INVERSE LIMITS AND HOMOGENEITY([1])

BY
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1. **Introduction.** In 1958 Jack Segal [4] announced some sufficient conditions on an inverse limit sequence for the limit space to be homogeneous. However, this result was not verified and in 1965 McCord [3] proved a weaker version of the theorem. In this paper we prove a partial converse to McCord's theorem which enables us to establish a counterexample to Segal's original theorem.

Segal announced: If, in an inverse limit sequence of topological spaces, the coordinate spaces are path-connected, locally path-connected, and homogeneous, and the bonding maps are covering maps, then the inverse limit space is homogeneous. McCord proved that if (X_n, π_m^n) is an inverse sequence of manifolds and each $\pi_1^n: X_n \to X_1$ is a regular covering map, then the limit space is homogeneous. Theorem 4 of this paper says roughly that if the bonding maps are badly non-regular, then the limit space is not homogeneous. An inverse limit of compact 2-manifolds is then constructed satisfying the conditions of Theorem 4. In [3, p. 208] McCord has conjectured the existence of such an example. Thus, we see that in spirit Segal's theorem was right, but curiously it needed the somewhat natural condition of regularity on the covering maps to force homogeneity on the limit space. The author thanks the referee for many suggestions and in particular for the formulation of Theorem 4.

2. **Preliminaries.** If X and Y are connected and locally path-connected spaces, then $p: X \to Y$ is a *covering map* if (1) p is a map from X onto Y and (2) for each point y in Y there exists a connected open set U containing y such that each component of $p^{-1}(U)$ is open in X and is mapped homeomorphically onto U by p. The set U is called a *canonical neighborhood* with respect to the covering map p. See [2] for the definition of a covering map to be regular. If $p: X \to Y$ is a covering map, $u: I \to Y$ is a path in Y, and $v: I \to X$ is a path in X, then we say that u *lifts* to v iff $p \circ v = u$. If X is a space, $u: I \to X$ is a path, and $x \in X$, then u is a *loop* on x iff $u(0) = u(1) = x$. A necessary and sufficient condition for a covering map p to be regular is that if w is any loop in Y on y that lifts to a loop for some point

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of $p^{-1}(y)$, then w lifts to a loop for all points of $p^{-1}(y)$. The map f_1^2 of Figure 3 is an example of a nonregular covering map.

The symbol (M_i, f_{i-1}^i) stands for an inverse limit sequence where if $i \geqq 1$, M_i is a space, if $i > 1$, f_{i-1}^i is a map from M_i into M_{i-1}, if $1 \leqq i \leqq j$, then $f_i^j = f_i^{i-1} \circ \cdots \circ f_{j-1}^j$, and if M is the limit space then $f_i : M \to M_i$ is the natural projection.

3. **Results.** The first two theorems are probably known, but are not to my knowledge in the literature.

THEOREM 1. *Let M be the limit of an inverse sequence (M_i, f_{i-1}^i) where each bonding map is a covering map. Two points $x = (x_i)$ and $y = (y_i)$ of M are in the same path component of M if and only if there exists a path*

$$v_1 : (I, 0, 1) \to (M_1, x_1, y_1)$$

such that for each $i > 1$, v_1 lifts to a path $v_i : (I, 0, 1) \to (M_i, x_i, y_i)$.

Proof. Assume x and y are in the same path-component of M and let $v : (I, 0, 1) \to (M, x, y)$. If $i \geqq 1$, define $v_i : (I, 0, 1) \to (M_i, x_i, y_i)$ such that $v_i = f_i \circ v$, and thus the collection $v_1, v_2, \cdots$ satisfies the condition.

Assume now that the condition holds and define $v : I \to M$ such that $v(t) = (v_i(t))$. Then v is a path in M from x to y and hence x and y are in the same path-component of M. ∎

Since a 1-fold covering map is a homeomorphism, from now on let us assume that all covering maps are at least 2-fold.

THEOREM 2. *If (M_i, f_{i-1}^i) is an inverse sequence of compact manifolds, where each bonding map is a covering map, then the limit space M has c path components. (In [3] McCord has shown this result for the case when each f_1^i is a regular covering map.)*

