q-COMPLETE SPACES AND COHOMOLOGY

BY

GIULIANO SORANI AND VINICIO VILLANI(1)

1. **Introduction.** q-complete complex spaces have important cohomological properties. It is not known whether these properties are sufficient to characterize them.

Since in many applications the q-completeness of a space X is of interest mostly because it helps to determine cohomological properties of X, we thought it would be useful to introduce the more general notion of a *cohomologically q-complete space*. The study of these spaces may be an initial step towards a possible characterization of complex spaces with respect to their degree of completeness.

The plan of the paper is as follows: in §2 we give some conditions for q-completeness of a complex space.

In §3 we give the definition of a cohomologically q-complete space and prove some criteria for cohomological q-completeness of a complex space.

In §4 some results in the theory of q-complete spaces are extended to cohomologically q-complete spaces.

In §5 we produce examples of q-complete or cohomologically q-complete spaces and study some of their properties.

We should like to thank Hugo Rossi for a suggestion which led to the proof of Lemma (2.11).

2. **q-complete spaces.** Let A be an analytic set defined on an open set U of C^N. Let $z=(z_1, \ldots, z_N)$ be local coordinates on U. A real-valued function ϕ, defined on A, is said to be *differentiable on A* if there exists on U a differentiable(2) function $\hat{\phi}$ such that $\hat{\phi}|A=\phi$.

The function ϕ is said to be *strongly q-plurisubharmonic* on A, if $\hat{\phi}$ can be chosen so that the Hermitian form in N variables:

$$\mathscr{L}(\hat{\phi}, x) = \sum_{\alpha,\beta=1}^{N} \left(\frac{\partial^2\hat{\phi}}{\partial z_\alpha\, \partial \bar{z}_\beta}\right)_x u_\alpha \bar{u}_\beta$$

has at least $N-q$ positive eigenvalues at every point $x \in U$(3). $\mathscr{L}(\hat{\phi}, x)$ is called the

Received by the editors April 7, 1966.

(1) The authors were partially supported by grants of the CNR (Gruppi di ricerca matematica). The second author was also partially supported by a grant from the National Science Foundation at Harvard University.

(2) By differentiable we always mean differentiable of class C^∞.

(3) There is no uniformity in the literature concerning this terminology. The function ϕ which we call strongly q-plurisubharmonic is sometimes called strongly $(q+1)$-plurisubharmonic. As a consequence also the complex spaces which we shall call q-complete are sometimes called $(q+1)$-complete.

Levi form of $\hat{\phi}$ and it will be written $\mathscr{L}(\hat{\phi})$ when there is no possibility of confusion.

Now let X be a reduced complex space (i.e., a complex space in the sense of Serre). A real-valued function ϕ, defined on X, is said to be *differentiable*, respectively *strongly q-plurisubharmonic on* X, if the restrictions of ϕ to a system of coordinate neighborhoods, which determine the given structure of X as a complex space, are differentiable, respectively strongly q-plurisubharmonic.

A complex space X is said to be *q-complete* (see for instance [1, p. 235]) if there exists a differentiable strongly q-plurisubharmonic function ϕ defined on X such that for every $c \in \boldsymbol{R}$ the sets

$$B_c = \{x \in X \mid \phi(x) < c\}$$

are relatively compact in X.

We recall the following theorem.

(2.1) THEOREM. *A complex space X is* 0*-complete if and only if it is holomorphically complete* (*i.e., is a Stein space*).

For the proof of this theorem see [9]. Note that this theorem generalizes to complex spaces the analogous theorem established in [7] for complex manifolds.

The following propositions give useful sufficient conditions for a complex space to be q-complete.

(2.2) PROPOSITION. *Let X be a complex space, q-complete with respect to a function* ϕ. *Then for every fixed real constant c the open subspace of X*:

$$B_c = \{x \in X \mid \phi(x) < c\}$$

is q-complete.

Proof. We consider on B_c the function $1/(c-\phi)$. For every $s \in \boldsymbol{R}$ the sets $B_s = \{x \in X \mid 1/(c-\phi(x)) < s\}$ are relatively compact in B_c. In every coordinate neighborhood which contains a point $x \in B_c$, the function $1/(c-\phi)$ is the trace of the function $1/(c-\hat{\phi})$. For every $y \in U$, we compute

$$\mathscr{L}\left(\frac{1}{c-\hat{\phi}}, y\right)(u) = \frac{1}{(c-\hat{\phi})^2}\mathscr{L}(\hat{\phi}, y)(u) + \frac{2}{(c-\hat{\phi})^3}|\text{grad}\,\hat{\phi}\times u|^2.$$

The last summand, when considered at the points where $\hat{\phi} < c$, is a positive semi-definite form. It follows that the number of positive eigenvalues of $\mathscr{L}(1/(c-\hat{\phi}), y)$ is not less than the number of positive eigenvalues of $\mathscr{L}(\hat{\phi}, y)$. Q.E.D.

(2.3) PROPOSITION. *Let X and Y be two complex spaces. Let X be p-complete and Y be q-complete*; *then* $X \times Y$ *is* $(p+q)$*-complete.*

Proof. By hypothesis there exists a function $\phi(x)$ strongly p-plurisubharmonic on X such that the sets $B_c(X) = \{x \in X \mid \phi(x) < c\}$ are relatively compact in X. There is also a function $\psi(y)$ strongly q-plurisubharmonic on Y such that the sets $B_c(Y) = \{y \in Y \mid \psi(y) < c\}$ are relatively compact in Y.

The function χ defined on $X\times Y$ by

$$\chi(x, y) = \phi(x)+\psi(y)$$

is clearly strongly $(p+q)$-plurisubharmonic on $X\times Y$.

Moreover, the sets:

$$B_c(X\times Y) = \{(x, y) \mid \chi(x, y) < c\}$$

are relatively compact in $X\times Y$, as it is easy to check, since the functions ϕ, ψ are bounded from below on X and Y respectively. Q.E.D.

