MARKOV PROCESSES WHOSE HITTING DISTRIBUTIONS ARE DOMINATED BY THOSE OF A GIVEN PROCESS⁽¹⁾

BY
CHUNG-TUO SHIH

Introduction. Let X be a Markov process. Let X^* be the process obtained from a random time change in a subprocess of X . Then obviously the hitting distributions of X^* are dominated by those of X . It has been naturally conjectured that the converse is true under broad conditions. The exact statement of the converse would be as follows: if two Markov processes are such that the hitting distributions of X dominate those of X^* , then there is a process Y obtained from a random time change in a subprocess of X that is equivalent to X^* . The conjecture is proved by Šur [6] in case X is a Brownian motion process and X^* is a standard process. This paper deals with the very general case where X and X^* are Hunt processes with a locally compact separable metric space as their common state space. (Our definition of a Hunt process agrees essentially with that of a standard process.) The conjecture is proved true under a trivially necessary condition, provided a slight change in the definition of a subprocess is allowed (see §1). Our method is to construct a multiplicative functional of X which induces a process with the same hitting distributions as X^* . When the state space of X^* is a proper subspace of that of X which is a locally compact separable metric space, it seems that no difficulty would arise in obtaining a similar result with the same techniques.

1. Preliminaries and the main result. Let E be a locally compact separable metric space. Let $E_\Delta = E \cup \{\Delta\}$ be the one-point compactification of E . Denote by $\mathcal{B}$ and $\mathcal{B}_\Delta$ respectively the Borel fields of E and E_Δ . A Hunt process X with E as its state space is a structure $(\Omega, \mathcal{M}, X(t), P^x, \theta_t)$ where Ω is a set (the sample space), $\mathcal{M}$ a σ -field on Ω , $X(t)$, $0 \leq t < \infty$, are random variables on $(\Omega, \mathcal{M})$ taking values in E_Δ , P^x , $x \in E_\Delta$, probability measures on $\mathcal{M}$ with $P^x(X(0)=x)=1$, and θ_t , $0 \leq t < \infty$, operators on Ω satisfying $X(s)(\theta_t) = X(s+t)$. For each $\omega \in \Omega$, the path function $t \rightarrow X_t(\omega) = X(t, \omega) = X(t)(\omega)$ is right continuous and has a left limit at every t , and $X(t, \omega) = \Delta$ if $t \geq \sigma(\omega) = \inf\{s \mid X(s) = \Delta\}$, the lifetime of ω . Let $\mathcal{G}_t$ be the σ -field generated by $X(s)$, $s \leq t$ and $\mathcal{G}$ that generated by $X(t)$, $t < \infty$. For each

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$\Lambda \in \mathcal{G}$, $x \rightarrow P^x(\Lambda)$ is $\mathcal{B}_\Delta$ -measurable. For a probability measure μ on $\mathcal{B}_\Delta$ (μ will always denote such a measure) P^μ is defined on $\mathcal{G}$ by $P^\mu(\Lambda) = \int P^x(\Lambda) \mu(dx)$. Let $\mathcal{G}^\mu$ be the P^μ -completion of $\mathcal{G}$, and let $\mathcal{F}$ be the intersection of $\mathcal{G}^\mu$ over all μ . Let $\mathcal{F}_t$ be the σ -field of sets Λ satisfying the condition that, for each μ , there exist $\Lambda_1 \in \mathcal{G}_t$ and $\Lambda_2 \in \mathcal{G}$ with $P^\mu(\Lambda_2) = 0$ and $(\Lambda - \Lambda_1) \cup (\Lambda_1 - \Lambda) \subset \Lambda_2$. A stopping time T is a function from Ω to $[0, \infty]$ such that $\{T < t\} \in \mathcal{F}_t$ for every t . For a stopping time T , $\mathcal{F}_T = \mathcal{F}(T)$ denotes the σ -field of sets $\Lambda \in \mathcal{F}$ satisfying $\Lambda \cap \{T < t\} \in \mathcal{F}_t$ for every t . If ϕ is a bounded real-valued $\mathcal{F}$ -measurable function and T a stopping time, then $E^x\{\phi(\theta_T); \Lambda \cap [T < \infty]\} = E^x\{E^{X(T)}(\phi); \Lambda \cap [T < \infty]\}$ for all $\Lambda \in \mathcal{F}(T)$. This is the strong Markov property of X . If T_n are stopping times increasing to T , then $X(T_n)$ converge to $X(T)$ a.e. on $\{T < \sigma\}$, where a.e. means a.e. P^x for every x . This is called the quasi-left continuity of X . Note that it is not required that X satisfies the stronger form of quasi-left continuity that the above convergence holds a.e. on $\{T < \infty\}$. Thus our definition of a Hunt process agrees with that of a standard process.

For an analytic subset A of E and $x \in E$, the hitting distribution $H_A(x, \cdot)$ is defined on $\mathcal{B}$ by $H_A(x, B) = P^x(X(T_A) \in B, T_A < \infty)$, where $T_A = \inf\{t > 0 \mid X(t) \in A\}$ is the hitting time of A . A point x is regular for A if $P^x(T_A = 0) = 1$, and irregular if otherwise. A point $x \in E$ is a holding point if it is irregular for $E - \{x\}$. The set of holding points will be denoted by H . The points in $I = E - H$ are called instantaneous points. A holding point is called a trap if $H_{E - \{x\}}(x, E) = 0$. Let T be a stopping time $\leq \sigma$ such that $T(\theta_t) = T - t$ on $\{t < T\}$. Let $Z(t) = X(t)$ on $\{t < T\}$ and $= \Delta$ elsewhere. Then $Z = (\Omega, \mathcal{M}, Z(t), P^x, \theta_t)$ is again a Hunt process, whose state space is $E' = \{x \in E \mid P^x(T > 0) = 1\}$. (If X is only a strong Markov process then so is Z .) Such a process is called a subprocess of X . A multiplicative functional M of X is a real-valued function on $[0, \infty) \times \Omega$ such that (i) $t \rightarrow M(t)(\omega) = M(t, \omega)$ is right continuous, nonincreasing and taking values in $[0, 1]$ for almost all ω , (ii) $M(t)$ is $\mathcal{F}(t)$ -measurable, and (iii) $M(s+t) = M(t)[M(s)(\theta_t)]$ a.e. If M is a multiplicative functional such that $M(t) = 0$ on $\{\sigma \leq t\}$, then one can define a Markov process Y in the manner of [5, p. 142] from X and M . Y has $E' = \{x \in E \mid P^x(M(0) = 1) = 1\}$ as its state space. The transition function $q(t, x, B)$, $x \in E'$, $B \in \mathcal{B} \cap E'$, of Y is given by $E^x\{M(t); X(t) \in B\}$ and its hitting distribution $H_A^Y(x, B)$, $x \in E$, $A \subset E'$, is equal to $E^x\{M(T_{A'}); X(T_{A'}) \in B, T_{A'} < \infty\}$ where $A' = (E - E') \cup A$. Such a process we will also call a subprocess of X .

For a summary of the basic definitions and facts of the theory of Hunt processes we refer the reader to §1 of [4] or §2 of [1]. A discussion of multiplicative functionals can be found in [3], [6] or Chapter 10 of [2].

Let us consider two Hunt processes X and X^* with E as their common state space. For notational convenience suppose X and X^* are defined on the same sample space $(\Omega, \mathcal{M})$ and by the same random variables $X(t)$. Thus we write $X^* = (\Omega, \mathcal{M}, X(t), *P^x, \theta_t)$. Let $*P^\mu$, $\mathcal{F}^*$, $\mathcal{F}_t^*$, $\mathcal{F}_T^*$, $H_A^*(x, B)$, H^* and I^* be defined from the probability measures $*P^x$ in the same manner as P^μ , $\mathcal{F}$, etc. are defined

from P^x ; a.e. $*$ will mean a.e. $*P^x$ for every x . If $\mathcal{S}$ is sub- σ -field of $\mathcal{F}$ ($\mathcal{F}^*$), $\mathcal{S}^\mu$ ($\mathcal{S}^{*\mu}$) will denote the σ -field of sets Λ such that there exist $\Lambda_1 \in \mathcal{S}$ and $\Lambda_2 \in \mathcal{F}$ ($\Lambda_2 \in \mathcal{F}^*$) with $P^\mu(\Lambda_2)=0$ ($*P^\mu(\Lambda_2)=0$) and $(\Lambda - \Lambda_1) \cup (\Lambda_1 - \Lambda) \subset \Lambda_2$. Note that this definition agrees with a previous one if $\mathcal{S} = \mathcal{G}$. For $\Lambda \subset \Omega$, $\mathcal{S} \cap \Lambda = \{\Lambda_1 \cap \Lambda \mid \Lambda_1 \in \mathcal{S}\}$. Let $\mathcal{C}_0$ be the set of real continuous functions on E vanishing at infinity.

Our basic hypotheses are:

A. $H_K^*(x, \cdot) \leq H_K(x, \cdot)$ for every compact subset K of E and $x \in E$.

B. If x is a trap for X^* , then $x \in H$.

The main result is the following

THEOREM. *Under hypotheses A and B there exists a multiplicative functional M of X such that $E^x\{M(T_A); X(T_A) \in B, T_A < \infty\} = H_A^*(x, B)$ for all $x \in E$, analytic subsets A of E and $B \in \mathcal{B}$.*

Obviously the multiplicative functional M in the theorem must satisfy $M(0)=1$ a.e. P^x for all $x \in E$, and we may assume $M(t)=0$ on $\{\sigma \leq t\}$. Thus the subprocess Y of X induced by M has E as its state space and has the same hitting distributions (on analytic subsets of E) as X^* . One can then appeal to the result of [1] to obtain a random time change in Y which induces a process equivalent to X^* .

A remark about the hypotheses: if the result of the theorem holds and if $x \in I$, then x is not a trap for X^* , since

$$H_{E-\{x\}}^*(x, E) = E^x\{M(T_{E-\{x\}}); X(T_{E-\{x\}}) \in E, T_{E-\{x\}} < \infty\} = 1.$$

Thus hypothesis B is necessary. Hypothesis A implies that $H_A^*(x, \cdot) \leq H_A(x, \cdot)$ for all analytic subsets A of E . This is a result of the fact that, in a Hunt process, the hitting time of an analytic set can be approximated by the hitting time of its compact subsets.

Let us choose a metric ρ on E_Δ . Let $\text{reg } A$ ($\text{reg}^* A$) be the set of regular points for A relative to X (X^*). We will now prove two immediate results of hypotheses A and B.

PROPOSITION 1.1. $H = H^*$.

Proof. If $x \in H$, then $H_{E-\{x\}}^*(x, \{x\}) \leq H_{E-\{x\}}(x, \{x\})=0$, and hence x cannot be in I^* . So $H \subset H^*$. Suppose $x \in H^*$. If $H_{E-\{x\}}^*(x, E-\{x\})=0$, then x is a trap for X^* and so $x \in H$ by hypothesis B. If $H_{E-\{x\}}^*(x, E-\{x\})>0$, then $H_{E-\{x\}}(x, E-\{x\})>0$ and so $x \notin I$. So $H^* \subset H$.

PROPOSITION 1.2. *If K is a compact subset of E , then $\text{reg } K = \text{reg}^* K$.*

Proof. If $x \in K$, then trivially $x \notin \text{reg } K$ and also $x \notin \text{reg}^* K$. Suppose $x \in K$. If $x \in H = H^*$, then of course $x \in \text{reg } K \cap \text{reg}^* K$. Suppose $x \in I = I^*$. Let

$$D_n = K \cup \{y \mid 1/n \leq \rho(x, y) \leq \rho(x, \Delta)/2\}.$$

Then we clearly have: $H_{D_n}(x, \{x\}) = 1$ for all n if and only if $x \in \text{reg } K$, $H_{D_n}^*(x, \{x\}) = 1$ for all n if and only if $x \in \text{reg}^* K$ and $H_{D_n}^*(x, D_n) \uparrow 1$ as $n \rightarrow \infty$. These conditions and hypothesis A imply that $x \in \text{reg } K$ if and only if $x \in \text{reg}^* K$. The proof is complete.

2. Partitioning of the state space and a family of stopping times. A pair $(\mathcal{U}, \mathcal{V})$ will be called a partition of E if $\mathcal{U} = \{U_1, \dots, U_p\}$ is a finite covering of E by open subsets of E , and $\mathcal{V} = \{V_1, \dots, V_p\}$ is an (ordinary) partition of E by Borel sets with $V_i \subset U_i$ for all i . A partition $(\mathcal{U}, \mathcal{V})$ of E is a refinement of another partition $(\mathcal{U}', \mathcal{V}')$ if, whenever $V_i \in \mathcal{V}$, $U'_j \in \mathcal{U}'$, and $V_i \cap U'_j \neq \emptyset$, we have $U_i \subset U'_j$. It is easily checked that being a refinement of is a transitive relation among partitions $(\mathcal{U}, \mathcal{V})$ of E . For a family $\mathcal{U}$ of subsets of E let $|\mathcal{U}|$ be the sup over $U \in \mathcal{U}$ of $\text{diam } U = \sup \{\rho(x, y) \mid x, y \in U\}$. The following observation is basic to our study.

