ELEMENTARY DIFFERENCES BETWEEN THE ISOLS AND THE CO-SIMPLE ISOLS[1]

BY

LOUISE HAY[2]

1. **Introduction.** Let E denote the nonnegative integers. For α, $\beta \subseteq E$, α is *recursively equivalent* to β if there is a 1-1 partial recursive function p with $\alpha \subseteq$ domain p and $p(\alpha)=\beta$; the equivalence class of α is denoted by $\langle\alpha\rangle$. A set α is *isolated* if it has no infinite recursively enumerable (r.e.) subset. The equivalence classes of isolated sets are called *isols*, and their collection is denoted by Λ. The elements of Λ can be considered an "effective" analogue of the Dedekind finite cardinals; their properties were extensively studied by Dekker, Myhill, and Nerode (see, e.g., [2] and [6]). Isols $\langle\alpha\rangle$ of sets α such that α' is r.e. are called *co-simple isols*, and their collection is denoted by Λ_z. The system Λ_z was shown in [3] to exhibit much of the behavior of Λ; this presumably reflects the "effectiveness" common to the definitions of recursive equivalence and recursive enumerability, which makes it possible in many instances, given the existence of an isol with certain properties, to construct an r.e. set α such that $\langle\alpha'\rangle$ has the required properties. The question of whether there exist elementary differences between Λ and Λ_z was left open in [3]. It is the purpose of this paper to exhibit differences in the first-order theories of addition and multiplication of Λ and Λ_z.

More precisely, let L denote a first-order functional calculus based on identity, addition, and multiplication, with individual variables $x_1, x_2, \ldots, x, y, z, \ldots$ and logical symbols $(\exists)$, $(\forall)$, $\wedge$, $\vee$, $\neg$, $\supset$. Given a system $(M, +, \cdot)$, a formula $B(x_1, \ldots, x_n)$ of L whose only free variables are $x_1, \ldots, x_n$ and elements $X_1, \ldots, X_n$ of M, we say $B(X_1, \ldots, X_n)$ is *true in M* if, when the quantified variables are interpreted as ranging over M, the result is a true statement in the theory of $(M, +, \cdot)$. The first-order theories of $(\Lambda, +, \cdot)$ and $(\Lambda_z, +, \cdot)$ are both expressible in L, and we propose to exhibit a class of (closed) sentences $\{S_\theta\}$ of L which are true in Λ but false in Λ_z. We shall follow the usual practice of identifying E with the finite elements of Λ and Λ_z and of thus considering the system $(E, +, \cdot)$ as a subsystem of $(\Lambda, +, \cdot)$ and $(\Lambda_z, +, \cdot)$.

To define the sentences S_θ we shall require a formula of L which defines E in Λ and in Λ_z, i.e., a formula with one free variable which, when interpreted in Λ and Λ_z respectively, is true of exactly the finite elements of those systems. Such a

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formula is given for Λ in [2], by the fact that the finite isols are exactly those comparable (under $\leqq$) to all other elements of Λ. It will be shown that this is also true in Λ_z, so that the same formula can be used to define E in Λ_z. The proof of this fact will constitute Part 3 of this paper.

2. **The sentences S_θ.** Let $P_0, P_1, \ldots$ be the sequence of positive primes. If $\theta: E \to E$ is any function and X is an isol, we follow [1] in calling the sequence $\{\theta(n)\}$ the *characteristic* of X if, for all n,

$$P_n^{\theta(n)} | X \quad \text{and not} \quad P_n^{\theta(n)+1} | X.$$

It is proved in [1] that for each θ, there is an isol X having $\{\theta(n)\}$ as its characteristic. It will be shown that this fact is expressible by a sentence S_θ of L when θ is arithmetically definable. It will also be shown that for a large class of such θ, no $X \in \Lambda_z$ has $\{\theta(n)\}$ as its characteristic. Then for such θ, S_θ will be true in Λ but false in Λ_z.

THEOREM 1. *Corresponding to each arithmetically definable function $\theta: E \to E$ whose representing predicate is not expressible in both 5-quantifier forms in the arithmetic hierarchy, there is a sentence S_θ of L which is true in $(\Lambda, +, \cdot)$ but false in $(\Lambda_z, +, \cdot)$.*

Proof. We shall henceforth assume that a formula of L written $B(x_1, \ldots, x_m)$ contains no free variables other than $x_1, \ldots, x_m$. Let $\theta: E \to E$ be arithmetically definable, and let $R_\theta(m, n)$ be its representing predicate. There is thus a prenex formula $B_\theta(y, z)$ of L such that, for all $m, n \in E$,

(D1) $$m = \theta(n) \leftrightarrow R_\theta(m, n) \leftrightarrow B_\theta(m, n) \text{ is true in } E.$$

Let $h: E \times E \to E$ be the recursive function with representing predicate $S(k, m, n)$ defined by

$$S(k, m, n) \leftrightarrow k = h(m, n) \leftrightarrow k = P_n^m.$$

By the arithmetic definability of recursive relations, there is a prenex formula $C(x, y, z)$ of L such that for all $k, m, n \in E$,

(D2) $$S(k, m, n) \leftrightarrow C(k, m, n) \text{ is true in } E.$$

(D3) Let $\mathrm{Fin}(x)$ denote the following formula of L:

$$(\forall y)(\exists z)(x + z = y \lor y + z = x).$$

A conjunction $\mathrm{Fin}(x_1) \land \cdots \land \mathrm{Fin}(x_m)$ will be abbreviated by $\mathrm{Fin}(x_1, \ldots, x_m)$.

(D4) For any prenex formula $B(x_1, \ldots, x_m)$ of L, define a corresponding formula $B^E(x_1, \ldots, x_m)$ of L by induction on the number of quantifiers, as follows:

(a) If $B(x_1, \ldots, x_m)$ has no quantifiers, let

$$B^E(x_1, \ldots, x_m) = B(x_1, \ldots, x_m).$$

(b) If $B(x_1, \ldots, x_m)$ is $(\forall y)C(x_1, \ldots, x_m, y)$ and $C^E(x_1, \ldots, x_m, y)$ has been defined, let

$$B^E(x_1, \ldots, x_m) = (\forall y)(\mathrm{Fin}(y) \supset C^E(x_1, \ldots, x_m, y)).$$

(c) If $B(x_1, \ldots, x_m)$ is $(\exists y)C(x_1, \ldots, x_m, y)$ and $C^E(x_1, \ldots, x_m, y)$ has been defined, let

$$B^E(x_1, \ldots, x_m) = (\exists y)(\mathrm{Fin}(y) \wedge C^E(x_1, \ldots, x_m, y)).$$

The definition of $B^E(x_1, \ldots, x_m)$ evidently serves to restrict the range of the quantifiers in $B(x_1, \ldots, x_m)$ to $\{y \mid \mathrm{Fin}(y)\}$.

