DISCONJUGACY OF COMPLEX DIFFERENTIAL SYSTEMS

BY

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1. **Introduction.** In his paper *On an inequality of Lyapunov* [7] Nehari considered a linear differential system

$$y_i'(x) = \sum_{k=1}^{n} a_{ik}(x)y_k(x), \qquad i = 1, \ldots, n, \tag{1}$$

where the real (or complex) functions $a_{ik}(x)$ are continuous on the interval $a \leqq x \leqq b$. From the coefficients $a_{ik}(x)$ Nehari built a matrix A with nonnegative constants as elements. Denoting the maximal characteristic value of A by $\lambda(A)$, he showed that $\lambda(A) \leqq 1$ ensures the disconjugacy of (1) in the sense defined below [7, Theorem III]. His proof, based on a variational property of $\lambda(A)$, holds also for the complex system

$$w_i'(z) = \sum_{k=1}^{n} a_{ik}(z)w_k(z), \qquad i = 1, \ldots, n, \tag{2}$$

where the $a_{ik}(z)$ are analytic functions in a bounded convex domain (Theorem 1 below). By means of Gronwall's inequality this result can be strengthened (Theorem 2).

The disconjugacy of system (2) can be interpreted in different ways; it is equivalent to a simple property of the determinant of any fundamental system of solutions of (2) (Theorem 3). This interpretation leads to applications which may be of independent interest. One application is a condition which implies that two analytic functions map a given domain onto disjoint domains (Theorem 4); the other one implies that $f(z)$ satisfies $\prod_{i=1}^{n} f(z_i) \neq 1$ for all sets $(z_1, \ldots, z_n)$ of a given domain (Theorem 5).

2. **Preliminaries and Theorem 1.** We shall write the system (2) in matrix notation as

$$w'(z) = A(z)w(z). \tag{2}$$

Here $A(z)$ is the matrix $(a_{ik}(z))_1^n$ and $w(z)$ is the column vector $[w_1(z), \ldots, w_n(z)]$. We consider only the case where the n^2 analytic functions $a_{ik}(z)$ are regular in a bounded simply connected domain D. We now define: *the differential system* (2) *is called disconjugate in D if, for every choice of n (not necessarily distinct) points*

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$z_1, \ldots, z_n$ *in* D, *the only solution of* (2) *which satisfies* $w_i(z_i)=0$, $i=1, \ldots, n$ *is the trivial one* $w(z)\equiv 0$ (*i.e.,* $w_i(z)=0$, $i=1, \ldots, n$ *and all* $z \in D$). Note that for $n=1$ every "system" is disconjugate.

A $(n \times n)$ matrix with constant elements $A=(\alpha_{ik})_1^n$ is nonnegative, $A \geqq 0$, if $\alpha_{ik} \geqq 0$, $i, k=1, \ldots, n$. We denote the maximal characteristic value of such a nonnegative matrix A by $\lambda(A)$ and we shall use the following variational property of $\lambda(A)$:

Lemma 1. *If A is a $(n \times n)$ nonnegative matrix and x a nonnegative nonvanishing n-dimensional vector* (*i.e.,* $A \geqq 0$, $x \geqq 0$, $x \neq 0$) *then* $Ax \geqq \lambda x$, $\lambda \geqq 0$ *implies* $\lambda(A) \geqq \lambda$.

This property of the Perron-Frobenius maximal characteristic value $\lambda(A)$ was first proved by Collatz [2] and Wielandt [10]. For recent independent proofs see Nehari [7] and Ostrowski [9]. (Lemma 1 is a slight modification of the lemma in [9, p. 82]; the corresponding modification of the elegant proof given there is obvious.)

After these preparations we now state

Theorem 1. *Let the analytic functions* $a_{ik}(z)$, $i, k=1, \ldots, n$, $n \geqq 2$, *be regular and bounded in the bounded convex domain* D. *Set*

$$\alpha_{ik} = \sup_{z \in D} |a_{ik}(z)|, \qquad i, k = 1, \ldots, n, \tag{3}$$

and let $\lambda(A)$ *be the maximal characteristic value of the matrix* $A=(\alpha_{ik})_1^n$. *Let* d *be the diameter of* D. *If*

$$d\lambda(A) < 1, \tag{4}$$

then the differential system (2) *is disconjugate in* D.

Proof. (Cf. [7].) Assume, to the contrary, that there exist points $\alpha_i \in D$, $i=1, \ldots, n$ and a nontrivial solution $w(z)=[w_1(z), \ldots, w_n(z)]$ of (2) such that $w_i(\alpha_i)=0$, $i=1, \ldots, n$. The n points α_i cannot all coincide, as this would imply $w(z)\equiv 0$. Their convex hull $H=H(\alpha_1, \ldots, \alpha_n)$ is thus either a segment or a closed convex polygon belonging to D. Set now

$$m_i = \max_{z \in H} |w_i(z)|, \qquad i = 1, \ldots, n. \tag{5}$$

The vector $m=[m_1, \ldots, m_n]$ is nonnegative and nonvanishing. If $m_i>0$, let β_i be a point in H (necessarily on the boundary of H if H is a polygon) satisfying $|w_i(\beta_i)|=m_i$. Integrating the ith component of (2) along the segment from α_i to β_i, we obtain

$$\begin{aligned} 0 < m_i = |w_i(\beta_i)-w_i(\alpha_i)| &= \left| \int_{\alpha_i}^{\beta_i} w_i'(z)\, dz \right| \leqq \int_{\alpha_i}^{\beta_i} |w_i'(z)\, dz| \\ &\leqq \sum_{k=1}^{n} \int_{\alpha_i}^{\beta_i} |a_{ik}(z) w_k(z)\, dz| < d \sum_{k=1}^{n} \alpha_{ik} m_k. \end{aligned} \tag{6}$$