PROPOSITION 2.1. *There exists a sequence $(\mathcal{Q}^{(n)}, \mathcal{V}^{(n)})$, $n = 1, 2, \dots$, of partitions of E such that (i) $|\mathcal{Q}^{(n)}| < 1/n$ for all n , (ii) $(\mathcal{Q}^{(n)}, \mathcal{V}^{(n)})$ is a refinement of $(\mathcal{Q}^{(m)}, \mathcal{V}^{(m)})$ if $m \leq n$, and (iii) $V_i^{(n)} = U_i^{(n)} - \bigcup_{j=1}^{i-1} U_j^{(n)}$ for all n and $i > 1$.*

Proof. For each n choose a finite covering $\mathcal{W}^{(n)}$ of E consisting of open subsets of E such that $|\mathcal{W}^{(n)}| < 1/n$ (note that ρ is a metric on E_Δ) and $\mathcal{W}^{(n)}$ is closed w.r.t. finite intersection. Define $\mathcal{Q}^{(n)}$ inductively by setting $\mathcal{Q}^{(1)} = \mathcal{W}^{(1)}$ and $\mathcal{Q}^{(n+1)} = \{U_1 \cap U_2 \mid U_1 \in \mathcal{Q}^{(n)} \text{ and } U_2 \in \mathcal{W}^{(n+1)}\}$ for $n \geq 1$. Clearly each $\mathcal{Q}^{(n)}$ is closed w.r.t. finite intersection. Enumerate numbers of $\mathcal{Q}^{(n)}$ as $U_1^{(n)}, \dots, U_p^{(n)}$ in such a manner that $i \leq j$ whenever $U_i^{(n)} \subset U_j^{(n)}$. Let $V_1^{(n)} = U_1^{(n)}$ and $V_i^{(n)} = U_i^{(n)} - \bigcup_{j=1}^{i-1} U_j^{(n)}$ for $i > 1$. (i) and (iii) are immediate. We now show $(\mathcal{Q}^{(n+1)}, \mathcal{V}^{(n+1)})$ refines $(\mathcal{Q}^{(n)}, \mathcal{V}^{(n)})$. Suppose $x \in V_i^{(n+1)} \cap U_j^{(n)}$. We need to show $U_i^{(n+1)} \subset U_j^{(n)}$. It is clear that $U_i^{(n+1)} = \bigcap \{U \in \mathcal{Q}^{(n+1)} \mid x \in U\}$. On the other hand $U_i^{(n+1)} = U_1 \cap U_2$ for some $U_1 \in \mathcal{Q}^{(n)}$ and $U_2 \in \mathcal{W}^{(n+1)}$. Since $x \in U_j^{(n)} \cap U_2 \in \mathcal{Q}^{(n+1)}$, we have $U_i^{(n+1)} \subset U_j^{(n)} \cap U_2 \subset U_j^{(n)}$. This proves $(\mathcal{Q}^{(n+1)}, \mathcal{V}^{(n+1)})$ is a refinement of $(\mathcal{Q}^{(n)}, \mathcal{V}^{(n)})$, and it follows from transitivity that $(\mathcal{Q}^{(n)}, \mathcal{V}^{(n)})$ refines $(\mathcal{Q}^{(m)}, \mathcal{V}^{(m)})$ if $m < n$. It remains to show $(\mathcal{Q}^{(n)}, \mathcal{V}^{(n)})$ is a refinement of itself. But this is obvious since if $x \in V_i^{(n)} \cap U_j^{(n)}$, then $U_i^{(n)} = \bigcap \{U \in \mathcal{Q}^{(n)} \mid x \in U\} \subset U_j^{(n)}$. We choose a sequence $\{(\mathcal{Q}^{(n)}, \mathcal{V}^{(n)})\}$ of partitions of E satisfying the conditions in the above proposition. Let us define a family of stopping times as follows. First, for each n let $T^{(n)} = T_{E - U_i^{(n)}}$ if $X(0) \in V_i^{(n)}$, and $= \infty$ if $X(0) = \Delta$. It is clear that each $T^{(n)}$ is a stopping time (for both processes X and X^* —as is every stopping time to be defined in the sequel). Let π be the first uncountable ordinal. For each n define inductively stopping times $T_\alpha^{(n)}$, $\alpha < \pi$, as follows:

$$T_0^{(n)} = 0; T_{\alpha+1}^{(n)} = T_\alpha^{(n)} + T^{(n)}(\theta_{T_\alpha^{(n)}}) \quad \text{if } T_\alpha^{(n)} < \infty, \text{ and } = \infty \text{ if } T_\alpha^{(n)} = \infty;$$

$$T_\alpha^{(n)} = \sup_{\gamma < \alpha} T_\gamma^{(n)} \quad \text{if } \alpha \text{ is a limit ordinal.}$$

The right continuity of path functions implies that $T^{(n)} > 0$ and hence $T_{\alpha+1}^{(n)} > T_\alpha^{(n)}$ provided $T_\alpha^{(n)} < \infty$. It follows that for each ω there is $\alpha < \pi$ such that $T_\alpha^{(n)}(\omega) = \infty$. Also, a standard argument shows that for each μ there exists $\beta < \pi$ such that $P^\mu(T_\beta^{(n)} < \infty) = *P^\mu(T_\beta^{(n)} < \infty) = 0$.

PROPOSITION 2.2. *Let $m < n$ and $\omega \in \Omega$. If $X(0, \omega) \in U_i^{(m)}$, then, for some α , $T_\alpha^{(n)}(\omega) = T_{E - U_i^{(m)}}(\omega)$.*

Proof. We may assume $T_{E - U_i^{(n)}}(\omega) < \infty$. Then $\gamma_0 = \sup\{\gamma \mid T_\gamma^{(n)}(\omega) \leq T_{E - U_i^{(n)}}(\omega)\} < \pi$ and $T_{\gamma_0}^{(n)}(\omega) \leq T_{E - U_i^{(m)}}(\omega)$. If the equality does not hold, then $x = X(T_{\gamma_0}^{(n)}(\omega), \omega) \in U_i^{(n)}$. (Note that $x = \Delta$ implies $T_{E - U_i^{(m)}}(\omega) = \infty$.) Suppose $x \in V_j^{(n)}$. Then since $(\mathcal{U}^{(n)}, \mathcal{V}^{(n)})$ is a refinement of $(\mathcal{U}^{(m)}, \mathcal{V}^{(m)})$ we have $U_j^{(n)} \subset U_i^{(m)}$. This implies $T^{(n)}(\theta_{T_{\gamma_0}^{(n)}}\omega) \leq T_{E - U_i^{(m)}}(\theta_{T_{\gamma_0}^{(n)}}\omega)$ and hence $T_{\gamma_0+1}^{(n)}(\omega) \leq T_{E - U_i^{(m)}}(\omega)$, a contradiction.

COROLLARY 2.3. *Let $m < n$, $\alpha < \pi$, and $\omega \in \Omega$. There is $\lambda < \pi$ satisfying*

$$T_\lambda^{(n)}(\omega) = T_\alpha^{(m)}(\omega).$$

Proof. The case $\alpha = 1$ follows immediately from the previous proposition and the definition of $T_1^{(m)}(\omega)$. Suppose the assertion is true if α is replaced by any smaller ordinal. If α has a predecessor γ and $T_\gamma^{(m)}(\omega) = T_{\lambda_1}^{(n)}(\omega)$, then assuming as we may that $T_\gamma^{(m)}(\omega) < \infty$, we can find λ_2 such that $T_{\lambda_2}^{(n)}(\theta_{T_{\lambda_1}^{(n)}}\omega) = T_{\lambda_2}^{(n)}(\theta_{T_\gamma^{(m)}}\omega) = T_1^{(m)}(\theta_{T_\gamma^{(m)}}\omega)$. Thus we have $T_{\lambda_1+\lambda_2}^{(n)}(\omega) = T_\alpha^{(m)}(\omega)$. If α is a limit ordinal, let $\lambda(\gamma)$ be such that $T_{\lambda(\gamma)}^{(n)}(\omega) = T_\gamma^{(m)}(\omega)$ for $\gamma < \alpha$ and let $\lambda = \sup\{\lambda(\gamma) \mid \gamma < \alpha\}$. Then $T_\lambda^{(n)}(\omega) = T_\alpha^{(m)}(\omega)$ by definition. The corollary then follows from induction.

The fact asserted in the above corollary, which may be called the interpolation property of the $T_\alpha^{(n)}$, is fundamental to this work. Let $\lambda(m, \alpha, n) = \inf\{\lambda < \pi \mid T_\lambda^{(n)} = T_\alpha^{(m)}\}$ for $m \leq n$. Note that $\lambda(m, \alpha, n)$ is a function from Ω into the set of countable ordinals, and $\{\lambda(m, \alpha, n) = \gamma\} = \{T_\alpha^{(m)} = T_\gamma^{(n)} > T_\delta^{(n)} \mid \text{for all } \delta < \gamma\}$ is in $\mathcal{F}(T_\alpha^{(m)}) \cap \mathcal{F}(T_\gamma^{(n)})$ and also in $\mathcal{F}(T_\alpha^{(m)})^* \cap \mathcal{F}(T_\gamma^{(n)})^*$.

COROLLARY 2.4. *Let $m < n$. If $T_\alpha^{(m)}(\omega) \geq t$, there exists $\gamma < \pi$ such that*

$$T_\alpha^{(m)}(\omega) = t + T_\gamma^{(n)}(\theta_t\omega).$$

Proof. The case $T_\alpha^{(m)}(\omega) = t$ is trivial and we may assume $T_\alpha^{(m)}(\omega) > t$. Let $\alpha_0 = \sup\{\alpha_1 \mid T_{\alpha_1}^{(m)}(\omega) \leq t\}$. Then $T_{\alpha_0}^{(m)}(\omega) \leq t < T_{\alpha_0+1}^{(m)}(\omega)$. Suppose $X(T_{\alpha_0}^{(m)}(\omega), \omega) \in V_i^{(m)}$. Then $X(t, \omega) \in U_i^{(m)}$ and $T_{\alpha_0+1}^{(m)} - t$ is the first time $\theta_t\omega$ hits $E - U_i^{(m)}$. Now Proposition 2.2 implies that $T_{\alpha_0+1}^{(m)}(\omega) - t = T_\gamma^{(n)}(\theta_t\omega)$ for some γ . This proves the case when $\alpha = \alpha_0 + 1$. The case when $\alpha > \alpha_0 + 1$ is then easy to see.

For a compact subset K of E , we can define stopping times in terms of $T_\alpha^{(n)}$ to approximate T_K . Let $T_n = \inf\{T_\alpha^{(n)} < \sigma \mid \text{if } X(T_\alpha^{(n)}) \in V_i^{(n)} \text{ then } U_i^{(n)} \cap K \neq \emptyset\}$ and $R_n = \inf\{T_\alpha^{(n)} \mid T_\alpha^{(n)} \geq T_K\}$.

PROPOSITION 2.5. *Each T_n is a stopping time. If $x \notin K$, then $T_n \uparrow T_K$ or σ a.e. P^x and a.e. $*P^x$ as $n \rightarrow \infty$.*

Proof. Let $F_n = \bigcup \{V_i^{(n)} \mid U_i^{(n)} \cap K \neq \emptyset\}$. From the fact that $(\mathcal{U}^{(n)}, \mathcal{V}^{(n)})$ refines itself one easily checks that $T_n = 0$ if $X(0) \in F_n$ and $T_n = T_{F_n}$ if otherwise. Hence T_n is a stopping time. The rest follows from the fact that $\rho(y, K) = \inf\{\rho(y, z) \mid z \in K\} < 1/n$ for $y \in F_n$ and the quasi-left continuity.

PROPOSITION 2.6. *Each R_n is a stopping time. If $K \subset I$, then $R_n \downarrow T_K$ a.e. and a.e.*.*

Proof. $\{R_n < t\} = \bigcup_{\alpha < \pi} \{R_n = T_\alpha^{(n)} < t\}$. Now $\{R_n = T_\alpha^{(n)}\} = \{T_\alpha^{(n)} \geq T_K > T_\gamma^{(n)} \text{ for all } \gamma < \alpha\} \in \mathcal{F}(T_\alpha^{(n)})$. Hence $\{R_n = T_\alpha^{(n)} < t\} \in \mathcal{F}(t)$. From the fact that for each μ there exists β such that $P^\mu(T_\beta^{(n)} < \infty) = 0$ we see that $\{R_n < t\} \in \mathcal{F}(t)$. Similarly $\{R_n < t\} \in \mathcal{F}(t)^*$. Hence R_n is a stopping time. Suppose $K \subset I = I^*$. It follows from the strong Markov property that, a.e. (a.e.*) on $\{T_K < \infty\}$, there is an arbitrarily small $\delta > 0$ such that $X(T_K) \neq X(T_K + \delta)$. Clearly if $X(t, \omega) \neq X(t + \delta, \omega)$ then there exists some (n, α) such that $t < T_\alpha^{(n)}(\omega) \leq t + \delta$. Hence $R_n \downarrow T_K$ a.e. and a.e.*.

3. Some measurability lemmas. For each positive integer n and countable ordinal γ , let $\mathcal{H}(n, \gamma)$ be the σ -field generated by sets of the form

$$\{X(T_\delta^{(n)}) \in B, T_\delta^{(n)} < \infty\}, \quad B \in \mathcal{B}_\Delta, \delta \leq \gamma.$$

Since $\{X(T_\delta^{(n)}) \in B, T_\delta^{(n)} < \infty\} \in \mathcal{F}(T_\delta^{(n)})$ and, for $\delta < \gamma$, $\mathcal{F}(T_\delta^{(n)}) \subset \mathcal{F}(T_\gamma^{(n)})$, $\mathcal{H}(n, \gamma)$ is a sub- σ -field of $\mathcal{F}(T_\gamma^{(n)})$. Also, $\mathcal{H}(n, \gamma) \subset \mathcal{H}(n, \gamma')$ whenever $\gamma < \gamma'$.

LEMMA 3.1. *If $m \leq n$, then $\{T_\alpha^{(m)} = T_\gamma^{(n)} < \infty\} \in \mathcal{H}(n, \gamma)$.*

Proof. We prove by induction on both γ and α . The induction hypothesis is that $\{T_{\gamma'}^{(n)} = T_{\alpha'}^{(m)} < \infty\} \in \mathcal{H}(n, \gamma')$ whenever $\gamma' < \gamma$ and $\alpha' < \pi$, or $\gamma' = \gamma$ and $\alpha' < \alpha$. Since the case $\alpha = 0$ is trivial we assume $\alpha > 0$. If α has a predecessor α' , then

$$\begin{aligned} \{T_\gamma^{(n)} = T_\alpha^{(m)} < \infty\} &= \bigcup_{\gamma' < \gamma} (\{T_{\gamma'}^{(n)} = T_\alpha^{(n)} < \infty\} \cap \bigcup_i \{X(T_{\gamma'}^{(n)}) \in V_i^{(m)}; \\ &\quad X(T_\delta^{(n)}) \in U_i^{(m)} \text{ for } \gamma' < \delta < \gamma; X(T_\delta^{(n)}) \in E - U_i^{(m)}, T_\delta^{(n)} < \infty\}) \\ &\in \mathcal{H}(n, \gamma). \end{aligned}$$

If α has no predecessor, then $T_\alpha^{(m)} = \sup\{T_{\alpha'}^{(m)} \mid \alpha' < \alpha\}$. Now for $\alpha' < \pi$,

$$\{T_{\alpha'}^{(m)} < T_\gamma^{(n)} < \infty\} = \{T_\gamma^{(n)} < \infty\} \cap \bigcup_{\gamma' < \gamma} \{T_{\alpha'}^{(m)} = T_{\gamma'}^{(n)} < \infty\} \in \mathcal{H}(n, \gamma).$$

Hence if $\alpha' < \pi$, then

$$\{T_\gamma^{(n)} < T_\alpha^{(m)}\} = \{T_\gamma^{(n)} < \infty\} - \{T_\alpha^{(m)} = T_\gamma^{(n)} < \infty\} - \{T_\alpha^{(m)} < T_\gamma^{(n)} < \infty\} \in \mathcal{H}(n, \gamma).$$

It follows that

$$\begin{aligned} \{T_\alpha^{(m)} = T_\gamma^{(n)} < \infty\} &= \{T_\gamma^{(n)} < \infty\} - \{T_\alpha^{(m)} < T_\gamma^{(n)} < \infty\} - \{T_\gamma^{(n)} < T_\alpha^{(m)}\} \\ &= \{T_\gamma^{(n)} < \infty\} - \{T_\alpha^{(m)} < T_\gamma^{(n)} < \infty\} - \bigcup_{\alpha' < \alpha} \{T_\gamma^{(n)} < T_{\alpha'}^{(m)}\} \in \mathcal{H}(n, \gamma). \end{aligned}$$

This completes the proof.