Proof. If $x_1 \in M_1$, the set $f_1^{-1}(x_1)$ is a Cantor set and thus has cardinality c. Let K be any path-component of M that intersects $f_1^{-1}(x_1)$. We will show that $K \cap f_1^{-1}(x_1)$ is countable. Let $x \in K \cap f_1^{-1}(x_1)$ and let $\pi(M_1, x_1)$ be the fundamental group of M_1 with base point x_1. If $\alpha \in \pi(M_1, x_1)$, then $\alpha = [v_1]$ where v_1 is a loop in M_1 on x_1. If $i > 1$, lift v_1 to $v_i : (I, 0) \to (M_i, x_i)$ the path in M_i starting at x_i. Thus $(v_1(1))$ is a point of $f_1^{-1}(x_1)$ and by Theorem 1 $(v_i(1)) \in K$. Define $\phi : \pi(M_1, x_1) \to K \cap f_1^{-1}(x_1)$ such that $\phi(\alpha) = (v_i(1))$. By [2, Theorem 6.5.9] ϕ is well defined and by Theorem 1 ϕ is onto. Hence, since $\pi(M_1, x_1)$ is countable, $K \cap f_1^{-1}(x_1)$ is countable. Thus, since the cardinality of $f_1^{-1}(x_1)$ is c, since each path-component of M intersects at most a countable subset of $f_1^{-1}(x_1)$, and since $a \cdot \aleph_0 = a_0$ for every infinite cardinal a, M has c path-components. ∎

For a compact metric space (M, d), let $\mathrm{Map}(M, M)$ be the separable space of all maps of M into M with metric $d(g, h) = \sup\{d(g(x), h(x)) : x \in M\}$.

THEOREM 3. *Let M be the limit of an inverse sequence (M_i, f_{i-1}^i) of compact manifolds where each bonding map is a covering map. Suppose M is homogeneous. Let $\varepsilon > 0$ and $r \in M$ be given. Then there exists a homeomorphism $h: M \to M$ such that $d(h, id) < \varepsilon$ and $h(r) \neq r$, but $f_1 h(r) = f_1(r)$.*

Proof. By Theorem 2 let S be an uncountable subset of M such that distinct members of S belong to different path components of M. By homogeneity of M, choose for each s in S a homeomorphism

$$h_s : (M, s) \to (M, r).$$

Now $\{h_s : s \in S\}$ is an uncountable subset of the separable metric space Map(M, M). Hence there are distinct points $s^0, s^1, s^2, \cdots$ in S, so that, setting $h_i = h_{s^i}$, the sequence $h_1, h_2, \cdots$ converges to h_0. Define $g_i: M \to M$ such that $g_i = h_i h_0^{-1}$. Since $s^0, s^1, s^2, \cdots$ all belong to different path-components of M, $r = h_0(s^0)$, $h_0(s^1)$, $h_0(s^2), \cdots$ also belong to different path components and $g_i(h_0(s^i)) = r$.

Let $U(r, \varepsilon/4)$ be the $\varepsilon/4$-sphere about r in M. There exists an integer n and open cell U_n in M containing r_n such that $f_n^{-1}(U_n) \subset U(r, \varepsilon/4)$. Since the $h_0(s^i)$ converge to r and the g_i converge uniformly to the identity, there exists an integer k such that $h_0(s^k) \in f_n^{-1}(U_n)$ and $d(g_k^{-1}, id) < \varepsilon/2$. We have $h_0(s^k)_n$ and r_n in U_n and thus we may let u_n be a homeomorphism of M_n onto itself that is the identity on the complement of U_n and takes $h_0(s^k)_n$ to r_n. For each $i > n$, the set $(f_n^i)^{-1}$ (U_n) is the disjoint union of open sets each mapped homeomorphically onto U_n by f_n^i. Thus u_n lifts to a homeomorphism $u_i: M_i \to M_i$ and hence induces a homeomorphism $u: M \to M$.

Since u is the identity on the complement of $f_n^{-1}(U_n)$ and diam$(f_n^{-1}(U_n)) < \varepsilon/2$, we have $d(u, id) < \varepsilon/2$. Now let $v_n: I \to U_n$ be a path in U_n from $h_0(s^k)_n$ to r_n. If $j > n$, lift v_n to v_j, the path in M_j starting at $h_0(s^k)_j$. Let t be the point of $f_n^{-1}(r_n)$ determined by the $v_j(1)$. Hence $u(h_0(s^k)) = t$ and by Theorem 1, t is in the same path component as $h_0(s^k)$ and thus $t \neq r$. Define h such that $h = u \circ g_k^{-1}$. Since $d(g_k^{-1}, id) < \varepsilon/2$ we have $d(h, id) < \varepsilon$ and $h(r) = t \neq r$. Also, since $t_n = r_n$ we have $f_1(h(r)) = f_1(r) = r_1$. ∎

In the following, let us call paths $\alpha_0, \alpha_1: I \to X$ *homotopic*, denoted $\alpha_0 \simeq \alpha_1$, only when there exists a homotopy $\alpha_t: I \to X$ from α_0 to α_1 leaving end points fixed. Thus $\alpha_s(0) = \alpha_t(0)$ and $\alpha_s(1) = \alpha_t(1)$ for all $s, t \in I$.