In terms of the formulas introduced in (D1)–(D4), let $A_\theta(x)$ be the following formula of L:

$$(\forall y)(\forall z)(\forall u)(\forall v)[(\mathrm{Fin}(y, z, u, v) \wedge B_\theta^E(y, z) \wedge C^E(u, y, z) \wedge C^E(v, y+1, z)) \supset ((\exists t)(x = tu) \wedge \neg(\exists t)(x = tv))].$$

Finally, let S_θ be the sentence $(\exists x)A_\theta(x)$. It remains to show that:

(1) S_θ is true in Λ.

(2) S_θ is true in Λ_z only if $R_\theta(m, n)$ is expressible in both 5-quantifier forms.

LEMMA 1.1. *Let* $X \in \Lambda$. *Then* $\mathrm{Fin}(X)$ *is true in* Λ *if and only if* $X \in E$.

Proof. This is shown in [2, p. 103].

LEMMA 1.2. *Let* $B(x_1, \ldots, x_m)$ *be a prenex formula of* L, *and let* $X_1, \ldots, X_m \in E$. *Then* $B(X_1, \ldots, X_m)$ *is true in* E *if and only if* $B^E(X_1, \ldots, X_m)$ *is true in* Λ.

Proof. By induction on the number of quantifiers in $B(x_1, \ldots, x_m)$:

(a) If $B(x_1, \ldots, x_m)$ has no quantifiers, then since $(E, +, \cdot)$ is a subsystem of $(\Lambda, +, \cdot)$, the following are equivalent:

(a1) $B(X_1, \ldots, X_m)$ is true in E,

(a2) $B(X_1, \ldots, X_m)$ is true in Λ,

(a3) $B^E(X_1, \ldots, X_m)$ is true in Λ.

(b) Assume that $B(x_1, \ldots, x_m)$ is $(\forall y)C(x_1, \ldots, x_m, y)$ and that the lemma holds for $C(x_1, \ldots, x_m, y)$. Then by Lemma 1.1, the following are equivalent:

(b1) $B(X_1, \ldots, X_m)$ is true in E,

(b2) for all $Y \in E$, $C(X_1, \ldots, X_m, Y)$ is true in E,

(b3) for all $Y \in E$, $C^E(X_1, \ldots, X_m, Y)$ is true in Λ,

(b4) $(\forall y)(\mathrm{Fin}(y) \supset C^E(X_1, \ldots, X_m, y))$ is true in Λ,

(b5) $B^E(X_1, \ldots, X_m)$ is true in Λ.

(c) Similarly if $B(x_1, \ldots, x_m)$ is $(\exists y)C(x_1, \ldots, x_m, y)$.

LEMMA 1.3. *Let* $X \in \Lambda$. *Then* $\{\theta(n)\}$ *is the characteristic of* X *if and only if* $A_\theta(X)$ *is true in* Λ.

Proof. By (D1) and (D2), Lemma 1.2, and Lemma 1.1, the following are equivalent:

(1) $\{\theta(n)\}$ is the characteristic of X,

(2) for all $Z \in E$, $Y = \theta(Z)$ and $U = P_Z^Y$ and $V = P_Z^{Y+1}$ together imply

$$[(\exists t)(X = tU) \wedge \neg(\exists t)(X = tV)]$$

is true in Λ,

(3) for all $Y, Z, U, V \in E$, $[B_\theta(Y, Z) \wedge C(U, Y, Z) \wedge C(V, Y+1, Z)]$ true in E implies $[(\exists t)(X=tU) \wedge \neg(\exists t)(X=tV)]$ true in Λ,

(4) for all $Y, Z, U, V \in E$, $[B_\theta^E(Y, Z) \wedge C^E(U, Y, Z) \wedge C^E(V, Y+1, Z)]$ true in Λ implies $[(\exists t)(X=tU) \wedge \neg(\exists t)(X=tV)]$ true in Λ,

(5) for all $Y, Z, U, V \in \Lambda$,

$$[\text{Fin}(Y, Z, U, V) \wedge B_\theta^E(Y, Z) \wedge C^E(U, Y, Z) \wedge C^E(V, Y+1, Z)]$$

true in Λ implies $[(\exists t)(X=tU) \wedge \neg(\exists t)(X=tV)]$ true in Λ,

(6) $A_\theta(X)$ is true in Λ.

LEMMA 1.4. (a) *If $X \in \Lambda_z$, $Y \in \Lambda$ and $(\exists t)(Y+t=X)$ is true in Λ then $Y \in \Lambda_z$.*

(b) *If $X \in \Lambda_z$ and $U \in E$, then $(\exists t)(X=tU)$ is true in Λ if and only if it is true in Λ_z.*

Proof. Part (a) is Theorem 56(b) of [2]. For (b), assume that $X \in \Lambda_z$ and $U \in E$. One direction is trivial. Now assume that for some $T \in \Lambda$, $X=TU$. If $U=0$, then $X=0$. If $U>0$, then $X=T+(U-1)T$, so that by (a), $T \in \Lambda_z$. In either case, $(\exists t)(X=tU)$ is true in Λ_z.

LEMMA 1.5. *Let $X \in \Lambda_z$. Then* Fin(X) *is true in Λ_z if and only if $X \in E$.*

Proof. Assume $X \in E$ and $Y \in \Lambda_z$. We require some $Z \in \Lambda_z$ such that $X+Z=Y$ or $Y+Z=X$. Now by Lemma 1.1, there is a $Z \in \Lambda$ such that $X+Z=Y$ or $Y+Z=X$. In the first case $Z \in \Lambda_z$ by Lemma 1.4(a); in the second case, $X \in E$ and $Z \leqq X$ implies $Z \in E \subseteq \Lambda_z$. Thus in either case, $Z \in \Lambda_z$ and Fin(X) is true in Λ_z.

The converse will follow from Theorem 2 of Part 3, in which it is proved that $X \in \Lambda_z - E$ implies Fin(X) is false in Λ_z.

LEMMA 1.6. *Let $B(x_1, \ldots, x_m)$ be a prenex formula of L, and let $X_1, \ldots, X_m \in E$. Then $B(X_1, \ldots, X_m)$ is true in E if and only if $B^E(X_1, \ldots, X_m)$ is true in Λ_z.*

Proof. Exactly like the proof of Lemma 1.2, replacing use of Lemma 1.1 by use of Lemma 1.5.

