ON THE ALTERNATING PROJECTIONS THEOREM AND BIVARIATE STATIONARY STOCHASTIC PROCESSES

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Summary. In this paper we shall first use the theorem of von Neumann on alternating projections to obtain an algorithm for finding the projection of an element x in a Hilbert space $\mathcal{H}$ onto the subspace spanned by $\mathcal{H}$ -valued orthogonally scattered measures ξ_1 and ξ_2 . We then specialize this algorithm to the case that ξ_1 and ξ_2 are the canonical measures of the components of a bivariate stationary stochastic process (SP), and thereby get an algorithm for finding the best linear predictor in the time domain.

1. Preliminary results. In this section we state some results which are used in later sections.

The following theorem is due to J. von Neumann (cf. [7, p. 55]).

THEOREM 1.1 (ALTERNATING PROJECTIONS). *Let P_1 , P_2 , and T be projection operators on a Hilbert space $\mathcal{H}$ onto the subspaces $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_1 \cap \mathcal{M}_2$. If T_n is the n th term of either of the sequences*

$$P_1, P_2P_1, P_1P_2P_1, P_2P_1P_2P_1, \dots,$$

$$P_2, P_1P_2, P_2P_1P_2, P_1P_2P_1P_2, \dots,$$

then $T_n \rightarrow T$ strongly⁽²⁾, as $n \rightarrow \infty$.

This at once yields the following corollary, which we need later.

COROLLARY 1.2. *With the notation of 1.1, if P is the projection operator on $\mathcal{H}$ onto the subspace $\mathfrak{S}(\mathcal{M}_1 + \mathcal{M}_2)$ which is spanned by $\mathcal{M}_1$ and $\mathcal{M}_2$, then*

$$P = P_1 + P_2 - P_1P_2 - P_2P_1 + P_1P_2P_1 + P_2P_1P_2 - \dots$$

the convergence being in the strong sense.

The next theorem is due to A. S. Besicovitch (cf. [1, p. 9]).

THEOREM 1.3. *Let (i) μ be a bounded, countably additive (c.a.), nonnegative measure defined on the family $\mathcal{B}$ of Borel subsets of the real line R .*

(ii) ν be a c.a., complex-valued measure on $\mathcal{B}$.

(iii) $D(\lambda, h) = \{\nu(\lambda - h, \lambda + h) / \mu(\lambda - h, \lambda + h)\} \chi_{\sigma(\mu)}(\lambda)$, where $\sigma(\mu)$ is the spectrum of μ , i.e., $\sigma(\mu) = \{\lambda: \text{for each } h > 0, \mu(\lambda - h, \lambda + h) > 0\}$. Then (a) the limit $D(\lambda, h)$, as $h \rightarrow 0$, exists (finite) almost everywhere with respect to μ (a.e. μ).

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⁽²⁾ I.e., for each x in $\mathcal{H}$, $|T_n x - Px| \rightarrow 0$, as $n \rightarrow \infty$, $| \cdot |$ being the norm in $\mathcal{H}$.

(b) If ν is absolutely continuous (a.c.) with respect to μ and if the Radon-Nikodym derivative of ν with respect to μ is $d\nu/d\mu$, then

$$(d\nu/d\mu)(\cdot) = \lim D(\cdot, h), \text{ as } h \rightarrow 0, \text{ a.e. } \mu.$$

We shall denote $\lim D(\cdot, h)$, as $h \rightarrow 0$, by $(D\nu/D\mu)(\cdot)$ and shall call it the *Besicovitch derivative* of ν with respect to μ .

The following is an immediate consequence of 1.3.

COROLLARY 1.4. Let (i) M_i ($i=1, 2$) be a σ -finite, c.a., nonnegative measure on the family $\mathcal{B}$ of Borel subsets of the real line R .

(ii) $\mathcal{B}_1 = \{B: B \in \mathcal{B}_1 \text{ and } M_1(B) < \infty\}$, M_{12} the product measure generated by M_1 and M_2 , and $\mathcal{B}_{12} = \{B: B \text{ is a Borel subset of the complex plane } R^2 \text{ and } M_{12}(B) < \infty\}$.

(iii) μ be a c.a. measure on the ring $\mathcal{B}_{12}$ which is a.c. with respect to M_{12} .

(iv) $d\mu/dM_{12}$ be any version of the Radon-Nikodym derivative of μ with respect to M_{12} , which is defined throughout R^2 .

(v) For each $B \in \mathcal{B}_1$ and each $t \in R$,

$$p(B, t) = \int_B (d\mu/dM_{12})(s, t) M_1(ds).$$

Then (a) for each $B \in \mathcal{B}_1$, $\mu_B(\cdot) = \mu(B \times \cdot)$ is a c.a. measure on $\mathcal{B}_2$ which is a.c. with respect to M_2 .

(b) For each $B \in \mathcal{B}_1$, $p(B, \cdot)$ is a version of $d\mu_B/dM_2$, the domain of which is R .

(c) For each $t \in R$, $p(\cdot, t)$ is a c.a. measure on $\mathcal{B}_1$, which is a.c. with respect to M_1 .

(d) Analogous results to (a), (b), and (c) hold for ν_B , where for each $B \in \mathcal{B}_2$, $\nu_B(\cdot) = \mu(\cdot \times B)$.

2. Orthogonally scattered, $\mathcal{H}$ -valued measures. We first state a few results concerning measures whose values are orthogonally scattered in a given Hilbert space. These are special versions of more general results contained in [9], [10], and [11].

DEFINITION 2.1. Let (i) M be a σ -finite, c.a., nonnegative measure on the family $\mathcal{B}$ of Borel subsets of the real line R .

(ii) $\mathcal{B}_0 = \{B: B \in \mathcal{B} \text{ and } M(B) < \infty\}$.

Then a function ξ on $\mathcal{B}_0$ into a given Hilbert space $\mathcal{H}$ such that for each pair of $B, C \in \mathcal{B}_0$,

$$(\xi(B), \xi(C)) = M(B \cap C)$$

is called an *orthogonally scattered (o.s.) $\mathcal{H}$ -valued measure* over $(R, \mathcal{B})$. M is called the *spectral measure* associated with ξ . It follows immediately that $(\xi(B), \xi(C)) = 0$ if B and C are disjoint sets in $\mathcal{B}_0$, and $\xi(\bigcup_k B_k) = \sum_k \xi(B_k)$ if B_k 's are disjoint sets in $\mathcal{B}_0$ whose union $\bigcup_k B_k$ is in $\mathcal{B}_0$.

