)}$, then $P(B_n) \leqq P(D^{(n)}) \leqq 2P(B_n)$.

Now let

$$f^{(n)} = \begin{cases} f & \text{on } B_n, \\ 0 & \text{otherwise}, \end{cases}$$

and $\mu(f^{(n)})$ be the absolutely continuous measure whose derivative is $f^{(n)}$.

Now define a discrete measure $\delta^{(n)}$ as a sum of point masses $\sum_r \delta_r^{(n)} = \delta^{(n)}$ where each $\delta_r^{(n)}$ is determined by setting

$$\delta_r^{(n)}(x) = \begin{cases} \displaystyle\int_{G_r^{(n)}} f^{(n)}(x)dx \text{ if } x = x_{n,r} & \text{where } x_{n,r} \in G_r^{(n)} \text{ such that} \\ & \qquad \delta^{(k)}(x_{n,r}) = 0 \text{ for } k < n, \\ 0 \quad \text{otherwise}. \end{cases}$$

Finally let

$$\mu_n = \mu(f^{(n)}) - \delta^{(n)}$$

and

$$\mu = \sum_{n=1}^{\infty} \mu_n.$$

Then

$$d\mu/dx = \sum_{n=1}^{\infty} f^{(n)} = f \text{ a.e.}$$

where the exceptional set consists of the countable set supporting the discrete measures $\delta^{(n)}$, $n = 1, 2, \cdots$.

Define coefficients

$$a_k = \sum_{n=1}^{\infty} \int u_k(x)\, d\mu_n.$$

For n sufficiently large, the sets $G_r^{(n)}$ are all subsets of the sets of constancy of u_k. Since

$$\int_{G_r^{(n)}} d\mu_n = 0$$

for all r and n, it follows that $\int u_k\, d\mu_n = 0$ for all $n \geqq N_k$. Therefore, the series defining a_k converges for each fixed $k > 0$.

Now it must be shown that the series $\sum_{k=1}^{\infty} a_k u_k$ converges to f almost everywhere. To this end, observe that for each fixed m,

$$\lim_{n\to\infty} E(f^{(m)} \,\|\, u_1, u_2, \cdots, u_n) = f^{(m)}$$

except on a set A_m of measure zero. Since $\sigma(u_1, u_2, \cdots)$ is nonatomic, the discrete measure $\delta^{(m)}$ vanishes almost everywhere. This implies that

$$\lim_{n\to\infty} E(\delta^{(m)} \,\|\, u_1, u_2, \cdots, u_n) = 0$$

except on a set B_m of measure zero. Let $E_m = A_m \cup B_m$. The exceptional set for the convergence of the series $\sum_{k=1}^{\infty} a_k u_k$ is to consist of (i) $\bigcup_{m=1}^{\infty} E_m$, (ii) the countable set supporting the discrete measure $\sum_{m=1}^{\infty} \delta^{(m)}$, and (iii) the set $\limsup D^{(m)}$. The set (i) is a countable union of sets of measure zero. The countable set (ii) has measure zero since the measure space in question is nonatomic. The set (iii) has measure zero by the Borel-Cantelli lemma, [4, p. 104] since

$$\sum_{m=1}^{\infty} P(D^{(m)}) \leqq 2 \sum_{m=1}^{\infty} P(B_m) < +\infty .$$

Let x be a point in complement of the exceptional set. Then we may show that

$$\sum_{k=1}^{n} a_k u_k(x) = \frac{1}{P(I_n)} \sum_{m=1}^{\infty} \int_{I_n} d\mu_m \tag{1}$$

where I_n is the atom of $\sigma(u_1, u_2, \cdots, u_n)$ that contains the point x. Equation (1) is formally correct: The left side of (1) is the Fourier series of the "measure" $\sum_{m=1}^{\infty} \mu_m$ and the right-hand side is the conditional expectation of the same measure relative to $\sigma(u_1, \cdots, u_n)$. The equality in (1) holds because $\{u_k\}_{k=1}^{\infty}$ is an H-system. We now show that this formalism may be justified.

Since x is not contained in the exceptional set, there is an m_x, depending on x, such that

$$x \notin D^{(m)} \qquad \text{for all } m \geqq m_x .$$

Consider the partial sum (1) for a fixed n, and the atom I_n appearing on the right-hand side of (1). For $m \geqq m_x$ let $G_r^{(m)}$ be one of the atoms supporting μ_m, as constructed above. Since I_n and $G_r^{(m)}$ are both atoms, they satisfy one and only one of the following conditions: (1) $I_n = G_r^{(m)}$; (2) $I_n \subset G_r^{(m)}$; (3) $I_n \supset G_r^{(m)}$; (4) $I_n \cap G_r^{(m)} = \varnothing$.

Conditions 1 and 2 are impossible for $m \geqq m_x$ since $I_n \subseteq G_r^{(m)}$ and $x \in G_n$ together imply $x \in G_r^{(m)}$ which contradicts the assumptions that $m \geqq m_x$.