LEMMA 1.7. *Let $X \in \Lambda_z$. Then $\{\theta(n)\}$ is the characteristic of X if and only if $A_\theta(X)$ is true in Λ_z.*

Proof. By Lemma 1.4(b), (D1), and (D2), Lemma 1.6, and Lemma 1.5, the following are equivalent:

(1) $\{\theta(n)\}$ is the characteristic of X,

(2) for all $Z \in E$, $Y=\theta(Z)$ and $U=P_Z^Y$ and $V=P_Z^{Y+1}$ together imply

$$[(\exists t)(X = tU) \wedge \neg(\exists t)(X = tV)]$$

is true in Λ,

(3) for all $Z \in E$, $Y=\theta(Z)$ and $U=P_Z^Y$ and $V=P_Z^{Y+1}$ together imply

$$[(\exists t)(X = tU) \wedge \neg(\exists t)(X = tV)]$$

is true in Λ_z,

(4) for all $Y, Z, U, V \in E$, $[B_\theta^E(Y, Z) \wedge C^E(U, Y, Z) \wedge C^E(V, Y+1, Z)]$ true in Λ_z implies $[(\exists t)(X = tU) \wedge \neg(\exists t)(X = tV)]$ true in Λ_z,

(5) $A_\theta(X)$ is true in Λ_z.

We now introduce some recursion-theoretic notation. This will be largely informal, and such formalism as we use derives from [4]. The notation ( ), (E), &, $\vee$, $\neg$, $\rightarrow$ of a first-order predicate calculus will be used for notational convenience, and is not to be confused with the formal symbolism of L. Let $q_0, q_1, \ldots$ be a Kleene enumeration of all partial recursive functions of one variable. If $w_k = \text{range } q_k$, then $w_0, w_1, \ldots$ is an enumeration of all r.e. sets and if $X \in \Lambda_z$, then $X = \langle w_e' \rangle$ for some e; we call e an *index* of X. Let $p_0, p_1, \ldots$ be an effective enumeration of all 1–1 partial recursive functions, given by a recursive function g such that $p_k \simeq q_{g(k)}$ for each k. We note that

$$\text{(A1)} \qquad z = p_k(x) \leftrightarrow z = q_{g(k)}(x) \leftrightarrow (Ey)(T_1^1(g(k), x, y) \;\&\; z = U(y)),$$

$$\begin{aligned} \text{(A2)} \qquad \langle \alpha \rangle = \langle \beta \rangle &\leftrightarrow (Ek)(\alpha \subseteq \text{domain}\, p_k \;\&\; p_k(\alpha) = \beta) \\ &\leftrightarrow [(x)(x \in \alpha' \vee (Ez)(z = p_k(x) \;\&\; z \in \beta)) \\ &\qquad \& \; (z)(z \in \beta' \vee (Ex)(x \in \alpha \;\&\; z = p_k(x)))]. \end{aligned}$$

LEMMA 1.8. *Let $Q(a, b, m, n)$ denote the number-theoretic predicate*:

$$\langle w_a' \rangle = P_n^m \cdot \langle w_b' \rangle.$$

Then $Q(a, b, m, n)$ is expressible in form EAEA in the arithmetic hierarchy.

Proof. We show that a defining expression for $Q(a, b, m, n)$ in terms of quantifiers and recursive predicates can be brought to *EAEA* prenex form by means of the Tarski-Kuratowski algorithm described in [7]. Let h be the recursive function defined by $h(m, n) = P_m^n$ and, for $i \in E$, let

$$\beta_i = \{2^i 3^x \mid x \in w_b'\},$$

$$\beta_{b,m,n} = \beta_0 \cup \cdots \cup \beta_{h(m,n)-1}.$$

Then $\langle \beta_i \rangle = \langle w_b' \rangle$ for each i, so that $\langle \beta_{b,m,n} \rangle = P_m^n \cdot \langle w_b' \rangle$ and

$$\begin{aligned} Q(a, b, m, n) \leftrightarrow \langle w_a' \rangle = P_m^n \cdot \langle w_b' \rangle &= \langle \beta_{b,m,n} \rangle \\ \leftrightarrow (Ek)[(x)(x \in w_a \vee (Ez)&(z = p_k(x) \;\&\; z \in \beta_{b,m,n})) \\ &\& \; (z)(z \in \beta_{b,m,n}' \vee (Ex)(x \in w_a' \;\&\; z = p_k(x)))]. \end{aligned}$$

by (A2). Now $x \in w_a$ has E form, since w_a is r.e., $z = p_k(x)$ has E form, by (A1), and $z \in \beta_{b,m,n} \leftrightarrow (Eu)(Ev)(v \in w_b' \;\&\; u < h(m, n) \;\&\; z = 2^u 3^v)$, which can be brought to form EA. Thus $Q(a, b, m, n)$ has form

$$E[A(E \vee E(E \;\&\; EA)) \;\&\; A(AE \vee E(A \;\&\; E))]$$

which by the algorithm can be reduced, in sequence, to

$$E[A(E \vee EA) \;\&\; A(AE \vee EA)], \quad E[AEA \;\&\; (AE \vee AEA)], \quad EAEA.$$

Lemma 1.9. *Let $X \in \Lambda_z$, and assume that $\{\theta(n)\}$ is the characteristic of X. Then the representing predicate of θ is expressible in both 5-quantifier forms in the arithmetic hierarchy.*

Proof. Let $X=\langle w'_a\rangle$ have characteristic $\{\theta(n)\}$, and let $R_\theta(m, n)$ be the representing predicate of θ, i.e.,

$$R_\theta(m, n) \leftrightarrow m = \theta(n) \qquad \text{for all } m, n \in E.$$

Then, since the characteristic of X is uniquely determined,

$$\begin{aligned} R_\theta(m, n) &\leftrightarrow P_n^m | X \ \& \ \neg P_n^{m+1} | X \\ &\leftrightarrow (EY)_{Y\in\Lambda}(X = P_n^m Y) \ \& \ \neg(EY)_{Y\in\Lambda}(X = P_n^{m+1} Y) \\ &\leftrightarrow (EY)_{Y\in\Lambda_z}(X = P_n^m Y) \ \& \ \neg(EY)_{Y\in\Lambda_z}(X = P_n^{m+1} Y), \end{aligned}$$

by Lemma 1.4(b)

$$\begin{aligned} &\leftrightarrow (Eb)(\langle w'_a\rangle = P_n^m\langle w'_b\rangle) \ \& \ \neg(Eb)(\langle w'_a\rangle = P_n^{m+1}\langle w'_b\rangle) \\ &\leftrightarrow (Eb)Q(a, b, m, n) \ \& \ (b)\neg Q(a, b, m+1, n), \end{aligned}$$

which by Lemma 1.8 has form $E(EAEA)$ & $A(AEAE)$ which reduces to $EAEA$ & $AEAE$. As is well known, this form is recursive in the 4-quantifier form of highest degree, or, equivalently, can be written in both 5-quantifier forms.