(We used (3), (5), and $|\beta_i-\alpha_i|<d$ for the last inequality sign.) Hence,

$$\sum_{k=1}^{n} \alpha_{ik} m_k \geqq \frac{1}{d} m_i,$$

which holds for all i (i.e., also if $m_i=0$). We thus obtained the vector inequality

$$Am \geqq \frac{1}{d} m, \qquad m \geqq 0, \qquad m \neq 0.$$

By Lemma 1 this implies $\lambda(A) \geqq 1/d$, which contradicts assumption (4). This completes the proof of the theorem.

We add some remarks.

(i) If we assume that the functions $a_{ik}(z)$ are regular in the closure $\bar{D}$ of the bounded convex domain D, then (4) implies the disconjugacy of (2) in $\bar{D}$. The proof is now simpler; (5) has to be replaced by

$$(5') \qquad m_i = \max_{z \in \bar{D}} |w_i(z)|, \qquad i = 1, \ldots, n,$$

and there is no need to consider closed subdomains of D.

(ii) The convexity of D was used only to obtain an upper bound, d, for the lengths of all paths which had to be considered. Let D be a domain bounded by a, not necessarily convex, rectifiable Jordan curve C of length L. It follows from the isoperimetric inequality that any two points in $\bar{D}$ $(=D \cup C)$ can be joined by a path in $\bar{D}$ of length smaller than $(2+\pi)L/2\pi$. Let d be the smallest number such that any pair of points in $\bar{D}$ can be connected by a path in $\bar{D}$ of length not larger than d. If the $a_{ik}(z)$ are regular in $\bar{D}$ and if (4) holds, then (2) is disconjugate in $\bar{D}$.

(iii) Theorem III of [7] is stronger than the restriction of Theorem 1 to an interval. Indeed, the elements of the nonnegative matrix used by Nehari for the system (1) are $\int_a^b |a_{ik}(x)|\,dx$ and are thus smaller than $(b-a)\max|a_{ik}(x)|$; the maximal characteristic value of his matrix is therefore smaller than $d\lambda(A)$ of Theorem 1 $(d=b-a)$. A similar idea can be used in the complex case; however the result in this case, which we are now going to state, is not necessarily stronger than Theorem 1. Let D be bounded by a rectifiable Jordan curve C. For any analytic function $f(z)$ regular in $\bar{D}$ and any pair of points $\alpha, \beta \in \bar{D}$, the inequality

$$|f(\beta)-f(\alpha)| \leqq \frac{1}{2}\int_C |f'(z)\,dz|,$$

holds [8, formula (16)]. Assume that the coefficients $a_{ik}(z)$ of (2) are regular in $\bar{D}$ and define m_i by (5′). Instead of (6), we now use

$$m_i = |w_i(\beta_i)-w_i(\alpha_i)| \leqq \frac{1}{2}\int_C |w_i'(z)\,dz| \leqq \frac{1}{2}\sum_{k=1}^{n} m_k \int_C |a_{ik}(z)\,dz|.$$

Set $C=(c_{ik})_1^n$, where $c_{ik}=\frac{1}{2}\int_C |a_{ik}(z)\,dz|$, $i, k=1,\ldots,n$. *If* $\lambda(C)<1$, *then the system* (2) *is disconjugate in* $\bar{D}$.

3. **Theorem 2.** To improve the former result we shall use the following

LEMMA 2. *Let $u(t)$ be a real continuous function for $0 \leqq t \leqq T$ and let $a \geqq 0$, $b > 0$. Then*

$$0 \leqq u(t) \leqq \int_0^t (a+bu(s))\, ds, \qquad 0 \leqq t \leqq T,$$

implies

$$u(t) \leqq \frac{a}{b}(e^{bt}-1), \qquad 0 \leqq t \leqq T.$$

This is a simple case of (well-known generalizations of) Gronwall's inequality and e.g., a special case of an inequality proved in [1, p. 37, problem 1]. For a direct proof, set $\phi(t)=\int_0^t (a+bu(s))\, ds$, which gives $\phi'(t) \leqq a+b\phi(t)$. Set now $\psi(t)=\phi(t)e^{-bt}$. Then $\psi'(t) \leqq ae^{-bt}$ and therefore $\psi(t) \leqq (a/b)(1-e^{-bt})$.

We now state our main result on disconjugacy of differential systems.