(2.4) PROPOSITION. *Let X, Y be two open subsets of the same complex space. Let X be 0-complete and Y be q-complete. Then $X\cap Y$ is q-complete.*

Proof. Let $\phi(x)$, $\psi(y)$ be the two functions already considered in Proposition (2.3). Define on $X\cap Y$ the function:

$$\xi(x) = \phi(x)+\psi(x), \qquad x\in X\cap Y.$$

Let W be a coordinate neighborhood of $x\in X\cap Y$ and let A be an analytic set, defined on an open set $U\subset C^N$, isomorphic to W. By definition there exist two functions $\hat{\phi}$ and $\hat{\psi}$, respectively, strongly 0- and q-plurisubharmonic on U such that $\hat{\phi}|A$ coincides with the image of ϕ in A and $\hat{\psi}|A$ coincides with the image of ψ in A.

The function $\hat{\xi}$ defined on U by

$$\hat{\xi}(z) = \hat{\phi}(z)+\hat{\psi}(z), \qquad z\in U,$$

has the property that $\hat{\xi}|A$ coincides with the image of ξ in A.

By hypothesis there exists a complex linear subspace L of C^N, of complex dimension $N-q$, such that the form $\mathscr{L}(\hat{\psi})(u)$ is positive definite if $u\in L$. Therefore the form $\mathscr{L}(\hat{\xi})(u)$ is positive definite if $u\in L$. This proves that $\mathscr{L}(\hat{\xi})$ has at least $N-q$ positive eigenvalues at every point of U; thus ξ is strongly q-plurisubharmonic on $X\cap Y$.

Moreover, it is easy to check that the sets:

$$B_c(X\cap Y) = \{x\in X\cap Y \mid \xi(x) < c\}$$

are relatively compact in $X\cap Y$. Q.E.D.

(2.5) REMARK. Proposition (2.4) can be easily generalized. In fact if X is p-complete and Y is q-complete then $X\cap Y$ is $(p+q)$-complete. Indeed, let L_1, L_2 be two complex linear subspaces of C^N, of dimensions $N-p$, $N-q$ where the forms $\mathscr{L}(\hat{\phi})$, $\mathscr{L}(\hat{\psi})$ are positive definite. Then the form $\mathscr{L}(\hat{\xi})$ is positive definite on the intersection $L_1\cap L_2$ which is a linear subspace of C^N of dimension $\geqq N-(p+q)$.

This result is, in general, of little interest, at least when it happens that

$$p+q \geqq \dim_C (X\cap Y).$$

Indeed, if $Z=X\cap Y$ is a manifold, and $n=\dim_C Z$, then the following stronger result holds: Z is n-complete [14].

However, if $p+q<\dim_C (X\cap Y)$, it can be shown that the result of the generalized proposition is the best possible, since $X\cap Y$ may fail to be s-complete for any $s<p+q$ (see §5, Example 5.4).

(2.6) PROPOSITION. *Let X be a 0-complete complex space. Let Y be a subspace of X representable in the form: $Y=\{f_1=\cdots=f_q=0\}$ with $f_1,\ldots,f_q\in\Gamma(X,\mathcal{O})$*(4)*. Then the space $X-Y$ is $(q-1)$-complete.*

Proof. Let $\phi(x)$ be a strongly 0-plurisubharmonic function on X such that the sets $B_c(X)=\{x\in X\mid \phi(x)<c\}$ are relatively compact in X. For $x\in X-Y$, put:

$$\psi(x)=\frac{1}{\left(\sum_{\alpha=1}^{q} f_\alpha(x)\bar{f}_\alpha(x)\right)^{q}}.$$

We shall prove that the function: $\lambda(x)=\phi(x)+\psi(x)$, $x\in X-Y$, is strongly $(q-1)$-plurisubharmonic at every point $x\in X-Y$.

Let $x_0\in X-Y$; we consider a regular embedding of an open neighborhood $U\subset X-Y$ of x_0 in a suitable open set of C^N. Let $z_1,\ldots,z_N$ be coordinates in C^N and let the embedding be defined by the equations:

$$\begin{aligned} z_1 &= h_1(x) \\ &\cdots\cdots\cdots \\ z_N &= h_N(x) \end{aligned} \qquad x\in U,$$

with $h_1,\ldots,h_N\in\Gamma(U,\mathcal{O})$.

The neighborhood U is also regularly embedded in an open set $W\subset C^{N+q}$ by the map $\tau\colon U\to C^{N+q}$ defined by the equations:

$$\begin{aligned} z_1 &= h_1(x), \\ &\cdots\cdots\cdots \\ z_N &= h_N(x), \\ z_{N+1} &= f_1(x), \\ &\cdots\cdots\cdots \\ z_{N+q} &= f_q(x), \end{aligned} \qquad x\in U.$$

Since the set

$$\{(z_1,\ldots,z_{N+q})\in\tau(U)\mid z_{N+1}=\cdots=z_{N+q}=0\}$$

is clearly empty, τ induces a regular embedding of U in

$$W_1 = W-\{z\in W\mid z_{N+1}=\cdots=z_{N+q}=0\}.$$

(4) $\Gamma(X,\mathcal{O})$ denotes as usual the group of sections of the sheaf $\mathcal{O}$ on X.

Let ϕ_1 be a strongly 0-plurisubharmonic function which extends ϕ to all of W, and let

$$\psi_1 = \frac{1}{\left(\sum_{j=1}^{q} z_{N+j}\bar{z}_{N+j}\right)^q}.$$

Then the function:

$$\lambda_1(x) = \phi_1(x)+\psi_1(x)$$

extends λ to all of W_1. We must prove that $\lambda_1(x)$ is strongly $(q-1)$-plurisubharmonic at every point $z_0 \in W_1$. An easy computation shows that:

$$\frac{\partial^2\psi_1(x)}{\partial z_{N+\alpha}\,\partial\bar{z}_{N+\alpha}} = \frac{q\left[(q+1)z_{N+\alpha}\bar{z}_{N+\alpha}-\sum_{j=1}^{q} z_{N+j}\bar{z}_{N+j}\right]}{\left(\sum_{j=1}^{q} z_{N+j}\bar{z}_{N+j}\right)^{q+2}}.$$

Let now $z^0=(z_1^0,\ldots,z_N^0,z_{N+1}^0,\ldots,z_{N+q}^0)\in W_1$ and let $\alpha_0\geqq 1$ be an index such that:

$$|z_{N+\alpha_0}^0| = \max_{j=1,\ldots,q} |z_{N+j}^0|.$$

Then the form $\mathscr{L}(\psi_1, z^0)$ is positive semidefinite on those vectors $(u_1,\ldots,u_{N+q})$ whose $q-1$ components $u_{N+1},\ldots,u_{N+\alpha_0-1},u_{N+\alpha_0+1},\ldots,u_{N+q}$ are zero.