There is a well-established theory of integration with respect to an o.s. $\mathcal{H}$ -valued measure ξ whose associated spectral measure is σ -finite (cf. [10] and [11]). We will briefly state the main result.

THEOREM 2.2. *If $\mathcal{S}$ is the set of all stochastic integrals $\int_R \phi d\xi$ then*

(a) $\int_R \phi d\xi$ exists iff $\int_R |\phi|^2 dM < \infty$.

(b) $\int_R (a\phi + b\psi) d\xi = a \int_R \phi d\xi + b \int_R \psi d\xi$, where a and b are complex numbers.

(c) $\mathcal{S}$ = the cyclic subspace spanned by ξ , i.e.,

$$\mathcal{S} = \mathfrak{C}\{\xi(B); B \in \mathcal{B}_0\}.$$

(d) $(\int_R \phi d\xi, \int_R \psi d\xi) = \int_R \phi \bar{\psi} dM$.

(e) The correspondence $\phi \rightarrow \int_R \phi d\xi$ is an isomorphism from $L_2(R, \mathcal{B}, M)$ onto $\mathcal{S}$ such that (a) and (b) hold, and therefore

$$\left| \int_R \phi d\xi \right|^2 = \int_R |\phi|^2 dM.$$

Let x be in $\mathcal{H}$ and $\{\xi(m)\}$ be an orthonormal basis for a subspace $\mathcal{M}$ of $\mathcal{H}$. Then $\sum_m |(x, \xi(m))|^2 < \infty$ and the orthogonal projection of x onto $\mathcal{M}$ is given by $\sum_m (x, \xi(m))\xi(m)$. The following theorem generalizes this result.

THEOREM 2.3. *Let (i) $x \in \mathcal{H}$ and ξ be an o.s. $\mathcal{H}$ -valued measure over $(R, \mathcal{B})$ whose associated spectral measure M is σ -finite.*

(ii) P be the projection operator on $\mathcal{H}$ onto the cyclic subspace $\mathcal{M}$ spanned by the measure ξ , i.e., $\mathcal{M} = \mathfrak{C}\{\xi(B); B \in \mathcal{B}_0\}$, where $\mathcal{B}_0 = \{B \in \mathcal{B} \text{ \& } M(B) < \infty\}$.

Then (a) the complex-valued functions (x, ξ) and (ξ, x) defined by $(x, \xi)(\cdot) = (x, \xi(\cdot))$ and $(\xi, x)(\cdot) = (\xi(\cdot), x)$ are c.a. measures over $(R, \mathcal{B})$ which are a.c. with respect to M .

(b) $(x, \xi) = (\xi, x)^*$ and $d(x, \xi)/dM = \{d(\xi, x)/dM\}^*$.

(c) $d(x, \xi)/dM$ and $d(\xi, x)/dM \in L_2(R, \mathcal{B}, M)$, and

$$Px = \int_R \{d(x, \xi)/dM\} d\xi.$$

Proof. Since the proof of (a) and (b) is trivial, we proceed to give the proof of (c). By 2.2(c), there exists a complex-valued function ϕ such that

$$(1) \quad Px = \int_R \phi d\xi \quad \& \quad \phi \in L_2(R, \mathcal{B}, M).$$

Now by (1) and 2.2(d), for each $B \in \mathcal{B}_0$

$$\begin{aligned} (x, \xi)(B) &= (x, \xi(B)) = (Px, \xi(B)) \\ &= \left(\int_R \phi d\xi, \int_R \chi_B d\xi \right) = \int_B \phi dM. \end{aligned}$$

It immediately follows from the Besicovitch Theorem 1.3 that

$$(2) \quad d(x, \xi)/dM = \phi \text{ a.e. } M.$$

By (b), (1), and (2) the result follows. (Q.E.D.)

COROLLARY 2.4. Let (i) ξ be an o.s. $\mathcal{H}$ -valued measure over $(R, \mathcal{B})$ whose associated spectral measure M is σ -finite.

(ii) $\phi \in L_2(R, \mathcal{B}, M)$.

Then (a) for each $x \in \mathcal{H}$, the complex-valued function (ξ, x) defined in 2.3 is a c.a. measure over $(R, \mathcal{B})$ which is a.c. with respect to M .

(b) $\phi \in L_1[R, \mathcal{B}, (\xi, x)]$, and

$$\left(\int_R \phi(\omega) \xi(d\omega), x \right) = \int_R \phi(\omega) (\xi, x)(d\omega).$$

Proof. The proof of (a) follows from 2.3(a). Let P denote the projection operator on $\mathcal{H}$ onto the cyclic subspace $\mathcal{M}$ spanned by the measure ξ . Since $\phi \in L_2(R, \mathcal{B}, M)$ and by 2.3(c), $d(\xi, x)/dM \in L_2(R, \mathcal{B}, M)$, therefore $\phi d(x, \xi)/dM \in L_1(R, \mathcal{B}, M)$ or equivalently $\phi \in L_1[R, \mathcal{B}, (\xi, x)]$. To prove the second part of (c), we observe that by 2.2(c), $\int_R \phi d\xi \in \mathcal{M}$ and therefore

$$(1) \quad \left(\int_R \phi d\xi, x \right) = \left(\int_R \phi d\xi, Px \right).$$

Now by 2.2(d), 2.3(b), and 2.3(c),

$$(2) \quad \begin{aligned} \left(\int_R \phi d\xi, Px \right) &= \left(\int_R \phi d\xi, \int_R \frac{d(x, \xi)}{dM} d\xi \right) = \int_R \phi(\omega) \frac{(d(x, \xi))^*}{dM}(\omega) M(d\omega) \\ &= \int_R \phi(\omega) \frac{d(\xi, x)}{dM}(\omega) M(d\omega) = \int_R \phi(\omega) (\xi, x)(d\omega). \end{aligned}$$

By (1) and (2) we get the result. (Q.E.D.)