LEMMA 1. *Let M be a compact manifold with metric d. There exists an $\varepsilon > 0$ such that if α and β are paths in M with $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$ and $d(\alpha, \beta) < \varepsilon$, then $\alpha \simeq \beta$.*

The proof is an immediate corollary to [2, Theorem 1.8.5]. ∎

THEOREM 4. *Let (M_i, f_{i-1}^i) be an inverse sequence of compact manifolds where each bonding map is a covering map. Suppose there exists a point $r \in M$*

with the following property. If $k > 1$ and s_k is any point of M_k distinct from r_k where $f_{k-1}^k(s_k) = r_{k-1}$, then there exists a loop $\alpha : I \to M$ on r and a path $\alpha_k' : I \to M_k$ with initial point s_k such that $f_1^k \alpha_k' = \alpha_1$, but α_k' is not a loop. Then M is not homogeneous.

Proof. Suppose M is homogeneous. Choose metrics d_1 on M_1 and d on M such that if $x, y \in M$, then $d_1(x_1, y_1) \leqq d(x, y)$. This is possible by the work of Fort and Segal in [1]. Choose $\varepsilon > 0$ for (M_1, d_1) as in Lemma 1. Let r be as above. By Theorem 3 we can choose a homeomorphism $h : M \to M$ such that $s = h(r) \neq r$, $s_1 = r_1$, and $d(h, id) < \varepsilon$. Since $r \neq s$ there exists $k > 1$ such that $r_k \neq s_k$ but $r_{k-1} = s_{k-1}$. Let α and α_k' be as above. If $\beta : I \to M$ is the path $\beta = h\alpha$, then $d_1(\alpha_1, \beta_1) < \varepsilon$ and hence by Lemma 1, $\alpha_1 \simeq \beta_1$. Now both α_k' and β_k are paths with initial point s_k and $f_1^k \alpha_k' = \alpha_1 \simeq \beta_1 = f_1^k \beta_k$. Hence, by [2, Theorem 6.9.5], $\alpha_k'(1) = \beta_k(1)$. However, $\beta_k(1) = f_k h\alpha(1) = f_k(s) = s_k$. Thus α_k' is indeed a loop (on s_k). This contradiction completes the proof. ∎

We shall construct an inverse sequence (M_i, f_{i-1}^i) and then verify that it satisfies the hypothesis of Theorem 4.

EXAMPLE. Let M_1 be a double torus or a sphere with two handles. If n is a positive integer, by an *n-handle* we mean a torus with n mutually disjoint open disks punched out. Thus M_1 is the union of two 1-handles, H_1 and H_1' whose intersection, B, is equal to the boundary of each of them. Let C_1'' and D_1'' be meridianal simple closed curves on H_1 and H_1', respectively. See Figure 1.

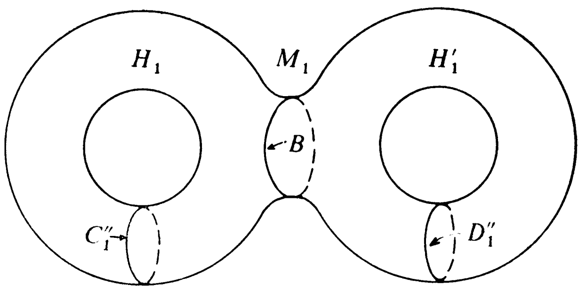

FIGURE 1

Suppose $k \geqq 1$ and assume that (1) the (sphere with $3^{k-1} + 1$ handles), M_k, has been defined, (2) two 2^{k-1}-handles, H_k and H_k', on M_k have been distinguished, and (3) C_k'' and D_k'' are simple closed curves on H_k and H_k', respectively. Let $M_k^* = M_k \backslash (C_k'' \cup D_k'')$ and let $h_k^* : M_k^* \to E^3$ be a homeomorphism that pulls the cut handles apart as shown in Figure 2 for the case when $k = 1$. Let $M_k' = \mathrm{Cl}[h_k^*(M_k^*)]$ and thus, the homeomorphism $h_k^{*-1} : h_k^*(M_k^*) \to M_k^*$ has a unique extension to a continuous function h_k from M_k' onto M_k. Let C_k and C_k' (D_k and D_k') denote the two simple closed curves that are boundary components of M_k' and that map onto $C_k''(D_k'')$ under h_k.