For the rest of this section let m and $\alpha > 0$ be fixed. For $n \geq 0$ and $\gamma < \pi$ let $R_\gamma^{(n)} = \min\{T_\gamma^{(m+n)}, T_\alpha^{(m)}\}$. Note that for each μ there exists $\beta < \pi$ such that $P^\mu(R_\beta^{(n)} < T_\alpha^{(m)}) = *P^\mu(R_\beta^{(n)} < T_\alpha^{(m)}) = 0$. Let $\mathcal{H}_\infty(m, \alpha)$ be the σ -field generated by sets

of the form $\{X(R_\gamma^{(n)}) \in B, R_\gamma^{(n)} < \infty\}$ or $\{X(T_\alpha^{(m)}) \in B, T_\alpha^{(m)} < \infty\}$, where $B \in \mathcal{B}_\Delta$ and $\gamma < \pi$. Let $\mathcal{H}_\infty(m, \alpha)$ be the smallest σ -field containing all $\mathcal{H}_n(m, \alpha)$, $n \geq 0$. Note that $\mathcal{H}_\infty(m, \alpha) \subset \mathcal{F}(T_\alpha^{(m)})$.

COROLLARY 3.2. $\mathcal{H}_n(m, \alpha) \cap \{T_\alpha^{(m)} = T_\gamma^{(m+n)} < \infty\} \subset \mathcal{H}(m+n, \gamma)$.

Proof. If $\Lambda = \{X(T_\alpha^{(m)}) \in B, T_\alpha^{(m)} < \infty\}$ then

$$\Lambda \cap \{T_\alpha^{(m)} = T_\gamma^{(m+n)} < \infty\} = \{X(T_\gamma^{(m+n)}) \in B, T_\alpha^{(m)} = T_\gamma^{(m+n)} < \infty\} \in \mathcal{H}(m+n, \gamma)$$

by Lemma 3.1. Let $\Lambda = \{X(R_\gamma^{(n)}) \in B, R_\gamma^{(n)} < \infty\}$. If $\gamma' \leq \gamma$ then

$$\Lambda \cap \{T_\alpha^{(m)} = T_\gamma^{(m+n)} < \infty\} = \{X(T_\gamma^{(m+n)}) \in B, T_\alpha^{(m)} = T_\gamma^{(m+n)} < \infty\} \in \mathcal{H}(m+n, \gamma).$$

If $\gamma < \gamma'$, then

$$\Lambda \cap \{T_\alpha^{(m)} = T_\gamma^{(m+n)} < \infty\} = \{X(T_\gamma^{(m+n)}) \in B, T_\alpha^{(m)} = T_\gamma^{(m+n)} < \infty\} \in \mathcal{H}(m+n, \gamma).$$

The corollary follows.

LEMMA 3.3. $\mathcal{H}_n(m, \alpha)^\mu \subset \mathcal{H}_{n+1}(m, \alpha)^\mu$ for every μ and n .

Proof. (i) The following equalities show that for arbitrary δ the sets

$$\{T_\delta^{(m+n+1)} < T_\alpha^{(m)}\} \quad \text{and} \quad \{T_\delta^{(m+n+1)} = T_\alpha^{(m)} < \infty\}$$

are in $\mathcal{H}_{n+1}(m, \alpha)$:

$$\begin{aligned} \{T_\delta^{(m+n+1)} < T_\alpha^{(m)}\} &= \{X(R_\delta^{(n+1)}) \neq X(R_{\delta+1}^{(n+1)}), R_{\delta+1}^{(n+1)} < \infty\} \\ &\quad \cup \{R_\delta^{(n+1)} < \infty, R_{\delta+1}^{(n+1)} = \infty\}, \end{aligned}$$

$$\{T_\delta^{(m+n+1)} = T_\alpha^{(m)} < \infty\} = \{X(R_\delta^{(n+1)}) \neq X(R_{\delta'}^{(n+1)}) \text{ for all } \delta' < \delta,$$

$$X(R_\delta^{(n+1)}) = X(R_{\delta+1}^{(n+1)}), R_{\delta+1}^{(n+1)} < \infty\}.$$

(ii) Let us show that, if $\Lambda \in \mathcal{H}(m+n+1, \delta)$, $\Lambda \cap \{T_\delta^{(m+n+1)} < T_\alpha^{(m)}\}$ and $\Lambda \cap \{T_\delta^{(m+n+1)} = T_\alpha^{(m)} < \infty\}$ are in $\mathcal{H}_{n+1}$. We need only to consider

$$\Lambda = \{X(T_{\delta'}^{(m+n+1)}) \in B, T_{\delta'}^{(m+n+1)} < \infty\}$$

where $B \in \mathcal{B}_\Delta$, $\delta' \leq \delta$. Now

$$\Lambda \cap \{T_\delta^{(m+n+1)} < T_\alpha^{(m)}\} = \{X(R_{\delta'}^{(n+1)}) \in B, R_{\delta'}^{(n+1)} < \infty\} \cap \{T_\delta^{(m+n+1)} < T_\alpha^{(m)}\},$$

$$\begin{aligned} \Lambda \cap \{T_\delta^{(m+n+1)} = T_\alpha^{(m)} < \infty\} &= \{X(R_{\delta'}^{(n+1)}) \in B, R_{\delta'}^{(n+1)} < \infty\} \\ &\quad \cap \{T_\delta^{(m+n+1)} = T_\alpha^{(m)} < \infty\}, \end{aligned}$$

and it follows from (i) that they are $\mathcal{H}_{n+1}$ -sets.

(iii) Suppose $\Lambda = \{X(R_\gamma^{(n)}) \in B, R_\gamma^{(n)} < \infty\}$ where $B \in \mathcal{B}_\Delta$ and $\gamma < \pi$. Then Λ is the union of

$$\Lambda_1 = \{X(T_\alpha^{(m)}) \in B, T_\alpha^{(m)} < \infty\} \cap \{T_\alpha^{(m)} \leq T_\gamma^{(m+n)}\}$$

and

$$\Lambda_2 = \{X(T_\gamma^{(m+n)}) \in B, T_\gamma^{(m+n)} < \infty\} \cap \{T_\gamma^{(m+n)} < T_\alpha^{(m)}\}.$$

Let β be such that $P^\mu(T_\beta^{(m+n+1)} < \infty) = 0$. We have

$$\begin{aligned}\Lambda_1 &= \Gamma_1 \cup \bigcup_{\delta < \beta} (\{X(T_\delta^{(m+n+1)}) \in B, T_\delta^{(m+n+1)} < \infty\} \cap \{T_\gamma^{(m+n)} \geq T_\delta^{(m+n+1)} < \infty\} \\ &\quad \cap \{T_\alpha^{(m)} = T_\delta^{(m+n+1)} < \infty\}), \\ \Lambda_2 &= \Gamma_2 \cup \bigcup_{\delta < \beta} (\{X(T_\delta^{(m+n+1)}) \in B, T_\delta^{(m+n+1)} < \infty\} \cap \{T_\gamma^{(m+n)} = T_\delta^{(m+n+1)} < \infty\} \\ &\quad \cap \{T_\delta^{(m+n+1)} < T_\alpha^{(m)}\}),\end{aligned}$$

where Γ_1 and Γ_2 are $\mathcal{F}$ -sets contained in $\{T_\beta^{(m+n+1)} < \infty\}$. Now

$$\{T_\gamma^{(m+n)} = T_\delta^{(m+n+1)} < \infty\}$$

and

$$\begin{aligned}\{T_\delta^{(m+n+1)} \leq T_\gamma^{(m+n)}, T_\delta^{(m+n+1)} < \infty\} &= \{T_\gamma^{(m+n)} = T_\delta^{(m+n+1)} < \infty\} \\ &\quad \cup \{T_\delta^{(m+n+1)} < T_\gamma^{(m+n)}\}\end{aligned}$$

are in $\mathcal{H}(m+n+1, \delta)$ (see Lemma 3.1). It follows from (ii) that Λ_1 and Λ_2 are in $\mathcal{H}_{n+1}^\mu$ and hence so is Λ . This proves $\mathcal{H}_n \subset \mathcal{H}_{n+1}^\mu$ and the lemma follows.

LEMMA 3.4. *Let K be a compact subset of E and let $R_m = \inf\{T_\alpha^{(m)} \mid T_\alpha^{(m)} \geq T_K\}$. If $x \notin K$, then $\{R_m = T_\alpha^{(m)} < \sigma\} \in \mathcal{H}_\infty(m, \alpha)^{\varepsilon_x}$. (ε_x is the unit mass at x .)*

Proof. Since $\{R_n = T_\alpha^{(n)} < \sigma\} = \{T_K \leq T_\alpha^{(n)} < \sigma\} - \bigcup_{\alpha' < \alpha} \{T_K \leq T_{\alpha'}^{(n)} < \sigma\}$, it suffices to show that $\{T_K \leq T_\alpha^{(n)} < \sigma\} \in \mathcal{H}_\infty(m, \alpha)^{\varepsilon_x}$ for all $\alpha' \leq \alpha$. Let $T_n = \inf\{T_\gamma^{(n)} < \sigma \mid \text{if } X(T_\gamma^{(n)}) \in V_i^{(n)} \text{ then } U_i^{(n)} \cap K \neq \emptyset\}$. By Proposition 2.5 $T_n \uparrow T_K$ or σ a.e. P^x . Hence we need only to show $\{T_n \leq T_\alpha^{(n)} < \sigma\} \in \mathcal{H}_\infty(m, \alpha)^{\varepsilon_x}$ for $\alpha' \leq \alpha$, $n \geq m$. Let $n \geq m$ and let β be such that $P^x(T_\beta^{(n)} < \infty) = 0$. We have

$$(3.1) \quad \{T_n \leq T_{\alpha'}^{(m)} < \sigma\} = \Gamma \cup \bigcup_{\gamma < \beta} \{T_n = T_\gamma^{(n)} \leq T_{\alpha'}^{(m)} < \sigma\}$$

where Γ is an $\mathcal{F}$ -set contained in $\{T_\beta^{(n)} < \infty\}$. Now $\{T_\gamma^{(n)} \leq T_{\alpha'}^{(m)} < \sigma\}$ is the union of $\bigcup_{\gamma \leq \delta < \beta} \{T_{\alpha'}^{(m)} = T_\delta^{(n)} < \sigma\}$ and a subset (in $\mathcal{F}$) of $\{T_\beta^{(n)} < \infty\}$. Since for $\alpha' < \alpha$

$$\begin{aligned}\{T_{\alpha'}^{(m)} = T_\delta^{(n)} < \sigma\} &= \{T_{\alpha'}^{(m)} = T_\delta^{(n)} < \sigma, T_\delta^{(n)} < T_\alpha^{(m)}\} \\ &\in \mathcal{H}(n, \delta) \cap \{T_\delta^{(n)} < T_\alpha^{(m)}\} \subset \mathcal{H}_\infty(m, \alpha)\end{aligned}$$

(see (ii) of the proof of Lemma 3.3), we have $\{T_\gamma^{(n)} \leq T_{\alpha'}^{(m)} < \sigma\} \in \mathcal{H}_\infty(m, \alpha)^{\varepsilon_x}$. Furthermore, $\{T_n = T_\gamma^{(n)} \leq T_{\alpha'}^{(m)}, T_\gamma^{(n)} < \sigma\}$ is clearly in $\mathcal{H}_\infty(m, \alpha)$ by the definition of T_n . It follows that $\{T_n = T_\gamma^{(n)} \leq T_{\alpha'}^{(m)} < \sigma\} \in \mathcal{H}_\infty(m, \alpha)^{\varepsilon_x}$. (3.1) then implies $\{T_n \leq T_{\alpha'}^{(m)} < \sigma\} \in \mathcal{H}_\infty(n, \alpha)^{\varepsilon_x}$. The proof is complete.

4. A martingale. In this section we make an additional hypothesis: (a) both X and X^* satisfy the stronger form of quasi-left continuity, i.e., if stopping times T_n increase to T , then $X(T_n) \rightarrow X(T)$ a.e. (a.e.*) on $\{T < \infty\}$, and (b) $X(t) \rightarrow \Delta$ a.e. and a.e.* as $t \rightarrow \infty$. However, the following proposition depends only on hypothesis A.

PROPOSITION 4.1. *Given a partition $(\mathcal{U}, \mathcal{V})$ of E , there is a $\mathcal{B} \times \mathcal{B}$ -measurable function $e: E \times E \rightarrow [0, 1]$ such that $H_{E-U_i}^*(x, B) = \int_B e(x, y) H_{E-U_i}(x, dy)$ if $x \in V_i$.*

Proof. For $x \in E$, $B \in \mathcal{B}$ let $q(x, B) = H_{E-U_i}(x, B)$ and $q^*(x, B) = H_{E^*-U_i}(x, B)$, where i is such that $x \in V_i$. For each B , $q(x, B)$ and $q^*(x, B)$ are $\mathcal{B}$ -measurable in x . (This seems to be a standard fact; however, it is a special case of Proposition 5.1.) By hypothesis A, $q^*(x, \cdot) \leq q(x, \cdot)$ as measures on $\mathcal{B}$ and hence $q^*(x, \cdot)$ is absolutely continuous with respect to $q(x, \cdot)$. Since $\mathcal{B}$ is countably generated, it follows from a well-known theorem that there is a real-valued function e defined on $E \times E$, measurable over $\mathcal{B} \times \mathcal{B}$, such that $q^*(x, B) = \int_B e(x, y)q(x, dy)$ for all $x \in E$, $B \in \mathcal{B}$. Obviously we may assume $0 \leq e \leq 1$.

Now for each n let $e_n: E \times E \rightarrow [0, 1]$ be a fixed $\mathcal{B} \times \mathcal{B}$ -measurable function satisfying

$$H_{E^*-U_i^{(n)}}^*(x, B) = \int_B e_n(x, y)H_{E-U_i^{(n)}}(x, dy)$$

whenever $x \in V_i^{(n)}$ and $B \in \mathcal{B}$. Since $H_{E-U_i^{(n)}}(x, U_i^{(n)}) = 0$ we may assume $e_n(x, x) = 1$ for all $x \in E$. For $n \geq 1$ and $0 < \gamma < \pi$ let $M(n, \gamma)$ be a function on Ω defined as follows:

$$\begin{aligned} M(n, \gamma) &= 0 && \text{if } T_\gamma^{(n)} \geq \sigma, \\ &= \prod_{\delta < \gamma} e_n(X(T_\delta^{(n)}), X(T_{\delta+1}^{(n)})) && \text{if } T_\gamma^{(n)} < \sigma, \end{aligned}$$

where the infinite product $\prod_{\delta < \gamma} e_n(X(T_\delta^{(n)}), X(T_{\delta+1}^{(n)}))$ denotes the infimum of all finite products $e_n(X(T_{\delta_1}^{(n)}), X(T_{\delta_1+1}^{(n)})) \cdots e_n(X(T_{\delta_k}^{(n)}), X(T_{\delta_k+1}^{(n)}))$, $\delta_1 < \cdots < \delta_k < \gamma$. Clearly $M(n, \gamma)$ is measurable over $\mathcal{H}(n, \gamma)$, and, if $\delta < \gamma$, $M(n, \gamma) = M(n, \delta) \times [M(n, \delta')(\theta_{T_\delta^{(n)}})]$ on $\{T_\delta^{(n)} < \sigma\}$, where $\delta + \delta' = \gamma$ (i.e., γ is the δ' th ordinal after δ).