3. The alternating projections theorem and $\mathcal{H}$ -valued measures. Let $\{\xi_1(m)\}$ and $\{\xi_2(m)\}$ be orthonormal bases for the subspaces $\mathcal{M}_1$ and $\mathcal{M}_2$ of a given Hilbert space $\mathcal{H}$. Let P_1 , P_2 , and P be projection operators on $\mathcal{H}$ onto the subspaces $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathfrak{S}(\mathcal{M}_1 + \mathcal{M}_2)$. From 1.2 it follows that for each x in $\mathcal{H}$

$$(A) \quad \begin{aligned} Px &= \sum_m (x, \xi_1(m)) \xi_1(m) + \sum_m (x, \xi_2(m)) \xi_2(m) \\ &- \sum_n \left\{ \sum_m (x, \xi_2(m)) (\xi_2(m), \xi_1(n)) \right\} \xi_1(n) \\ &- \sum_n \left\{ \sum_m (x, \xi_1(m)) (\xi_1(m), \xi_2(n)) \right\} \xi_2(n) \\ &+ \dots \end{aligned}$$

Now an orthonormal basis is just an o.s. $\mathcal{H}$ -valued measure over $(R, \mathcal{B})$ which is concentrated on the set of integers. In many situations it is more natural to think of a subspace $\mathcal{M}$ as being spanned by such a measure than by some basis, and to represent a vector in $\mathcal{H}$ by an integral rather than by a sum. (This happens, for instance, in the theory of the Fourier integral.)

We shall generalize equation (A) to the case in which $\mathcal{M}_1$ and $\mathcal{M}_2$ are spanned by such measures ξ_1 and ξ_2 . We shall show that under certain conditions we get instead of (A):

$$(B) \quad \begin{aligned} Px = & \int_R \frac{d(x, \xi_1)}{dM_1}(s) \xi_1(ds) + \int_R \frac{d(x, \xi_2)}{dM_2}(s) \xi_2(ds) \\ & - \int_R \left\{ \int_R \frac{d(x, \xi_2)}{dM_2}(s) \frac{d(\xi_2, \xi_1)}{dM_{21}}(s, t) M_2(ds) \right\} \xi_1(dt) \\ & - \int_R \left\{ \int_R \frac{d(x, \xi_1)}{dM_1}(s) \frac{d(\xi_1, \xi_2)}{dM_{12}}(s, t) M_1(ds) \right\} \xi_2(dt) + \dots \end{aligned}$$

where M_{12}, M_{21} are the product measures $M_1 \times M_2, M_2 \times M_1$, and $d(\cdot)/d$ denotes the Radon-Nikodym derivative.

NOTATION 3.1. $\mathcal{B}$ will denote the family of Borel subsets of the real line R . $\mathcal{H}$ will denote a fixed Hilbert space. ξ_i ($i=1, 2$) will denote an o.s. $\mathcal{H}$ -valued measure over $(R, \mathcal{B})$ whose associated spectral measure M_i ($i=1, 2$) is σ -finite. We set

$$\mathcal{B}_i = \{B : B \in \mathcal{B} \text{ \& } M_i(B) < \infty\}, \quad 1 \leq i \leq 2,$$

$$\mathcal{B} \times \mathcal{B} = \text{the family of Borel subsets of the complex plane } R^2.$$

$$\mathcal{B}_i \times \mathcal{B}_j = \{B \times C : B \in \mathcal{B}_i \text{ \& } C \in \mathcal{B}_j\}, \quad 1 \leq i, j \leq 2,$$

$$\mathcal{R}_{ij} = \text{the ring generated by } \mathcal{B}_i \times \mathcal{B}_j, \quad 1 \leq i, j \leq 2,$$

$$M_{ij} = \text{the product measure } M_i \times M_j \text{ on } \mathcal{B} \times \mathcal{B}, \quad 1 \leq i, j \leq 2,$$

$$\mathcal{B}_{ij} = \{B : B \in \mathcal{B} \times \mathcal{B} \text{ \& } M_{ij}(B) < \infty\}, \quad 1 \leq i, j \leq 2.$$

To be able to carry out our work we need to study the complex-valued measure (ξ_i, ξ_j) which is generated by ξ_i and ξ_j . Our definition of this measure will be given in several steps.

DEFINITION 3.2. We define (ξ_i, ξ_j) on $\mathcal{B}_i \times \mathcal{B}_j$ by:

$$(\xi_i, \xi_j)(B \times C) = (\xi_i(B), \xi_j(C)).$$

A usual argument may be used to prove the following lemma.

LEMMA 3.2. Let $\{B_k\}_{k=1}^m$ and $\{C_l\}_{l=1}^n$ be finite sequences of sets in $\mathcal{B}_i \times \mathcal{B}_j$ such that $\bigcup_{k=1}^m B_k = \bigcup_{l=1}^n C_l$. Then

$$\sum_{k=1}^m (\xi_i, \xi_j)(B_k) = \sum_{l=1}^n (\xi_i, \xi_j)(C_l).$$

Since every $B \in \mathcal{R}_{ij}$ is a finite, disjoint union of sets in $\mathcal{B}_i \times \mathcal{B}_j$ (cf. [3, p. 139]), from 3.2 it follows that (ξ_i, ξ_j) can be defined on $\mathcal{R}_{ij}$ in the following way.

DEFINITION 3.3. Let $B \in \mathcal{R}_{ij}$. Then we define

$$(\xi_i, \xi_j)(B) = \sum_{k=1}^n (\xi_i, \xi_j)(B_k),$$

where the B_k 's are any finite sequence of disjoint sets in $\mathcal{B}_i \times \mathcal{B}_j$ such that $B = \bigcup_{k=1}^n B_k$.

The next lemma whose proof is easily seen shows that (ξ_i, ξ_j) is finitely additive on $\mathcal{R}_{ij}$.

LEMMA 3.4. (a) (ξ_i, ξ_j) is finitely additive on $\mathcal{R}_{ij}$ and is a.c. with respect to M_{ij} on $\mathcal{R}_{ij}$.

(b) (ξ_i, ξ_j) is the unique finitely additive extension of the measure defined in 3.2 on $\mathcal{B}_i \times \mathcal{B}_j$ to $\mathcal{R}_{ij}$.

To proceed further, some restriction has to be imposed on ξ_1 and ξ_2 .

ASSUMPTION 3.5. There exist functions φ_{12} and φ_{21} on R^2 such that

- (1) $\varphi_{ij}(\cdot, t) \in L_2(R, \mathcal{B}, M_i)$ a.e. $t(M_j)$,
- M_j -ess. $\text{lub}_{t \in R} |\varphi_{ij}(\cdot, t)|_{2, M_i} < \infty$ ⁽³⁾.
- (2) For each $B \in \mathcal{B}_{ij}$, $\chi_B \varphi_{ij} \in L_1(R^2, \mathcal{B} \times \mathcal{B}, M_{ij})$.
- (3) For each $B \times C \in \mathcal{B}_i \times \mathcal{B}_j$

$$(\xi_i(B), \xi_j(C)) = \iint_{B \times C} \varphi_{ij}(s, t) M_{ij}(ds, dt).$$

We will see later (cf. 4.5) that in many situations Assumption 3.5 is satisfied.