The above remarks may be used to compute the integral $\int_{I_n} d\mu_m$. Recall that the measure μ_m is supported in the set $D^{(m)} = \bigcup_{r=1}^{\infty} G_r^{(m)}$ and defined so that $\int_{G_r^{(m)}} d\mu_m = 0$ for $r = 1, 2, \cdots$. Therefore

$$\int_{I_n} d\mu_m = \int_{I_n \cap D^{(1)}} d\mu_m + \int_{I_n \cap \bar{D}^{(m)}} d\mu_m = \mathrm{I} + \mathrm{II}$$

where $\bar{D}^{(m)}$ denotes the complement of $D^{(m)}$. The integral I consists of a sum of integrals $\int_{G_r^{(m)}} d\mu_m$ where the sum may be restricted to those indices for which condition 3 holds. Condition 3 and the definition of μ_m imply that

$$\int_{I_n \cap G_r^{(m)}} d\mu_m = \int_{G_r^{(m)}} d\mu_m = 0$$

for each $r = 1, 2, \cdots$ so that the integral I vanishes. The integral II vanishes because the measure μ_m vanishes on the set $\bar{D}^{(m)}$. Therefore, the expression (1) may be written

$$\begin{aligned} \sum_{k=1}^{n} a_k u_k(x) &= \frac{1}{P(I_n)} \sum_{m=1}^{m} \int_{I_n} d\mu_m \\ &= \sum_{m=1}^{m} E(f^{(m)} \| u_1, u_2, \cdots, u_n)(x) - \sum_{m=1}^{m} \frac{\delta^{(m)}(I_n)}{P(I_n)}. \end{aligned}$$

Then

$$\lim_{n \to \infty} \sum_{m=1}^{m} E(f^{(m)} \| u_1, u_2, \cdots, u_n)(x) = \sum_{m=1}^{m} f^{(m)}(x)$$

and

$$\lim_{n \to \infty} \sum_{m=1}^{m} \frac{\delta^{(m)}(I_n)}{P(I_n)} = \lim_{n \to \infty} \sum_{m=1}^{m} E(\delta^{(m)} \| u_1, u_2, \cdots, u_n)(x) = 0$$

since the point x belongs to the complement of the exceptional set. Notice also that,

$$\sum_{m=1}^{m} f^{(m)}(x) = f(x)$$

since $f^{(m)}(x) = 0$ except possibly for that value m such that $f^{(m)}(x) = f(x)$. We may now conclude that

$$\begin{aligned} &\lim_{n \to \infty} \sum_{k=1}^{n} a_k u_k(x) \\ &= \lim_{n \to \infty} \left[\sum_{m=1}^{m} E(f^{(m)} \| u_1, u_2, \cdots, u_n)(x) - \sum_{m=1}^{m} \frac{\delta^{(m)}(I_n)}{P(I_n)} \right] \\ &= \lim_{n \to \infty} \sum_{m=1}^{m} E(f^{(m)} \| u_1, u_2, \cdots, u_n)(x) - \lim_{n \to \infty} \sum_{m=1}^{m} E(\delta^{(m)} \| u_1, u_2, \cdots, u_n)(x) \\ &= \sum_{m=1}^{m} f^{(m)}(x) = f(x) \end{aligned}$$

for all x in the complement of the exceptional set. This concludes the proof of the lemma.

REMARK. It is clear that the measure $\mu = \sum_{n=1}^{\infty}\mu_n$, constructed in the proof of Lemma 2.2 is not uniquely determined by f. For example, for any fixed integer N, we may choose the sets $G_r^{(n)}$ to be atoms of $\sigma(u_1,u_2,\cdots,u_{k_n})$ where $k_n \geqq N+n$. Then, if the measures $\{\mu_n\}_{n=1}^{\infty}$ are constructed as indicated in the lemma, we have

$$\int_{E_j} d\mu_n = 0, \; n = 1,2,\cdots$$

where E_j, $j = 1,2,\cdots,N$ are the atoms of $\sigma(u_1,u_2,\cdots,u_N)$. In other words, we may find a series representation for f such that $a_k = 0$ for $k \leqq N$. This remark will be useful in proving part (b) of Theorem 2.1. (See Lemma 2.3 below.)

Proof of Part (a) of Theorem 2.1. Let $\{u_k\}_{k=1}^{\infty}$ be a complete H-system. It follows from Definition 1.2 that $\sigma(u_1,u_2,\cdots)$ is equivalent to the Borel sets of $[0,1]$, so that, in particular, $\sigma(u_1,u_2,\cdots)$ is nonatomic. Therefore, any Lebesgue measurable function that is finite a.e. is equal a.e. to a function measurable on $\sigma(u_1,u_2,\cdots)$. Part (a) of Theorem 2.1 is, therefore, a consequence of Lemma 2.2.

The following strengthened version of Lemma 2.2 facilitates the proof of Part (b) of Theorem 2.1.

LEMMA 2.3. *Let $\{u_k\}_{k=1}^{\infty}$ be any complete H-system and E_N any set measurable with respect to $\sigma(u_1,u_2,\cdots,u_n)$ where N is an arbitrary fixed integer. Let f be finite a.e., and suppose $f = 0$ on the complement of E_N. Then f has a series representation $\sum_{k=1}^{\infty} a_k u_k$ such that:*

(i) *$a_k = 0$ for $k \leqq N$,*

(ii) *the summands $a_k u_k$ vanish identically on the complement of E_N.*

Proof. The proof of assertion (i) follows from Lemma 2.2 and the remark following it. To prove Part (ii), consider any series representation $\sum_{k=1}^{\infty} a_k u_k$ such that $a_k = 0$ for $k \leqq N$. Since the set E_N belongs to $\sigma(u_1,u_2,\cdots,u_N)$, the set where $|a_{N+k}u_{N+k}| > 0$ is either contained in E_N or disjoint from it. Therefore, we may modify the coefficients a_{N+k}, if necessary, by setting $a_{N+k} = 0$ if $\{|a_{N+k}u_{N+k}| > 0\} \cap E_N = \varnothing$. This modification of the series meets the requirement (ii) without damaging the representation of f on E_N. The proof of the lemma is complete.