THEOREM 2. *Let the analytic functions $a_{ik}(z)$, $i, k=1, \ldots, n$, $n \geqq 2$, be regular and bounded in the bounded convex domain D of diameter d. Let α_{ik}, $i, k=1, \ldots, n$, be defined by* (3) *and set*

$$\begin{aligned} &b_{ii} = 0, && && i = 1, \ldots, n, \\ (7) \qquad &b_{ik} = \frac{\alpha_{ik}}{\alpha_{ii}}(\exp(\alpha_{ii}d)-1) && \text{if } \alpha_{ii} \neq 0, && i \neq k;\, i, k = 1, \ldots, n. \\ &b_{ik} = \alpha_{ik}d && \text{if } \alpha_{ii} = 0, && \end{aligned}$$

Let $\lambda(B)$ be the maximal characteristic value of the matrix $B=(b_{ik})_1^n$. If

$$\lambda(B) < 1, \tag{8}$$

then the differential system (2) *is disconjugate in D.*

Proof. We start as in the proof of Theorem 1; i.e., we assume the existence of n points $\alpha_i \in D$ and of a nontrivial solution $w(z)$ of (2) such that $w_i(\alpha_i)=0$, $i=1, \ldots, n$. We define H, m_i (by (5)) and, if $m_i>0$, β_i as before. If $m_i>0$, we integrate $w_i'(z)$ along the segment from α_i to β_i. For any point z on this segment we obtain

$$\begin{aligned} |w_i(z)| &= \left| \int_{\alpha_i}^z w_i'(\zeta)\, d\zeta \right| \leqq \int_{\alpha_i}^z |w_i'(\zeta)\, d\zeta| \\ &\leqq \int_{\alpha_i}^z \Big(\sum_{k \neq i} \alpha_{ik} m_k + \alpha_{ii}|w_i(\zeta)| \Big)\, |d\zeta|. \end{aligned} \tag{9}$$

We now choose the arc length as parameter:

$$\alpha_i + \frac{\beta_i - \alpha_i}{|\beta_i - \alpha_i|}\, t = z, \qquad \alpha_i + \frac{\beta_i - \alpha_i}{|\beta_i - \alpha_i|}\, s = \zeta,$$

and denote $u_i(t)=|w_i(z)|$, $u_i(s)=|w_i(\zeta)|$. (9) yields

$$(10)\qquad 0 \leqq u_i(t) \leqq \int_0^t \Big(\sum_{k\neq i} \alpha_{ik}m_k+\alpha_{ii}u_i(s)\Big)\, ds, \qquad 0 \leqq t \leqq |\beta_i-\alpha_i|.$$

If $\alpha_{ii}\neq 0$, it follows from Lemma 2 that

$$(11)\qquad m_i = |w_i(\beta_i)| = u_i(|\beta_i-\alpha_i|) \leqq \sum_{k\neq i} \frac{\alpha_{ik}}{\alpha_{ii}}\, m_k\, (\exp(\alpha_{ii}|\beta_i-\alpha_i|)-1).$$

If $\alpha_{ii}=0$, (10) implies

$$(11')\qquad m_i \leqq |\beta_i-\alpha_i| \sum_{k\neq i} \alpha_{ik}m_k.$$

(7), (11), (11′) and $|\beta_i-\alpha_i|<d$ give the vector inequality

$$Bm \geqq m, \qquad m \geqq 0, \qquad m \neq 0.$$

By Lemma 1 this implies $\lambda(B)\geqq 1$ and we thus obtained the desired contradiction to assumption (8).

We now prove that Theorem 2 is stronger than Theorem 1. To do this we show that $\lambda(B)\geqq 1$ implies $d\lambda(A)\geqq 1$. Let x be the nonnegative eigenvector of B which corresponds to the maximal characteristic value $\lambda(B)$ [3, p. 66]. $\lambda(B)\geqq 1$ implies

$$(12)\qquad Bx \geqq x, \qquad x \geqq 0, \qquad x \neq 0.$$

If $\alpha_{ii}\neq 0$, then the ith component of (12) is

$$\alpha_{ii}^{-1}(\exp(\alpha_{ii}d)-1) \sum_{k\neq i} \alpha_{ik}x_k \geqq x_i.$$

This gives

$$\sum_{k\neq i} \alpha_{ik}x_k \geqq x_i\alpha_{ii}(\exp(\alpha_{ii}d)-1)^{-1} \geqq x_i\Big(\frac{1}{d}-\alpha_{ii}\Big).$$

Hence,

$$(13)\qquad \sum_{k=1}^{n} \alpha_{ik}x_k \geqq \frac{x_i}{d}.$$

If $\alpha_{ii}=0$, then the ith component of (12) is $d\sum_{k\neq i} \alpha_{ik}x_k\geqq x_i$. (13) is thus valid for all i, $i=1,\dots,n$. By Lemma 1 this implies $d\lambda(A)\geqq 1$.

The weaker Theorem 1 has some obvious merits: its elements α_{ik} are simpler than the b_{ik} and do not depend explicitly on d. We mention that remarks similar to (i) and (ii) of the end of the last section hold also for Theorem 2.

Theorem 2 is sharp in the following sense.