Proof of Theorem 1. Let θ be any arithmetically definable function whose representing predicate $R_\theta(m, n)$ is not expressible in both 5-quantifier forms. Then:

(1) By [1, Theorem T1] there is an $X \in \Lambda$ which has $\{\theta(n)\}$ as its characteristic. By Lemma 1.3, this implies $A_\theta(X)$ is true in Λ, so that S_θ is true in Λ.

(2) Assume S_θ is true in Λ_z; then for some $X \in \Lambda_z$, $A_\theta(X)$ is true in Λ_z. By Lemma 1.7, this implies $\{\theta(n)\}$ is the characteristic of X, from which it follows by Lemma 1.9 that $R_\theta(m, n)$ is expressible in both 5-quantifier forms. Since this is a contradiction, we conclude that S_θ is false in Λ_z.

Remark. The sentences S_θ chosen for Theorem 1 are merely illustrative of a type of sentence which can serve to distinguish between the first-order theories of $(\Lambda, +, \cdot)$ and $(\Lambda_z, +, \cdot)$. Theorem T1 of [1], which was applied above, is a special case of the "extended Chinese remainder theorem for isols" [5, Theorem 4.5]. Other instances of the latter could be similarly used to yield elementary differences between Λ and Λ_z.

3. **First-order characterization of E in Λ_z.** It remains to show that the finite isols are the only elements of Λ_z comparable to all other elements of Λ_z. This requires the direct construction of r.e. sets, for which the natural tool is the "priority" method in its various manifestations [8]. A scheme for adapting this method to the construction of co-simple isols was described in [3]; unfortunately it does not appear to be sufficiently general to handle the present case.

THEOREM 2. Assume $X \in \Lambda_z - E$. Then there is a $Y \in \Lambda_z$ such that $X \nleqq Y$ and $Y \nleqq X$.

Proof. The recursion-theoretic notation is that introduced for the proof of Lemma 1.8. We assume an effective procedure for simultaneously generating all r.e. sets, and denote by w_k^u the finite set of elements of w_k generated at stages $t \leqq u$. Let $R_0, R_1, \ldots$ be a partition of E into infinite disjoint recursive sets. We adopt the following notation for purposes of abbreviation:

(1) $$\alpha \prec_k \beta \equiv \alpha \subseteq \text{domain } p_k \text{ and } p_k(\alpha) \subseteq \beta.$$

Thus $\langle \alpha \rangle \leqq \langle \beta \rangle$ only if $(Ek)(\alpha \prec_k \beta)$.

(2) $$\begin{aligned} \pi_k^t(x) &= 1 + U((\mu y)_{y \leqq t} T_1^1(g(k), x, y)), \quad \text{if } (Ey)_{y \leqq t} T_1^1(g(k), x, y), \\ &= 0 \quad \text{otherwise,} \end{aligned}$$

(3) $$\begin{aligned} p_k^t(x) &= \pi_k^t(x) - 1, \quad \text{if } \pi_k^t(x) > 0, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

p_k^t is thus a recursive function whose value corresponds roughly to the result of performing t steps in the computation of $p_k(x)$. It is evident that $\pi_k^t(x)$ and $p_k^t(x)$ are bounded, nondecreasing functions of t and that

$$\begin{aligned} \lim_t p_k^t(x) &= p_k(x), \quad \text{if } x \in \text{domain } p_k, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Now assume $X \in \Lambda_z - E$, so that $X = \langle w'_a \rangle$ where w'_a is immune. Instructions will be given for generating an r.e. set γ which will satisfy the following "requirements," for each k:

(1_k) w_k infinite $\rightarrow w_k \cap \gamma \neq \varnothing$,

(2_k) not $\gamma' \prec_k w'_a$,

(3_k) not $w'_a \prec_k \gamma'$.

This will evidently give $Y = \langle \gamma' \rangle \in \Lambda_z$, with $Y \nleqq X$ and $X \nleqq Y$. We will define by simultaneous induction on t, recursive functions $F(k, t)$, $G(k, t)$, and $H(k, t)$, and will generate γ by putting into it at each stage t the values $F(k, t)$, $G(k, t)$, $H(k, t)$ for $k \leqq t$. The gist of the construction is as follows: To satisfy requirement (2_k), we try to keep in γ' some x for which $p_k(x) \notin w'_a$; to satisfy requirement (1_k), we try to put into γ an element of w_k; and to satisfy requirement (3_k), we try to put into γ the number $p_k(z)$ for some $z \in w'_a$. These are evidently conflicting requirements, and we resolve this conflict by the following device:

At stage t, we "tag" the least $x \in R_k - \gamma^{t-1}$ for which it appears that $p_k(x) \notin w'_a$, and put into γ, $G(k, t)$ = the next larger element of $R_k - \gamma^{t-1}$. The fact that w'_a is immune and that hence there is no infinite r.e. sequence $\{x_i\}$ such that $p_k(x_i) \in w'_a$ for all i, will insure that (i) only finitely many elements of R_k are ever "tagged,"

(ii) for some "tagged" $x \in R_k$, $p_k(x) \notin w'_a$ and x is never put in γ, and (iii) $R_k \cap \gamma'$ is finite.

Since the "naïve" attempts at satisfying requirements (1_k) and (3_k) described above would interfere with the termination of the "tagging" process, we modify them slightly as follows: The value of $F(k, t)$ represents an attempt to put into γ an element of $w_k - \bigcup_{j \leq k} R_j$, and that of $H(k, t)$ an attempt to put into γ, $p_k(z)$ for some $z \in w'_a$ for which $p_k(z) \notin \bigcup_{j<k} R_j$. These attempts will fail only if (1) $w_k \subseteq \bigcup_{j \leq k} R_j$ or (2) $p_k(w'_a) \subseteq \bigcup_{j<k} R_j$. In the first case, $w_k \cap \gamma' \subseteq (\bigcup_{j \leq k} R_j) \cap \gamma'$ which is finite, so that either w_k is finite or $w_k \cap \gamma \neq \varnothing$; in the second case, $p_k(w'_a) \cap \gamma' \subseteq (\bigcup_{j<k} R_j) \cap \gamma'$ which is finite, so that either $p_k(z)$ is undefined for some $z \in w'_a$ or $p_k(z) \in \gamma$ for some $z \in w'_a$. Thus in either case, the relevant requirement is satisfied anyway.