The assertion of Part (b) of Theorem 2.1 follows from Part (a) of Theorem 2.2.

Proof of Part (a) of Theorem 2.2. Let $\{u_k\}_{k=1}^{\infty}$ be a complete H^*-system such that $\liminf_{k\to\infty} p_k = 0$. Such a system exists by Theorem 2.3. Let f be any measurable function, written as a sum $f = \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3$ where f_1 is finite a.e., f_2 is $+\infty$ on a set of positive measure, zero otherwise and f_3 is $-\infty$ on a set of positive measure, zero otherwise. Each coefficient α_i is either one or zero.

A series representation for f_1 in terms of $\{u_k\}_{k=1}^{\infty}$ follows from Lemma 2.2.

The theorem will be proved if we can show that f_2 and f_3 have series representations. It clearly suffices to show that f_2 has a representation.

First, we construct a series converging to $+\infty$ a.e.. Let $\{\rho\}_{k=1}^{\infty}$ be the sequence of independent binomial functions from which the H^*-system is constructed. Select a subsequence $\{\rho_{k_i}\}_{i=1}^{\infty}$ such that the associated parameter sequence $\{p_{k_i}\}_{i=1}^{\infty}$ has the property $\sum_{i=1}^{\infty} p_{k_i} < +\infty$. Define a sequence of coefficients $\{c_{k_i}\}_{i=1}^{\infty}$ such that

$$P\{c_{k_i}\rho_{k_i} = 1\} = 1 - p_{k_i}.$$

An application of the Borel-Cantelli lemma shows that $\sum_{i=1}^{\infty} c_{k_i}\cdot\rho_{k_i} = \infty$ a.e., since $\sum_{i=1}^{\infty} p_{k_i} < \infty$ implies that $P\{c_{k_i}\rho_{k_i} \neq 1 \text{ infinitely often}\} = 0$. It follows from the definition of H^*-system $\{u_k\}_{k=1}^{\infty}$ that the series $\sum_{i=1}^{\infty} c_{k_i}\rho_{k_i}$ may be expressed as a series $\sum_{k=1}^{\infty} a_k u_k$, and that $\sum_{k=1}^{n} a_k u_k = S_n$ converges to $+\infty$ a.e.

Now let f be a function that is $+\infty$ a.e. on E and vanishes on $\bar{E}$. (Henceforth the complement of a set will be denoted with a bar.) We will modify the tails of the above series $\sum_{k=1}^{\infty} a_k u_k$ so that the modified series $\sum_{k=1}^{\infty} \hat{a}_k u_k$ converges to $+\infty$ a.e. on E and to zero a.e. on E.

Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of nonnegative functions such that each f_k is measurable with respect to $\sigma(u_1,\cdots,u_k)$ and $\lim_{k\to\infty} f_k = \chi_E$, the characteristic function of E, almost everywhere. Lemma 2.1 insures the existence of such a sequence. The collection of sets $E_n = \{x : \sup_{n\leq k} f_k(x) \geqq \frac{1}{2}\}$ is monotone decreasing with $\lim_{n\to\infty} P(E_n) = P(E)$, and each set E_n is a countable union of disjoint atoms.

We first sketch the proof. The series $\sum_{k=1}^{\infty} a_k u_k$ is to undergo a succession of modifications such that (i) after the nth modification the series $\sum_{k=1}^{\infty} a_k\ u_k$ converges to $+\infty$ a.e. on E_n and to zero a.e. on $\bar{E}_n$, (ii) $a_k^{(n)} \equiv a_k^{(n-1)}$, $k = 1,2,\cdots,N_n$ where $\{N_n\}_{n=1}^{\infty}$ increases with n. (iii) $\sum_{k=1}^{\infty} a_k^{(n)} u_k = \sum_{k=1}^{\infty} a_k^{(n-1)} u_k$ on $\bar{E}_{n-1}$. Requirement (ii) insures that $\tilde{a}_k = \lim_{n\to\infty} a_k^{(n)}$ exists for each k. Since $\bar{E}_1 \subset \bar{E}_2 \subset \cdots \subset \mathrm{E}$ and $\lim_{n\to\infty} P(\mathrm{E}) = P(\mathrm{E})$, requirement (iii) implies that $\sum_{k=1}^{\infty} \tilde{a}_k u_k$ converges to zero a.e. on $\bar{E}$. The requirement (i) will be fulfilled by the construction.

Let $\{\varepsilon_k\}_{k=1}^{\infty}$ be a sequence of positive numbers such that $\sum_{k=1}^{\infty} \varepsilon_k < P(E)/2$.