Let κ be any given constant larger than 1. *Let Δ be the diameter of the convex domain D and set $d=\Delta/\kappa$. Let the α_{ik} be defined by* (3) *and let the b_{ik} be defined by* (7) (*using d, not Δ*). *Then* (8) *does, in general, not imply the disconjugacy of the system* (2) *in D.*

To prove this, let the matrix $A(z)$, defining the system (2), be the constant matrix

$$A(z) = \begin{pmatrix} p & 1 \\ 1 & -p \end{pmatrix}, \qquad p \geqq 0. \tag{14}$$

The general solution of (2) is in this case given by

$$w_1(z) = c_1(p+(p^2+1)^{1/2}) \exp ((p^2+1)^{1/2}z) - c_2 \exp(-(p^2+1)^{1/2}z),$$
$$w_2(z) = c_1 \exp ((p^2+1)^{1/2}z) + c_2(p+(p^2+1)^{1/2}) \exp(-(p^2+1)^{1/2}z).$$

$w_1(z_1)=w_2(z_2)=0$, $w(z)\not\equiv 0$, imply

$$\exp [2(p^2+1)^{1/2}(z_2-z_1)] = -(p+(p^2+1)^{1/2})^2.$$

Hence,

$$|z_2-z_1| = \frac{1}{2(p^2+1)^{1/2}} [\log^2 (p+(p^2+1)^{1/2})^2 + (2n-1)^2\pi^2]^{1/2}, \tag{15}$$

here $n=1, 2, \ldots$, and log denotes the principal branch of the logarithm. For any $p \geqq 0$ we define $\Delta(p) > 0$ by

$$\Delta(p)^2 = \frac{1}{4(p^2+1)} [4 \log^2 (p+(p^2+1)^{1/2}) + \pi^2]. \tag{16}$$

(15) and (16) imply that (2), with $A(z)$ given by (14), is disconjugate in every convex domain of diameter $\Delta(p)$, but that for any $\varepsilon > 0$ there exist convex domains of diameter $\Delta(p)+\varepsilon$ for which (2) is not disconjugate.

On the other hand, the matrix B of Theorem 2 corresponding to our $A(z)$ becomes, for $p>0$,

$$B = \begin{pmatrix} 0 & \frac{1}{p}(e^{pd}-1) \\ \frac{1}{p}(e^{pd}-1) & 0 \end{pmatrix}. \tag{17}$$

Hence,

$$\lambda(B) = \frac{1}{p}(e^{pd}-1), \qquad p > 0. \tag{18}$$

For given $p>0$, we denote the root of the equation

$$\lambda(B) = 1 \tag{8'}$$

by $d(p)$. Therefore,

$$d(p) = \frac{1}{p} \log (p+1). \tag{19}$$

Theorem 2 gives that for any $p>0$

$$d(p) \leqq \Delta(p). \tag{20}$$

(16) and (19) imply

$$\lim_{p\to\infty} \frac{d(p)}{\Delta(p)} = 1. \tag{21}$$

It follows from (20) and (21) that for any given κ, $\kappa > 1$, we can find p large enough and $\varepsilon > 0$ small enough such that

$$\Delta(p)+\varepsilon = \kappa(d(p)-\varepsilon). \tag{22}$$

Consider now the differential system (2), with $A(z)$ given by (14), such that the constant p of (14) satisfies (22). By the above, there exist convex domains D of diameter $\Delta = \Delta(p)+\varepsilon$ such that this system is not disconjugate in D. On the other hand, if the constant d, appearing in the definition (7) of the corresponding B, is given by $d = d(p)-\varepsilon = \Delta/\kappa$, then the inequality (8) holds. This proves the italicized sharpness statement.

We did not prove that the constant 1 on the right-hand side of (8) is the best possible constant. However, it is easily seen that this constant, both in (4) and (8), cannot be replaced by any constant larger than $\pi/2$. This follows by considering (14) for $p=0$. In this case (17) has to be replaced by

$$B = \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}, \tag{17'}$$

and it follows that $d\lambda(A) = \lambda(B) = d$. Using the former notation, we thus have $d(0) = 1$. On the other hand, $\Delta(0) = \pi/2$.

To satisfy (22) we had to take p large, so $\Delta(p)$ and $d(p)$ became small. This smallness can be avoided by using the invariance of the theorems under the transformation $z^* = az$, $a \neq 0$. Indeed, if we define $w^*(z^*)$ in D^* by $w^*(z^*) = w(z)$, then (2) becomes

$$\frac{dw^*}{dz^*} = A^*(z^*)w(z^*), \qquad z^* \in D^*, \tag{2*}$$

where $A^*(z^*) = a^{-1}A(z)$. For the corresponding nonnegative matrices of Theorem 1 it follows that $A^* = |a|^{-1}A$ (and therefore $\lambda(A^*) = |a|^{-1}\lambda(A)$). As we now have to use $d^* = |a|d$ in the definition of the matrix B^*, we obtain $B^* = B$. To return to our example, if instead of (14) we use

$$A^*(z^*) = \begin{pmatrix} 1 & \frac{1}{p} \\ \frac{1}{p} & -1 \end{pmatrix}, \qquad p > 0, \tag{14*}$$

then it follows for the corresponding system (2*) that $d^*(p)$ $(=pd(p))$ and $\Delta^*(p)$ $(=p\Delta(p))$ tend, together with p, to infinity.