LEMMA 3.6. Let (i) ξ_1 and ξ_2 be o.s. $\mathcal{H}$ -valued measures over $(R, \mathcal{B})$ satisfying Assumption 3.5. Then (a) the measure (ξ_i, ξ_j) on $\mathcal{R}_{ij}$, introduced in 3.3, has a c.a. extension μ_{ij} to $\mathcal{B}_{ij}$ such that for each $B \in \mathcal{B}_{ij}$,

$$\mu_{ij}(B) = \iint_B \varphi_{ij}(s, t) M_{ij}(ds, dt).$$

- (b) μ_{ij} is a.c. with respect to M_{ij} and $d\mu_{ij}/dM_{ij} = \varphi_{ij}$ a.e. M_{ij} .

DEFINITION 3.7. We define (ξ_i, ξ_j) on $\mathcal{B}_{ij}$ to be the μ_{ij} of the last lemma.

In the next theorem we state some consequences which occur when we have a c.a. measure (ξ_i, ξ_j) as above. The proof is immediate from 1.4.

THEOREM 3.8. With the notation of 3.1, if (ξ_i, ξ_j) is any c.a. measure on $\mathcal{B}_{ij}$ which is a.c. with respect to M_{ij} such that for each $B \times C \in \mathcal{B}_i \times \mathcal{B}_j$, $(\xi_i, \xi_j)(B \times C) = (\xi_i(B), \xi_j(C))$. Then (a) for each $B \in \mathcal{B}_j$, $(\xi_i, \xi_j(B))$ is a c.a. measure on $\mathcal{B}_i$ which is a.c. with respect to M_i .

- (b) For each $B \in \mathcal{B}_j$, $\int_B \{d(\xi_i, \xi_j)/dM_{ij}\}(\cdot, t) M_j(dt)$ is a version of $d(\xi_i, \xi_j(B))/dM_i$.
- (c) For almost all $s(M_i)$, $\{d(\xi_i, \xi_j(\cdot))/dM_i\}(s)$ is a c.a. measure on $\mathcal{B}_j$ which is a.c. with respect to M_j .

- (d) For each $B \in \mathcal{B}_i$ similar results hold for the measure $(\xi_i(B), \xi_j)$.

We shall now state the main theorem of this section.

⁽³⁾ By definition $|\varphi_{ij}(\cdot, t)|_{2, M_i} = \left[\int_R |\varphi_{ij}(s, t)|^2 M_i(ds) \right]^{1/2}$.

THEOREM 3.9. *With the notation of 3.1, if ξ_i ($i=1, 2$) are o.s. $\mathcal{H}$ -valued measures over $(R, \mathcal{B})$ satisfying Assumption 3.5, and if (ξ_i, ξ_j) is the c.a. measure on $\mathcal{B}_{ij}$ given by 3.7, then for each x in $\mathcal{H}$*

$$\begin{aligned}
 (a) \quad & d(x, \xi_i)/dM_i \in L_2(R, \mathcal{B}, M_i), \\
 & \frac{d(x, \xi_i)}{dM_i}(\cdot) \frac{d(\xi_i, \xi_j)}{dM_{ij}}(\cdot, t) \in L_1(R, \mathcal{B}, M_i) \text{ a.e. } t(M_j), \\
 & \int_R \frac{d(x, \xi_i)}{dM_i}(s) \frac{d(\xi_i, \xi_j)}{dM_{ij}}(s, \cdot) M_i(ds) \in L_2(R, \mathcal{B}, M_j), \\
 & \left\{ \int_R \frac{d(x, \xi_i)}{dM_i}(s) \frac{d(\xi_i, \xi_j)}{dM_{ij}}(s, \cdot) M_i(ds) \right\} \frac{d(\xi_j, \xi_i)}{dM_{ji}}(\cdot, u) \in L_1(R, \mathcal{B}, M_j) \text{ a.e. } u(M_i), \\
 & \int_R \left\{ \int_R \frac{d(x, \xi_i)}{dM_i}(s) \frac{d(\xi_i, \xi_j)}{dM_{ij}}(s, t) M_i(ds) \right\} \frac{d(\xi_j, \xi_i)}{dM_{ji}}(t, \cdot) M_j(dt) \in L_2(R, \mathcal{B}, M_i), \dots
 \end{aligned}$$

(b) *If P_i ($i=1, 2$) is the projection operator onto the cyclic subspaces $\mathcal{M}_i$ ($i=1, 2$) which is spanned by ξ_i ($i=1, 2$), then*

$$\begin{aligned}
 P_i x &= \int_R \frac{d(x, \xi_i)}{dM_i}(s) \xi_i(ds), \\
 P_j P_i x &= \int_R \left\{ \int_R \frac{d(x, \xi_i)}{dM_i}(s) \frac{d(\xi_i, \xi_j)}{dM_{ij}}(s, t) M_i(ds) \right\} \xi_j(dt), \\
 P_i P_j P_i x &= \int_R \left\{ \int_R \left\{ \int_R \frac{d(x, \xi_i)}{dM_i}(s) \frac{d(\xi_i, \xi_j)}{dM_{ij}}(s, t) \right\} \frac{d(\xi_j, \xi_i)}{dM_{ji}}(t, u) M_j(du) \right\} \xi_i(du), \dots
 \end{aligned}$$

Proof. For simplicity let $i=1$ and $j=2$. By 2.3,

$$(1) \quad \frac{d(x, \xi_1)}{dM_1} \in L_2(R, \mathcal{B}, M_1) \quad \text{and} \quad P_1 x = \int_R \frac{d(x, \xi_1)}{dM_1}(s) \xi_1(ds).$$

Replacing x by $P_1 x$ and ξ_1 by ξ_2 in (1) we get

$$(2) \quad \frac{d(P_1 x, \xi_2)}{dM_2} \in L_2(R, \mathcal{B}, M_2) \quad \text{and} \quad P_2 P_1 x = \int_R \frac{d(P_1 x, \xi_2)}{dM_2}(t) \xi_2(dt).$$