PROPOSITION 4.2. For every $x \in E$ and for every $\Lambda \in \mathcal{H}(n, \gamma) \cap \{T_\gamma^{(n)} < \sigma\}$

$$(4.1) \quad *P^x(\Lambda) = E^x\{M(n, \gamma); \Lambda\}.$$

Proof. (i) We first prove (4.1) for Λ of the form $\{X(T_0^{(n)}) \in B_0, X(T_\gamma^{(n)}) \in B_1, T_\gamma^{(n)} < \sigma\}$. If $x \notin B_0$, both sides of (4.1) are zero. Hence we assume $x \in B_0$, in which case it suffices to let $\Lambda = \{X(T_\gamma^{(n)}) \in B_1, T_\gamma^{(n)} < \sigma\}$. Let us define for $x \in E$, $0 < \delta < \pi$ measures $\nu_{x, \delta}$ and $\nu_{x, \delta}^*$ on $\mathcal{B}$ by setting $\nu_{x, \delta}(B) = E^x\{M(n, \delta); X(T_\delta^{(n)}) \in B, T_\delta^{(n)} < \sigma\}$ and $\nu_{x, \delta}^*(B) = *P^x(X(T_\delta^{(n)}) \in B, T_\delta^{(n)} < \sigma)$. We show inductively $\nu_{x, \delta} = \nu_{x, \delta}^*$ for all x and δ . For $\delta = 1$ we have, from the previous proposition,

$$\begin{aligned} \nu_{x, 1}^*(B) &= H_{E^*-U_i^{(n)}}^*(x, B) = \int_B e_n(x, y)H_{E-U_i^{(n)}}(x, dy) \\ &= E^x(e_n(X(T_0^{(n)}), X(T_1^{(n)})); X(T_1^{(n)}) \in B, T_1^{(n)} < \sigma) = \nu_{x, 1}(B), \end{aligned}$$

i being such that $x \in V_i^{(n)}$.

(ii) Suppose $\nu_{y, \delta'} = \nu_{y, \delta'}^*$ for all $y \in E$ and $0 < \delta' < \delta$. If δ has a predecessor $\delta' > 0$, then

$$\begin{aligned} \nu_{x, \delta}^*(B) &= *E^x\{ *P^{X(T_\delta^{(n)})}(X(T_1^{(n)}) \in B, T_1^{(n)} < \sigma); T_1^{(n)} < \sigma\} \\ &= \int \nu_{y, 1}^*(B) \nu_{x, \delta'}^*(dy) = \int \nu_{y, 1}(B) \nu_{x, \delta'}(dy) \\ &= E^x\{M(n, \delta')E^{X(T_\delta^{(n)})}[M(n, 1); X(T_1^{(n)}) \in B, T_1^{(n)} < \sigma]; T_\delta^{(n)} < \sigma\} \\ &= E^x\{M(n, \delta')[M(n, 1)(\theta_{T_\delta^{(n)}})]; X(T_\delta^{(n)}) \in B, T_\delta^{(n)} < \sigma\} = \nu_{x, \delta}(B). \end{aligned}$$

If δ has no predecessor, let $\delta_1 < \dots < \delta_j \uparrow \delta$. Then $T_{\delta_j}^{(n)} \uparrow T_\delta^{(n)}$ as $j \rightarrow \infty$. Hence $X(T_{\delta_j}^{(n)}) \rightarrow X(T_\delta^{(n)})$ a.e. and a.e.* on $\{T_\delta^{(n)} < \sigma\}$. Furthermore, the additional hypothesis implies that $X(T_{\delta_j}^{(n)}) \rightarrow \Delta$ a.e. and a.e.* on $\{T_{\delta_j}^{(n)} < \sigma \text{ for all } j, T_\delta^{(n)} = \sigma\}$ as $j \rightarrow \infty$. Hence for $f \in \mathcal{C}_0$ we have by the bounded convergence theorem, as $j \rightarrow \infty$,

$$\begin{aligned} \int f dv_{x, \delta_j}^* &= {}^*E^x\{f(X(T_{\delta_j}^{(n)})); T_{\delta_j}^{(n)} < \sigma\} \rightarrow {}^*E^x\{f(X(T_\delta^{(n)})); T_\delta^{(n)} < \sigma\} = \int f dv_{x, \delta}^* \\ \int f dv_{x, \delta_j} &= E^x\{f(X(T_{\delta_j}^{(n)}))M(n, \delta_j); T_{\delta_j}^{(n)} < \sigma\} \\ &\rightarrow E^x\{f(X(T_\delta^{(n)}))M(n, \delta); T_\delta^{(n)} < \sigma\} = \int f dv_{x, \delta}, \end{aligned}$$

since $M(n, \delta_j) \downarrow M(n, \delta)$ on $\{T_\delta^{(n)} < \sigma\}$. The condition $\nu_{x, \delta_j}^* = \nu_{x, \delta}$ for all j then implies $\int f dv_{x, \delta}^* = \int f dv_{x, \delta}$ for all $f \in \mathcal{C}_0$. Hence $\nu_{x, \delta}^* = \nu_{x, \delta}$. This completes the proof that (4.1) holds for Λ of the form $\{X(T_\gamma^{(n)}) \in B, T_\gamma^{(n)} < \sigma\}$.

(iii) We now prove (4.1) for $\Lambda = \{X(T_{\gamma_j}^{(n)}) \in B_j, j = 1, \dots, k; X(T_\gamma^{(n)}) \in B, T_\gamma^{(n)} < \sigma\}$ where $\gamma_1 < \dots < \gamma_k < \gamma$ and $B, B_j \in \mathcal{B}$. We may assume $\gamma_1 > 0$ for otherwise (4.1) is trivial or reduces to this case. The proof is an induction on k . Note that the case $k = 0$ is established already. Thus we assume $k > 0$ and (4.1) holds when γ is replaced by any $\delta > 0$ and Λ is replaced by any set of the form $\{X(T_{\delta_j}^{(n)}) \in A_j, j = 1, \dots, k-1; X(T_\delta^{(n)}) \in A, T_\delta^{(n)} < \sigma\}$, where $0 < \delta_1 < \dots < \delta_{k-1} < \delta$ and $A, A_j \in \mathcal{B}$. Now with $\delta_j, j = 1, \dots, k-1$, and δ being such that $\gamma_1 + \delta_j = \gamma_{1+j}$ and $\gamma_1 + \delta = \gamma$, and with $\Lambda' = \{X(T_{\delta_j}^{(n)}) \in B_{j+1}, j = 1, \dots, k-1; X(T_\delta^{(n)}) \in B, T_\delta^{(n)} < \sigma\}$, we have

$$\begin{aligned} {}^*P^x(\Lambda) &= {}^*E^x\{{}^*P^{X(T_{\gamma_1}^{(n)})}(\Lambda'); X(T_{\gamma_1}^{(n)}) \in B_1, T_{\gamma_1}^{(n)} < \sigma\} \\ &= \int_{B_1} \nu_{x, \gamma_1}^*(dy) {}^*P^y(\Lambda') = \int_{B_1} \nu_{x, \gamma_1}(dy) E^y\{M(n, \delta); \Lambda'\} \\ &= E^x\{M(n, \gamma_1) E^{X(T_{\gamma_1}^{(n)})}[M(n, \delta); \Lambda']; X(T_{\gamma_1}^{(n)}) \in B_1, T_{\gamma_1}^{(n)} < \sigma\} \\ &= E^x\{M(n, \gamma_1)[M(n, \delta)(\theta_{T_{\gamma_1}^{(n)}})]; X(T_{\gamma_j}^{(n)}) \in B_j, j = 1, \dots, k, \\ &\quad X(T_\gamma^{(n)}) \in B, T_\gamma^{(n)} < \sigma\} \\ &= E^x\{M(n, \gamma); \Lambda\}. \end{aligned}$$

Hence we have proved (4.1) for a big enough class of sets to guarantee that (4.1) holds for all $\Lambda \in \mathcal{H}(n, \gamma) \cap \{T_\gamma^{(n)} < \sigma\}$.

For each m and $\alpha > 0$ we define a sequence $\{M^{(n)}(m, \alpha), n = 0, 1, 2, \dots\}$ of functions on Ω by setting

$$M^{(n)}(m, \alpha) = M(m+n, \lambda(m, \alpha, m+n)).$$

Recall that $\lambda(m, \alpha, m+n) = \inf\{\gamma \mid T_\gamma^{(m+n)} = T_\alpha^{(m)}\}$. Note that $M^{(n)}(m, \alpha) = 0$ on $\{T_\alpha^{(m)} \geq \sigma\}$.

LEMMA 4.3. $M^{(n)}(m, \alpha)$ is measurable over $\mathcal{H}_n(m, \alpha)^\mu$ for every μ .

Proof. $M^{(n)}(m, \alpha) = M(m+n, \gamma)$ on $\Lambda_\gamma = \{T_\alpha^{(m)} = T_\gamma^{(m+n)} < \infty\}$. Now

$$\Lambda_\gamma = \{R_{\gamma+1}^{(n)} < \infty, X(R_{\gamma+1}^{(n)}) = X(R_\gamma^{(n)}) \neq X(R_\delta^{(n)}) \text{ for all } \delta < \gamma\} \in \mathcal{H}_n(m, \alpha).$$

Hence $H(m+n, \gamma) \cap \Lambda_\gamma \in \mathcal{H}_n(m, \alpha)$. Since $M(m+n, \gamma)$ is $\mathcal{H}(m+n, \gamma)$ -measurable, its restriction to Λ_γ is measurable over $\mathcal{H}(m+n, \gamma) \cap \Lambda_\gamma$. It follows that the function $\sum_{\gamma \leq \beta} M^{(n)}(m, \alpha) I_{\Lambda_\gamma}$ is $\mathcal{H}_n(m, \alpha)$ -measurable for every β . (I_Λ is the indicator function of Λ .) But this function differs from $M^{(n)}(m, \alpha)$ on a subset of $\{R_\beta^{(n)} < T_\alpha^{(m)}\}$. Hence the lemma follows from the fact that there exists β such that $\{R_\beta^{(n)} < T_\alpha^{(m)}\}$ is a P^μ -null set.

COROLLARY 4.4. $M^{(n)}(m, \alpha)$ is measurable over $\mathcal{F}(T_\alpha^{(m)})$.

Proof. This follows from the previous lemma since $\mathcal{H}_n(m, \alpha) \subset \mathcal{F}(T_\alpha^{(m)})$ and $\mathcal{F}(T_\alpha^{(m)}) = \bigcap_\mu \mathcal{F}(T_\alpha^{(m)})^\mu$.

LEMMA 4.5. For every $x \in E$ and $\Lambda \in \mathcal{H}_n(m, \alpha) \cap \{T_\alpha^{(m)} < \sigma\}$,

$$*P^x(\Lambda) = E^x\{M^{(n)}(m, \alpha); \Lambda\}.$$

Proof. Let $\beta < \pi$ be such that $P^x(R_\beta^{(n)} < T_\alpha^{(m)}) = *P^x(R_\beta^{(n)} < T_\alpha^{(m)}) = 0$. Then Λ is the disjoint union of the sets $\Lambda \cap \{T_\alpha^{(m)} = T_\gamma^{(m+n)} < \infty\}$, $\gamma \leq \beta$, and a set Γ with $P^x(\Gamma) = *P^x(\Gamma) = 0$. Hence

$$*P^x(\Lambda) = \sum_{\gamma \leq \beta} *P^x(\Lambda \cap \{T_\alpha^{(m)} = T_\gamma^{(m+n)} < \infty\})$$

and

$$E^x\{M^{(n)}(m, \alpha); \Lambda\} = \sum_{\gamma \leq \beta} E^x\{M^{(n)}(m, \alpha); \Lambda \cap [T_\alpha^{(m)} = T_\gamma^{(m+n)} < \infty]\}.$$

Now $\Lambda \cap \{T_\alpha^{(m)} = T_\gamma^{(m+n)} < \infty\} \in \mathcal{H}(m+n, \gamma)$ by Corollary 3.2, and on

$$\Lambda \cap \{T_\alpha^{(m)} = T_\gamma^{(m+n)} < \infty\} \subset \{T_\alpha^{(m)} = T_\gamma^{(m+n)} < \sigma\},$$

$M^{(n)}(m, \alpha) = M(m+n, \gamma)$ since $\lambda(m, \alpha, m+n) = \gamma$. Thus from Proposition 4.2 we have $*P^x(\Lambda \cap \{T_\alpha^{(m)} = T_\gamma^{(m+n)} < \infty\}) = E^x\{M^{(n)}(m, \alpha); \Lambda \cap [T_\alpha^{(m)} = T_\gamma^{(m+n)} < \infty]\}$. It follows that $*P^x(\Lambda) = E^x\{M^{(n)}(m, \alpha); \Lambda\}$.

We now prove the basic

THEOREM 4.6. For every x , the sequence of functions $M^{(n)}(m, \alpha)$, $n=0, 1, 2, \dots$, is a martingale with respect to the σ -fields $\mathcal{H}_n(m, \alpha)^{e_x}$, $n=0, 1, 2, \dots$, and the measure P^x .

Proof. We have seen that $M^{(n)} = M^{(n)}(m, \alpha)$ is measurable over $\mathcal{H}_n(m, \alpha)^{e_x}$ and $\mathcal{H}_n(m, \alpha)^{e_x} \subset \mathcal{H}_{n+1}(m, \alpha)^{e_x}$. Trivially $E^x\{M^{(n)}\} < \infty$ for all n . It remains to show that, for every $\Lambda \in \mathcal{H}_n(m, \alpha)$, $E^x\{M^{(n)}; \Lambda\} = E^x\{M^{(n+1)}; \Lambda\}$. Since $M^{(n)}(m, \alpha) = 0$ on $\{T_\alpha^{(m)} \geq \sigma\}$, we may assume $\Lambda \subset \{T_\alpha^{(m)} < \sigma\}$. In view of Lemma 3.3 there exist Λ_1 and Λ_2 in $\mathcal{H}_{n+1}(m, \alpha)$ such that $\Lambda_1 \subset \Lambda \subset \Lambda_2 \subset \{T_\alpha^{(m)} < \sigma\}$ and $P^x(\Lambda_1) = P^x(\Lambda_2)$. By Lemma 4.5 we have

$$\begin{aligned} E^x\{M^{(n)}; \Lambda\} &= *P^x(\Lambda) \leq *P^x(\Lambda_2) = E^x\{M^{(n+1)}; \Lambda_2\} = E^x\{M^{(n+1)}; \Lambda\} \\ &= E^x\{M^{(n+1)}; \Lambda_1\} = *P^x(\Lambda_1) \leq *P^x(\Lambda) = E^x\{M^{(n)}; \Lambda\}. \end{aligned}$$

The theorem is proved.

For each pair (m, α) , $\alpha > 0$, let $M^\infty(m, \alpha)$ (interchangeably $M_{m, \alpha}^\infty$) be defined by

$$M^\infty(m, \alpha) = \liminf_{n \rightarrow \infty} M^{(n)}(m, \alpha).$$

COROLLARY 4.7. (i) For every x , $M^{(n)}(m, \alpha) \rightarrow M^\infty(m, \alpha)$ a.e. P^x as $n \rightarrow \infty$; (ii) $0 \leq M^\infty(m, \alpha) \leq 1$ and $M^\infty(m, \alpha) = 0$ on $\{T_\alpha^{(m)} \geq \sigma\}$; (iii) $M^\infty(m, \alpha)$ is $\mathcal{F}(T_\alpha^{(m)})$ -measurable; (iv) for $\Lambda \in \mathcal{H}_\infty(m, \alpha)^{e_x} \cap \{T_\alpha^{(m)} < \sigma\}$, $E^x\{M^\infty(m, \alpha); \Lambda\} = *P^x(\Lambda)$.