Hence the form $\mathscr{L}(\phi_1+\psi_1, z^0)$ is positive definite on the same vectors. Since the given vectors form a vector subspace of codimension $q-1$, it follows that $\phi_1+\psi_1=\lambda_1$ is strongly $(q-1)$-plurisubharmonic at the point z^0.

To complete the proof of Proposition (2.6) it suffices to observe that, by construction, the sets:

$$B_c(X-Y) = \{x\in X-Y \mid \lambda(x) < c\}$$

are relatively compact in $X-Y$. Q.E.D.

(2.7) Remark. If Y is a complete intersection in X then the integer q which appears in Proposition (2.6) coincides with the complex codimension of Y, with respect to X, at every point of Y. The example (5.1) that we will study in §5 will show that, corresponding to every nonnegative integer j, there exist complex spaces, indeed even manifolds X, holomorphically complete, and analytic subsets Y of codimension q ($q\geqq 2$) at every point, such that the manifold $X-Y$ is not $(q+j)$-complete.

However, if X is a 0-complete manifold, under suitable hypotheses on Y it is still possible to establish a relation between the codimension and the degree of completeness (see Proposition (2.12) below).

We first consider the following situation: let X be a complex manifold, holomorphically complete, of complex dimension n, and let Y be a submanifold of X

of codimension q, *regularly embedded in X by means of a finite number* (but arbitrarily large) *of functions* $f_1, \dots, f_r \in \Gamma(X, \mathcal{O})$; i.e., assume

$$Y = \{x \in X \mid f_1(x) = \cdots = f_r(x) = 0\}$$

and assume that the Jacobian of $f_1, \dots, f_r$ with respect to the local coordinates of X has rank q at every point of Y. Then we have the following:

(2.8) LEMMA. *Let X and Y satisfy the above hypotheses. Then the manifold $X-Y$ is q-complete.*

Proof. Let ψ be a strongly 0-plurisubharmonic function on X which determines the 0-completeness of X. With no loss of generality we may assume $\psi \geqq 0$ on X, since otherwise it would be sufficient to consider the function $\psi + k$ (k suitable real constant) or e^{ψ} which also determine the 0-completeness of X.

For $m = 1, 2, \dots$ we put:

$$X_m = \{x \in X \mid \psi(x) < m\}$$

and

$$Y_m = X_m \cap Y.$$

Fix an arbitrary point $y \in Y \cap \bar{X}_m$. There exists a coordinate neighborhood U_0 of y in X such that on U_0 the first q coordinates $z_1, \dots, z_q$ are q of the r functions $f_1, \dots, f_r$, while the last $n-q$ coordinates $z_{q+1}, \dots, z_n$ form a system of local coordinates of Y, when restricted to $U_0 \cap Y$. Let y be the origin of this system of coordinates.

For simplicity of notation we shall assume that $z_1 = f_1, \dots, z_q = f_q$. There exists a neighborhood U_1 of y, possibly smaller than U_0, in which the remaining functions $f_{q+1}, \dots, f_r$ are linear combinations, with holomorphic coefficients, of the functions $z_1, \dots, z_q$:

$$f_{q+j} = z_1 a_{1,j}(z_1, \dots, z_n) + \cdots + z_q a_{q,j}(z_1, \dots, z_n) \qquad (j = 1, \dots, r-q).$$

We consider on $X-Y$ the function:

$$\phi = -\log \sum_{\alpha=1}^{r} f_\alpha \bar{f}_\alpha$$

and shall prove that there exists a neighborhood U_2 of y, possibly smaller than U_1, such that at every point $z^0 = (z_1^0, \dots, z_n^0) \in U_2 - Y$ the restriction of the Levi form $\mathscr{L}(\phi, z^0)$, to the space:

$$z_1 = z_1^0, \dots, z_q = z_q^0,$$

has all its $n-q$ eigenvalues (positive and negative) bounded in absolute value by a suitable constant $K(U_2)$.

Putting $\rho=\sum_{\alpha=1}^{r} f_\alpha \bar{f}_\alpha$ one has:

$$\frac{\partial^2\phi}{\partial z_{q+\alpha}\,\partial \bar{z}_{q+\beta}} = -\frac{1}{\rho}\sum_{j=1}^{r-q}\left[\left(\sum_{i=1}^{q} z_i \frac{\partial a_{i,j}}{\partial z_{q+\alpha}}\right)\left(\sum_{i=1}^{q} \bar{z}_i \frac{\partial \bar{a}_{i,j}}{\partial \bar{z}_{q+\beta}}\right)\right] + \frac{1}{\rho^2}\left[\sum_{j=1}^{r-q}\left(\sum_{i=1}^{q} z_i \frac{\partial a_{i,j}}{\partial z_{q+\alpha}}\right)\bar{f}_{q+j}\right]\left[\sum_{j=1}^{r-q}\left(\sum_{i=1}^{q} \bar{z}_i \frac{\partial \bar{a}_{i,j}}{\partial \bar{z}_{q+\beta}}\right)f_{q+j}\right].$$

Take as U_2 any relatively compact subset of U_1 containing y. Then there exists a constant M such that at every point of U_2 one has:

$$\left|\frac{\partial a_{i,j}}{\partial z_{q+\alpha}}\right| < M,$$

for every $i=1,\dots,q$, for every $j=1,\dots,r-q$, and for every $\alpha=1,\dots,r-q$. Moreover, there exists a constant N such that at every point of U_2,

$$|a_{i,j}| < N,$$

for every $i=1,\dots,q$ and for every $j=1,\dots,r-q$. It follows that:

$$\begin{aligned}\left|\frac{\partial^2\phi}{\partial z_{q+\alpha}\,\partial \bar{z}_{q+\beta}}\right| &\leqq \frac{r-q}{\rho}\left[\left(M\sum_{i=1}^{q}|z_i|\right)\left(M\sum_{i=1}^{q}|\bar{z}_i|\right)\right] \\ &\quad + \frac{1}{\rho^2}\left[\sum_{j=1}^{r-q}\left(M\sum_{i=1}^{q}|z_i|\right)|\bar{f}_{q+j}|\right]\left[\sum_{j=1}^{r-q}\left(M\sum_{i=1}^{q}|\bar{z}_i|\right)|f_{q+j}|\right] \\ &\leqq \frac{M^2}{\rho}(r-q)\left(\sum_{i=1}^{q}|z_i|\right)\left(\sum_{i=1}^{q}|\bar{z}_i|\right) \\ &\quad + \frac{M^2}{\rho^2}\left[N(r-q)\left(\sum_{i=1}^{q}|z_i|\right)\left(\sum_{i=1}^{q}|\bar{z}_i|\right)\right]\left[N(r-q)\left(\sum_{i=1}^{q}|z_i|\right)\left(\sum_{i=1}^{q}|\bar{z}_i|\right)\right].\end{aligned}$$