Let $B \in \mathcal{B}_2$. Then we have

$$\begin{aligned}
 (P_1 x, \xi_2(B)) &= \left(\int_R \frac{d(x, \xi_1)}{dM_1}(s) \xi_1(ds), \xi_2(B) \right) && \text{by (1)} \\
 &= \int_R \frac{d(x, \xi_1)}{dM_1}(s) (\xi_1(ds), \xi_2(B)) && \text{by 2.4} \\
 (3) \quad &= \int_R \frac{d(x, \xi_1)}{dM_1}(s) \frac{d(\xi_1, \xi_2(B))}{dM_1}(s) M_1(ds) && \text{by 3.8(a)} \\
 &= \int_R \frac{d(x, \xi_1)}{dM_1}(s) \left\{ \int_B \frac{d(\xi_1, \xi_2)}{dM_{12}}(s, t) M_2(dt) \right\} M_1(ds) && \text{by 3.8(b)} \\
 &= \int_B \left\{ \int_R \frac{d(x, \xi_1)}{dM_1}(s) \frac{d(\xi_1, \xi_2)}{dM_{12}}(s, t) M_1(ds) \right\} M_2(dt),
 \end{aligned}$$

where in the last step, the change of order of integration can be justified as follows:

By the Schwarz inequality, we have

$$\begin{aligned} \int_R \left| \frac{d(x, \xi_1)}{dM_1}(s) \right| \left| \frac{d(\xi_1, \xi_2)}{dM_{12}}(s, t) \right| M_1(ds) \\ \leq \left[\int_R \left| \frac{d(x, \xi_1)}{dM_1}(s) \right|^2 M_1(ds) \right]^{1/2} \times \left[\int_R \left| \frac{d(\xi_1, \xi_2)}{dM_{12}}(s, t) \right|^2 M_1(ds) \right]^{1/2} \leq CK, \end{aligned}$$

where by (1), $C = [\int_R |(d(x, \xi_1)/dM_1)(s)|^2 M_1(ds)]^{1/2} < \infty$, and by our hypothesis and Assumption 3.5, $K = M_2 - \text{ess. lub.}_{t \in R} |(d(\xi_1, \xi_2)/dM_{12})(\cdot, t)|_{2, M_1} < \infty$. Then

$$\int_B \left\{ \int_R \left| \frac{d(x, \xi_1)}{dM_1}(s) \right| \left| \frac{d(\xi_1, \xi_2)}{dM_{12}}(s, t) \right| M_2(dt) \right\} \leq CK M_2(B) < \infty.$$

Hence Fubini's Theorem (cf. [3, p. 148]) can be applied.

Since M_2 is σ -finite, there exist countably many disjoint sets N_i such that $\sigma_{M_2} = \bigcup N_i$, where σ_{M_2} is the spectrum of M_2 (cf. 1.3(iii)), and $M_2(N_i) < \infty$. If $t \in N_i$, taking in (3) $B = N_i \cap \Delta_{t,h}$, where $\Delta_{t,h} = \{s: t-h < s < t+h\}$ we get

$$(P_1 x, \xi_2(N_i \cap \Delta_{t,h})) = \int_{N_i \cap \Delta_{t,h}} \left\{ \int_R \frac{d(x, \xi_1)}{dM_1}(s) \frac{d(\xi_1, \xi_2)}{dM_{12}}(s, u) M_1(ds) \right\} M_2(du).$$

Hence by the Besicovitch Theorem 1.3

$$\begin{aligned} (4) \quad \frac{d(P_1 x, \xi_2)}{dM_2}(t) &= \lim_{h \rightarrow 0} \frac{(P_1 x, \xi_2(N_i \cap \Delta_{t,h}))}{M_2(N_i \cap \Delta_{t,h})}, \quad \text{as } h \rightarrow 0 \\ &= \int_R \frac{d(x, \xi_1)}{dM_1}(s) \frac{d(\xi_1, \xi_2)}{dM_{12}}(s, t) M_1(ds) \quad \text{a.e. } t \in N_i(M_2). \end{aligned}$$

Since $\{N_i\}$ is a countable collection, and $M_2(R - \sigma_{M_2}) = 0$, (4) holds a.e. $t(M_2)$. By (2), we see that the function on the right-hand side of (4) is in $L_2(R, \mathcal{B}, M_2)$ and that

$$P_2 P_1 x = \int_R \left\{ \int_R \frac{d(x, \xi_1)}{dM_1}(s) \frac{d(\xi_1, \xi_2)}{dM_{12}}(s, t) M_1(ds) \right\} \xi_2(dt).$$

This completes the proof of the first relation in (a) and the first two relations in (b).

To obtain the expression for $P_2 P_1 x$ given by (b), we essentially made use of the fact that

$$M_2 - \text{ess. lub.}_{t \in R} \left| \frac{d(\xi_1, \xi_2)}{dM_{12}}(\cdot, t) \right|_{2, M_1} < \infty.$$

$$\frac{d(x, \xi_1)}{dM_1}(\cdot) \in L_2(R, \mathcal{B}, M_1),$$

$$P_1 x = \int_R \frac{d(x, \xi_1)}{dM_1} d\xi_1.$$

Now since

$$M_1 - \text{ess. lub.}_{t \in R} \left| \frac{d(\xi_2, \xi_1)}{dM_{21}}(\cdot, u) \right|_{2, M_2} < \infty,$$

$$\phi(\cdot) = \int_R \frac{d(x, \xi_1)}{dM_1}(s) \frac{d(\xi_1, \xi_2)}{dM_{12}}(s; \cdot) M_1(ds) \in L_2(R, \mathcal{B}, M_2),$$

$$P_2 P_1 x = \int_R \phi \, d\xi_2,$$

therefore in exactly the same way as before we can obtain the expression for $P_1 P_2 P_1 x$ given in (b). The rest of the proof may be completed by induction. (Q.E.D.)

The following corollary is an immediate consequence of this theorem and 1.2.