4. **Simultaneous disconjugacy.** Until now a given differential system (2) was considered and conditions for its disconjugacy were obtained. We now could consider not only (2), but also the permuted system $\hat{w}'(z)=\hat{A}(z)\hat{w}(z)$; here the permutation $\hat{A}(z)$ is obtained by a permutation of the rows of the given matrix $A(z)$ combined with the same permutation of its columns [3, p. 50]. Clearly, (2) and the permuted system are simultaneously disconjugate or not disconjugate. But this elementary remark is worthless for applications. Indeed, in the notation of Theorems 1 and 2, obviously $\lambda(A)=\lambda(\hat{A})$ and $\lambda(B)=\lambda(\hat{B})$. The case which we are going to consider is also elementary, but useful. The nonnegative matrices, built according to the two theorems, for the two simultaneously disconjugate systems (2) and ($\tilde{2}$) (below) will, in general, have different maximal characteristic values. The following lemma may thus be applied to improve the estimate for the domain of disconjugacy of the given system (2), (see proof of Theorem 5 below).

LEMMA 3. *Let the analytic functions $a_{ik}(z)$ and $\sigma_i(z)$, $i, k=1,\dots,n$, be regular in a bounded simply connected domain D and assume that $\sigma_i(z)\neq 0$, for $i=1,\dots,n$ and all $z\in D$. Set $A(z)=(a_{ik}(z))_1^n$, and $\tilde{A}(z)=(\tilde{a}_{ik}(z))_1^n$, where*

$$\tilde{a}_{ik}(z) = a_{ik}(z)\frac{\sigma_k(z)}{\sigma_i(z)}-\delta_{ik}\frac{\sigma_k'(z)}{\sigma_i(z)}, \qquad i, k = 1,\dots,n. \tag{23}$$

The systems (2) *and*

$$\tilde{w}'(z) = \tilde{A}(z)\tilde{w}(z) \tag{$\tilde{2}$}$$

are together disconjugate or not disconjugate in D.

To prove this, let $w(z)=[w_1(z),\dots,w_n(z)]$ be a solution of (2) and define $\tilde{w}_i(z)$ by

$$w_i(z) = \sigma_i(z)\tilde{w}_i(z), \qquad i = 1,\dots,n. \tag{24}$$

(2) and (24) give that $\tilde{w}(z)=[\tilde{w}_1(z),\dots,\tilde{w}_n(z)]$ satisfies ($\tilde{2}$). Conversely, if $\tilde{w}(z)$ is a solution of ($\tilde{2}$) and $w(z)$ is defined by (24), then $w(z)$ satisfies (2). As $\sigma_i(z)\neq 0$, $i=1,\dots,n$, (24) gives the assertion.

We mention two special cases. (i) For any given matrix $A(z)$ and any analytic function $s(z)$ we can always choose $\tilde{A}(z)=A(z)-s(z)I$, $(I=(\delta_{ik})_1^n)$. This follows from (23) by setting $\sigma_i(z)=\exp\int_{z_0}^z s(\zeta)\,d\zeta$, $i=1,\dots,n$. (ii) We can always choose $\tilde{A}(z)$ so that $\tilde{a}_{ii}(z)=0$, $i=1,\dots,n$. This follows by setting $\sigma_i(z)=\exp\int_{z_0}^z a_{ii}(\zeta)\,d\zeta$, $i=1,\dots,n$. Theorem 2 reduces in this case to the simpler Theorem 1.

5. **A property of determinants equivalent to disconjugacy.** We now interpret disconjugacy of the linear homogeneous system (2) in terms of the determinant of n independent solutions, i.e., in terms of the determinant of a fundamental matrix [1, p. 69]. In view of the applications which we shall give in the next sections, it seems preferable to define the system (2) by one of its fundamental matrices (and not vice versa).

THEOREM 3. *Let the analytic functions* $w_{ik}(z)$, $i, k=1,\dots,n$, $n\geqq 2$, *be regular in the bounded simply connected domain* D *and assume that the determinant* $|W(z)|$ *of the matrix* $W(z)=(w_{ik}(z))_1^n$ *satisfies*

$$(25)\qquad |W(z)| = |w_{ik}(z)|_1^n = \begin{vmatrix} w_{11}(z) & w_{12}(z) & \cdots & w_{1n}(z) \\ w_{21}(z) & w_{22}(z) & \cdots & w_{2n}(z) \\ \cdot & \cdot & \cdots & \cdot \\ w_{n1}(z) & w_{n2}(z) & \cdots & w_{nn}(z) \end{vmatrix} \neq 0, \qquad \textit{for all } z \in D.$$

Let the matrix $A(z)=(a_{ik}(z))_1^n$ *be defined by*

$$(26)\qquad A(z) = W'(z)W^{-1}(z).$$

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$$(2)\qquad w'(z) = A(z)w(z)$$

$(w(z)=[w_1(z),\dots,w_n(z)])$ *in* D *is equivalent to*

$$(27)\qquad |w_{ik}(z_i)|_1^n = \begin{vmatrix} w_{11}(z_1) & w_{12}(z_1) & \cdots & w_{1n}(z_1) \\ w_{21}(z_2) & w_{22}(z_2) & \cdots & w_{2n}(z_2) \\ \cdot & \cdot & \cdots & \cdot \\ w_{n1}(z_n) & w_{n2}(z_n) & \cdots & w_{nn}(z_n) \end{vmatrix} \neq 0,$$

for every choice of n (*not necessarily distinct*) *points* $z_1,\dots,z_n$ *in* D.