On the other hand, $\rho \geqq (\sum_{i=1}^{q}|z_i|)^2/q^2$ and therefore at every point $z^0 \in U_2 - Y$ one has:

$$\left|\frac{\partial^2\phi}{\partial z_{q+\alpha}\,\partial \bar{z}_{q+\beta}}\right| \leqq (r-q)q^2M^2 + (r-q)^2q^4M^2N^2.$$

This proves that all the coefficients of the Levi form $\mathscr{L}(\phi, z^0)$, restricted to the space of the last $n-q$ variables, are bounded on $U_2 - Y$. Hence also the eigenvalues of the same form are bounded on $U_2 - Y$. Therefore there exists, as we claimed, a constant $K(U_2)$ which, at every point $z^0 \in U_2 - Y$, majorizes the absolute values of these eigenvalues.

We now cover $Y \cap \bar{X}_m$ by a finite number of coordinate neighborhoods $U_2^{(i)}$, with the same properties as U_2 and let $K_m = \max_i K(U_2^{(i)})$.

Using the function ψ which determines the 0-completeness of X, we define $\mu(U_2^{(i)})$ = infimum of the values taken by the eigenvalues of $\mathscr{L}(\psi, z)$ computed at the points $z \in U_2^{(i)}$.

Let $\mu_m = \min_i \mu(U_2^{(i)})$; μ_m is a positive number.

Let T_m be a constant such that $T_m\mu_m > K_m$ and such that $\mathscr{L}(\phi)+T_m\mathscr{L}(\psi)$ is positive definite on the compact set $\bar{X}_m - \bigcup_i U_2^{(i)}$.

Let δ_m = infimum of the values of ϕ on $X_m - Y_m$. Clearly $\delta_m > -\infty$.

Corresponding to each nonnegative integer m we choose now a new constant k_m, such that:

(i) $k_m \geqq T_{m+1}$,

(ii) $k_m > m+1-\delta_{m+2}$,

(iii) $k_{m+1} \geqq k_m$.

We may now use a technique of Andreotti-Narasimhan [2] in order to replace the function ψ by a function $\tilde{\psi}$ such that the function $\phi+\tilde{\psi}$ determines the q-completeness of $X-Y$. To do this we consider a differentiable function $h(t)$, of the real variable t, such that:

$$h(t) = 0 \quad \text{if } t \leqq -1,$$

$$h(t) > k_m \quad \text{if } m \leqq t \leqq m+1,$$

$$h'(t) > 0 \quad \text{if } t > -1.$$

Next we define a function $\chi: \boldsymbol{R} \to \boldsymbol{R}$ by:

$$\chi(\lambda) = \int_{-\infty}^{\lambda} h(t)\, dt$$

and let $\tilde{\psi}(x) = \chi(\psi(x))$.

We shall show that the function $\phi_0 = \phi+\tilde{\psi}$ has the required property. First of all, we shall show that ϕ_0 is strongly q-plurisubharmonic on $X-Y$. To do this, observe that:

$$\mathscr{L}(\phi_0) = \mathscr{L}(\phi)+\mathscr{L}(\tilde{\psi}) \geqq \mathscr{L}(\phi)+k_{m-1}\mathscr{L}(\psi)$$

on $(X_m - X_{m-1}) - Y$.

Hence from $k_{m-1} \geqq T_m$ it follows that:

$$\mathscr{L}(\phi_0) \geqq \mathscr{L}(\phi)+T_m\mathscr{L}(\psi).$$

Therefore $\mathscr{L}(\phi_0)$ has at least $n-q$ positive eigenvalues at every point of $(X_m - X_{m-1}) - Y$. Applying the same reasoning for every m, it follows that ϕ_0 is strongly q-plurisubharmonic at every point of $X-Y$.

In order to prove that $X-Y$ is q-complete with respect to ϕ_0 it remains to show that the sets:

$$B_c^0 = \{x \in X-Y \mid \phi_0(x) < c\}$$

are relatively compact in $X-Y$ for all $c \in \boldsymbol{R}$. Actually it will suffice to prove this statement for integral values of c. Assume then that c is an integer. We shall prove that $\bar{B}_c^0$ (closure of B_c^0 in X) has empty intersection with Y and that $\bar{B}_c^0$ is contained in the compact set $\bar{X}_c$. Let y be a point of Y. There exists an open neighborhood $U \subset X$ of y such that $\phi(x) > c - \tilde{\psi}(x)$ for all $x \in U - Y$ (we may choose U to be relatively compact in X; then $\tilde{\psi}(x)$ is bounded from below on U). Hence no point of U belongs to B_c^0, so $y \notin \bar{B}_c^0$, and this proves precisely that $\bar{B}_c^0 \cap Y = \varnothing$.

Finally, $\bar{B}_c^0$ is contained in $\bar{X}_c$, for if $x \notin X_c - Y$, then there exists a nonnegative integer s such that $x \in (X_{c+s+1} - X_{c+s}) - Y$.

Then one has:

$$\psi(x) \geqq c+s$$

whence

$$\tilde{\psi}(x) \geqq k_{c+s-1} > c+s-\delta_{c+s+1} \geqq c+s-\phi(x)$$

so that

$$\phi_0(x) = \phi(x)+\tilde{\psi}(x) > c+s \geqq c$$

and this implies that $x \notin \bar{B}_c^0$.

Thus $X - Y$ is q-complete with respect to ϕ_0. Q.E.D.