COROLLARY 3.10. *With the notation of 3.1, if ξ_i ($i=1, 2$) are o.s. $\mathcal{H}$ -valued measures satisfying Assumption 3.5 and if P_i ($i=1, 2$) and P are the projection operators on $\mathcal{H}$ onto the cyclic subspaces $\mathcal{M}_i = \mathfrak{C}\{\xi_i(B), B \in \mathcal{B}_i\}$ ($i=1, 2$) and $\mathfrak{C}\{\mathcal{M}_1 + \mathcal{M}_2\}$, then for each $x \in \mathcal{H}$,*

$$\begin{aligned} Px = & \int_R \frac{d(x, \xi_1)}{dM_1}(s) \xi_1(ds) + \int_R \frac{d(x, \xi_2)}{dM_2}(s) \xi_2(ds) \\ & - \int_R \left\{ \int_R \frac{d(x, \xi_2)}{dM_2}(s) \frac{d(\xi_2, \xi_1)}{dM_{21}}(s, t) M_2(ds) \right\} \xi_1(dt) \\ & - \int_R \left\{ \int_R \frac{d(x, \xi_1)}{dM_1}(s) \frac{d(\xi_1, \xi_2)}{dM_{12}}(s, t) M_1(ds) \right\} \xi_2(dt) \\ & + \int_R \left\{ \int_R \left\{ \int_R \frac{d(x, \xi_1)}{dM_1}(s) \frac{d(\xi_1, \xi_2)}{dM_{12}}(s, t) M_1(ds) \right\} \frac{d(\xi_2, \xi_1)}{dM_{21}}(t, u) M_2(dt) \right\} \xi_1(du) \\ & + \int_R \left\{ \int_R \left\{ \int_R \frac{d(x, \xi_1)}{dM_1}(s) \frac{d(\xi_2, \xi_1)}{dM_{21}}(s, t) M_2(ds) \right\} \frac{d(\xi_1, \xi_2)}{dM_{12}}(t, u) M_1(dt) \right\} \xi_2(du) \\ & - \dots \end{aligned}$$

4. Application of the alternating projections theorem to bivariate stationary stochastic processes. Let $\mathbf{x}(t) = (x_i(t))_{i=1}^q \in \mathcal{H}^q$ be a q -variate, stationary SP with the shift group $(U_t, t \in R)$. Here $\mathcal{H}^q$ is the cartesian product of a complex Hilbert space $\mathcal{H}$ with itself q -times, and $(U_t, t \in R)$ is a strongly continuous group of unitary operators on $\mathcal{H}$ such that $x_i(t+s) = U_t x_i(s)$ for all $s, t \in R$ ($1 \leq i \leq q$). By Stone's Theorem (cf. [8, p. 383]), there exists a spectral measure E on the family $\mathcal{B}$ of Borel subsets of the real line R such that $U_t = \int_{-\infty}^{\infty} e^{-it\lambda} E(d\lambda)$. The $q \times q$ non-negative, hermitian matrix-valued function $F(\lambda) = 2\pi(E(-\infty, \lambda] \mathbf{x}(0), \mathbf{x}(0))$ is called the spectral distribution of the SP. If F is absolutely continuous with respect to the Lebesgue measure (Leb), its derivative F' is called the *spectral density* of the SP.

Let $\mathcal{M}(t)$ be the past-present subspace of the SP $(\mathbf{x}(t), t \in R)$ up to time t in $\mathcal{H}^q$, i.e., $\mathcal{M}(t) = \mathcal{S}\{\mathbf{x}(s), s \leq t\}$. Let $\mathcal{M}(-\infty)$ be its remote past subspace in $\mathcal{H}^q$, i.e., $\mathcal{M}(-\infty) = \bigcap_{t \leq 0} \mathcal{M}(t)$.

An important result is that if $\mathcal{M}(-\infty) = \{0\}$, then $\mathbf{F}$ is a.c. with respect to Leb, and $\mathbf{x}(t) = \int_{-\infty}^t \mathbf{C}(t-s)\xi(ds)$, where $\mathbf{C}(\cdot)$ and $\mathbf{F}'(\cdot)$ are related by a matrix-valued function Φ on R such that

$$\Phi(\lambda)\Phi^*(\lambda) = \mathbf{F}'(\lambda) \text{ a.e. } \lambda \quad \text{and} \quad \Phi(\lambda) = \int_0^\infty \mathbf{C}(t)e^{i\lambda t} dt \in L_2(R, \mathcal{B}, \text{Leb}),$$

and if Φ^+ denotes the holomorphic extension of Φ to the upper half-plane, $\Phi^+(i)$ is nonnegative hermitian, and

$$\det \Phi^+(i) = \exp \frac{1}{2\pi} \int_{-\infty}^\infty \log \det \mathbf{F}'(\lambda) \frac{d\lambda}{1+\lambda^2}.$$

The function Φ is called the *generating function* of the SP. It is not hard to show (cf. [5, p. 1372] and [11, p. 33]) that for each bounded interval $(a, b]$

$$\xi(a, b] = 2^{-1/2} \left\{ \mathbf{h}(b) - \mathbf{h}(a) + \int_a^b \mathbf{h}(s) ds \right\},$$

where $(\mathbf{h}(t), t \in R)$ is the weakly Markovian SP which is associated with the SP $(\mathbf{x}(t), t \in R)$.

We will need the following theorem, the proof of which is omitted and may be found in [11].

THEOREM 4.1. *Let $(\mathbf{x}(t), t \in R)$ be a q -variate, stationary SP with spectral distribution $\mathbf{F}$. Then (a) $(\mathbf{x}(t), t \in R)$ is purely nondeterministic, i.e., $\mathcal{M}(-\infty) = \{0\}$ iff $\mathbf{F}$ is absolutely continuous with respect to the Lebesgue measure. If (a) holds, then*

(b) $\mathcal{S}\{\xi(a, b], -s < a \leq b < t\} = \mathcal{S}\{\mathbf{x}(u): s < u < t\}$.

(c) ξ can be uniquely extended to the family $\mathcal{B}$ of Borel subsets of the real line R . If the SP is of rank q , or equivalently $\log \det \mathbf{F}'(\lambda)/(1+\lambda^2) \in L_1(R, \mathcal{B}, \text{Leb})$, then for each Borel set B of finite Lebesgue measure this extension has the representation

$$\xi(B) = (2\pi)^{1/2} \int_{-\infty}^\infty \hat{\chi}_B(\lambda) \Phi^{-1}(\lambda) \mathbf{E}(d\lambda) \mathbf{x}(0),$$

where $\hat{\chi}_B(\lambda) = (2\pi)^{-1/2} \int_B e^{-i\lambda t} dt$.

(d) The extension ξ is an o.s. $\mathcal{H}^q$ -valued measure over $(R, \mathcal{B})$, whose associated spectral measure is $\text{Leb}(\cdot) \mathbf{I}_q$, where $\mathbf{I}_q$ is the identity matrix of order q .

DEFINITION 4.2. ξ as defined in 4.1(c) is called the *canonical measure* of the SP $(\mathbf{x}(t), t \in R)$.