Let $E_1^{N_1} \subset E_1$ be a set measurable with respect to $\sigma(u_1, u_2, \cdots, u_{N_1})$ such that $P(E_1 - E_1^{N_1}) < \varepsilon_1$. Such a set certainly exists since E_1 is the countable union of disjoint atoms. Optionally stop the series $\sum_{k=1}^{\infty} a_k u_k$ with a stopping time τ_1, so that

$$\sum_{k=1}^{\tau_1} a_k u_k = \begin{cases} \sum_{k=1}^{\infty} a_k u_k & \text{on } E_1^{N_1}, \\ \\ \sum_{k=1}^{N_1} a_k u_k & \text{on } \bar{E}_1^{N_1}. \end{cases}$$

The series $\sum_{k=1}^{\tau_1} a_k u_k$ is finite on $\bar{E}_1^{N_1}$. Therefore, by Lemma 2.3 we may define another series $\sum_{k>N_1} b_k u_k$ that vanishes identically on $E_1^{N_1}$ and converges to $-\sum_{k=1}^{N_1} a_k u_k$ on $E_1^{N_1}$. Since $b_k \equiv 0$ for $k \leqq N_1$ and for all k such that u_k is supported on $E_1^{N_1}$, the two series $\sum^{\tau_1} a_k u_k$ and $\sum_{k>N_1} b_k u_k$ may be combined into a single series $\sum c_k u_k$ that converges to $+\infty$ a.e. on $E_1^{N_1}$ and to zero a.e. on $\bar{E}_1^{N_1}$. The series $\sum c_k u_k$ may be further modified on the remaining collection of disjoint atoms $\bigcup_{k=1}^{\infty} A_k^{(1)}$ of E_1 contained in the set $E_1 - E_1^{N_1}$: the tails of the original series $\sum_{k>N} a_k u_k$ may be substituted for the tails $\sum_{k>N} c_k u_k$ on the sets $A_k^{(1)}$, $k = 1, 2, \cdots$.

The series $\sum_{k=1}^{\infty} a_k^{(1)} u_k$ obtained in this way has the following properties: (i) on $\bar{E}_1$, it agrees with the original series $\sum a_k u_k$ for $k \leqq N_1$, then is defined so that $\sum^{N_1} a_k^{(1)} u_k + \sum_{k>N_1} a_k^{(1)} u_k = 0$ a.e.; (ii) on $E_1 - E_1^{N_1}$, it agrees with $\sum a_k u_k$ for $k \leqq N_1$, then agrees with $\sum b_k u_k$ as defined above, for a single, finite block of terms beginning with the index $N_1 + 1$, then agrees with the tail of the original series; (iii) on $E_1^{N_1}$, it agrees entirely with the original series. Thus, $\sum_{k=1}^{\infty} a_k^{(1)} u_k$ fails to agree with the original series only on a subset of E_1 of measure less than ε_1, and on this subset, it fails to agree only in a single, finite block of terms. Therefore $\sum_{k=1}^{\infty} a_k^{(1)} u_k$ converges to $+\infty$ a.e. on E_1 and by (i) to zero a.e. on $\bar{E}_1$.

The second modification of the series is made entirely on the set E_1 so that $\sum_{k=1}^{\infty} a_k^{(1)} u_k = \sum_{k=1}^{\infty} a_k^{(2)} u_k$ on $\bar{E}_1$. We find a set $E_2^{N_2} \subset E_2$, measurable with respect to $\sigma(u_1, \cdots, u_{N_2})$ such that $P(E_2 \cap E_1^{N_1} \cap \bar{E}_2^{N_2}) < \varepsilon_1/2$ and $P(E_2 \cap (E_1 - E_1^{N_1}) \cap \bar{E}_2^{N_2}) < \varepsilon_2$.

Since the series $\sum_{k=1}^{\infty} a_k^{(1)} u_k$ converges to $+\infty$ a.e. on E_1, we may proceed as before, with the exception that all considerations will be restricted to the set E_1.

Define a stopping time τ_2 that is finite on the set $E_1 - E_2^{N_2}$ and infinite on $E_2^{N_2}$. Using Lemma 2.3 again, we may continue the series $\sum_{k=1}^{\tau_2} a_k^{(1)} u_k$ on $E_2 - E_2^{N_2}$ so that it converges to zero a.e. on this set. Finally, the tails of the original series $\sum_{k=1}^{\infty} a_k u_k$ may be replaced on the set $E_2 - E_2^{N_2}$ so that the resulting series $\sum_{k=1}^{\infty} a_k^{(2)} u_k$ converges to $+\infty$ a.e. on E_2 and to zero a.e. on $E_1 - E_2$. Since the series $\sum_{k=1}^{\infty} a_k^{(2)} u_k$ is obtained from $\sum_{k=1}^{\infty} a_k^{(1)} u_k$ by modification only on the set E_1, $\sum_{k=1}^{\infty} a_k^{(2)} u_k = \sum_{k=1}^{\infty} a_k^{(1)} u_k = 0$ a.e. on $\bar{E}_1$. It follows that the series $\sum_{k=1}^{\infty} a_k^{(2)} u_k = 0$ a.e. on $\bar{E}_2 = \bar{E}_1 \cup (E_1 - E_2)$. Furthermore (i) the series $\sum_{k=1}^{\infty} a_k^{(2)} u_k$ coincides with the original series on $E_1^{N_1} \cap E_2^{N_2}$; (ii) coincides with the original series except for a single finite block of terms on the sets $(E_1 - E_1^{N_1}) \cap E_2^{N_2}$ and $E_1^{N_1} \cap \bar{E}_2^{N_2}$, of measure less than $\varepsilon_1 + \varepsilon_1/2$, (iii) coincides with original series except for two finite blocks of terms on $(E_1 - E_1^{N_1}) \cap (E_2 - E_2^{N_2})$ of measure less than ε_2.