(2.9) LEMMA. *Let Y be a submanifold of codimension q, regularly embedded in X*; $\dim_C X = n$. *Then there exists a regular embedding of Y in X by a finite number of functions of* $\Gamma(X, \mathcal{O})$.

Proof. The submanifold Y, as a set, can be represented as the zeroes of a finite number of global holomorphic functions $f_1, \ldots, f_k$. (Actually k could be taken equal to n [6].)

Let $\mathcal{I}$ be the sheaf of ideals of Y in X. Let F be the Fréchet space direct sum of q copies of $\Gamma(X, \mathcal{I})$. Then an element of F is a q-tuple of global holomorphic functions on X, vanishing on Y.

Let $y \in Y$. Let F_y be the subset of F of those q-tuples $(f_1, \ldots, f_q) \in F$ whose Jacobian with respect to a system of local coordinates $z_1, \ldots, z_n$ of X near y, has rank q at y.

For the proof we need the fact that the set $F - F_y$ is of the first category in F[5] as we shall prove in the next lemma.

Let N_0 be a countable subset everywhere dense in Y. Then, $\bigcup_{y \in N_0} (F - F_y) = F - \bigcap_{y \in N_0} F_y$ is a set of the first category in F, but F is not of the first category in F, hence $\bigcap_{y \in N_0} F_y$ is not empty. Therefore there exists a q-tuple

$$f_{0,1}, \ldots, f_{0,q} \in \Gamma(X, \mathcal{I})$$

such that the Jacobian of $f_{0,1}, \ldots, f_{0,q}$ with respect to the chosen system of local coordinates of X near y, has rank q at every point $y_0 \in N_0$.

(5) A set N is said to be of the first category if $N = \bigcup_{\alpha=1,2,\ldots}^{\infty} N_\alpha$ with $(\bar{N}_\alpha)^0 = \varnothing$.

Now Y, as a set, may also be represented as:

$$Y = \{x \in X \mid f_1 = \cdots = f_k = f_{0,1} = \cdots = f_{0,q} = 0\};$$

moreover the embedding defined by these functions is regular outside a closed analytic subset $Y_0 \subset Y$ which is nowhere dense in Y. Hence $\dim_C Y_0 < \dim_C Y$.

Now let N_1 be a countable subset everywhere dense in Y_0. By the same argument, there exist functions $f_{1,1}, \ldots, f_{1,q} \in \Gamma(X, \mathscr{I})$, $(f_{1,1}, \ldots, f_{1,q}) \in \bigcap_{y \in N_1} F_y$. Furthermore:

$$Y = \{x \in X \mid f_1 = \cdots = f_k = f_{0,1} = \cdots = f_{0,q} = f_{1,1} = \cdots = f_{1,q} = 0\}$$

and the embedding defined by these functions is regular outside a closed analytic subset $Y_1 \subset Y_0$ with $\dim_C Y_1 < \dim_C Y_0$.

After at most $n-q+1$ steps one has:

$$\text{(2.10)} \quad Y = \{x \in X \mid f_1 = \cdots = f_k = f_{0,1} = \cdots = f_{0,q} = \cdots = f_{n-q,1} = \cdots = f_{n-q,q} = 0\}$$

and these functions define a regular embedding of Y in X. Q.E.D.
It remains to prove the following lemma.

(2.11) LEMMA. *$F - F_y$ is a set of the first category in F.*

Proof. It suffices to prove that F_y is open and everywhere dense in F.

First of all we shall prove that F_y is not empty. In fact, since the embedding of Y in X is regular, an open set $U \subset X$, $y \in U$, and a q-tuple $(\phi_1, \ldots, \phi_q) \in \Gamma(U, \mathscr{I})$ exist such that:

$$\operatorname{rank} \left(\frac{\partial(\phi_1, \ldots, \phi_q)}{\partial(z_1, \ldots, z_n)} \right)_y = q.$$

Now from Theorem A (Cartan-Serre) it follows that there exist functions $h_1, \ldots, h_d \in \Gamma(X, \mathscr{I})$ such that:

$$\phi_\alpha = \sum_{\beta=1}^{d} a_{\alpha\beta} h_\beta \qquad (\alpha = 1, \ldots, q)$$

with $a_{\alpha\beta} \in \Gamma(V, \mathscr{O})$ where V is an open set containing y, $V \subset U$.

Moreover, putting $A_y = (a_{\alpha\beta}(y))$, one has:

$$\left(\frac{\partial(\phi_1, \ldots, \phi_q)}{\partial(z_1, \ldots, z_n)} \right)_y = A_y \left(\frac{\partial(h_1, \ldots, h_d)}{\partial(z_1, \ldots, z_n)} \right)_y.$$

Hence there are functions $h_{i_1}, \ldots, h_{i_q}$, among the $h_1, \ldots, h_d$ such that:

$$\operatorname{rank} \left(\frac{\partial(h_{i_1}, \ldots, h_{i_q})}{\partial(z_1, \ldots, z_n)} \right)_y = q,$$

which shows that F_y is not empty.

Next, F_y is clearly open in F.

Finally, we prove that F_y is everywhere dense in F. With no loss of generality we may assume that:

$$\operatorname{rank}\left(\frac{\partial(h_1,\ldots,h_q)}{\partial(z_1,\ldots,z_q)}\right)_y = q.$$

Let $(g_1,\ldots,g_q)\in F$. We consider $(g_1+\lambda h_1,\ldots,g_q+\lambda h_q)\in F$ where λ is any complex number, and prove that for sufficiently small $\lambda\neq 0$ the rank of

$$J_y(\lambda) = \left(\frac{\partial(g_1+\lambda h_1,\ldots,g_q+\lambda h_q)}{\partial(z_1,\ldots,z_q)}\right)_y$$

is equal to q. This will prove density.

In fact $\det J_y(\lambda)$ is a polynomial of degree q in λ which does not vanish identically, since the coefficient of λ^q is

$$\det\left(\frac{\partial(h_1,\ldots,h_q)}{\partial(z_1,\ldots,z_q)}\right)_y,$$

which is different from zero.

Therefore all the values of λ which are not roots of the equation $\det J_y(\lambda)=0$ give elements of F_y. Q.E.D.