Proof. (26) is equivalent to

$$(28)\qquad W'(z) = A(z)W(z).$$

The given matrix $W(z)$ is, by (25), a fundamental solution of the matrix differential equation (28). Keeping thus $W(z)$ fixed, $|W(z)|\neq 0$, the solution vectors $w(z)$ of (2) are given by

$$(29)\qquad w(z) = W(z)c, \qquad w(z) = [w_1(z),\dots,w_n(z)], \qquad c = [c_1,\dots,c_n].$$

$c=0$ implies $w(z)\equiv 0$, i.e., $w(z)$ is the trivial solution of (2). Conversely, if $w(z)\equiv 0$ then $c=0$. We called (2) disconjugate if $w_i(z_i)=0$, $i=1,\dots,n$, always implies $w(z)\equiv 0$. (2) is thus disconjugate if, for any set $z_1,\dots,z_n$ in D,

$$w_i(z_i) = \sum_{k=1}^{n} w_{ik}(z_i)c_k = 0, \qquad i = 1,\dots,n,$$

implies $c_1=\cdots=c_n=0$. But this holds if, and only if, $|w_{ik}(z_i)|_1^n\neq 0$. Disconjugacy of (2) in D is thus equivalent to the validity of (27) for all sets of n points in D.

We mention here another interpretation of disconjugacy, which, however, will not be used in the sequel. Disconjugacy of the homogeneous system (2) in D is equivalent to the existence and uniqueness of a solution for the nonhomogeneous system

$$(30)\qquad w'(z) = A(z)w(z)+b(z), \quad b(z) = [b_1(z),\dots,b_n(z)], \qquad \text{all } b_i(z) \text{ regular in } D,$$

under the initial condition $w_i(z_i)=d_i$, $z_i \in D$, $i=1,\ldots,n$. This follows by (27) and the well-known relation: general solution of (30) equals particular solution of (30) and general solution of (2) (given by (29)).

6. **Mappings onto disjoint domains.** As first application we obtain

THEOREM 4. *Let the analytic functions $f(z)$ and $g(z)$ be regular in the bounded convex domain D of diameter d and assume that*

$$f(z) \neq g(z), \qquad \text{for all } z \in D. \tag{31}$$

Assume also that the following suprema are finite:

$$F = \sup_{z\in D}\left|\frac{f'(z)}{f(z)-g(z)}\right|, \qquad G = \sup_{z\in D}\left|\frac{g'(z)}{f(z)-g(z)}\right|. \tag{32}$$

If

$$(e^{Fd}-1)(e^{Gd}-1) < 1, \tag{33}$$

then

$$f(z_1) \neq g(z_2), \qquad \text{for all pairs } z_1 \in D,\ z_2 \in D. \tag{34}$$

Proof. If one of the functions is constant, then (33) holds trivially; but in this case (31) and (34) are equivalent. We therefore assume that $F>0$, $G>0$. Set

$$W(z) = \begin{pmatrix} f(z) & 1 \\ g(z) & 1 \end{pmatrix};$$

(31) is thus $|w_{ik}(z)|_1^2 \neq 0$. Define

$$A(z) = W'(z)W^{-1}(z) = \begin{pmatrix} \dfrac{f'}{f-g} & -\dfrac{f'}{f-g} \\ \dfrac{g'}{f-g} & -\dfrac{g'}{f-g} \end{pmatrix}, \tag{26'}$$

and consider the corresponding differential system

$$w'(z) = A(z)w(z), \qquad \text{with } A(z) \text{ given by (26')}. \tag{2'}$$

The matrix $A=(\alpha_{ik})_1^2$ of Theorem 1 is in this case

$$A = \begin{pmatrix} F & F \\ G & G \end{pmatrix},$$

and (as $F>0$, $G>0$) the matrix B of Theorem 2 is

$$B = \begin{pmatrix} 0 & e^{Fd}-1 \\ e^{Gd}-1 & 0 \end{pmatrix}.$$

$\lambda(B)^2=(e^{Fd}-1)(e^{Gd}-1)$ and (33) is therefore equivalent to $\lambda(B)<1$. Theorem 2 implies that (2′) is disconjugate in D. This is, by Theorem 3, equivalent to

$$|w_{ik}(z_i)|_1^2 = \begin{vmatrix} f(z_1) & 1 \\ g(z_2) & 1 \end{vmatrix} \neq 0, \tag{27′}$$

which is just the conclusion (34) of the theorem.

We add again some remarks.

(i) We do not claim that Theorem 4 is sharp. All we know is that the constant 1 on the right-hand side of (33) cannot be replaced by any constant larger than $(e-1)^2=2.95\ldots$. This follows by choosing $f(z)=z$ and $g(z)=z+1$. $F=G=1$ and for any $d>1$ there exist convex domains of diameter d for which (34) is invalidated.

(ii) A direct proof gives the following result. Set

$$F^* = \sup_{z\in D} |f'(z)| \Big/ \inf_{z\in D} |f(z)-g(z)|, \qquad G^* = \sup_{z\in D} |g'(z)| \Big/ \inf_{z\in D} |f(z)-g(z)|.$$

Then $\min(F^*d, G^*d) \leqq 1$ implies (34). This follows by supposing that $f(z_1)=g(z_2)$. Then

$$f(z_2)-g(z_2) = f(z_2)-f(z_1) = \int_{z_1}^{z_2} f'(\zeta)\, d\zeta,$$

which gives $F^*d>1$; $G^*d>1$ follows similarly. The example in (i) shows that this result is sharp.