We proceed to the formal construction and the proof (by induction on k) that all requirements are eventually satisfied. Define $F(k, t)$, $G(k, t)$, $H(k, t)$ and auxiliary recursive functions $n(k, t)$, $r(k, t)$, $x(i, k, t)$, $z(i, t)$, $v(k, t)$, $s(k, t)$ as follows:

Stage 0. For all k and i, let $F(k, 0) = G(k, 0) = H(k, 0) = n(k, 0) = r(k, 0) = x(i, k, 0) = z(i, 0) = v(k, 0) = s(k, 0) = 0$. Let $\gamma^0 = \{0\}$.

Stage $t > 0$. For $k > t$ and all i, let $F(k, t) = G(k, t) = H(k, t) = n(k, t) = r(k, t) = x(i, k, t) = v(k, t) = s(k, t) = 0$.

(a) For $0 \leq k \leq t$, let

$$v = v(k, t) = (\mu v)_{v \leq t}\Big(\Big(w_k^t - \bigcup_{j \leq k} R_j = \varnothing \;\&\; v = 0\Big) \vee w_k^v - \bigcup_{j \leq k} R_j \neq \varnothing\Big),$$

$$F(k, t) = (\mu x)\Big(\Big(w_k^v - \bigcup_{j \leq k} R_j = \varnothing \;\&\; x = 0\Big) \vee x \in w_k^v - \bigcup_{j \leq k} R_j\Big).$$

(b) For $0 \leq k \leq t$, let

$$x(0, k, t) = (\mu x)(x \in R_k - \gamma^{t-1}),$$

$$x(i+1, k, t) = (\mu x)(x > x(i, k, t) \;\&\; x \in R_k - \gamma^{t-1}) \qquad i = 0, 1, \ldots,$$

$$n = n(k, t) = (\mu i)_{i \leq t}(\pi_k^t x(i, k, t) = 0 \vee (\pi_k^t x(i, k, t) > 0 \;\&\; p_k^t x(i, k, t) \in w_a^t) \vee i = t),$$

$$G(k, t) = x(n+1, k, t) \quad \text{if } \pi_k^t x(n, k, t) = 0 \vee (\pi_k^t x(n, k, t) > 0 \;\&\; p_k^t x(n, k, t) \in w_a)$$
$$= 0 \quad \text{otherwise.}$$

(c) Let

$$z(0, t) = (\mu z)(z \notin w_a^t),$$

$$z(i+1, t) = (\mu z)(z > z(i, t) \;\&\; z \notin w_a^t) \qquad i = 0, 1, \ldots,$$

$$H(0, t) = p_k^t z(0, t).$$

For $0<k\leqq t$, let

$$r = r(k, t) = 1+\sum_{j<k}(1+n(j, t)),$$

$$s = s(k, t) = (\mu i)_{i\leqq r}\Big(\pi^t_k z(i, t) = 0 \vee \Big(\pi^t_k z(i, t) > 0 \ \& \ p^t_k z(i, t) \in \bigcap_{j<k} R'_j\Big) \vee i = r\Big),$$

$$H(k, t) = p^t_k z(s, t) \quad \text{if } \pi^t_k z(s, t) > 0 \ \& \ p^t_k z(s, t) \in \bigcap_{j<k} R'_j,$$
$$= 0 \quad \text{otherwise.}$$

Finally, let $\gamma^t=\gamma^{t-1} \cup (\bigcup_{k\leqq t}\{F(k, t), G(k, t), H(k, t)\})$.

REMARK. The following easily verified facts are noted here for future reference:

(R1) $z(i, t)$ and $x(i, k, t)$ are strictly increasing functions of i.

(R2) $F(j, t)\neq 0 \rightarrow F(j, t) \in \bigcap_{k<j} R'_k$.

(R3) $H(j, t)\neq 0 \rightarrow H(j, t) \in \bigcap_{k<j} R'_k$.

(R4) $(G(j, t)\neq 0 \ \& \ n=n(j, t)) \rightarrow (\pi^t_k x(n, j, t)=0 \vee p^t_k x(n, j, t) \in w^t_a)$.

(R5) $G(j, t)\neq 0 \rightarrow G(j, t) \in R_j$.

LEMMA 2.1. *For each* k,

(a) $\lim_t F(k, t)$ *exists*,

(b) *if* $w_k-\bigcup_{j\leqq k} R_j\neq \varnothing$, *then* $w_k \cap \gamma\neq \varnothing$.

Proof. *Case* 1. $w_k-\bigcup_{j\leqq k} R_j= \varnothing$.
Then for all t, $w^t_k-\bigcup_{j\leqq k} R_j= \varnothing$, so that $v(k, t)=F(k, t)=0$. This proves (a), and (b) holds trivially.

Case 2. Otherwise. Let

$$t^* = (\mu t)\Big(t \geqq k \ \& \ w^t_k - \bigcup_{j\leqq k} R_j \neq \varnothing\Big),$$

$$x^* = (\mu x)\Big(x \in w^{t^*}_k - \bigcup_{j\leqq k} R_j\Big).$$

Then for all $t\geqq t^*$, $v(k, t)=t^*$ and $F(k, t)=x^*$, which proves (a). Part (b) follows since $x^* \in w_k$ and $x^*=F(k, t^*) \in \gamma^{t^*}\subseteq\gamma$.

LEMMA 2.2. (a) $\lim_t H(0, t)$ exists,

(b) not $w'_a \prec_0 \gamma'$.

Proof. Let $z^*=(\mu z)(z \in w'_a)$, so that $(z)_{z<z^*}(Et)(z \in w^t_a)$. Let

$$t^* = (\mu t)(z)_{z<z^*}(z \in w^t_a).$$

Then for all $t\geqq t^*$, $z(0, t)=z^*$.

Case 1. $z^* \notin \text{domain } p_0$. Then $\pi^t_0(z^*)=0$ for all t, so that for all $t\geqq t^*$, $H(0, t) =\pi^t_0(z^*)=0$, which proves (a). Part (b) holds since $z^* \in w'_a$ implies $w'_a \not\subseteq \text{domain } p_0$.

Case 2. Otherwise. Then for some u, $\pi^u_0(z^*)>0$ and $p^u_0(z^*)=p_0(z^*)$. Let $u^* =\max(t^*, u)$. Then for all $t\geqq u^*$, $H(0, t)=p^t_0(z^*)=p_0(z^*)$, which proves (a). Part (b) holds since $z^* \in w'_a$ while $p_0(z^*)=H(0, u^*) \in \gamma^{u^*}\subseteq\gamma$.

In the following, $c(\alpha)$ denotes cardinality α.

LEMMA 2.3. *If* $\lim_t H(j, t)$ *exists for all* $j \leqq k$, *then*

(a) $\lim_t n(k, t)$ *exists*,

(b) $c(R_k \cap \gamma') = 1 + \lim_t n(k, t)$,

(c) *not* $\gamma' \prec_k w'_a$.