The third series $\sum_{k=1}^{\infty} a_k^{(3)} u_k$ is obtained by the modification procedure outlined above, restricted to the set E_2. That is, the series $\sum_{k=1}^{\infty} a_k^{(3)} u_k$ has the following properties (i) $\sum_{k=1}^{\infty} a_k^{(3)} u_k = \sum_{k=1}^{\infty} a_k^{(2)} u_k$ on $\bar{E}_2$, (ii) $\sum_{k=1}^{\infty} a_k^{(3)} u_k$ converges to $+\infty$ a.e. on E_3 and to zero a.e. on $\bar{E}_3$. As before, we choose a set $E_3^{N_3} \subset E_3$ so that $\sum_{k=1}^{\infty} a_k^{(3)} u_k \equiv \sum_{k=1}^{\infty} a_k^{(2)} u_k$ on $E_3^{N_3}$. The set $E_3^{N_3}$ is chosen large enough so that the series $\sum_{k=1}^{\infty} a_k^{(3)} u_k$ agrees with the original series (i) except for a single finite block of terms on a set of measure less than $\varepsilon_1 + \varepsilon_1/2 + \varepsilon_1/2^2$, (ii) except for two finite blocks of terms on a set of measure less than $\varepsilon_2 + \varepsilon_2/2$, (iii) except for three finite blocks of terms on a set of measure less than ε_3.

The nth modification is carried out similarly so that (i) $\sum_{k=1}^{\infty} a_k^{(n)} u_k \equiv \sum_{k=1}^{\infty} a_k^{(n-1)} u_k$ on $\bar{E}_{n-1}$, (ii) $\sum_{k=1}^{\infty} a_k^{(n)} u_k$ tends to zero a.e. on $\bar{E}_n$ and to $+\infty$ a.e. on E_n. The series $\sum_{k=1}^{\infty} a_k^{(n)} u_k$ agrees with the original series except for K finite blocks of terms on a set of measure less than

$$\varepsilon_K + \sum_{j=1}^{n} \varepsilon_K/2^j < 2\varepsilon_K.$$

Finally, the series $\sum_{k=1}^{\infty} \tilde{a}_k u_k$, with $\tilde{a}_k = \lim_{n\to\infty} a_k^{(n)}$, converges to $+\infty$ a.e. on E and zero a.e. on $\bar{E}$. In fact, the series $\sum_{k=1}^{\infty} \tilde{a}_k u_k$ agrees entirely with the original series $\sum_{k=1}^{\infty} a_k u_k$ on E except possibly on a subset of E of measure less than $2 \cdot \sum_{k=1}^{\infty} \varepsilon_k < P(E)$. More generally, the series $\sum_{k=1}^{\infty} \tilde{a}_k u_k$ agrees with the original series except for at most K finite blocks of terms everywhere on E except possibly on a subset of E of measure less than $2\sum_{j=K+1}^{\infty} \varepsilon_j$. It follows that the series $\sum_{k=1}^{\infty} \tilde{a}_k u_k$ converges to $+\infty$ a.e. on E. The series $\sum_{k=1}^{\infty} \tilde{a}_k u_k$ converges to zero on $\bar{E}$ since $\sum_{k=1}^{\infty} \tilde{a}_k u_k = \sum_{k=1}^{\infty} a_k^{(n)} u_k = 0$ a.e. on $\bar{E}_n \subset \bar{E}$ where the sequence of sets $\bar{E}_n$ increases to $\bar{E}$.

This completes the proof of Part (a) of Theorem 2.2, and consequently, Part (b) of Theorem 2.1.

We turn now to a proof of Part (b) of Theorem 2.2.

DEFINITION 2.1. An increasing sequence of atomic σ-fields $\sigma(n)$ is called regular if for any two atoms E_n belonging to $\sigma(n)$ and E_{n+1} belonging to $\sigma(n+1)$ with $E_n \supseteq E_{n+1}$ we have $0 < \delta \leqq P(E_{n+1})/P(E_n)$ for some $\delta > 0$ and all $n = 1, 2, \cdots$.

Part (b) of Theorem 2.2 follows from a submartingale convergence theorem due to Chow [2, Corollary 3]. (See also Doob [3].) The martingale version of Chow's theorem is as follows:

THEOREM 2.4 (CHOW). *Let S_n be a martingale with respect to a regular sequence of σ-fields $\{\sigma(n)\}_{n=1}^{\infty}$, such that $E(|S_n|) < +\infty$ for each n. (That is, $E(S_{n+1} \| \sigma(n)) = S_n$ for each n.) Then, $\lim S_n$ exists a.e. and is finite a.e. on the union of the sets where $\liminf S_n > -\infty$ and $\limsup S_n < +\infty$.*

If $\{u_k\}_{k=1}^{\infty}$ is an H^*-system such that $\liminf_{k\to\infty} p_k > 0$ then the sequence of partial sums of any series $\sum_{k=1}^{\infty} a_k u_k$ is a martingale with respect to the regular sequence of σ-fields $\sigma(u_1, u_2, \cdots, u_n)$. Assume that there is a series $\sum_{k=1}^{\infty} a_k u_k$ converging to $+\infty$ on a set of positive measure E. Then certainly $\liminf \sum_{k=1}^{n} a_k u_k > -\infty$ a.e. on E. Applying Theorem 2.4, we conclude that $\lim_{n\to\infty} \sum_{k=1}^{n} a_k u_k$ exists and is finite a.e. on E, contradicting our previous assumption. This proves Part (b) of Theorem 2.2.