(iii) Pairs of univalent functions mapping $|z|<1$ onto disjoint domains were considered by Nehari [6]. The necessary conditions obtained by him were generalized by M. Lavie [5] to nonunivalent functions. See also [4, pp. 123, 124].

We finally note that we could modify our theorem according to the remarks at the end of §2. Such a modification, to nonconvex domains, will be convenient for the second application.

7. **Products not taking a fixed value.** As second application we derive the following result.

THEOREM 5. *Let D be the interior of a piecewise smooth Jordan curve C and let the positive number d be such that any pair of points in $\bar{D}=D\cup C$ can be joined by a path in $\bar{D}$ of length not larger than d. Let the analytic function $f(z)$ be regular in $\bar{D}$ and assume that for a given integer n, $n\geqq 2$, the nth power of $f(z)$ satisfies*

$$f^n(z) \neq 1, \qquad \textit{for all } z\in \bar{D}. \tag{35}$$

Denote

$$F_i = \max_{z\in \bar{D}} \left| \frac{f'(z)f^i(z)}{f^n(z)-1} \right|, \qquad i=0,\ldots,n-2. \tag{36}$$

If

$$d\sum_{i=0}^{n-2} F_i < 1, \tag{37}$$

then

$$\prod_{i=1}^{n} f(z_i) \neq 1, \qquad \textit{for all sets } (z_1, \ldots, z_n) \subset \bar{D}. \tag{38}$$

Proof. We first prove the case $n=2$. Set

$$W(z) = \begin{pmatrix} f(z) & 1 \\ 1 & f(z) \end{pmatrix}. \tag{39}$$

The assumption $f^2(z)\neq 1$ is thus equivalent to $|w_{ik}(z)|_1^2\neq 0$. Define

$$A(z) = W'(z)W^{-1}(z) = \frac{f'(z)}{f^2(z)-1}\begin{pmatrix} f(z) & -1 \\ -1 & f(z) \end{pmatrix}, \tag{40}$$

and

$$\tilde{A}(z) = A(z) - \frac{f'(z)f(z)}{f^2(z)-1} I = \frac{f'(z)}{f^2(z)-1}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \tag{41}$$

It follows from Lemma 3 (see cases (i) or (ii) at the end of §4), that the corresponding differential systems (2) and ($\tilde{2}$) are together disconjugate (or not disconjugate) in $\bar{D}$. The constant matrix $\tilde{A}$, whose elements are the maxima, for $z \in \bar{D}$, of the absolute values of the elements of $\tilde{A}(z)$, is

$$\tilde{A} = \begin{pmatrix} 0 & F_0 \\ F_0 & 0 \end{pmatrix}. \tag{42}$$

Hence, $\lambda(\tilde{A}) = F_0$. By Theorem 1, modified according to remark (ii) following its proof, our assumption $(d\lambda(\tilde{A}) =)\ dF_0 < 1$ implies that ($\tilde{2}$) is disconjugate in $\bar{D}$. By the above, system (2) is then also disconjugate in $\bar{D}$. By Theorem 3, disconjugacy of (2) is equivalent to

$$|w_{ik}(z_i)|_1^2 = \begin{vmatrix} f(z_1) & 1 \\ 1 & f(z_2) \end{vmatrix} \neq 0.$$

But this is the desired result for $n=2$.

For general n, the $(n\times n)$ matrix $W(z)$ has only $2n$ elements different from zero: $f(z)$ in the diagonal, 1 in the first superdiagonal and $(-1)^n$ in the lower left corner:

$$W(z) = \begin{pmatrix} f & 1 & & & \\ & f & 1 & & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ (-1)^n & & & & f \end{pmatrix}. \tag{39'}$$

$|W(z)| = |w_{ik}(z)|_1^n \neq 0$ is the assumption (35). Set

(40′)
$$A(z) = \frac{f'}{f^n-1}\begin{pmatrix} f^{n-1} & -f^{n-2} & f^{n-3} & \cdots & (-1)^{n-1} \\ -1 & f^{n-1} & -f^{n-2} & \cdots & (-1)^n f \\ f & -1 & f^{n-1} & \cdots & (-1)^{n-1}f^2 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ (-1)^{n-1}f^{n-2} & (-1)^n f^{n-3} & (-1)^{n-1}f^{n-4} & \cdots & f^{n-1} \end{pmatrix}.$$

(For even n $A(z)$ is a circulant.) The validity of $W'(z) = A(z)W(z)$ is readily checked. $\tilde{A}(z)$ is again obtained from $A(z)$ by replacing the elements in the diagonal by zeros:

(41′)
$$\tilde{A}(z) = A(z) - \frac{f'(z)f^{n-1}(z)}{f^n(z)-1} I.$$

The corresponding nonnegative matrix $\tilde{A}$ is a circulant (hence generalized stochastic):

(42′)
$$\tilde{A} = \begin{pmatrix} 0 & F_{n-2} & F_{n-3} & \cdots & F_0 \\ F_0 & 0 & F_{n-2} & \cdots & F_1 \\ F_1 & F_0 & 0 & \cdots & F_2 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ F_{n-2} & F_{n-3} & F_{n-4} & \cdots & 0 \end{pmatrix}.$$

Clearly, $\lambda(\tilde{A}) = F_0 + F_1 + \cdots + F_{n-2}$. This and (37) give, by Theorem 1, that the system $(\tilde{2})$ is disconjugate in $\bar{D}$. By Lemma 3, the same holds for (2). Theorem 3 and (39′) now yield

$$|w_{ik}(z_i)|_1^n = \prod_{i=1}^n f(z_i) - 1 \neq 0.$$

This completes the proof of Theorem 5.