Proof. This will be divided into steps (2.3.1)–(2.3.5), each of which has as hypothesis

$$\lim_t H(j, t) \text{ exists for all } j \leqq k.$$

(2.3.1) There is a stage $u > k$ such that

$$(x)[(x \in R_k \ \& \ (Et)_{t>u}(Ej)(x = F(j, t) \vee x = H(j, t))) \rightarrow x \in \gamma^u]$$

(i.e., after stage u no new elements of R_k are added to γ to satisfy requirements (1_j) or (3_j) for any j.)

Proof. Let u be chosen so large that for all $j \leqq k$ and all $t \geqq u$,

$$H(j, t) = \lim_t H(j, t) = H_j \quad \text{(which exists by the hypothesis)},$$

$$F(j, t) = \lim_t F(j, t) = F_j \quad \text{(which exists by Lemma 2.1)}.$$

If $x = 0$, the conclusion holds trivially since $x \in \gamma^u$ for all u. If $x \neq 0$, $x \in R_k$ and $(x = F(j, t) \vee x = H(j, t))$ then by (R2) and (R3) it follows that $k \geqq j$, while by choice of u, $t > u$ implies

$$x = F(j, t) = F_j = F(j, u) \vee x = H(j, t) = H_j = H(j, u).$$

This, together with $j \leqq k < u$, implies $x \in \gamma^u$.

Let u_k = the least u satisfying (2.3.1). We define by simultaneous induction on t two partial recursive functions $t_k(i)$ and $f_k(i)$, as follows:

$$t_k(0) = u_k + 1, \qquad f_k(0) = (\mu x)(x \in R_k - \gamma^{u_k}).$$

Now assume that for all $j < i$, $t_k(j)$ and $f_k(j)$ have been defined.

Case 1. $(Et)(t > t_k(i-1) \ \& \ (j)_{j<i}(\pi^t_k f_k(j) > 0 \ \& \ p^t_k f_k(j) \notin w^t_a))$. Then define

$$t_k(i) = \text{the least such } t,$$
$$f_k(i) = (\mu x)(x > f_k(i-1) \ \& \ x \in R_k - \gamma^{t_k(i)-1}).$$

Case 2. Otherwise. Then $t_k(i)$ and $f_k(i)$ are undefined. It is evident that t_k and f_k are strictly increasing functions of i, that $t_k(i) > i$ for all $i \in$ domain t_k, and that domain t_k = domain f_k is an initial segment of E.

Let σ_k denote this common domain,

(2.3.2) σ_k is finite and nonvoid.

Proof. $\sigma_k \neq \varnothing$ since $f_k(0)$ and $t_k(0)$ are defined, so that $0 \in \sigma_k$. We will show that if σ_k is infinite, then w'_a has an infinite r.e. subset. Now if σ_k is infinite, then $\sigma_k = E$ and $f_k(i)$ is defined for all $i \in E$. We claim the following then hold:

(a) $p_k f_k(i)$ is defined for each $i \in E$,

(b) $p_k f_k(i) \in w'_a$ for each $i \in E$.

To prove (a), assume that $p_k f_k(i^*)$ is undefined for some i^*. Then for all t, $\pi_k^t f_k(i^*)=0$, so that for $i=i^*+1$, Case 1 of the definition of $f_k(i)$ fails to occur and $f_k(i)$ is undefined, contradicting the assumption that $\sigma_k=E$. To prove (b), assume that for some i^*, $p_k f_k(i^*)\in w_a$. Then for all sufficiently large t, $\pi_k^t f_k(i^*)>0$ and $p_k^t f_k(i^*)\in w_a^t$. Let

$$t^* = (\mu t)(t > t_k(i^*)\ \&\ \pi_k^t f_k(i^*) > 0\ \&\ p_k^t f_k(i^*)\in w_a^t).$$

Now $i^*<t_k(i^*)<t^*$, and $t>t_k(t^*-1)>t^*-1\rightarrow t\geqq t^*$, so that

$$(t)(t > t_k(t^*-1)\rightarrow (Ej)_{j<t^*}(p_k f_k(j)\in w_a^t)).$$

Thus for $i=t^*$, Case 1 above fails to occur and $f_k(t^*)$ is undefined, again contradicting the assumption that $\sigma_k=E$.

The set $\{p_k f_k(i)\mid i\in E\}$ is evidently r.e.; it is infinite because of (a) and the fact that p_k is 1–1 and f_k strictly increasing; and by (b), it is a subset of w_a'. Hence σ_k being infinite is inconsistent with the hypothesis that w_a' is immune.

Let M_k denote the largest element of σ_k.

(2.3.3) $$(Ej)_{j\in\sigma_k}(f_k(j)\notin \text{domain}\, p_k\ \vee\ p_k f_k(j)\in w_a).$$

Proof. Assume not. Then for all $j\leqq M_k$, $f_k(j)\in\text{domain}\, p_k$ and $p_k f_k(j)\notin w_a$. Choose $t>t_k(M_k)$ so large that $\pi_k^t f_k(j)>0$ for all $j\leqq M_k$. Then for this t, we have

$$(j)_{j\leqq M_k}(\pi_k^t f_k(j) > 0\ \&\ p_k^t f_k(j)\notin w_a^t),$$

so that Case 1 of the definition of $f_k(i)$ occurs for $i=M_k+1$. Then $M_k+1\in\sigma_k$, which is a contradiction.

Let $m_k=(\mu j)_{j\in\sigma_k}(f_k(j)\notin\text{domain}\, p_k\vee p_k f_k(j)\in w_a)$.

(2.3.4) $$(i)_{i\leqq m_k}(t)_{t\geqq t_k(i)}(j)_{j\leqq i}(x(j,k,t) = f_k(j))$$

(i.e., for all sufficiently large t,

$$R_k-\gamma^{t-1} = \{f_k(0),\ldots,f_k(i), x(i+1,k,t), x(i+2,k,t),\ldots\}).$$

Proof. By induction on i and $t-t_k(i)$:

(a_0) For $t=t_k(0)$, we have

$$\begin{aligned} f_k(0) &= (\mu x)(x\in R_k-\gamma^{u_k})\\ &= x(0,k,u_k+1) = x(0,k,t_k(0)). \end{aligned}$$

(b_0) Now assume that $f_k(0)=x(0,k,t)$ for some $t\geqq t_k(0)$. Then

$$f_k(0) = (\mu x)(x\in R_k-\gamma^{t-1})$$

and, since $R_k-\gamma^t\subseteq R_k-\gamma^{t-1}$, proving $f_k(0)=x(0,k,t+1)$ reduces to showing that $f_k(0)\notin\gamma^t-\gamma^{t-1}$. Assume otherwise. Then $f_k(0)\in\bigcup_{j\leqq t}\{F(j,t), G(j,t), H(j,t)\}$.