Lemma (2.9) proves that the hypotheses of Lemma (2.8) are always satisfied for any regular embedding of Y in X. Hence:

(2.12) Proposition. *Let X be a holomorphically complete manifold. Let Y be a regularly embedded submanifold of X of codimension q. Then the manifold $X-Y$ is q-complete.*

As a consequence of Lemma (2.9) we have the following proposition which we wish to note because of its intrinsic interest.

(2.13) Proposition. *Let X be a holomorphically complete manifold. Let Y be a regularly embedded submanifold of X and let $\mathscr{I}$ be the sheaf of ideals of Y in X. Then $\Gamma(X,\mathscr{I})$ is a finitely generated $\Gamma(X,\mathcal{O})$-module.*

Proof. We denote by $s_1,\ldots,s_p$ the functions defining the regular embedding of Y in X in (2.10). The sequence:

$$0\longrightarrow \operatorname{Ker}\alpha\longrightarrow \mathcal{O}^p\xrightarrow{\alpha}\mathscr{I}\longrightarrow 0$$

where $\alpha(v_i)=s_i$, v_i being the ith unit vector in $\mathcal{O}^p$, is exact. We thus have the exact sequence:

$$\cdots\longrightarrow\Gamma(X,\mathcal{O}^p)\xrightarrow{\alpha_*}\Gamma(X,\mathscr{I})\longrightarrow H^1(X,\operatorname{Ker}\alpha)\longrightarrow\cdots.$$

Since $\operatorname{Ker}\alpha$ is a coherent analytic sheaf, it follows from Theorem B (Cartan-Serre) that $H^1(X,\operatorname{Ker}\alpha)=0$. Hence the homomorphism α_* is surjective. Q.E.D.

3. **Cohomologically q-complete spaces.** An important result in the theory of q-complete spaces is expressed by the following theorem.

(3.1) THEOREM [1, p. 250]. *Let X be a q-complete complex space, let $\mathscr{F}$ be any coherent analytic sheaf on X. Then*

$$H^i(X, \mathscr{F}) = 0 \qquad \text{for } i \geqq q+1. \tag{3.2}$$

(3.3) DEFINITION. *A complex space X will be called cohomologically q-complete if for every coherent analytic sheaf $\mathscr{F}$ on X the property* (3.2) *holds.*

A cohomologically q-complete space is clearly also cohomologically $(q+1)$-complete.

(3.4) THEOREM [10]. *A complex space of complex dimension n which is countable at infinity is cohomologically n-complete.*

Clearly a q-complete space is cohomologically q-complete by Theorem (3.1). J.-P. Serre has shown that the converse is true when $q=0$:

(3.5) THEOREM. *A complex space X, which is cohomologically* 0*-complete is necessarily* 0*-complete.*

The proof is divided into two parts. First, one proves that every cohomologically 0-complete space is holomorphically complete. Indeed, when X is a manifold, this result can be found in [5, p. 53]; and a similar proof can be given in the general case. The theorem now follows from Theorem (2.1).

(3.6) PROPOSITION. *Let X be a complex space. Let $X=\bigcup_{m=1,2,\dots} B_m$, where $\{B_m\}$ is an increasing sequence of open subsets of X. If each B_m is cohomologically q-complete, then X is cohomologically $(q+1)$-complete.*

For the proof of this proposition, see [4]. Some similar results are also proved in [1] (see in particular §20).

(3.7) PROPOSITION. *Let X be a complex space; let $X=X_1 \cup X_2$, with X_1, X_2 open subspaces of X such that for some fixed integer $q>0$:*

(i) *X_1, X_2 are cohomologically q-complete,*

(ii) *$X_1 \cap X_2$ is cohomologically $(q-1)$-complete. Then X is cohomologically q-complete.*

Proof. Let $\mathscr{F}$ be a coherent analytic sheaf on X. We shall denote by $\mathscr{F}$ also the restrictions $\mathscr{F}|X_1$, $\mathscr{F}|X_2$, $\mathscr{F} \mid X_1 \cap X_2$. They too are coherent analytic sheaves. Then the following Mayer-Vietoris sequence holds:

$$\cdots \to H^{i-1}(X_1 \cap X_2, \mathscr{F}) \to H^i(X, \mathscr{F}) \to H^i(X_1, \mathscr{F}) \oplus H^i(X_2, \mathscr{F}) \to \cdots.$$

Hence from hypotheses (i) and (ii) it follows:

$$0 \to H^i(X, \mathscr{F}) \to 0 \qquad \text{if } i \geqq q+1. \qquad \text{Q.E.D.}$$

The latter proposition has the following generalization:

(3.8) PROPOSITION. *Let X be a complex space. Let $X=\bigcup_{j=1}^{h} X_j$, with X_j open subspaces such that*:

(i) *X_j is cohomologically q-complete* ($j=1,\dots,h$),

(ii) $(X_1\cup\cdots\cup X_j)\cap X_{j+1}$ *is cohomologically* $(q-1)$*-complete* ($j=1,\dots,h-1$).
Then X is cohomologically q-complete.

Proof. Applying Proposition (3.7) to the triple $(X_1\cup X_2, X_1, X_2)$ it follows that $X_1\cup X_2$ is cohomologically q-complete. Therefore Proposition (3.7) may be applied to triple $\left(\bigcup_{j=1}^{3} X_j, \bigcup_{j=1}^{2} X_j, X_3\right)$. After $h-1$ steps the proof is complete. Q.E.D.

4. **Some properties of cohomologically q-complete manifolds.** Theorem 2 of [12] can be extended to cohomologically q-complete manifolds. Namely, one has:

(4.1) THEOREM. *Let X be a cohomologically q-complete manifold. Then every complex-valued differential form, d-closed, of degree $n+q$ on X is cohomologous to a differential form of type (n,q), $\bar\partial$-closed (and therefore d-closed).*

Putting:

$$H^{p,q}(X,C)=\{\xi\in H^{p+q}(X,C)\mid \xi \text{ can be represented in the de Rham isomorphism by a form of type } (p,q)\}, \tag{4.2}$$

Theorem (4.1) shows that if X is cohomologically q-complete, then:

$$H^{n+q}(X,C)\cong H^{n,q}(X,C).$$

The proof is the same as in [12] since only cohomological properties are used.

(4.3) REMARK. Let X_0 be a complex manifold of complex dimension n, and let $X\subset X_0$ be an open submanifold such that $\bar X=X_0$.