We now show that *for any given even n, the constant* 1 *on the right-hand side of* (37) *cannot be replaced by any constant larger than* $\pi(n-1)/n$. To prove this, we choose $f(z) = z$ and consider circular arcs $C(\varepsilon)$, $\varepsilon > 0$, defined by

$$C(\varepsilon) = \{z : |z-1| = \varepsilon, |z| \leqq 1\}.$$

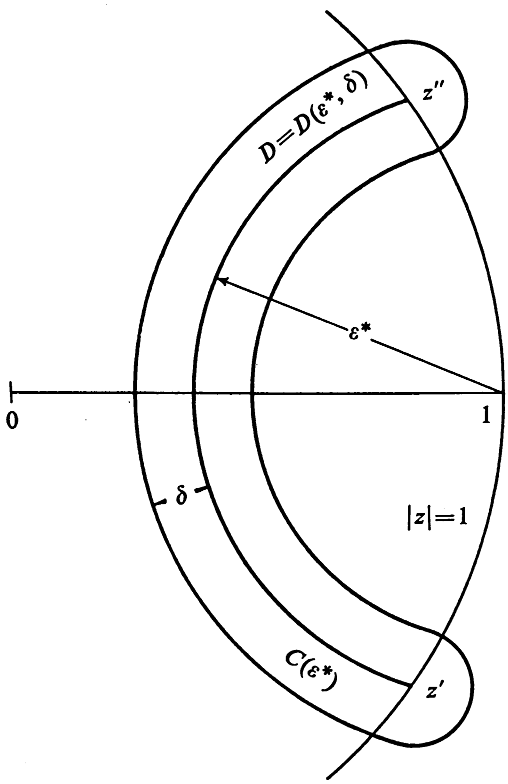
$\varepsilon \to 0$ implies

$$|z^n - 1| = n\varepsilon + O(\varepsilon^2), \qquad z \in C(\varepsilon),$$

which holds uniformly in z, $z \in C(\varepsilon)$. It follows, that each of the $n-1$ fractions, appearing in (36), satisfies

(43)
$$\left|\frac{f'(z)f^i(z)}{f^n(z)-1}\right| = \left|\frac{z^i}{z^n-1}\right| \leqq \frac{1}{|z^n-1|} = \frac{1}{n\varepsilon} + O(1), \qquad i = 0, \ldots, n-2,$$

which again holds uniformly in z, $z \in C(\varepsilon)$, as $\varepsilon \to 0$. Let $l(\varepsilon)$ be the length of



$C(\varepsilon)$. Then $l(\varepsilon)<\pi\varepsilon$. This and (43) imply that, for any given $\eta>0$, we can find a small enough ε^*, $\varepsilon^*>0$, such that

$$l(\varepsilon^*)\sum_{i=0}^{n-2}\max_{z\in C(\varepsilon^*)}\left|\frac{z^i}{z^n-1}\right| < \frac{n-1}{n}\pi+\eta. \tag{44}$$

For any δ, $0<\delta<\varepsilon^*$, we now define the δ-neighborhood $D=D(\varepsilon^*,\delta)$ of $C(\varepsilon^*)$ by $D=D(\varepsilon^*,\delta)=\{z: |z-w|<\delta,\ w\in C(\varepsilon^*)\}$. (See figure.) For this domain we set

$$d = d(\varepsilon^*,\delta) = l(\varepsilon^*)+2\delta. \tag{45}$$

This d can serve as the bound for the lengths of all necessary paths in $\bar{D}$. By (44), (45), and the continuity of the appearing fractions, it follows that we can choose δ so small, $\delta>0$, that

$$d\sum_{i=0}^{n-2}F_i < \frac{n-1}{n}\pi+\eta, \tag{46}$$

where

$$F_i = \max_{z\in\bar{D}}\left|\frac{z^i}{z^n-1}\right|, \qquad i=0,\ldots,n-2, \quad D = D(\varepsilon^*,\delta).$$

If ε^* is small enough, then

$$f^n(z) = z^n \neq 1, \qquad \text{for all } z \in \bar{D}. \tag{35'}$$

On the other hand, denote the endpoints of $C(\varepsilon^*)$ by z' and z'' and set

$$z_1 = \cdots = z_{n/2} = z'; \quad z_{n/2+1} = \cdots = z_n = z''.$$

Then

$$\prod_{i=1}^{n} f(z_i) = \prod_{i=1}^{n} z_i = 1.$$

This proves that, for even n, the assumption (37) of the theorem cannot be replaced by (46).

The class of analytic functions $f(z)$ regular in $|z|<1$ and such that $f(0)=0$, $f(z_1)f(z_2)\neq 1$, $|z_1|, |z_2|<1$ was studied extensively. These are the Bieberbach-Eilenberg functions [4].

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