Proof. (i) follows from Theorem 4.6 and the martingale convergence theorem, (ii) is trivial, and (iii) is a result of Corollary 4.4. If $\Lambda \in \mathcal{H}_n(m, \alpha)^{e_x} \cap \{T_\alpha^{(m)} < \sigma\}$, then $E^x\{M^{(n')}(m, \alpha); \Lambda\} = *P^x(\Lambda)$ for all $n' \geq n$. As $n' \rightarrow \infty$ we obtain from (i) that $E^x\{M^\infty(m, \alpha); \Lambda\} = *P^x(\Lambda)$. (ii) follows from this and the fact that $\mathcal{H}_n(m, \alpha)^{e_x}$ increases with n .

5. Removal of the additional hypothesis in 4. In this section we will obtain functions $M^\infty(m, \alpha)$ satisfying Corollary 4.7 without the additional hypothesis in the previous section.

LEMMA 5.1. Let ϕ be a bounded real-valued $\mathcal{H}(n, \gamma)$ -measurable function. Then $E^x\{\phi; T_\gamma^{(n)} < \sigma\}$ and $*E^x\{\phi; T_\gamma^{(n)} < \sigma\}$ are $\mathcal{B}_\Delta$ -measurable in x .

Proof. Let $W(m, i) = \{y \in E \mid \rho(y, E - U_i^{(n)}) < 1/m\}$. Let $\bar{Q}_m = T_{W(m, i)}$ if $X(0) \in V_i^{(n)}$ and $= \infty$ if $X(0) = \Delta$. $\bar{Q}_m$ is $\mathcal{G}$ -measurable; hence so is the increasing limit Q_1 of $\bar{Q}_m$ as $m \rightarrow \infty$. Since $\{X(Q_1) \in B, Q_1 < \infty\} \in \mathcal{G}$ for $B \in \mathcal{B}_\Delta$, it follows from the way Q_1 is defined that Q_2 defined by $Q_2 = Q_1 + Q_1(\theta_{Q_1})$ for $Q_1 < \infty$ and $= \infty$ for $Q_1 = \infty$ is $\mathcal{G}$ -measurable. In fact, let $\bar{Q}'_m = \inf\{t \geq Q_1 \mid X(t) \in W(m, i)\}$ if $X(Q_1) \in V_i^{(n)}$, and $= \infty$ if otherwise; then $\bar{Q}'_m$ are $\mathcal{G}$ -measurable and increase to Q_2 . From the same reasoning we see that all the Q_γ , $\gamma \leq \pi$, defined below are $\mathcal{G}$ -measurable:

$$\begin{aligned} Q_{\gamma+1} &= Q_\gamma + Q_1(\theta_{Q_\gamma}) && \text{if } Q_\gamma < \infty, \\ &= \infty && \text{if otherwise,} \\ Q_\gamma &= \sup_{\delta < \gamma} Q_\delta && \text{if } \gamma \text{ is a limit ordinal.} \end{aligned}$$

Now it follows from the quasi-left continuity that $\{T_1^{(n)} < \sigma\} \subset \{Q_1 < \sigma\}$ and $T_1^{(n)} = Q_1$ a.e. and a.e.* on $\{Q_1 < \sigma\}$. By the definitions of $T_\alpha^{(n)}$ and Q_α we then have $T_\alpha^{(n)} = Q_\alpha$ a.e. and a.e.* on $\{T_\alpha^{(n)} < \sigma\} \cup \{Q_\alpha < \sigma\}$. The lemma immediately follows.

Let n be fixed for a while. Denote by γ the set of limit ordinals $\leq \gamma$.

We will define inductively functions $\xi_\gamma^{(n)}: \Omega \rightarrow [0, 1]$, γ a limit ordinal, satisfying:

- (i) $\xi_\gamma^{(n)}$ is measurable over $\mathcal{H}(n, \gamma)$,
(ii) if $\delta + \gamma' = \gamma$ then $\xi_\gamma^{(n)}(\theta_{T_\delta^{(n)}}) = \xi_{\gamma'}^{(n)}$ a.e. on $\{T_\delta^{(n)} < \sigma\}$,
(5.1) (iii) for every $x \in E$ and $\Lambda \in \mathcal{H}(n, \gamma) \cap \{T_\gamma^{(n)} < \sigma\}$, $*P^x(\Lambda) = E^x\{M(n, \gamma); \Lambda\}$,

where $M(n, \gamma)$ is defined as follows (for every $\gamma < \pi$):

$$\begin{aligned} M(n, \gamma) &= \left[\prod_{\delta < \gamma} e_n(X(T_\delta^{(n)}), X(T_{\delta+1}^{(n)})) \right] \left[\prod_{\delta \in \gamma} \xi_\delta^{(n)} \right], && T_\gamma^{(n)} < \sigma, \\ &= 0, && T_\gamma^{(n)} \geq \sigma, \end{aligned}$$

$\prod_{\delta \in \gamma} \xi_\delta^{(n)}$ meaning 1 if γ is finite. If $\gamma' = \gamma + p$, p a finite ordinal, then we can obtain from (iii) of (5.1) that

$$(5.2) \quad *P^x(\Lambda) = E^x\{M(n, \gamma'); \Lambda\} \quad \text{for every } x \in E \text{ and } \Lambda \in \mathcal{H}(n, \gamma') \cap \{T_{\gamma'}^{(n)} < \sigma\}.$$

This is obvious from the proof of Proposition 4.2. Hence in order to define $\xi_\gamma^{(n)}$, γ a limit ordinal, we may suppose that $\xi_\delta^{(n)}$ are defined and satisfy (5.1) for $\delta \in \Upsilon - \{\gamma\}$ and (5.2) holds for all $\gamma' < \gamma$. Let $\gamma_1 < \dots < \gamma_m \uparrow \gamma$. Let $M'(n, \gamma) = \lim_m M(n, \gamma_m)$.

LEMMA 5.2. *For each x , $*P^x(X(T_\gamma^{(n)}) \in dy, T_\gamma^{(n)} < \sigma) = E^x\{M'(n, \gamma); X(T_\gamma^{(n)}) \in dy, T_\gamma^{(n)} < \sigma\}$ as measures on $\mathcal{B}$.*

Proof. Suppose as $m \rightarrow \infty$ $X(T_{\gamma_1}^{(n)}) \rightarrow X(T_\gamma^{(n)}) \in V_i^{(n)}$. Then there exists m (m depending on ω) such that $X(T_{\gamma'}^{(n)}) \in U_i^{(n)}$ for $\gamma_m \leq \gamma' < \gamma$. Necessarily $X(T_{\gamma'}^{(n)}) \in V_j^{(n)} \cap U_i^{(n)}$ for some $j \neq i$ if $\gamma_m \leq \gamma' < \gamma$. Now if $V_j^{(n)} \cap U_i^{(n)} \neq \emptyset$ and $j \neq i$, then $U_j^{(n)} \cap V_i^{(n)} = \emptyset$ by conditions (ii) and (iii) of Proposition 2.1. This means that for all large m , $T_\gamma^{(n)} = T_{V_i^{(n)}}(\theta_{T_{\gamma_m}^{(n)}}) + T_{\gamma_m}^{(n)}$ on $\{X(T_\gamma^{(n)}) \in V_i^{(n)}, T_\gamma^{(n)} < \sigma\}$. Let

$$O_k = \{y \in U_i^{(n)} \mid 0 < \rho(y, V_i^{(n)}) \leq 1/k\}.$$

Now for a Borel $B \subset V_i^{(n)}$ we have from the induction hypothesis

$$\begin{aligned} & *E^x\{H_{(E - U_i^{(n)}) \cup V_i^{(n)}}(X(T_{\gamma_m'}^{(n)}), B); X(T_{\gamma'}^{(n)}) \in O_k \text{ for } \gamma_m \leq \gamma' \leq \gamma_m', T_{\gamma_m'}^{(n)} < \sigma\} \\ & \leq E^x\{M(n, \gamma_m')H_{(E - U_i^{(n)}) \cup V_i^{(n)}}(X(T_{\gamma_m'}^{(n)}), B); X(T_{\gamma'}^{(n)}) \in O_k \text{ for } \gamma_m \leq \gamma' \leq \gamma_m', T_{\gamma_m'}^{(n)} < \sigma\}. \end{aligned}$$

Let $m' \rightarrow \infty$ to obtain by the bounded convergence theorem

$$\begin{aligned} & *E^x\{H_{(E - U_i^{(n)}) \cup V_i^{(n)}}^*(X(T_\gamma^{(n)}), B); [X(T_\gamma^{(n)}) \in O_k, T_\gamma^{(n)} < \sigma] \cap \Lambda(m, k)\} \\ & + *P^x(\{X(T_\gamma^{(n)}) \in B, T_\gamma^{(n)} \in B, T_\gamma^{(n)} < \sigma\} \cap \Lambda(m, k)) \\ & \leq E^x\{M'(n, \gamma)H_{(E - U_i^{(n)}) \cup V_i^{(n)}}(X(T_\gamma^{(n)}), B); [X(T_\gamma^{(n)}) \in O_k, T_\gamma^{(n)} < \sigma] \cap \Lambda(m, k)\} \\ & + E^x\{M'(n, \gamma); [X(T_\gamma^{(n)}) \in B, T_\gamma^{(n)} < \sigma] \cap \Lambda(m, k)\} \end{aligned}$$

where $\Lambda(m, k) = \{X(T_{\gamma'}^{(n)}) \in O_k \text{ for } \gamma_m \leq \gamma' < \gamma, T_\gamma^{(n)} < \sigma\}$. As $m \rightarrow \infty$, we obtain in the limit

$$\begin{aligned} & *E^x\{H_{(E - U_i^{(n)}) \cup V_i^{(n)}}^*(X(T_\gamma^{(n)}), B); X(T_\gamma^{(n)}) \in O_k, \gamma_m \leq \gamma' \leq \gamma, \text{ for some } m, T_\gamma^{(n)} < \sigma\} \\ & + *P^x(X(T_\gamma^{(n)}) \in B, T_\gamma^{(n)} < \sigma) \\ & \leq E^x\{M'(n, \gamma)H_{(E - U_i^{(n)}) \cup V_i^{(n)}}(X(T_\gamma^{(n)}), B); \\ & \quad X(T_\gamma^{(n)}) \in O_k, \gamma_m \leq \gamma' \leq \gamma, \text{ for some } m, T_\gamma^{(n)} < \sigma\} \\ & + E^x\{M'(n, \gamma); X(T_\gamma^{(n)}) \in B, T_\gamma^{(n)} < \sigma\}. \end{aligned}$$

Let $k \rightarrow \infty$. Then the first terms on both sides of the above inequality approach 0. Thus we have $*P^x(X(T_\gamma^{(n)}) \in B, T_\gamma^{(n)} < \sigma) \leq E^x\{M'(n, \gamma); X(T_\gamma^{(n)}) \in B, T_\gamma^{(n)} < \sigma\}$ for $B \subset V_i^{(n)}$. The lemma follows.

LEMMA 5.3. *There exists a function $g_\gamma: E \times E \rightarrow [0, 1]$, measurable over $\mathcal{B} \times \mathcal{B}$, such that*

$$*P^x(X(T_\gamma^{(n)}) \in B, T_\gamma^{(n)} < \sigma) = E^x\{g_\gamma(X(0), X(T_\gamma^{(n)}))M'(n, \gamma); X(T_\gamma^{(n)}) \in B, T_\gamma^{(n)} < \sigma\}$$

for all $x \in E, B \in \mathcal{B}$.

Proof. It follows from Lemma 5.1 that

$$*P^x(X(T_\gamma^{(n)}) \in B, T_\gamma^{(n)} < \sigma)$$

and

$$E^x\{M'(n, \gamma); X(T_\gamma^{(n)}) \in B, T_\gamma^{(n)} < \sigma\}$$

are $\mathcal{B}$ -measurable in x for each $B \in \mathcal{B}$.

Lemma 5.2 then implies the existence of a function $g_\gamma: E \times E \rightarrow [0, 1]$, measurable over $\mathcal{B} \times \mathcal{B}$, such that

$$\begin{aligned} *P^x(X(T_\gamma^{(n)}) \in B, T_\gamma^{(n)} < \sigma) &= \int_B g_\gamma(x, y) E^x\{M'(n, \gamma); X(T_\gamma^{(n)}) \in dy, T_\gamma^{(n)} < \sigma\} \\ &= E^x\{g_\gamma(X(0), X(T_\gamma^{(n)}))M'(n, \gamma); X(T_\gamma^{(n)}) \in B, T_\gamma^{(n)} < \sigma\} \\ &\quad \text{for all } x \in E \text{ and } B \in \mathcal{B}. \end{aligned}$$

Let δ_m be such that $\gamma_m + \delta_m = \gamma$. Suppose g_{δ_m} is also defined and satisfies Lemma 5.3 with δ_m replacing γ throughout. Let $\eta_m = g_{\delta_m}(X(T_{\gamma_m}^{(n)}), X(T_\gamma^{(n)}))$ on $\{T_\gamma^{(n)} < \sigma\}$ and $= 0$ elsewhere. Let $\mathcal{H}'(n, \gamma_m)$ be the σ -field generated by $\mathcal{H}(n, \gamma_m)$ and sets of the form $\{X(T_\gamma^{(n)}) \in B, T_\gamma^{(n)} < \sigma\}$.