Case 1. $(Ej)_{j\leqq t}(f_k(0)=F(j,t)\vee f_k(0)=H(j,t))$. Then, since $f_k(0)\in R_k$ and $t\geqq t_k(0)>u_k$ we deduce from (2.3.1) and the definition of u_k that $f_k(0)\in\gamma^{u_k}\subseteq\gamma^{t-1}$, which contradicts the induction hypothesis.

Case 2. Otherwise. Then $(Ej)_{j\leqq t}(f_k(0)=G(j,t))$. Now $0\in\gamma^{t-1}$ and $f_k(0)=x(0,k,t)\in R_k-\gamma^{t-1}$ implies $f_k(0)\neq 0$, so that by (R5), $f_k(0)\in R_j$. Since $R_j\cap R_k=\varnothing$ for $j\neq k$, this implies $j=k$. So $f_k(0)=G(k,t)=x(n+1,k,t)$ where $n=n(k,t)\geqq 0$. Then by (R1), $f_k(0)=x(n+1,k,t)>x(0,k,t)=f_k(0)$, which is a contradiction.

Now assume that $0<i\leqq m_k$ and that the statement holds for $i-1$, i.e., that $(t)_{t\geqq t_k(i-1)}(j)_{j<i}(x(j,k,t)=f_k(j))$.

(a_i) For $t=t_k(i)$, recall that $x(i,k,t)=(\mu x)(x>x(i-1,k,t)\ \&\ x\in R_k-\gamma^{t-1})$. Since $t_k(i)>t_k(i-1)$, the induction hypothesis yields $x(i-1,k,t)=f_k(i-1)$, so that

$$x(i,k,t_k(i)) = (\mu x)(x>f_k(i-1)\ \&\ x\in R_k-\gamma^{t_k(i)-1}) = f_k(i)$$

by definition of $f_k(i)$.

(b_i) Now assume that $x(i,k,t)=f_k(i)$ for some $t\geqq t_k(i)$. Then

$$\begin{aligned} f_k(i) &= (\mu x)(x>x(i-1,k,t)\ \&\ x\in R_k-\gamma^{t-1}) \\ &= (\mu x)(x>f_k(i-1)\ \&\ x\in R_k-\gamma^{t-1}) \end{aligned}$$

by the induction hypothesis (on i) and the fact that $t\geqq t_k(i)>t_k(i-1)$. Then to prove that $f_k(i)=x(i,k,t+1)$ it again suffices to show that $f_k(i)\notin\gamma^t-\gamma^{t-1}$. Assume the contrary; then

$$f_k(i)\in\bigcup_{j\leqq t}\{F(j,t),G(j,t),H(j,t)\}.$$

Case 1. $(Ej)_{j\leqq t}(f_k(i)=F(j,t)\vee f_k(i)=H(j,t))$. Then since $t\geqq t_k(i)>t_k(0)=u_k$ we obtain as in (b_0) above that $f_k(i)\in\gamma^{u_k}\subseteq\gamma^{t-1}$, which is a contradiction.

Case 2. Otherwise. Then $(Ej)_{j\leqq t}(f_k(i)=G(j,t))$. Again as in ($b_0$) above, we conclude that $j=k$ and that $f_k(i)=G(k,t)=x(n+1,k,t)$ where $n=n(k,t)$. From $f_k(i)=x(i,k,t)$ we then deduce $i=n+1$ or $n(k,t)=i-1$. By (R4) above, $G(k,t)=f_k(i)\neq 0$, which, together with the induction hypothesis that

$$f_k(i-1) = x(i-1,k,t) = x(n,k,t),$$

implies that $\pi_k^t f_k(i-1)=0\vee p_k^t f_k(i-1)\in w_a^t$. We show this leads to a contradiction. Now by definition of $t_k(i)$, $\pi_k^{t_k(i)}f_k(i-1)>0$ and, since π_k^t is a nondecreasing function of t, $t\geqq t_k(i)$ implies $\pi_k^t f_k(i-1)>0$. It follows that $p_k f_k(i-1)=p_k^t f_k(i-1)\in w_a^t\subseteq w_a$; then by definition of m_k, $m_k\leqq i-1$, which contradicts the assumption that $0<i\leqq m_k$. This completes the induction and the proof of (2.3.4).

(2.3.5)

(a) $\lim_t n(k,t) = m_k$,

(b) $R_k\cap\gamma' = \{f_k(0),\ldots,f_k(m_k)\}$.

Proof. (2.3.4) for $i=m_k$ yields

$$(j)_{j\leqq m_k}(t)_{t\geqq t_k(m_k)}(x(j,k,t) = f_k(j)).$$

Recall that $m_k = (\mu i)(f_k(i) \notin \text{domain } p_k \vee p_k f_k(i) \in w_a)$, and choose v so large that

(i) $\pi_k^v f_k(j) > 0$ for $0 \leqq j < m_k$,

(ii) $v \geqq t_k(m_k) > m_k$,

(iii) if $f_k(m_k) \in \text{domain } p_k$, then $\pi_k^v f_k(m_k) > 0$ and $p_k^v f_k(m_k) \in w_a^v$.

It is then evident from the definition of $n(k, t)$ that $n(k, t) = m_k$ for all $t \geqq v$, which proves (a). To prove (b), note that by (2.3.4), $v \geqq t_k(m_k)$ implies

$$(j)_{j \leqq m_k}(t)_{t \geqq v}(f_k(j) = x(j, k, t)),$$

so that $\{f_k(0), \ldots, f_k(m_k)\} \subseteq \bigcap_{t \geqq v} R_k - \gamma^t = \bigcap_t R_k - \gamma^t = R_k \cap \gamma'$. To prove the converse inclusion, assume $x \in R_k \cap \gamma'$. Then $x \in R_k - \gamma^{v-1}$, which implies $x = x(j, k, v)$ for some j. But it is easily seen by induction on p that if

$$x = x(m_k + p + 1, k, v) \qquad \text{then} \qquad x = G(k, v+p) \in \gamma^{v+p} \subseteq \gamma.$$

So $x = x(j, k, v) \in R_k - \gamma$ only if $j \leqq m_k$, in which case $x \in \{f_k(0), \ldots, f_k(m_k)\}$.

Proof of Lemma 2.3. Part (a) follows from (2.3.5a). Part (b) follows from (2.3.5b). To prove (c), note that by (2.3.5b), $f_k(m_k) \in \gamma'$ while by definition of m_k, either $f_k(m_k) \notin \text{domain } p_k$ or $p_k f_k(m_k) \in w_a$.