Corollary 2.1 (Bari-Talalyan-Arutyunyan). *There is a Haar series representation for f if and only if f is finite a.e.*

Corollary 2.2 (Talalyan-Arutyunyan [12].) *There is no Walsh series converging to $+\infty$ on a set of positive measure.*

Proof. Suppose such a series could be defined. Then, the sequence of partial sums $\sum_{k=1}^{n} a_k\chi_k$ is a martingale with respect to the regular sequence of σ-fields $\sigma(\chi_1,\chi_2,\cdots,\chi_n)$ where χ_k are the Haar functions. An application of Theorem 2.4 as indicated above concludes the proof.

Corollary 2.3. *There is no divergent series $\sum_{k=1}^{\infty} a_k u_k$ that oscillates boundedly a.e. when $\{u_k\}_{k=1}^{\infty}$ is an H^*-system with $\liminf p_k > 0$.*

Proof. This is a direct application of Theorem 2.4.

Corollary 2.3 contrasts with the example of Marcinkiewicz which shows that divergent trigonometric series may oscillate boundedly a.e. [17, p. 308].

3. **A necessary and sufficient condition for pointwise convergence of series.** In this section, we consider H-systems such that the σ-fields $\sigma(u_1,u_2,\cdots,u_n)$ are regular in the sense of Definition 2.1. The Haar system qualifies as a regular system, as does any H^*-system such that $\liminf_{k\to\infty} p_k > 0$. For these systems, we have the following theorem.

Theorem 3.1. *Let $\{u_k\}_{k=1}^{\infty}$ be an H-system such that $\{\sigma(u_1,u_2,\cdots,u_n)\}_{n=1}^{\infty}$ is a regular sequence of σ-fields in the sense of Definition* 2.1. *Then any series $\sum_{k=1}^{\infty} a_k u_k$ converges a.e. on a set E if and only if $\sum_{k=1}^{\infty}(a_k u_k)^2 < +\infty$ a.e. on E.*

The following preliminary lemma is required.

Lemma 3.1. *Let $\{u_k\}_{k=1}^{\infty}$ be an H-system satisfying the conditions of Theorem* 3.1. *If $|a_k u_k| > \lambda$ on a set of positive measure, then $|a_k u_k| > \lambda(\delta/(1-\delta))$ on the set $\{|a_k u_k| > 0\}$ where δ is the lower bound of all ratios $\mathbf{P}(E_{n+1})/\mathbf{P}(E_n)$ such that $E_{n+1} \subseteq E_n$, and E_n, E_{n+1} are atoms,*

$$E_n \in \sigma(u_1,u_2,\cdots,u_n)$$

and

$$E_{n+1} \in \sigma(u_1,u_2,\cdots,u_{n+1}).$$

Proof. If $|a_k u_k| > \lambda$ almost everywhere on the set $\{|a_k u_k| > 0\}$ there is nothing to prove. Therefore, assume that $|a_k u_k|$ takes two positive values, α, β with $0 < \alpha \leqq \lambda < \beta$. Let $E_k = \{|a_k u_k| = \beta\}$ and $E_{k-1} = \{|a_k u_k| > 0\}$. Notice that E_{k-1} is measurable with respect to $\sigma(u_1,u_2,\cdots,u_{k-1})$ since $\{u_k\}_{k=1}^{\infty}$ is an H-system. The conditions of Theorem 3.1 taken together with the fact that $a_k u_k$ is a martingale difference, implies

$$\begin{aligned}\lambda \cdot \delta &\leqq \beta \cdot P\{|a_k u_k| = \beta \| \, |a_k u_k| > 0\},\\ &= \beta P(E_k)/P(E_{k-1}),\\ &= \alpha(1 - P(E_k)/P(E_{k-1})),\\ &\leqq \alpha(1-\delta)\end{aligned}$$

or $\lambda(\delta(1-\delta)) \leqq \alpha \leqq |a_k u_k|$ when $|a_k u_k| > 0$. The proof of the lemma is complete.

In the proof of the theorem, we will use the notion of a *minimal stopping time*: Given a stopping time τ, define $\bar{\tau}$ as

$$\bar{\tau} = \begin{cases}\min(k: \{\tau = n\} \in \sigma(u_1, u_2, \cdots, u_k) \text{ on } \{\tau = n\} \text{ for } n < \infty),\\ +\infty \text{ on } \{\tau = +\infty\}.\end{cases}$$

Notice that $\bar{\tau}$ is a stopping time such that $\sigma(\bar{\tau}) = \sigma(\tau)$ and is the smallest such stopping time with this property on the set where $\tau < +\infty$.