PROPOSITION 5.4. *For every x , $\{\eta_m, m=1, 2, \dots\}$ is a martingale with respect to the measure $M'(n, \gamma)dP^x$ and the σ -fields $\mathcal{H}'(n, \gamma_m)$.*

Proof. It suffices to show that for $\Lambda = \Lambda_1 \cap \{X(T_\gamma^{(n)}) \in B, T_\gamma^{(n)} < \sigma\}$, where $\Lambda_1 \in \mathcal{H}(n, \gamma_m) \cap \{T_{\gamma_m}^{(n)} < \sigma\}$ and $B \in \mathcal{B}$, $E^x\{\eta_m M'(n, \gamma); \Lambda\} = *P^x(\Lambda)$. From the previous proposition and the induction hypotheses we have

$$\begin{aligned} E^x\{\eta_m M'(n, \gamma); \Lambda\} &= E^x\{M(n, \gamma_m) E^{X(T_{\gamma_m}^{(n)})}[g_{\delta_m}(X(0), X(T_\gamma^{(n)}))M'(n, \gamma); \\ &\quad X(T_\gamma^{(n)}) \in B, T_\gamma^{(n)} < \sigma]; \Lambda_1\} \\ &= *E^x\{*P^{X(T_{\gamma_m}^{(n)})}(X(T_\gamma^{(n)}) \in B, T_\gamma^{(n)} < \sigma); \Lambda_1\} \\ &= *P^x(\Lambda_1 \cap \{X(T_\gamma^{(n)}) \in B, T_\gamma^{(n)} < \sigma\}) = *P^x(\Lambda). \end{aligned}$$

Let $\xi_\gamma^{(n)} = \liminf_{m \rightarrow \infty} \eta_m$. Then by the martingale convergence theorem $\eta_m \rightarrow \xi_\gamma^{(n)}$ a.e. as $m \rightarrow \infty$. We claim that $\xi_\gamma^{(n)}$ satisfies (5.1). (i) of (5.1) is obvious, and (iii) is a result of the facts that $\eta_m \rightarrow \xi_\gamma^{(n)}$ a.e. and that $E^x\{\eta_m M'(n, \gamma); \Lambda\} = *P^x(\Lambda)$ for $\Lambda \in \mathcal{H}'(n, \gamma_p) \cap \{T_\gamma^{(n)} < \sigma\}$, $p \leq m$. To show (ii), note that

$$\xi_\gamma^{(n)}(\theta_{T_\delta^{(n)}}) = \liminf_{m \rightarrow \infty} g_{\delta_m}(X(T_{\delta+\gamma_m}^{(n)}), X(T_\gamma^{(n)}))$$

on $\{T_\gamma^{(n)} < \sigma\}$, where $\gamma'_1 < \dots < \gamma'_m \uparrow \gamma'$ and $\gamma'_m + \delta'_m = \gamma'$. Let $\{\gamma_m''\}$ be the increasing sequence formed by $\{\gamma_m\}$ and $\{\delta + \gamma_m'\}$ and let $\gamma_m'' + \delta_m'' = \gamma$. Then Proposition 5.4 implies $g_{\delta_m''}(X(T_{\gamma_m''}^{(n)}), X(T_\gamma^{(n)}))$ converges a.e. on $\{T_\gamma^{(n)} < \sigma\}$. This proves $\xi_\gamma^{(n)}(\theta_{T_\delta^{(n)}}) = \xi_\gamma^{(n)}$ a.e. on $\{T_\gamma^{(n)} < \sigma\}$.

Thus by induction we have the following proposition, in which γ need not be a limit ordinal because of a previous remark.

PROPOSITION 5.5. $M(n, \gamma)$ is $\mathcal{H}(n, \gamma)$ -measurable; for each x and $\Lambda \in \mathcal{H}(n, \gamma) \cap \{T_\gamma^{(n)} < \sigma\}$, $E^x\{M(n, \gamma); \Lambda\} = *P^x(\Lambda)$. Let $M^{(n)}(m, \alpha) = M(m+n, \lambda(m, \alpha, m+n))$ for $n \geq 0$ and

$$M^\infty(m, \alpha) = \liminf_{n \rightarrow \infty} M^{(n)}(m, \alpha).$$

Then Lemma 4.3, Corollary 4.4, Lemma 4.5, Theorem 4.6 and finally Corollary 4.7 again hold, since their proofs depend only on the properties of $M(n, \gamma)$ stated in the above proposition.

6. Properties of the functions $M^\infty(m, \alpha)$.

LEMMA 6.1. For any two pairs (m, α) and (m', α') , $M^\infty(m, \alpha) = M^\infty(m', \alpha')$ on $\{T_\alpha^{(m)} = T_{\alpha'}^{(m')}\}$.

Proof. Suppose $m \leq m'$. If $T_\alpha^{(m)} = T_{\alpha'}^{(m')}$, then

$$\begin{aligned} \lambda(m', \alpha', m' + n) &= \inf \{\gamma \mid T_\gamma^{(m' + n)} = T_{\alpha'}^{(m')}\} \\ &= \inf \{\gamma \mid T_\gamma^{(m + m' - m + n)} = T_\alpha^{(m)}\} = \lambda(m, \alpha, m + (m' - m + n)). \end{aligned}$$

Hence

$$\begin{aligned} M^{(m' - m + n)}(m, \alpha) &= M(m + (m' - m + n), \lambda(m, \alpha, m + (m' - m + n))) \\ &= M(m' + n, \lambda(m', \alpha', m' + n)) \\ &= M^{(n)}(m', \alpha') \end{aligned}$$

for all n on $\{T_\alpha^{(m)} = T_{\alpha'}^{(m')} < \sigma\}$, while on $\{T_\alpha^{(m)} = T_{\alpha'}^{(m')} \geq \sigma\}$,

$$M^{(n)}(m, \alpha) = M^{(n)}(m', \alpha') = 0.$$

The lemma then follows from the definitions of $M^\infty(m, \alpha)$ and $M^\infty(m', \alpha')$.

For $\omega \in \Omega$, let $J(\omega) = \{t \in [0, \infty] \mid t = T_\alpha^{(m)}(\omega) \text{ for some } (m, \alpha)\}$. Let $M^\infty(m, 0) \equiv 1$ on Ω for all m . For $\omega \in \Omega$ let $M(\cdot, \omega)$ be defined on $J(\omega)$ by

$$M(t, \omega) = M_{m, \alpha}^\infty(\omega),$$

where (m, α) is any pair satisfying $T_\alpha^{(m)}(\omega) = t$. $M(\cdot, \omega)$ is well defined because of Lemma 6.1. For any random variable T with $T(\omega) \in J(\omega)$ for all ω , the function $M(T)$, interchangeably M_T , is defined by $M(T)(\omega) = M(T(\omega), \omega)$. In particular we have $M(T_\alpha^{(m)}) \equiv M^\infty(m, \alpha)$. We will investigate some properties of $M(\cdot, \omega)$ before setting up the desired multiplicative functional.

LEMMA 6.2. For each ω , $M(\cdot, \omega)$ is nonincreasing on $J(\omega)$.

Proof. Let $0 < \alpha < \alpha'$. It follows from definition that $M^{(n)}(m, \alpha) \geq M^{(n)}(m, \alpha')$. Hence $M^\infty(m, \alpha) \geq M^\infty(m, \alpha')$. Since $M^\infty(m, 0) \equiv 1$ and $M^\infty(m, \alpha) \leq 1$, we have that, for each ω , $M(\cdot, \omega)$ is nonincreasing on $J_m(\omega) = \{t \in [0, \infty] \mid t = T_\alpha^{(m)}(\omega) \text{ for some } \alpha\}$. The fact that $J_m(\omega) \uparrow J(\omega)$ implies the lemma.

LEMMA 6.3. Let $x \in I$. Then $M(T_1^{(m)}) \rightarrow 1$ a.e. P^x as $m \rightarrow \infty$.

Proof. Since $I = I^*$ (Proposition 1.1), x is also in I^* . Let $i(m)$ be such that $x \in V_{i(m)}^{(m)}$. Then, since $U_{i(m)}^{(m)} \downarrow \{x\}$, $*P^x(T_1^{(m)} < \infty) = *P^x(T_{E - U_{i(m)}^{(m)}} < \infty) \uparrow 1$ as $m \rightarrow \infty$. Now by Corollary 4.7 $E^x\{M(T_1^{(m)}); T_1^{(m)} < \infty\} = *P^x(T_1^{(m)} < \infty)$. It then follows from the previous lemma that $M(T_1^{(m)}) \rightarrow 1$ a.e. P^x .

LEMMA 6.4. $M(T_{\alpha+\alpha'}^{(m)}) = M(T_{\alpha}^{(m)})[M(T_{\alpha'}^{(m)})(\theta_{T_{\alpha}^{(m)}})]$ a.e. on $\{T_{\alpha}^{(m)} < \infty\}$ for arbitrary m, α and α' .

Proof. Obviously we may assume $\alpha, \alpha' > 0$. On $\{T_{\alpha}^{(m)} < \infty\}$, $M^{(n)}(m, \alpha + \alpha') = M^{(n)}(m, \alpha)[M^{(n)}(m, \alpha')(\theta_{T_{\alpha}^{(m)}})]$. If both $\lim_n M^{(n)}(m, \alpha + \alpha')$ and $\lim_n M^{(n)}(m, \alpha)$ exist, in which case they are equal to $M(T_{\alpha+\alpha'}^{(m)})$ and $M(T_{\alpha}^{(m)})$ respectively, then so does $\lim_n [M^{(n)}(m, \alpha')(\theta_{T_{\alpha}^{(m)}})]$ and it is equal to $M(T_{\alpha'}^{(m)})(\theta_{T_{\alpha}^{(m)}})$. Hence the lemma follows from (i) of Corollary 4.7.

COROLLARY 6.5. For arbitrary pairs (m, α) and (m', α') ,

$$M(T_{\alpha}^{(m)} + T_{\alpha'}^{(m')}(\theta_{T_{\alpha}^{(m)}})) = M(T_{\alpha}^{(m)})[M(T_{\alpha'}^{(m')})(\theta_{T_{\alpha}^{(m)}})]$$

a.e. on $\{T_{\alpha}^{(m)} < \infty\}$.

Proof. This follows from Lemma 6.5 and the facts that, with $m'' = \max\{m, m'\}$, $T_{\alpha}^{(m)}(\omega) = T_{\gamma}^{(m'')}(\omega)$ and $T_{\alpha}^{(m')}(\theta_{T_{\alpha}^{(m)}}\omega) = T_{\gamma'}^{(m'')}(\theta_{T_{\gamma}^{(m'')}}\omega)$ (if $T_{\alpha}^{(m)}(\omega) < \infty$) for some γ and γ' , and that for a fixed x , there is a β such that $P^x\{T_{\beta}^{(m'')} < \infty\} = 0$.

PROPOSITION 6.6. $M(\cdot, \omega)$ is right continuous on $J(\omega)$ a.e. $P^x(d\omega)$ for every x .

Proof. We show first that for every x , $M(\cdot, \omega)$ is right continuous at 0 a.e. $P^x(d\omega)$. If $x \in I$, this follows from Lemmas 6.3 and 6.4. If $x \in H$, it follows from the fact that, a.e. $P^x(d\omega)$, there exists $\delta > 0$ such that $T_1^{(m)}(\omega) > \delta$ for all m . Now let (m, α) be fixed. The strong Markov property and the above fact imply that $M(T_1^{(m')})(\theta_{T_{\alpha}^{(m)}}) \rightarrow 1$ a.e. on $\{T_{\alpha}^{(m)} < \infty, T_1^{(m')}(\theta_{T_{\alpha}^{(m)}}) \downarrow 0 \text{ as } m' \rightarrow \infty\}$. It then follows from Corollary 6.5 that $M(\cdot, \omega)$ is right continuous at $T_{\alpha}^{(m)}(\omega)$ a.e. $P^x(d\omega)$ for every x . Since for a fixed x there is a β such that $P^x(T_{\beta}^{(m)} < \infty) = 0$ for all m , the proposition follows.

7. The multiplicative functional—when the processes have no holding points. In this section we assume $H = H^* = \emptyset$, i.e., both processes have no holding points. For $t \in [0, \infty)$ let $\Phi_t = \{\omega \in \Omega \mid \text{for every } \delta > 0 \text{ there exists } (m, \alpha) \text{ such that } t \leq T_{\alpha}^{(m)}(\omega) < t + \delta \text{ or } X(t, \omega) = \Delta\}$. We now extend $M(\cdot, \omega)$ from $J(\omega)$ to $[0, \infty]$ by setting

$$\begin{aligned} M(t, \omega) &= \sup \{M(T_{\alpha}^{(m)}(\omega), \omega) \mid t \leq T_{\alpha}^{(m)}(\omega)\} \quad \text{if } \omega \in \Phi_t, \\ &= \inf \{M(T_{\alpha}^{(m)}(\omega), \omega) \mid t > T_{\alpha}^{(m)}(\omega)\} \quad \text{if } \omega \in \Omega - \Phi_t. \end{aligned}$$

Lemma 6.2 guarantees that this is indeed an extension of the original $M(\cdot, \omega)$ for every ω . M is now a function on $[0, \infty] \times \Omega$. We will denote by $M(T)$ (or M_T), for any $T: \Omega \rightarrow [0, \infty]$, the function defined by $M(T)(\omega) = M(T(\omega), \omega)$. Note that

$0 \leq M \leq 1$ and that $M(t, \omega)$ is nonincreasing in t for every ω . Moreover, if $M(\cdot, \omega)$ is right continuous on $J(\omega)$ then it is so on $[0, \infty]$. Hence Proposition 6.6 implies that $M(t)$ is right continuous in t a.e.

PROPOSITION 7.1. $M(t)$ is measurable over $\mathcal{F}_t$ for $0 \leq t < \infty$.

Proof. We show first that $\Phi_t \in \mathcal{F}_t$. For an arbitrary μ let β be such that $P^\mu\{T_\beta^{(m)} < \infty\} = 0$ for all m . Then

$$\Phi_t = \{X(t) = \Delta\} \cup \left(\bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{\alpha < \beta} \{t \leq T_\alpha^{(m)} < t + 1/k\} \right) \cup \Gamma,$$

where Γ is a subset of $\bigcup_{m=1}^{\infty} \{T_\beta^{(m)} < \infty\}$, a P^μ -null set in $\mathcal{F}$. Since

$$\bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{\alpha < \beta} \{t \leq T_\alpha^{(m)} < t + 1/k\} \in \bigcap_{k=1}^{\infty} \mathcal{F}(t + 1/k) = \mathcal{F}(t),$$

the arbitrariness of μ implies $\Phi_t \in \mathcal{F}_t$. Now define a sequence of random variables R_n as follows:

$$\begin{aligned} R_n &= \inf \{T_\gamma^{(n)} \mid T_\gamma^{(n)} \geq t\} \quad \text{on } \Phi_t, \\ &= \sup \{T_\gamma^{(n)} \mid T_\gamma^{(n)} < t\} \quad \text{on } \Omega - \Phi_t. \end{aligned}$$

Note that inf and sup above can be replaced by min and max respectively. Clearly as $n \rightarrow \infty$, $R_n \downarrow t$ on $\Phi_t - \{X(t) = \Delta\}$ and $R_n \uparrow \sup\{T_\alpha^{(n)} \mid T_\alpha^{(n)} < t\}$ on $\Omega - \Phi_t$. It then follows that $M(R_n) \rightarrow M(t)$ as $n \rightarrow \infty$. Now for a positive integer k let $M^{(k)}(R_n) = M(R_n)I_{(R_n < t + 1/k)}$. Then $M^{(k)}(R_n) \rightarrow M(t)$ as $n \rightarrow \infty$. We will show that $M^{(k)}(R_n)$ is $\mathcal{F}(t + 1/k)$ -measurable. This would imply that $M(t)$ is $\mathcal{F}(t + 1/k)$ -measurable for every k and thus $\mathcal{F}(t)$ -measurable. To show $M^{(k)}(R_n)$ is measurable over $\mathcal{F}(t + 1/k)$, again let μ be arbitrary and choose β such that $P^\mu(T_\beta^{(n)} < \infty) = 0$ for every n . For each pair (n, γ) let $\Phi_t(n, \gamma) = \Phi_t \cap \{T_\gamma^{(n)} \geq t; T_\delta^{(n)} < t \text{ for all } \delta < \gamma\}$ and $\Psi_t(n, \gamma) = \{T_\gamma^{(n)} < t \leq T_{\gamma+1}^{(n)}\} - \Phi_t$. Clearly $\Phi_t(n, \gamma)$ and $\Psi_t(n, \gamma)$ are in $\mathcal{F}(t)$, and $R_n = T_\gamma^{(n)}$ on $\Phi_t(n, \gamma) \cup \Psi_t(n, \gamma)$. Since $M(T_\gamma^{(n)})$ is $\mathcal{F}(T_\gamma^{(n)})$ -measurable, this implies that $M(R_n)I_{\Phi_t(n, \gamma) \cap \{T_\gamma^{(n)} < t + 1/k\}}$ and $M(R_n)I_{\Psi_t(n, \gamma)}$ are $\mathcal{F}(t + 1/k)$ -measurable. Thus the function

$$\sum_{\gamma < \beta} (M(R_n)I_{\Phi_t(n, \gamma) \cap \{T_\gamma^{(n)} < t + 1/k\}} + M(R_n)I_{\Psi_t(n, \gamma)})$$

is $\mathcal{F}(t + 1/k)$ -measurable. Since this function differs from $M^{(k)}(R_n)$ only on a subset of $\Omega - \bigcup_{\gamma < \beta} (\Phi_t(n, \gamma) \cup \Psi_t(n, \gamma)) \subset \{T_\beta^{(n)} < \infty\}$, $M^{(k)}(R_n)$ is measurable over $\mathcal{F}(t + 1/k)^\mu$. μ being arbitrary, $M^{(k)}(R_n)$ is measurable over $\mathcal{F}(t + 1/k)$. The proof is thus complete.