LEMMA 2.4. *Assume that* $0 < k$ *and that for all* $j < k$, $\lim_t n(j, t) = m_j$ *exists and* $c(R_j \cap \gamma') = 1 + m_j$. *Then*

(a) $\lim_t H(k, t)$ *exists*,

(b) *not* $w_a' \prec_k \gamma'$.

Proof. Let $R = 1 + \sum_{j<k} 1 + m_j$ and let $z_0, \ldots, z_R$ be the least $R+1$ elements of w_a'. Choose v so large that

(i) $(t)_{t \geqq v}(j)_{j<k}(n(j, t) = m_j)$,

(ii) $(z)_{z \leqq z_R}(z \in w_a \rightarrow z \in w_a^v)$.

Then $(t)_{t>v}(i)_{i \leqq R}(r(k, t) = R \ \& \ z(i, t) = z_i)$.

Case 1. $(i)_{i \leqq R}(z_i \in \text{domain } p_k \ \& \ p_k(z_i) \in \bigcup_{j<k} R_j)$. Let

$$v^* = (\mu t)_{t>v}(i)_{i \leqq R}(\pi_k^t(z_i) > 0).$$

Then for all $t \geqq v^*$, $s(k, t) = R$ and $H(k, t) = 0$, which proves (a). To prove (b), note that the hypothesis implies that

$$c\Big(\Big(\bigcup_{j<k} R_j\Big) \cap \gamma'\Big) = c\Big(\bigcup_{j<k} (R_j \cap \gamma')\Big) = \sum_{j<k} 1 + m_j = R.$$

But $\{p_k(z_0), \ldots, p_k(z_R)\}$ is an $(R+1)$-element subset of $\bigcup_{j<k} R_j$, which then cannot be contained in γ', i.e., for some $z_i \in w_a'$, $p_k(z_i) \in \gamma$.

Case 2. Otherwise. Let $i^* = (\mu i)_{i \leqq R}(z_i \notin \text{domain } p_k \vee p_k(z_i) \in \bigcap_{j<k} R_j')$.

Subcase 2.1. $z_{i^*} \notin \text{domain } p_k$. Let $v^* = (\mu t)_{t>v}(i)_{i<i^*}(\pi_k^t(z_i) > 0)$. Then for all $t \geqq v^*$, $s(k, t) = i^*$ and $\pi_k^t z(i^*, t) = 0$, so that $H(k, t) = 0$, which proves (a). Part (b) holds, since $z_{i^*} \in w_a'$, so that $w_a' \nsubseteq \text{domain } p_k$.

Subcase 2.2. Otherwise. Then $z_{i^*} \in \text{domain } p_k$ and $p_k(z_{i^*}) \in \bigcap_{j<k} R_j'$. Let $v^* = (\mu t)_{t>v}(i)_{i \leqq i^*}(\pi_k^t(z_i) > 0)$. Then for all $t \geqq v^*$, $s(k, t) = i^*$ and $H(k, t) = p_k(z_{i^*})$, which proves (a). Part (b) follows since $z_{i^*} \in w_a'$ while $p_k(z_{i^*}) = H(k, v^*) \in \gamma^{v^*} \subseteq \gamma$.

LEMMA 2.5. *For each k, the following hold*:

(A_k) *not* $w'_a \prec_k \gamma'$,

(B_k) *not* $\gamma' \prec_k w'_a$,

(C_k) $\lim_t H(k, t)$ *exists*,

(D_k) $\lim_t n(k, t)$ *exists*,

(E_k) $c(R_k \cap \gamma') = 1 + \lim_t n(k, t)$.

Proof. By induction on k. Lemma 2.2 gives (A_0) and (C_0). Lemma 2.3 then gives (D_0), (E_0), and (B_0), which completes the base step. Now assume $0 < k$ and that (A_j), (B_j), (C_j), (D_j), and (E_j) hold for all $j < k$. Then (C_k) and (A_k) follow by Lemma 2.4, which by Lemma 2.3 implies (D_k), (E_k), and (B_k). The conclusions thus hold for all k.

LEMMA 2.6. *γ' has no infinite r.e. subset.*

Proof. Let w_k be an infinite r.e. set.

Case 1. $w_k \subseteq \bigcup_{j<k} R_j$. Then $w_k \cap \gamma' \subseteq \bigcup_{j \leqq k} (R_j \cap \gamma')$ which by parts (D) and (E) of Lemma 2.5 is a finite union of finite sets. So $w_k \cap \gamma'$ is finite, which implies $w_k \cap \gamma \neq \varnothing$.

Case 2. Otherwise. Then $w_k - \bigcup_{j \leqq k} R_j \neq \varnothing$, which by Lemma 2.1(b) implies $w_k \cap \gamma \neq \varnothing$.

End of proof of Theorem 2. Let $X = \langle w'_a \rangle \in \Lambda_z - E$. The set γ constructed above is evidently r.e., since effective instructions were given for generating it. Let $Y = \langle \gamma' \rangle$; then by Lemma 2.5(A) and (B), $Y \nleqq X$ and $X \nleqq Y$. By Lemma 1.1, this implies $Y \notin E$, so that γ' is infinite. Then by Lemma 2.6, γ' is immune and $Y \in \Lambda_z$. This completes the proof of the theorem.

Note that the instructions for generating γ are uniform in a, so that an index of Y can be effectively computed given one of X; i.e., there is a recursive function h such that if $X = \langle w'_a \rangle \in \Lambda_z - E$ and $Y = \langle w'_{h(a)} \rangle$, then $Y \in \Lambda_z$ and Y is incomparable to X. In [3], a sequence $X_1, X_2, \ldots$ of elements of Λ_z is called *r.e.* if there is a recursive function f such that $X_i = \langle w'_{f(i)} \rangle$ for each $i \in E$, and a Gödel number of f is called an *index* of $\{X_i\}$. A set P of elements of Λ_z is called *productive* if there is a recursive function f such that if $\{X_i\}$ is an r.e. sequence of elements of P with index e, then $\langle w'_{f(e)} \rangle \in P - \{X_i\}$. Using standard methods of "interweaving priorities," the techniques of Theorem 2 can be modified so as to effectively produce, given an r.e. sequence $\{X_i\} \subseteq \Lambda_z - E$, an element $Y \in \Lambda_z$ which is incomparable to X_i for each i. This leads to a proof of

THEOREM 3. *Every maximal set of mutually incomparable elements of $\Lambda_z - E$ is productive.*

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MOUNT HOLYOKE COLLEGE,
SOUTH HADLEY, MASSACHUSETTS