Proof of Theorem 3.1. Sufficiency. Suppose $\sum_{k=1}^{\infty}(a_k u_k)^2 < +\infty$ a.e. on E. Let $\tau = \min(n: \sum_{k=1}^{n}(u_k a_k)^2 \geqq N)$ and $\bar{\tau}$ be the minimal stopping time associated with τ. Then, if

$$S_n^{\bar{\tau}} = \begin{cases}S_n \text{ if } n \leqq \bar{\tau},\\ S_{\bar{\tau}} \text{ if } n > \bar{\tau}\end{cases}$$

it can be shown that

$$\begin{aligned}\int |S_n^{\bar{\tau}}|^2 &= \int \sum_{k=1}^{\min(n,\bar{\tau})} (a_k u_k)^2\\ &\leqq N + N/\delta^2 \quad \text{for all } n = 1, 2, \cdots.\end{aligned} \tag{1}$$

Suppose for the moment that (1) has been established. Inequality (1) implies that $\{S_n^{\bar{\tau}}\}_{n=1}^{\infty}$ is a uniformly integrable martingale and, therefore, converges by the standard martingale convergence theorem. Since $S_n^{\bar{\tau}} = S_n$ for all n on the set E_N where $\sum_{k=1}^{\infty}(a_k u_k)^2 < N$, $\lim S_n$ exists almost everywhere on E_N. The sets E_N increase, with increasing N, to the set $E_\infty = \{\sum_{k=1}^{\infty}(a_k u_k)^2 < +\infty\}$. Consequently, $\lim S_n$ exists almost everywhere on E_∞.

To prove inequality (1), notice that the left-hand equality is a consequence of the orthogonality of the functions u_k. For the right-hand inequality, estimate the sum $\sum_{k=1}^{\min(n,\bar{\tau})}(a_k u_k)^2$ as follows. If $\min(n, \bar{\tau}) = n < \bar{\tau} \leqq \tau$ then

$$\sum_{k=1}^{n}(a_k u_k)^2 < N \leqq \sum_{k=1}^{\tau}(a_k u_k)^2$$

so that inequality (1) is certainly satisfied. If $\min(n, \bar{\tau}) = n = \bar{\tau}$, consider two cases:

(i) Suppose that on the set $\{\tau = m\}$, $|a_m u_m| < N^{1/2}/\delta$ where $m = \tau \geqq \bar{\tau}$. Then

$$\sum_{k=1}^{\bar{\tau}} (a_k u_k)^2 \leqq \sum_{k=1}^{m} (a_k u_k)^2 \leqq \sum_{k=1}^{m-1} (a_k u_k)^2 + \frac{N}{\delta^2} \leqq N + \frac{N}{\delta^2}.$$

Therefore, inequality (1) is satisfied under condition (i).

(ii) Now, suppose that somewhere on the set $\{\tau = m\}$, $|a_m u_m| > N^{1/2}/\delta$. Then the condition of Lemma 3.1 is satisfied for $\lambda = N^{1/2}/\delta$; it follows that $|a_m u_m| \geqq N^{1/2}/(1-\delta) \geqq N^{1/2}$ (or $(a_m u_m)^2 \geqq N$) everywhere on the set where $|a_m u_m| > 0$. This imples that

$$\{\tau = m\} = \left\{ \sum_{k=1}^{m-1} (a_k u_k)^2 < N;\ |a_m u_m| > 0 \right\}.$$

This, in turn, implies that the set $\{\tau = m\}$ is measurable on $\sigma(u_1, u_2, \cdots, u_{m-1})$ which means $\bar{\tau} \leqq m - 1$. Therefore we have

$$\sum_{k=1}^{\bar{\tau}} (a_k u_k)^2 \leqq \sum_{k=1}^{m-1} (a_k u_k)^2 < N$$

so that (1) is certainly satisfied in this case.

Cases (i) and (ii) exhaust the possible alternatives so that inequality (1) is established.

Necessity. Suppose that S_n is convergent on a set of positive measure. The limit is necessarily finite by Theorem 2.4 of the previous section. Let τ be the stopping time $\tau = \min(n : |S_n| \geqq N)$ and $\bar{\tau}$ the minimal stopping time associated with τ. Arguing as before, we observe that $\bar{\tau} \leqq n - 1$ on the set where $\tau = n$ and $|a_n u_n| > 2N/\delta$ somewhere on $\{\tau = n\}$. Therefore,

$$|S_n^{\bar{\tau}}| \leqq N + 2N/\delta \text{ and } \int |S_n^{\bar{\tau}}|^2 dP = \int \Sigma^{\min(n,\bar{\tau})} (a_k u_k)^2 dP,$$
$$\leqq (N + 2N/\delta)^2 \text{ for every } n$$

so that $\sum_{k=1}^{\infty} (a_k u_k)^2 < \infty$ a.e. on the set where $\sup |S_n| < N$. Since N is arbitrary, $\sum_{k=1}^{\infty} (a_k u_k)^2 < +\infty$ a.e. on the set where $\lim S_n$ exists.

COROLLARY 3.1. *A series $\sum_{k=1}^{\infty} a_k u_k$ from an H^*-system such that* $\liminf_{k\to\infty} p_k > 0$ *converges a.e. on a set E if and only if $\sum_{k=1}^{\infty} (a_k u_k)^2 < \infty$ a.e. on E.*

Proof. The σ-fields $\sigma(u_1, u_2, \cdots, u_n)$ form a regular sequence since $0 < \delta = \liminf_{k\to\infty} p_k \leqq P(E_{n+1})/P(E_n)$ for any two sets, E_{n+1} belonging to $\sigma(u_1, u_2, \cdots, u_{n+1})$, E_n belonging to $\sigma(u_1, u_2, \cdots, u_n)$ such that $E_{n+1} \subseteq E_n$. Therefore, Theorem 3.1 applies, and the corollary is proved.