We will now show that the restriction of M to $[0, \infty) \times \Omega$, which we will still call M , is a multiplicative functional of X and has the desired property. The assumption $H = \emptyset$ implies that for every x and t , $P^x(\{X(t) \in E\} - \Phi_t) = 0$. This fact will be strongly used in the next proof.

THEOREM 7.2. M is a multiplicative functional of X .

Proof. We have seen that $0 \leq M \leq 1$, $M(t)$ is nonincreasing and right continuous in t a.e., and $M(t)$ is $\mathcal{F}(t)$ -measurable. It remains to show that for arbitrary $s, t \geq 0$ the equality

$$(7.1) \quad M(s+t) = M(t)[M(s)(\theta_t)]$$

holds a.e. Obviously we need only to show the equality holding a.e. on $\{s+t < \sigma\}$. Since $\{s+t < \sigma\} = \{X(s+t) \in E\} = \{X(s+t) \in E\} \cap \{X(t) \in E\}$,

$$P^x(\{s+t < \sigma\} - \Phi_t \cap \Phi_{s+t}) \leq P^x(\{X(t) \in E\} - \Phi_t) + P^x(\{X(s+t) \in E\} - \Phi_{s+t}) = 0.$$

Thus it suffices to show that (7.1) holds a.e. on $\Phi_s \cap \Phi_{s+t} \cap \{s+t < \sigma\}$.

For each n define R_n as in the proof of Proposition 7.1, and define R'_n as R_n except that t is replaced by $s+t$, i.e., $R'_n = \inf\{T_\gamma^{(n)} \mid T_\gamma^{(n)} \geq s+t\}$ on Φ_{s+t} and $= \sup\{T_\gamma^{(n)} \mid T_\gamma^{(n)} < s+t\}$ elsewhere. On $\Phi_t \cap \Phi_{s+t}$, $R_n \leq R'_n$, although both may be infinite. Let $S_n = R'_n - R_n$ on $\{R_n < \infty\}$. Then

$$M(R'_n) = M(R_n)[M(S_n)(\theta_{R_n})]$$

a.e. on $\{R_n < \infty\} \cap \Phi_t \cap \Phi_{s+t}$ ($M(S_n)(\theta_{R_n})(\omega) = M(S_n(\omega), \theta_{R_n}\omega)$). This follows easily from Lemma 6.4. Now $M(R_n) \rightarrow M(t)$ and $M(R'_n) \rightarrow M(s+t)$ as $n \rightarrow \infty$. Since $\{R_n < \infty, t < \sigma\} \uparrow \{t < \sigma\}$ we need only to show

$$(7.2) \quad M(S_n)(\theta_{R_n}) \rightarrow M(s)(\theta_t) \quad \text{as } n \rightarrow \infty$$

a.e. on $\Phi_t \cap \Phi_{s+t} \cap \{s+t < \sigma\}$ (regard $M(S_n)(\theta_{R_n})$ as zero if $R_n = \infty$). In view of Corollary 2.4, for each n and $\omega \in \Phi_t \cap \Phi_{s+t}$ there exist $\delta_n(\omega) \leq \delta'_n(\omega) \leq \pi$ such that $R_n(\omega) = t + T_{\delta_n(\omega)}^{(n+1)}(\theta_t\omega)$ and $R'_n(\omega) = t + T_{\delta'_n(\omega)}^{(n+1)}(\theta_t\omega)$. For a fixed x choose β such that $P^x(T_\beta^{(n+2)} < \infty) = 0$. It is easy to see that if $T_\beta^{(n+2)}(\omega) = \infty$ then $\delta_n(\omega)$ and $\delta'_n(\omega)$ can be chosen $\leq \beta$. Now from Lemma 6.4 and the Markov property we have

$$M(T_\gamma^{(n+1)})(\theta_t) = [M(T_\gamma^{(n+1)})(\theta_t)][M(T_\gamma^{(n+1)})(\theta_{T_\gamma^{(n+1)}})(\theta_t)]$$

a.e. P^x on $\{T_\gamma^{(n+1)}(\theta_t) < \infty\}$ whenever $\gamma + \gamma'' = \gamma'$. It follows that

$$(7.3) \quad M(T_{\delta_n}^{(n+1)})(\theta_t) = [M(T_{\delta_n}^{(n+1)})(\theta_t)][M(T_{\delta_n}^{(n+1)})(\theta_{T_{\delta_n}^{(n+1)}})(\theta_t)]$$

a.e. P^x on $\{R_n \leq \infty\} \cap \Phi_t \cap \Phi_{s+t}$, where $\delta_n''(\omega)$ is defined by $\delta_n(\omega) + \delta_n''(\omega) = \delta'_n(\omega)$.

Now the second factor on the right side of (7.3) is exactly $M(S_n)(\theta_{R_n})$. It is clear that $T_{\delta_n}^{(n+1)}(\theta_t) = R_n - t \downarrow 0$ and $T_{\delta_n}^{(n+1)}(\theta_t) = R_n - t \downarrow s$ on $\Phi_t \cap \Phi_{s+t} \cap \{s+t < \sigma\}$. Since by the Markov property $M(t')(\theta_t)$ is right continuous in t' a.e. P^x , $M(T_{\delta_n}^{(n+1)})(\theta_t) \rightarrow 1$ and $M(T_{\delta_n}^{(n+1)})(\theta_t) \rightarrow M(s)(\theta_t)$ a.e. P^x on $\Phi_t \cap \Phi_{s+t} \cap \{s+t < \sigma\}$ as $n \rightarrow \infty$. Thus (7.2) follows from (7.3), and this completes the proof of (7.1) and hence of the theorem.

LEMMA 7.3. *Let K be a compact subset of E . For each m let $R_m = \inf\{T_\alpha^{(m)} \mid T_\alpha^{(m)} \geq T_K\}$. If $x \notin K$, then $E^x\{M(R_m); X(R_m) \in B, R_m < \infty\} = *P^x(X(R_m) \in B, R_m < \infty)$ for all $B \in \mathcal{B}_\Delta$.*

Proof. Let β be such that $P^x(T_\beta^{(m)} < \infty) = {}^*P^x(T_\beta^{(m)} < \infty) = 0$. Then for $B \in \mathcal{B}_\Delta$

$$E^x\{M(R_n); X(R_n) \in B, R_n < \infty\} = \sum_{\alpha < \beta} E^x\{M(T_\alpha^{(m)}); X(T_\alpha^{(m)}) \in B, R_m = T_\alpha^{(m)} < \infty\}$$

and

$${}^*P^x(X(R_m) \in B, R_m < \infty) = \sum_{\alpha < \beta} {}^*P^x(X(T_\alpha^{(m)}) \in B, R_m = T_\alpha^{(m)} < \infty).$$

Now it follows from Lemma 3.4 that $\{X(T_\alpha^{(m)}) \in B, R_m = T_\alpha^{(m)} < \infty\} \in \mathcal{H}_\infty(m, \alpha)^{e_x}$. Since obviously $R_m < \sigma$ if $R_m < \infty$, we have

$$E^x\{M(T_\alpha^{(m)}); X(T_\alpha^{(m)}) \in B, R_m = T_\alpha^{(m)} < \infty\} = {}^*P^x(X(T_\alpha^{(m)}) \in B, R_m = T_\alpha^{(m)} < \infty)$$

by Corollary 4.7. The desired equality follows.

THEOREM 7.4. For every $x \in E$, compact subset K of E , and $B \in \mathcal{B}$

$$(7.4) \quad E^x\{M(T_K); X(T_K) \in B, T_K < \infty\} = H_K^*(x, B).$$

Proof. (i) Suppose first $x \notin K$. Define R_m as in the previous proof. We have shown in Proposition 2.6 that $R_m \downarrow T_K$ a.e. P^x and a.e. ${}^*P^x$. It then follows from the a.e. right continuity of $M(t)$ and the previous lemma that, for $f \in \mathcal{C}_0$,

$$\begin{aligned} {}^*E^x\{f(X(T_K)); T_K < \infty\} &= \lim_m {}^*E^x\{f(X(R_m)); R_m < \infty\} \\ &= \lim_m E^x\{M(R_m)f(X(R_m)); R_m < \infty\} \\ &= E^x\{M(T_K)f(X(T_K)); T_K < \infty\}. \end{aligned}$$

The arbitrariness of f implies (7.4) for all $B \in \mathcal{B}$.

(ii) Suppose now $x \in K$. By Proposition 1.2 $x \in \text{reg } K$ if and only if $x \in \text{reg}^* K$. Hence if $x \in \text{reg } K$, $T_K = 0$ a.e. P^x and a.e. ${}^*P^x$, and it follows that

$$\begin{aligned} E^x\{M(T_K); X(T_K) \in B, T_K < \infty\} &= {}^*P^x(X(T_K) \in B, T_K < \infty) = 1 \quad \text{if } x \in B, \\ &= 0 \quad \text{if } x \notin B. \end{aligned}$$

If $x \notin \text{reg } K$, then $x \notin \text{reg}^* K$. Let $F_n = \{y \mid \rho(x, \Delta)/2 \geq \rho(x, y) \geq 1/n\}$. We have $T_{F_n} \downarrow 0$ a.e. P^x and a.e. ${}^*P^x$. Hence $P^x(T_{F_n} < T_K) \uparrow 1$ and ${}^*P^x(T_{F_n} < T_K) \uparrow 1$ as $n \rightarrow \infty$. Now it follows from the fact established in (i) that, for all n ,

$$\begin{aligned} {}^*E^x\{{}^*P^{X(T_{F_n})}(X(T_K) \in B, T_K < \infty); X(T_{F_n}) \notin K, T_{F_n} < \infty\} \\ = E^x\{M(T_{F_n})E^{X(T_{F_n})}[M(T_K); X(T_K) \in B, T_K < \infty]; X(T_{F_n}) \notin K, T_K < \infty\}. \end{aligned}$$

Clearly the left side converges to ${}^*P^x(X(T_K) \in B, T_K < \infty)$ as $n \rightarrow \infty$. Since on the set $\{F_n < T_K\}$, $M(T_K) = M(T_{F_n})[M(T_K)(\theta_{T_{F_n}})]$ a.e. P^x , the right side must converge to $E^x\{M(T_K); X(T_K) \in B, T_K < \infty\}$ as $n \rightarrow \infty$. Hence (7.4) follows, and the proof is complete.

8. The multiplicative functional—when the processes have holding points. If $H = H^*$ is nonempty, the functional defined in the previous section does not have

the desired property (in fact it is not necessarily a multiplicative functional), and we have to make a different definition. Let $R = \inf_m T_1^{(m)} = \inf\{t < \sigma \mid X(t) \neq X(0)\}$. For each t set $M(t) = \sup\{M(T_\alpha^{(n)}) \mid T_\alpha^{(n)} \geq t\}$ on Φ_t . Note that $M(t + R(\theta_t))$ is defined for every t . We then set on $\Omega - \Phi_t$

$$\begin{aligned} M(t) &= \min \{M(t + R(\theta_t))/M(R)(\theta_t), \inf \{M(T_\alpha^{(n)}) \mid T_\alpha^{(n)} < t\}\} \\ &\quad \text{if } \sup \{T_\alpha^{(n)} \mid T_\alpha^{(n)} < t\} \neq T_\gamma^{(m)} \text{ for any } (m, \gamma), \\ &= \inf \{M(T_\alpha^{(n)}) \mid T_\alpha^{(n)} < t\} \text{ otherwise.} \end{aligned}$$

We will show that this M (restricted to $[0, \infty) \times \Omega$) is a desired multiplicative functional. It is obvious that $M(t)$ is nonincreasing in t everywhere. If $M(\cdot, \omega)$ is right continuous on $J(\omega)$ then it is so on $[0, \infty]$. Hence $M(t)$ is right continuous a.e.

For a while let K be a fixed compact subset of H . Define stopping times R_γ and S_γ , $\gamma < \pi$, inductively as follows: $R_0 = S_0 = 0$; $S_{\gamma+1} = R_\gamma + T_K(\theta_{R_\gamma})$ if $R_\gamma < \infty$ and $= \infty$ if otherwise; $R_\gamma = S_\gamma + R(\theta_{S_\gamma})$ if $S_\gamma < \infty$ and $= \infty$ if otherwise; $S_\gamma = \sup_{\delta < \gamma} R_\delta = \sup_{\delta < \gamma} S_\delta$ for a limit ordinal γ . Obviously, $X(S_\gamma) \in K$ a.e. and a.e.*; $R_\gamma > S_\gamma$ a.e. and a.e.* on $\{S_\gamma < \infty\}$ for $\gamma > 0$, so for each μ there exists β such that $P^\mu(R_\beta < \infty) = {}^*P^\mu(R_\beta < \infty) = 0$; for each μ , $R_\gamma = \text{some } T_\alpha^{(m)}$ (m and α depending on ω) a.e. and a.e.*. For simplicity we assume that for each γ and ω $R_\gamma(\omega) = T_\alpha^{(m)}(\omega)$ for some (m, α) .

Let $T_n = \inf \{T_\alpha^{(n)} < \sigma \mid \text{if } X(T_\alpha^{(n)}) \in V_i^{(n)} \text{ then } K \cap U_i^{(n)} \neq \emptyset\}$. Let $\mathcal{J}_n$ be the σ -field generated by sets of the form $\{X(T_\alpha^{(m)}) \in B, T_\alpha^{(m)} \leq T_n, T_\alpha^{(m)} < \sigma\}$ or $\{X(S_1) \in B, S_1 < \sigma\}$. Let $\mathcal{J}_\infty$ be the smallest σ -field containing all $\mathcal{J}_n$. Let $\mathcal{K}$ be the σ -field generated by sets of the form $\{X(T_\alpha^{(m)}) \in B, T_\alpha^{(m)} \leq R_1, T_\alpha^{(m)} < \sigma\}$. Let γ be an arbitrary limit ordinal and let $\gamma_1 < \dots < \gamma_n \uparrow \gamma$. Let $\mathcal{J}_n$ be the σ -field generated by sets of the form $\{X(T_\alpha^{(m)}) \in B, T_\alpha^{(m)} \leq R_{\gamma_n}, T_\alpha^{(m)} < \sigma\}$ or $\{X(S_\gamma) \in B, S_\gamma < \infty\}$. Let $\mathcal{J}_\infty$ be the smallest σ -field containing all $\mathcal{J}_n$. Let $\mathcal{K}_\gamma$ be the σ -field generated by sets of the form $\{X(T_\alpha^{(m)}) \in B, T_\alpha^{(m)} \leq R_\gamma, T_\alpha^{(m)} < \sigma\}$. We will omit the proofs of the following two lemmas.