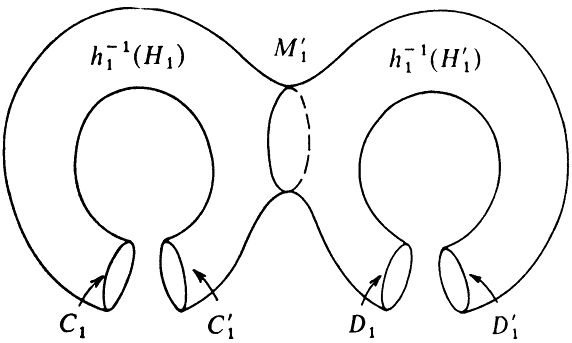

FIGURE 2

Let $\{1,2,3\}$ be a discrete space on three symbols and if $i=1,2$, or 3, let $M_{ki} = M' \times i$, $C_{ki} = C_k \times i$, $C'_{ki} = C'_k \times i$, $D_{ki} = D_k \times i$, $D'_{ki} = D'_k \times i$, $H_{ki} = h_k^{-1}(H_k) \times i$, and $H'_{ki} = h_k^{-1}(H'_k) \times i$. Thus, if $i = 1,2$, or 3, h_k induces the map h_{ki} from M_{ki} onto M_k such that if $(x,i) \in M_{ki}$, then $h_{ki}(x,i) = h_k(x)$. Define $u_{k1}: C_{k1} \to C'_{k1}$ such that if $(x,1) \in C_{k1}$, then $u_{k1}(x,1) = h_{k1}^{-1} h_k(x) \cap C'_{k1}$, $u_{k2}: C_{k2} \to C_{k3}$ such that if $(x,2) \in C_{k2}$, then $u_{k2}(x,2) = h_{k3}^{-1} h_k(x) \cap C'_{k3}$, and $u_{k3}: C_{k3} \to C'_2$ such that if $(x,3) \in C_{k3}$, then $u_{k3}(x,3) = h_{k2}^{-1} h_k(x) \cap C'_{k2}$. Define $v_{k1}: D_{k1} \to D'_{k2}$, $v_{k2}: D_{k2} \to D'_{k1}$, and $v_{k3}: D_{k3} \to D'_{k3}$ similarly. Let $A_k = \bigcup_{i=1}^3 (C_{ki} \cup D_{ki})$, $B_k = \bigcup_{i=1}^3 (C'_{ki} \cup D'_{ki})$, and $u_k: A_k \to B_k$ such that $u_k \mid C_{ki} = u_{ki}$ and $u_k \mid D_{ki} = v_{ki}$.

Let $M_{k+1} = (\bigcup_{i=1}^3 M_{ki})/u_k$, the identification space where each point of A_k is identified with its image under u_k and whose topology is the identification topology. M_{k+1} is a sphere with $3^k + 1$-handles. If $(x,i) \in M_{ki}$, let $[x,i]$ be the point of M_{k+1} which is the equivalence class containing (x,i). Define $f_k^{k+1}([x,i]) = h_k(x)$. This function is well defined and is a three fold covering map of M_k. Define H_{k+1} and H'_{k+1} to be the 2^k-handles $[H_{k2} \cup H_{k3}]$ and $[H'_{k1} \cup H'_{k2}]$, respectively. Let C''_{k+1} and D''_{k+1} be the simple closed curves $[C_{k2}] = [C'_{k3}]$ and $[D_{k1}] = [D'_{k2}]$ on H_{k+1} and H'_{k+1}, respectively. Note that C''_{k+1} connects the M_{k2} and M_{k3} layers of M_{k+1}, D_{k+1} connects the M_{k1} and M_{k2} layers of M_{k+1}, $f_k^{k+1}(C''_{k+1}) = C''_k$, and $f_k^{k+1}(D''_{k+1}) = D''_k$. See Figure 3 for the 1-dimensional counterparts of M_1, M_2, and M_3.

Thus, for each positive integer i, we have defined a 2-manifold M_i and a three fold covering map $f_i^{i+1}: M_{i+1} \to M_i$. Let $(M_i, f_{i-1})^i$ be the corresponding inverse limit sequence and let M be the limit of this sequence. Thus M is the inverse limit of homogeneous spaces where the bonding maps are covering maps. ∎

THEOREM 5. *The limit space M is not homogeneous.*

Proof. We must show that the conditions of Theorem 4 are satisfied. To say that the point x of M_k *lies* on layer i of M_k means that x has a representative