The problem of finding a representation of a holomorphic function on X_0 by an integral over an $(n+q)$-dimensional cycle of X, $(0\leqq q\leqq n-1)$, is connected with the study of the cohomology group $H^{n+q}(X,C)$. On the other hand, if ϕ^{n+q} represents, via de Rham's theorem, an element of $H^{n+q}(X,C)$, the condition $d\phi^{n+q}=0$ does not necessarily imply that $d(f\phi^{n+q})=0$ for every $f\in\Gamma(X_0,\mathcal{O})$.

But if ϕ^{n+q} is of type (n,q) then certainly $d(f\phi^{n+q})=0$ for every $f\in\Gamma(X_0,\mathcal{O})$. Therefore the problem is ultimately reduced to the study of the group $H^{n,q}(X,C)$.

These two aspects of the problem are related under the hypotheses of Theorem (4.1), for if X is q-complete, then the cohomology class of ϕ^{n+q} always contains a differential form of type (n,q).

With respect to the homology of cohomologically q-complete manifolds one has the following result, similar to the corollary of [11, p. 304]:

(4.4) THEOREM. *Let X be a cohomologically q-complete manifold of complex dimension n. Then*:

$$H_{n+i}(X,C)=0 \qquad \textit{for } i\geqq q+1.$$

Proof. One has:

$$H^i(X, \Omega^n) = 0 \qquad \text{if } i \geqq q+1.$$

This means, by the Dolbeault isomorphism, that every differential form $\phi^{n,i}$ of type (n, i), $\bar{\partial}$-closed, is of the form:

$$\phi^{n,i} = \bar{\partial}\psi^{n,i-1}.$$

On the other hand, if $\phi^{n,i}$ is any d-closed differential form of type (n, i), $\phi^{n,i}$ is also $\bar{\partial}$-closed. Furthermore, $\phi^{n,i} = \bar{\partial}\psi^{n,i-1}$ implies that:

$$\phi^{n,i} = d\psi^{n,i-1}.$$

This equality shows that the groups $H^{n,i}(X, C)$, defined in (4.2), are zero. Hence from the isomorphism proved in Theorem (4.1) it follows that:

$$H^{n+i}(X, C) = 0 \qquad \text{for } i \geqq q+1.$$

Therefore also $\operatorname{Hom}(H_{n+i}(X, C), C) = H^{n+i}(X, C) = 0$. This implies that $H_{n+i}(X, C) = 0$ for $i \geqq q+1$. Q.E.D.

5. **Examples.**

(5.1) We consider in C^{2n} $(n \geqq 2)$, the two linear complex subspaces of dimension n: $C^n(z_1, \ldots, z_n)$ given by the equations $z_{n+1} = \cdots = z_{2n} = 0$, $C^n(z_{n+1}, \ldots, z_{2n})$ given by the equations $z_1 = \cdots = z_n = 0$. We put:

$$X_1 = C^{2n} - C^n(z_1, \ldots, z_n), \; X_2 = C^{2n} - C^n(z_{n+1}, \ldots, z_{2n}),$$
$$X = X_1 \cup X_2, \; Y = C^n(z_1, \ldots, z_n) \cup C^n(z_{n+1}, \ldots, z_{2n}).$$

Then one has:

$$X = C^{2n} - \{0\}, \qquad X_1 \cap X_2 = C^{2n} - Y.$$

Let $\mathscr{F}$ be a coherent analytic sheaf on X. Since X_1 and X_2 are $(n-1)$-complete (see Proposition (2.6)), it follows from the Mayer-Vietoris sequence that:

$$0 \to H^i(X_1 \cap X_2, \mathscr{F}) \to H^{i+1}(X, \mathscr{F}) \to 0 \qquad \text{for } i \geqq n.$$

Thus we have the isomorphisms:

(5.2) $$H^i(X_1 \cap X_2, \mathscr{F}) \cong H^{i+1}(X, \mathscr{F}) \qquad (i \geqq n).$$

Now take $\mathscr{F} = \Omega^{2n}$, the sheaf of germs of holomorphic forms of maximum degree over X. We shall prove that $H^{2n-1}(X, \Omega^{2n}) \neq 0$.

Indeed, X is $(2n-1)$-complete; hence from Theorem (4.1) it follows that:

$$H^{2n,2n-1}(X, C) \cong H^{4n-1}(X, C).$$

Since $X = C^{2n} - \{0\}$ is contractible to the sphere S^{4n-1}, then

$$H^{4n-1}(X, C) \cong H^{4n-1}(S^{4n-1}, C) \cong C \neq 0.$$

Now $H^{2n,2n-1}(X, C) \cong C$ is in a natural way, a quotient of $H^{2n-1}(X, \Omega^{2n})$ and thus it follows that $H^{2n-1}(X, \Omega^{2n}) \neq 0$.

Hence from the isomorphism (5.2), for $i=2n-2$, it follows that:

$$H^{2n-2}(X_1 \cap X_2, \Omega^{2n}) \neq 0.$$

This proves that *for any nonnegative integer $j<n-2$, the space obtained by removing the subset Y of complex codimension n from C^{2n} is not $(n+j)$-complete.* This example shows that the hypothesis that Y be regularly embedded in X can not be removed in Proposition (2.12).

With a more subtle argument, it can be proved (see for instance [3]), that $H^{2n-1}(X, \Omega^{2n})$ is a complex vector space of infinite dimension; hence from the isomorphism (5.2), it follows that $\dim_C H^{2n-2}(X_1 \cap X_2, \Omega^{2n}) = +\infty$. This proves that the space $C^{2n}-Y$ not only is not q-complete, for $q<2n-2$, but it is not even *strongly q-pseudoconvex*(6) for $q<2n-2$, as one could have conjectured.

(5.3) REMARK. In example (5.1), Y has codimension $q \geqq 2$ at each of its points. It is not possible to give similar examples with $q=1$. In fact, if X is a 0-complete manifold and one removes from X any analytic subset Y of codimension 1 at each of its points, then the manifold $X-Y$ is 0-complete.