LEMMA 8.1. (i) For each μ , $\mathcal{J}_1^\mu \subset \dots \subset \mathcal{J}_n^\mu \subset \dots \subset \mathcal{J}_\infty^\mu \subset \mathcal{K}^\mu$; (ii) for each μ $M(T_n)$ is $\mathcal{J}_n^\mu$ -measurable; (iii) for every x and $\Lambda \in \mathcal{J}_n \cap \{T_n < \sigma\}$, ${}^*P^x(\Lambda) = E^x\{M(T_n); \Lambda\}$; (iv) $X(T_n)$ is $\mathcal{J}_n^\mu$ -measurable for each μ .

LEMMA 8.2. (i) For each μ , $\mathcal{J}_1^\mu \subset \dots \subset \mathcal{J}_n^\mu \subset \dots \subset \mathcal{J}_\infty^\mu \subset \mathcal{K}_\gamma^\mu$; (ii) for each μ $M(R_{\gamma_n})$ is $\mathcal{J}_n^\mu$ -measurable; (iii) for each x and $\Lambda \in \mathcal{J}_n \cap \{R_{\gamma_n} < \sigma\}$, ${}^*P^x(\Lambda) = E^x\{M(R_{\gamma_n}); \Lambda\}$; (iv) $X(R_{\gamma_n})$ is $\mathcal{J}_n^\mu$ -measurable for each μ .

Let $h: E \times E \rightarrow [0, 1]$ be $\mathcal{B} \times \mathcal{B}$ -measurable and satisfy

$$H_K^*(x, B) = \int_B h(x, y) H_K(x, dy)$$

for $x \in E$ and $B \in \mathcal{B}$. Let $\phi_n = h(X(T_n), X(S_1))$ on $\{S_1 < \sigma\}$ and $= 0$ elsewhere.

LEMMA 8.3. For each x , $\{\phi_n M(T_n), n = 1, 2, \dots\}$ is a martingale with respect to the σ -fields $\mathcal{J}_n^{e_x}$, $n = 1, 2, \dots$, and the measure P^x .

Proof. One shows that for $\Lambda \in \mathcal{J}_n \cap \{S_1 < \sigma\}$, $*P^x(\Lambda) = E^x\{\phi_n M(T_n); \Lambda\}$.

Let $N(S_1) = \liminf_n \phi_n M(T_n)$. Then we have

COROLLARY 8.4. For each x and $\Lambda \in \mathcal{K} \cap \{S_1 < \sigma\}$, $E^x\{N(S_1); \Lambda\} = *P^x(\Lambda)$.

LEMMA 8.5. (i) $*P^x(X(R) \in B, R < \sigma) = E^x\{M(R); X(R) \in B, R < \sigma\}$ for every x and $B \in \mathcal{B}$; (ii) $M(R_1) = N(S_1)[M(R)(\theta_{S_1})]$ a.e. on $\{S_1 < \infty\}$.

Proof. (i) is easy to see. Both sides of (ii) are $\mathcal{K}^{\varepsilon_x}$ -measurable, and it follows from (i) and Corollary 8.4 that for $\Lambda \in \mathcal{K}$, $E^x\{M(R_1); \Lambda\} = E^x\{N(S_1)[M(R)(\theta_{S_1})]; \Lambda\} = *P^x(\Lambda \cap \{R_1 < \sigma\})$.

Let $T = \inf\{t \mid X(t) \neq X(0)\}$ and for each p let $A_p = \{y \in K \mid E^y(T) > 1/p\}$. Then $A_p \uparrow K$ as $p \rightarrow \infty$. From Lemma 8.2 we obtain for $B \in \mathcal{B}$

$$*E^x\{H_{A_p}^*(X(R_{\gamma_n}), B); R_{\gamma_n} < \sigma\} \leq E^x\{M(R_{\gamma_n})H_{A_p}(X(R_{\gamma_n}), B); R_{\gamma_n} < \sigma\}.$$

It is not hard to see that $*P^x(X(R_\delta) \in A_p \text{ for some } \delta, \gamma_n \leq \delta < \gamma) \rightarrow 0$ and $P^x(X(R_\delta) \in A_p \text{ for some } \delta, \gamma_n \leq \delta < \gamma) \rightarrow 0$ as $n \rightarrow \infty$. Hence as $n \rightarrow \infty$ we obtain from the above inequality

$$\begin{aligned} &*P^x(X(S_\gamma) \in A_p \cap B, S_\gamma < \sigma) + *E^x\{H_{A_p}^*(X(S_\gamma), B); X(S_\gamma) \notin A_p, S_\gamma < \sigma\} \\ &\leq E^x\left\{\lim_n M(R_{\gamma_n}); X(S_\gamma) \in A_p \cap B, S_\gamma < \sigma\right\} \\ &\quad + E^x\left\{\lim_n M(R_{\gamma_n})H_{A_p}(X(S_\gamma), B); X(S_\gamma) \notin A_p, S_\gamma < \sigma\right\}. \end{aligned}$$

As $p \rightarrow \infty$ we obtain

$$*P^x(X(S_\gamma) \in K \cap B, S_\gamma < \sigma) \leq E^x\left\{\lim_n M(R_{\gamma_n}); X(S_\gamma) \in K \cap B, S_\gamma < \sigma\right\}.$$

Thus we have

LEMMA 8.6. For every x and $B \in \mathcal{B}$,

$$*P^x(X(S_\gamma) \in B, S_\gamma < \sigma) \leq E^x\left\{\lim_n M(R_{\gamma_n}); X(S_\gamma) \in B, S_\gamma < \sigma\right\}.$$

It is not hard to show that both sides of the above inequality are $\mathcal{B}_\Delta$ -measurable in x . Hence there exists $c_\gamma: E \times E \rightarrow [0, 1]$, $\mathcal{B} \times \mathcal{B}$ -measurable, such that

$$*P^x(X(S_\gamma) \in B, S_\gamma < \sigma) = E^x\left\{c_\gamma(X(0), X(S_\gamma)) \lim_n M(R_{\gamma_n}); X(S_\gamma) \in B, S_\gamma < \sigma\right\}$$

for all $x \in E$ and $B \in \mathcal{B}$. Let δ_n be such that $\gamma_n + \delta_n = \gamma$ and assume that c_{δ_n} are defined. Let $\psi_n = c_{\delta_n}(X(R_{\gamma_n}), X(R_\gamma))$ on $\{R_\gamma < \sigma\}$ and $= 0$ elsewhere.

LEMMA 8.7. For each x , $\{\psi_n, n=1, 2, \dots\}$ is a martingale with respect to $\{\mathcal{J}_n^{\varepsilon_x}, n=1, 2, \dots\}$ and the measure $\lim_m M(R_{\gamma_m}) dP^x$.

Proof. Applying Lemma 8.2 one shows that for $\Lambda \in \mathcal{J}_n \cap \{S_\gamma < \sigma\}$,

$$*P^x(\Lambda) = E^x\left\{\psi_n \lim_m M(R_{\gamma_m}); \Lambda\right\}.$$

Let $\psi = \liminf_n \psi_n$. Let $N(S_\gamma) = \psi \lim_m M(R_{\gamma_m})$. Then we have

COROLLARY 8.8. For $\Lambda \in \mathcal{J}_\infty \cap \{S_\gamma < \sigma\}$, $E^x\{N(S_\gamma); \Lambda\} = {}^*P^x(\Lambda)$.

LEMMA 8.9. $M(R_\gamma) = N(S_\gamma)[M(R)(\theta_{S_\gamma})]$ a.e. on $\{S_\gamma < \infty\}$.

Proof. Both sides are $\mathcal{K}_\gamma^{\varepsilon_x}$ -measurable for each x . One shows that, for $\Lambda \in \mathcal{K}_\gamma$,

$$E^x\{M(R_\gamma); \Lambda\} = E^x\{N(S_\gamma)[M(R)(\theta_{S_\gamma})]; \Lambda\} = {}^*P^x(\Lambda \cap \{S_\gamma < \sigma\}).$$

For each γ let $N(S_{\gamma+1}) = M(R_\gamma)[N(S_1)(\theta_{R_\gamma})]$ on $\{R_\gamma < \sigma\}$ and $= 0$ elsewhere. Then it follows from Lemma 8.5 that $M(R_{\gamma+1}) = N(S_{\gamma+1})[M(R)(\theta_{S_{\gamma+1}})]$ a.e. on $\{S_{\gamma+1} < \infty\}$.

LEMMA 8.10. For every γ , $M(t) = N(S_\gamma)$ a.e. on $\{S_\gamma \leq t < R_\gamma\}$.

Proof. This is easily deduced from Lemmas 8.5 and 8.9, the remark preceding the lemma, and the definition of $M(t)$ (note that $M(R)(\theta_t) = M(R)(\theta_{S_\gamma})$ on $\{S_\gamma \leq t < R_\gamma\}$).

PROPOSITION 8.11. $M(t)$ is $\mathcal{F}(t)$ -measurable for every t .

Proof. According to the proof of Proposition 7.1 $M(t)I_{\{X(t) \in E_{\Delta-H}\}}$ is $\mathcal{F}(t)$ -measurable. For a given μ there exist increasing compact subsets K_n of H such that $P^\mu(X(t) \in K_n) \uparrow P^\mu(X(t) \in H)$. It follows that to prove $M(t)I_{\{X(t) \in H\}}$ is $\mathcal{F}(t)$ -measurable it suffices to show $M(t)I_{\{X(t) \in H\}}$ is $\mathcal{F}(t)$ -measurable for all compact subsets K of H . Construct $N(S_\gamma)$ as before. Since $N(S_\gamma)$ is clearly $\mathcal{F}(S_\gamma)$ -measurable it follows from Lemma 8.10 that $M(t)I_{\{S_\gamma \leq t < R_\gamma\}}$ is $\mathcal{F}(t)$ -measurable. This implies $M(t)I_{\{X(t) \in H\}}$ is $\mathcal{F}(t)$ -measurable since for a fixed μ there exists β such that $P^\mu(R_\beta < \infty) = 0$.

PROPOSITION 8.12. For every s and t , $M(s+t) = M(t)[M(s)(\theta_t)]$ a.e.

Proof. We need only to show that the equality holds a.e. on $\{X(t) \in K \text{ or } X(s+t) \in K\}$, K a compact subset of H . And it suffices to prove the equality holding a.e. on

$$\begin{aligned} \Lambda_1 &= \{S_\gamma \leq t < R_\gamma, S_{\gamma'} \leq s+t < R_{\gamma'}\} \quad (\text{we may assume } \gamma < \gamma'), \\ \Lambda_2 &= \{S_\gamma \leq t < R_\gamma, X(s+t) \in I\}, \text{ and } \Lambda_3 = \{X(t) \in I, S_\gamma \leq s+t < R_\gamma\}. \end{aligned}$$

We show only $M(s+t) = M(t)[M(s)(\theta_t)]$ a.e. on Λ_1 as the rest follows from similar arguments together with the proof of Theorem 7.2. Suppose $\gamma + \delta = \gamma'$. Then

$$\begin{aligned} M(s+t) &= N(S_{\gamma'}) = M(R_\gamma)[N(S_\delta)(\theta_{R_\gamma})] = N(S_\gamma)[M(R)(\theta_{S_\gamma})][N(S_\delta)(\theta_{R_\gamma})] \\ &= N(S_\gamma)[N(S_\delta)(\theta_{S_\delta})] = M(t)[M(s)(\theta_t)] \text{ a.e.} \end{aligned}$$

These equalities are not hard to see. For the last equality note that $M(t)$ depends only on the trajectory of the path.

Thus we have proved that M is a multiplicative functional of X . The following theorem will establish its desired property.

THEOREM 8.13. For $x \in E$ and compact $K \subset E$, $E^x\{M(T_K); X(T_K) \in B, T_K < \infty\} = H_K^*(x, B)$ for all $B \in \mathcal{B}$.

Proof. If $K \subset I$, the theorem follows from the proof of Theorem 7.4. If $K \subset H$, it is a result of Corollary 8.4 and the fact that $M(T_K) = M(S_1) = N(S_1)$ a.e. In general we find compact $K_m \subset K \cap H$ and compact $K'_m \subset K \cap I$ such that $T_{K_m} \downarrow T_{K \cap H}$ and $T_{K'_m} \downarrow T_{K \cap I}$ a.e. P^x and a.e. $*P^x$ so that $T_{K_m \cup K'_m} \downarrow T_K$ a.e. P^x and a.e. $*P^x$ as $m \rightarrow \infty$. It then suffices to show

$$(8.1) \quad E^x\{M(T_{K_m \cup K'_m}); X(T_{K_m \cup K'_m}) \in B, T_{K_m \cup K'_m} < \infty\} = H_{K_m \cup K'_m}^*(x, B)$$

for all B . Let $T_n = \inf\{T_\alpha^{(n)} < \sigma \mid X(T_\alpha^{(n)}) \in V_i^{(n)} \Rightarrow U_i^{(n)} \cap (K_m \cup K'_m) \neq \emptyset\}$. Let N be so large that $\rho(K_m, K'_m) = \inf\{\rho(y, z) \mid y \in K_m, z \in K'_m\} > 1/2N$, and let

$$F = \{y \in E \mid \rho(y, K_m) \leq 1/N\} \quad \text{and} \quad F' = \{y \in E \mid \rho(y, K'_m) \leq 1/N\}.$$

If $n \geq N$, then we have

$$(8.2) \quad \begin{aligned} & *E^x\{H_{K_m}^*(X(T_n), B); X(T_n) \in F, T_n < \sigma\} \\ & + *E^x\{H_{K'_m}^*(X(T_n), B); X(T_n) \in F', T_n < \sigma\} \\ & = E^x\{M(T_n)E^{X(T_n)}\{M(T_{K_m}); X(T_{K_m}) \in B, T_{K_m} < \infty\}; X(T_n) \in F, T_n < \sigma\} \\ & + E^x\{M(T_n)E^{X(T_n)}\{M(T_{K'_m}); X(T_{K'_m}) \in B, T_{K'_m} < \infty\}; X(T_n) \in F', T_n < \sigma\}. \end{aligned}$$

Now it follows from the quasi-left continuity that $P^x(T_{K_m} < T_{K'_m}, X(T_n) \in F) \rightarrow 1$ and $P^x(T_{K'_m} < T_{K_m}, X(T_n) \in F') \rightarrow 1$ as $n \rightarrow \infty$, with similar convergences holding when $*P^x$ replaces P^x . Applying the bounded convergence theorem we obtain (8.1) from (8.2). Thus the theorem is proved.

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UNIVERSITY OF WASHINGTON,
SEATTLE, WASHINGTON
CORNELL UNIVERSITY,
ITHACA, NEW YORK