In particular, Corollary 3.1 applies to the Haar series, since $p_k = \frac{1}{2}$ for $k = 1, 2, \cdots$. From the result for Haar series, we may deduce the following result for Walsh series.

COROLLARY 3.2. *Let $\{W_{2^n}\}_{n=1}^{\infty}$ be the sequence of the 2^nth partial sums of a*

Walsh series. Then, $\lim W_{2^n}$ *exists a.e. if and only if* $\sum_{n=1}^{\infty}(W_{2^n} - W_{2^{n-1}})^2 < +\infty$ *a.e.*

Proof. It is pointed out in the first section that for every Walsh series, there is a Haar series such that $H_{2^n} = W_{2^n}$. Then $H_{2^n} - H_{2^{n-1}} = W_{2^n} - W_{2^{n-1}}$ for every n. However, $H_{2^n} - H_{2^{n-1}} = \sum_{k=2^{n-1}+1}^{2^n}(a_k\chi_k)$ has the property that the summands $a_k\chi_k, 2^{n-1} < k \leqq 2^n$ are supported on disjoint sets. This permits us to conclude that (a) $\lim H_n$ exists a.e. on E if and only if $\lim H_{2^n}$ exists a.e. on E; and (b) $(\sum_{k=2^{n-1}+1}^{2^n} a_k\chi_k)^2 = \sum_{k=2^{n-1}+1}^{2^n}(a_k\chi_k)^2$. Therefore, if $\lim W_{2^n}$ exists a.e. on a set E, then $\lim W_{2^n} = \lim H_{2^n} = \lim H_n$ exists a.e. (and is finite by Theorem 2.4). This implies that

$$\sum_{k=1}^{\infty}(a_k\chi_k)^2 = \sum_{n=1}^{\infty}\sum_{k=2^{n-1}+1}^{2^n}(a_k\chi_k)^2$$

$$= \sum_{n=1}^{\infty}(W_{2^n} - W_{2^{n-1}})^2 < \infty \text{ a.e.}$$

on E, so that one half of Corollary 3.2 is proved. The other half is proved similarly.

When the sequence W_{2^n} is the sequence of partial sums of a Walsh-Fourier series, Corollary 3.2 may be sharpened. The fundamental inequality of Walsh-Fourier series, proved by R. E. A. C. Paley [11] states that if $f \in L^p$, $1 < p < \infty$ and $\{W_n\}_{n=1}^{\infty}$ is the sequence of partial sums of Walsh-Fourier series, then the function $F = \sum_{n=1}^{\infty}(W_{2^n} - W_{2^{n-1}})^2$ is related to f by the inequality

$$A_p \int_0^1 F^{p/2}dx \leqq \int_0^1 f^p dx \leqq B_p \int_0^1 F^{p/2}dx.$$

For functions $f \in L^1$, a "weak type" bound on the function f has been obtained by Yano [16]. The observation that the above inequality holds for the Haar-Fourier series has been made by Marcinkiewicz [8].

The following two corollaries are of some interest in the study of divergent Fourier series [14].

COROLLARY 3.3. *Let* $\sum_{k=1}^{\infty} a_k u_k$ *be any a.e. convergent series from an H-system satisfying the condition of Theorem* 3.1. *Then, any series* $\sum_{k=1}^{\infty}\delta_k a_k u_k$ *converges a.e. when* $|\delta_n| \leqq B < \infty$ *uniformly for* $n = 1, 2, \cdots$.

Proof. The convergence of $\sum_{k=1}^{\infty} a_k u_k$ implies that $\sum_{k=1}^{\infty}(a_k u_k)^2 < \infty$ a.e. by Theorem 3.1, so that $\sum_{k=1}^{\infty}(\delta_k a_k u_k)^2 \leqq B^2 \sum_{k=1}^{\infty}(a_k u_k)^2 < \infty$ a.e. Another application of Theorem 3.1 shows that $\sum_{k=1}^{\infty}\delta_k a_k u_k$ converges a.e.

A series of functions is said to be *unconditionally convergent in measure* if the series converges in measure for every rearrangement of its terms.

Corollary 3.4. *Every a.e. convergent Haar series is unconditionally convergent in measure.*

Proof. A result of Orlicz [10] states that a series $\sum_{k=1}^{\infty} f_k$ is unconditionally convergent in measure if and only if $\sum_{k=1}^{\infty} \delta_k \cdot f_k$ converges in measure for every sequence of unit factors $\{\delta_k\}_{k=1}^{\infty}$. This theorem, in conjunction with Corollary 3.3, gives Corollary 3.4.

Remark. Actually, we have proved more than unconditional convergence in measure. In the terminology of Ul'yanov [14] we have shown that every convergent Haar series is weakly unconditionally convergent a.e.

Corollary 3.3 also has some interest for martingale theory.

Corollary 3.5. *Let $\sum_{k=1}^{\infty} a_k u_k$ be an almost everywhere convergent series from an H-system satisfying the condition of Theorem* 3.1. *Then, any optional skipping scheme defined on the martingale of partial sums of the series gives an a.e. convergent martingale.*

Proof. Optional skipping does not, in general, preserve a.e. convergence of a martingale. In the case at hand, however, we note that any optional skipping scheme is generated by a multiplier transformation $\sum_{k=1}^{\infty} \delta_k a_k u_k$, by Proposition 1.2 of the first section. An appeal to Corollary 3.3 finishes the proof.

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