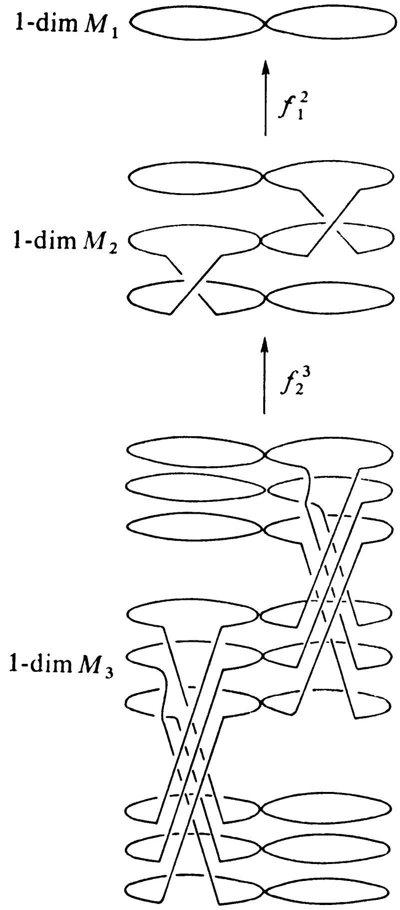

FIGURE 3

(y, i) where $(y, i) \in M_{(k-1)i}$. If H is an n-handle, J a closed interval, and $w: J \to H$ a path, then to say that w *goes around* H means that $w(J)$ is a simple closed curve that is longitudinal in H.

Let $r \in M$ such that for each $k \geqq 1$ r_k lies on layer 1 of M_k. Let $k > 1$ and let s_k be a point of M_k distinct from r_k where $f_{k-1}^k(s_k) = r_{k-1}$. Since r_k is on layer 1 of M_k, then s_k is either on layer 2 or layer 3. These are completely analogous situations and thus assume s_k lies on layer 2.

Let $p \in H_{k-1}$, the 2^{k-2}-handle of M_{k-1} that contains the simple closed curve C'_{k-1}. Let $\alpha_{k-1}: I \to M \setminus D''_{k-1}$ be a loop on r_{k-1} missing D''_{k-1} such that (1) from time 0 to 1/3 α_{k-1} goes from $r_{k-1} = s_{k-1}$ to p, (2) $\alpha_{k-1} \mid [1/3, 2/3]$ goes around H_{k-1}, and (3) from time 2/3 to 1, $\alpha_{k-1}(t) = \alpha_{k-1}(1-t)$. For each $i \geqq k$ lift α_{k-1} to the path $\alpha_i: I \to M_i$ with initial point r_i. In the construction of M_k, $C_{(k-1)1}$ was identified with $C'_{(k-1)1}$ and since α_{k-1} misses D''_{k-1} the identification of the $D_{(k-1)i}$

have no effect, and thus the loop α_{k-1} lifts to the loop α_k. Furthermore, for each $i > k$, the loop α_{k-1} lifts to the loop α_i and hence the loop α_{k-1} lifts to the loop $\alpha: I \to M$ on r. This is the loop α of Theorem 4.

We now define the loop α'_k of Theorem 4. Lift α_{k-1} to the path $\alpha'_k: I \to M_k$ with initial point s_k. Since (1) $\alpha_k \mid [1/3, 2/3]$ goes around H_{k-1}, (2) s_k and $\alpha'_k(1/3)$ are on layer 2, and (3) $C_{(k-1)2}$ is identified with $C'_{(k-1)3}$, then $\alpha'_k(2/3)$ is on layer 3 and so $\alpha'(1/3) \neq \alpha'_k(2/3)$. From this it follows that $s_k = \alpha'_k(0) \neq \alpha'_k(1)$. Thus α'_k is not a loop and since $f_1^k \alpha'_k = \alpha_1$ all conditions of Theorem 4 have been verified. Hence M is not homogeneous. ∎

Conjecture. Suppose in (M_i, f_{i-1}^i) each M_i is a closed manifold, each bonding map is a covering map, and M is homogeneous. Then there exists an integer m such that for all $n > m$, the covering map $f_m^n: M_n \to M_m$ is regular. This would be the converse to the theorem of McCord quoted in the introduction.

REFERENCES

1. M. K. Fort and J. Segal, *Local connectedness of inverse limit spaces*, Duke Math J. **28** (1961), 253–260.

2. P. J. Hilton and S. Wylie, *Homology theory*, Cambridge Univ. Press, Cambridge, 1960.

3. M. C. McCord, *Inverse limit sequences with covering maps as bounding maps*, Trans. Amer. Math. Soc. **114** (1965), 197-209.

4. J. Segal, *Homogeneity of inverse limit spaces*, Notices Amer. Math. Soc. **5** (1958), 687.

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