Now let $(\mathbf{x}(t), t \in R)$ be a bivariate, purely nondeterministic SP. Let $\mathbf{T}$ be the projection operator on $\mathcal{H}^2$ onto the subspace $\mathcal{M}(0)$, and P be the projection operator on $\mathcal{H}$ onto the subspace $\mathcal{S}\{\mathcal{M}_1(0) + \mathcal{M}_2(0)\}$, where $\mathcal{M}_1(0)$ and $\mathcal{M}_2(0)$ are the past-present subspaces of the component processes $x_1(t)$ and $x_2(t)$. It is easy to see (cf. [12, I, p. 131]) that the components of $\mathbf{T}\mathbf{x}(t)$ are precisely $Px_1(t)$ and

$Px_2(t)$. Applying the results of §3 we will be able to obtain convergent infinite series expressions for $Px_1(t)$ and $Px_2(t)$, and hence the best linear prediction $Tx(t)$ is determinable.

Since $(x(t), t \in R)$ is purely nondeterministic, so are its component processes $(x_1(t), t \in R)$ and $(x_2(t), t \in R)$ (cf. [11, p. 79]). Accordingly ξ_1 and ξ_2 will denote the canonical measures of the processes $(x_1(t), t \in R)$ and $(x_2(t), t \in R)$. Also $\mathcal{M}_1(t)$ and $\mathcal{M}_2(t)$ will denote the past-present subspaces of $(x_1(t), t \in R)$ and $(x_2(t), t \in R)$ up to time t in $\mathcal{H}$.

We will be able to effect our algorithm under the following assumption.

ASSUMPTION 4.3. The spectral density $F' = [F'_{ij}]$, $1 \leq i, j \leq 2$, of the bivariate, purely nondeterministic SP $(x(t), t \in R)$ satisfies the condition

$$F'_{12}/(F'_{11})^{1/2}(F'_{22})^{1/2} \in L_1(R, \mathcal{B}, \text{Leb})^{(4)}.$$

REMARK 4.4. We assert that the SP has rank 2. For if its rank were less than 2, then $F'_{11}F'_{22} - F'_{12}F'_{21} = \det F' = 0$ a.e. on R (cf. [11, p. 36]). Hence $F'_{12}/(F'_{11})^{1/2}(F'_{22})^{1/2} = 1 \notin L_1(R, \mathcal{B}, \text{Leb})$ which contradicts Assumption 4.3.

The component processes $(x_1(t), t \in R)$ and $(x_2(t), t \in R)$ are purely nondeterministic, and hence have the generating function ϕ_1, ϕ_2 . We also note that since $\det F' \geq 0$, $F'/(F'_{11})^{1/2}(F'_{22})^{1/2} \leq 1$. Hence by Assumption 4.3,

$$F'_{12}/(F'_{11})^{1/2}(F'_{22})^{1/2} \in L_\delta(R, \mathcal{B}, \text{Leb}), \quad 1 \leq \delta \leq \infty.$$

THEOREM 4.5. Let (i) the SP $(x(t), t \in R)$ satisfy Assumption 4.3.

(ii) ξ_i ($i=1, 2$) be the canonical measure of the component process $(x_i(t), t \in R)$ ($i=1, 2$). Then Assumption 3.5 is satisfied, i.e., there exist Borel measurable functions φ_{12} and φ_{21} on R^2 such that

$$(1) \quad \begin{aligned} \varphi_{ij}(\cdot, t) &\in L_2(R, \mathcal{B}, \text{Leb}) \text{ a.e. } t, \\ \text{ess. lub.}_{t \in R} |\varphi_{ij}(\cdot, t)|_{2, \text{Leb}} &< \infty. \end{aligned}$$

$$(2) \text{ For each } B \in \mathcal{B}_{ij}, \chi_B \varphi_{ij} \in L_1(R^2, \mathcal{B} \times \mathcal{B}, \text{Leb} \times \text{Leb}).$$

$$(3) \text{ For each } B \times C \in \mathcal{B}_i \times \mathcal{B}_j,$$

$$(\xi_i(B), \xi_j(C)) = \iint_{B \times C} \varphi_{ij}(s, t) ds dt.$$

In fact if

$$\gamma_{ij}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-it\lambda} \frac{F'_{ij}(\lambda)}{\phi_i(\lambda)\phi_j(\lambda)} d\lambda^{(5)},$$

then

$$\varphi_{ij}(s, t) = \gamma_{ij}(s-t),$$

where ϕ_i ($i=1, 2$) is the generating function of the component process $(x_i(t), t \in R)$ ($i=1, 2$).

⁽⁴⁾ By Cramér's Theorem (cf. [2, p. 221]) it is easy to see that this assumption is not void.

⁽⁵⁾ Since $|\phi_i|^2 = F'_{ii}$ ($i=1, 2$), by 4.4, $F'_{ij}/\phi_i\phi_j$ is in $L_\delta(R, \mathcal{B}, \text{Leb})$, $1 \leq \delta \leq \infty$.

Proof. Let $\gamma_{ij}(t) = (1/2\pi) \int_{-\infty}^{\infty} e^{-it\lambda} (F'_{ij}(\lambda)/\phi_i(\lambda)\bar{\phi}_j(\lambda)) d\lambda$. We define φ_{ij} on R^2 by

$$\varphi_{ij}(s, t) = \gamma_{ij}(s-t).$$

Since $|\phi_i|^2 = F'_{ii}$, by (i) and 4.4, $F'_{ij}/\phi_i\bar{\phi}_j \in L_2(R, \mathcal{B}, \text{Leb})$. Hence by Plancherel's Theorem we have $\int_{-\infty}^{\infty} |\varphi_{ij}(s, t)|^2 ds = \int_{-\infty}^{\infty} |\gamma_{ij}(s-t)|^2 ds = (1/2\pi) \|F'_{ij}/\phi_i\bar{\phi}_j\|_{2, \text{Leb}}^2 < \infty$, so that $\text{ess. lub.}_{t \in R} \|\varphi_{ij}(\cdot, t)\|_{2, \text{Leb}} = (1/2\pi) \|F'_{ij}/\phi_i\bar{\phi}_j\|_{2, \text{Leb}} < \infty$. Hence (1) is proved.

Also by (i) and 4.4, $\varphi_{ij} \in L_{\infty}(R^2, \mathcal{B} \times \mathcal{B}, \text{Leb} \times \text{Leb})$ and hence (2) is satisfied.