(5.4) Example (5.1) may be modified in the following way: consider in C^{n+m} $(n, m \geqq 2)$, the two linear complex subspaces $C^n(z_1, \ldots, z_n)$ represented by the equations $z_{n+1} = \cdots = z_{n+m} = 0$ and $C^m(z_{n+1}, \ldots, z_{n+m})$, by the equations $z_1 = \cdots = z_n = 0$. Put:

$$X_1 = C^{n+m} - C^n(z_1, \ldots, z_n), \quad X_2 = C^{n+m} - C^m(z_{n+1}, \ldots, z_{n+m}),$$

$$X = X_1 \cup X_2, \quad Y = C^n(z_1, \ldots, z_n) \cup C^m(z_{n+1}, \ldots, z_{n+m}).$$

By the same argument as in example (5.1) one proves that, for any integer $j<n+m-2$, the space $C^{n+m}-Y$ is not j-complete.

Noting that $C^{n+m}-Y = X_1 \cap X_2$ and that X_1, X_2 are $(m-1)$- and $(n-1)$-complete, this example proves that the result of (2.5) can not be improved.

(5.5) In C^n $(n>1)$, consider the two following domains:

$$X_1 = \{(z_1, \ldots, z_n) \mid 1/2 < |z_1| < 1,\ |z_2|^2 + \cdots + |z_n|^2 < 1\},$$

$$X_2 = \{(z_1, \ldots, z_n) \mid |z_1| < 1,\ |z_2|^2 + \cdots + |z_n|^2 < 1/2\}.$$

The *domain $X = X_1 \cup X_2$ is cohomologically* 1-*complete.* In fact, consider the domain $X_1 \cap X_2$ defined by:

$$X_1 \cap X_2 = \{(z_1, \ldots, z_n) \mid 1/2 < |z_1| < 1,\ |z_2|^2 + \cdots + |z_n|^2 < 1/2\}.$$

Then it is clear that $X_1, X_2, X_1 \cap X_2$ are holomorphically complete and therefore 0-complete. Hence from Proposition (3.7), it follows that X is cohomologically 1-complete.

(6) For the definition of a strongly q-pseudoconvex space see, for instance, [1].

(5.6) REMARK. The domain X of the preceding example occurs in some proofs of Hartog's theorem. This theorem shows, among other things, that the domain X is not holomorphically complete, since every holomorphic function on X can be continued outside X.

(5.7) Finally consider in C^n the linear complex hyperplanes C_j^{n-1} defined by the equation $z_j=0$. For a fixed integer q ($q=0, 1, \ldots, n-1$), consider the following domains:

$$X_1 = C^n - C_1^{n-1},$$
$$X_2 = C^n - C_2^{n-1},$$
$$\cdots\cdots$$
$$X_q = C^n - C_q^{n-1},$$
$$X_{q+1} = C^n - \bigcup_{j=q+1}^{n} C_j^{n-1}.$$

We shall show that *the domain* $X=\bigcup_{j=1}^{q+1} X_j$ *is q-complete*. In fact one has $X=C^n-A$, where:

$$A = \{z_1 = 0, \ldots, z_q = 0, z_{q+1}\cdot \cdots \cdot z_n = 0\}.$$

Since A is represented as locus of zeroes of $q+1$ global holomorphic functions, it follows from Proposition (2.6) that X is q-complete.

(5.8) REMARK. The domain X in example (5.7) is q-complete; moreover, $\dim_C H^{n+q}(X, C)=1$. According to Remark (4.3), this guarantees the existence of differential forms $\phi^{n,q}$ on X such that every function f, holomorphic on C^n, can be evaluated at the origin by means of the integral of $f\phi^{n,q}$ over an $(n+q)$-dimensional cycle of X.

The explicit determination of the forms $\phi^{n,q}$, together with the study of the general topological aspects of the cycles to be used in the integration was carried out by E. Martinelli [8], for a domain Y which is the intersection of a finite number of domains of the same type as the domain X considered above. For such a domain Y, one has $\dim_C H^{n+q}(Y, C)>1$; the resulting integral formulae are symmetric in the n variables involved [15].

A similar study was subsequently carried out in [13] for the domain X itself.

BIBLIOGRAPHY

1. A. Andreotti and H. Grauert, *Théorèmes de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France **90** (1962), 193–259.

2. A. Andreotti and R. Narasimhan, *Oka's Heftungs lemma and the Levi problem for complex spaces*, Trans. Amer. Math. Soc. **111** (1964), 345–366.

3. A. Andreotti and F. Norguet, *Problème de Levi pour les classes de cohomologie*, C. R. Acad. Sci. Paris **258** (1964), 778–781.

4. A. Andreotti and E. Vesentini, *Les théorèmes fondamentaux de la théorie des espaces holomorphiquement complets*, Sém. tenu a Paris, 1962.

5. H. Cartan, *Variétés analytiques complexes et cohomologie*, Colloque C.B.R.M. (fonctions analytiques de plusieurs variables (1953)); pp. 41–55.

6. O. Forster and K. J. Ramspott, *Über die Darstellung analytischer Mengen*, S.-B. Bayer. Akad. Wiss. (1963), 88–99.

7. H. Grauert, *On Levi's problem and the embedding of real analytic manifolds*, Ann. of Math. **68** (1958), 460–472.

8. E. Martinelli, *Sulle estensioni della formula integrale di Cauchy alle funzioni analitiche di più variabili complesse*, Ann. Mat. Pura Appl. **34** (1953), 277–347.

9. R. Narasimhan, *The Levi problem for complex spaces*, Math. Ann. **142** (1961), 355–365.

10. H. J. Reiffen, *Prolongement de Riemann concernant les classes de cohomologie a supports compacts*, C. R. Acad. Sci. Paris **259** (1964), 2333–2335.

11. G. Sorani, *Omologia degli spazi q-pseudoconvessi*, Ann. Scuola Norm. Sup. Pisa **16** (1962), 299–304.

12. ———, *Une propriété topologique des q-paires de Runge*, C. R. Acad. Sci. Paris **256** (1963), 3944–3945.

13. ———, *Integral representations of holomorphic functions*, Amer. J. Math, (1966).

14. V. Villani, *Su alcune proprietà coomologiche dei fasci coerenti su uno spazio complesso*, Rend. Sem. Mat. Univ. Padova **35** (1964), 47–55.

15. ———, *Alcuni problemi di natura coomologica sulle varietà complesse*, Mimeographed notes, Pisa, 1964.

Northeastern University,
Boston, Massachusetts
Harvard University,
Cambridge, Massachusetts