By 4.1, for each $B \in \mathcal{B}_i$ ($i = 1, 2$) we have

$$\xi_i(B) = (2\pi)^{1/2} \int_{-\infty}^{\infty} \hat{\chi}_B(\lambda) \phi_i^{-1}(\lambda) E(d\lambda) x_i(0) \quad (i = 1, 2).$$

From (i) it follows that $F'_{ij}/\phi_i\bar{\phi}_j \in L_1(R, \mathcal{B}, \text{Leb})$, so that by Fubini's Theorem for each $B \times C \in \mathcal{B}_i \times \mathcal{B}_j$,

$$\begin{aligned} (\xi_i(B), \xi_j(C)) &= 2\pi \left(\int_{-\infty}^{\infty} \hat{\chi}_B(\lambda) \phi_i^{-1}(\lambda) E(d\lambda) x_i(0), \int_{-\infty}^{\infty} \hat{\chi}_C(\lambda) \phi_j^{-1}(\lambda) E(d\lambda) x_j(0) \right) \\ &= 2\pi \int_{-\infty}^{\infty} \hat{\chi}_B(\lambda) \overline{\hat{\chi}_C(\lambda)} \phi_i^{-1}(\lambda) \overline{\phi_j^{-1}(\lambda)} E(d\lambda) x_i(0), x_j(0) \\ &= \int_{-\infty}^{\infty} \hat{\chi}_B(\lambda) \overline{\hat{\chi}_C(\lambda)} \cdot \frac{F'_{ij}(\lambda)}{\phi_i(\lambda)\bar{\phi}_j(\lambda)} d\lambda \quad (\text{cf. [11, p. 82]}) \\ &= \frac{1}{2\pi} \iint_{B \times C} \left\{ \int_{-\infty}^{\infty} e^{-i(s-t)\lambda} \frac{F'_{ij}(\lambda)}{\phi_i(\lambda)\bar{\phi}_j(\lambda)} d\lambda \right\} ds dt \\ &= \iint_{B \times C} \gamma_{ij}(s, t) ds dt. \end{aligned} \quad (\text{Q.E.D.})$$

We now state the main theorem of this section.

MAIN THEOREM 4.6. Let (i) the bivariate SP $(x(t), t \in R)$ satisfy Assumption 4.3.

(ii) $\mathcal{M}_i(0)$ ($i = 1, 2$) be the past-present subspace of the component process $(x_i(t), t \in R)$ ($i = 1, 2$), and let P_1 , P_2 , and P be the projection operators on $\mathcal{H}^{(6)}$ onto the subspaces $\mathcal{M}_1(0)$, $\mathcal{M}_2(0)$, and $\mathfrak{S}\{\mathcal{M}_1(0) + \mathcal{M}_2(0)\}$.

(iii) $\phi_i(\lambda) = \int_0^{\infty} e^{i\lambda t} c_i(t) dt$,

$$\gamma_{ij}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-it\lambda} \frac{F'_{ij}(\lambda)}{\phi_i(\lambda)\bar{\phi}_j(\lambda)} ds,$$

where ϕ_i ($i = 1, 2$) is the generating function of the component process $(x_i(t), t \in R)$, $1 \leq i \leq 2$.

(iv) ξ_i ($i = 1, 2$) be the canonical measure of the component process $(x_i(t), t \in R)$ ($i = 1, 2$).

⁽⁶⁾ $\mathcal{H}$ is a Hilbert space such that $x_i(t) \in \mathcal{H}$ for all $t \in R$, $i = 1, 2$.

Then for each $\tau \geq 0$,

$$Px_i(\tau) = \int_0^\infty c_i(\tau+s)\xi_i(-ds) + \int_0^\infty \left\{ \int_{-\tau}^0 c_i(\tau+s)\gamma_{ij}(t-s)ds \right\} \xi_j(-ds) \\ - \int_0^\infty \left\{ \int_0^\infty \left\{ \int_{-\tau}^0 c_i(\tau+s)\gamma_{ij}(t-s)ds \right\} \gamma_{ji}(u-t)dt \right\} \xi_i(-du) + \dots$$

Proof. By the third paragraph of §4 and 4.1(b),

$$(1) \quad x_i(\tau) = \int_{-\tau}^\infty c_i(\tau+s)\xi_i(-ds), \\ \mathcal{M}_i(0) = \mathfrak{S}\{\xi_i(B), B \subseteq (-\infty, 0] \text{ and } B \in \mathcal{B}_i\}.$$

By 4.5(c), $(d(\xi_i, \xi_j)/dM_{ij})(s, t) = \gamma_{ij}(s-t)$ and Assumption 3.5 is satisfied. Hence by (i) and 3.9, we have

$$P_i x_i(\tau) = \int_0^\infty c_i(\tau+s)\xi_i(-ds), \\ P_j x_i(\tau) = \int_0^\infty \left\{ \int_{-\tau}^\infty c_i(\tau+s)\gamma_{ij}(t-s)ds \right\} \xi_j(-dt), \\ P_i P_j x_i(\tau) = \int_0^\infty \left\{ \int_0^\infty \left\{ \int_{-\tau}^\infty c_i(\tau+s)\gamma_{ij}(t-s)ds \right\} \gamma_{ji}(u-t) \right\} \xi_i(-du), \\ P_j P_i x_j(\tau) = \int_0^\infty \left\{ \int_0^\infty c_i(\tau+s)\gamma_{ij}(t-s)ds \right\} \xi_j(-dt).$$

Hence, after some simplification, we have the result by 3.10. (Q.E.D.)

The following is an immediate corollary to this theorem.

COROLLARY 4.7. *With the notation of 4.6, if Q is the projection operator on $\mathcal{H}$ onto the orthogonal complement of $\mathfrak{S}\{\mathcal{M}_1(0) + \mathcal{M}_2(0)\}$ in $\mathcal{H}$ then for each $\tau \geq 0$*

$$Qx_i(\tau) = \int_{-\tau}^0 c_i(\tau+s)\xi_i(ds) - \int_0^\infty \left\{ \int_{-\tau}^0 c_i(\tau+s)\gamma_{ij}(t-s)ds \right\} \xi_j(-dt) \\ + \int_0^\infty \left\{ \int_0^\infty \left\{ \int_{-\tau}^0 c_i(t+s)\gamma_{ij}(t-s)ds \right\} \gamma_{ji}(u-t)dt \right\} \xi_i(-du) - \dots$$

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