AN ALGEBRAIC THEORY OF LOCAL KNOTTEDNESS. I

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0. Introduction.

A. *Summary.* A significant problem in topology is the classification of wild arcs in S^3 . The first fundamental paper in this area was written by Fox and Artin in 1948 [13]. There the authors developed rigorous algebraic means for demonstrating the nontameness of arcs. Some time later (1962) Fox and Harrold [14] succeeded in completely classifying a special class of wild arcs, the Wilder arcs. In 1960 Brody [6] developed invariants of infinitely generated modules and used such invariants to distinguish wild knots. Finally, in 1962 ([1], [2]) Alford and Ball constructed a geometric invariant, the penetration index, capable of distinguishing a large class of wild arcs.

In this paper we develop algebraic means for distinguishing wild arcs in S^3 which go significantly beyond previously known methods. Unlike most of the above authors we consider the problem from a local rather than a global standpoint.

The first three sections of this paper are devoted to extending and refining Brody's algebraic techniques [6].

In §IV, a fundamental invariance theorem (IV.B.3) is proven, i.e., (roughly) if p is an isolated singular interior point of an arc k and U is an arbitrary suitably nice neighborhood of p , then the fundamental group $\Pi(U-k)$ of $U-k$ modulo an equivalence relation, called local equivalence, is an invariant of the singularity p . Thus, the invariants of $\Pi(U-k)$ modulo local equivalence, i.e., the module $\mathfrak{M}$ (associated with $\Pi(U-k)$) modulo an equivalence relation (III.B.2, III.B.7, IV.A.3, IV.C.1), the k th divisor chains of $\Pi(U-k)$ (II.A.4, III.C.3, IV.C.2), the k th local topologies $\Lambda_n(k, p)$ of $\Pi(U-k)$ (I.D.3, II.A.1, II.A.5), etc., also become invariants of the singularity.

In §V some applications are considered. First (V.A) the algebraic invariants of Wilder arcs are found to have some pleasant properties (V.A.5). Moreover, the algebraic analogue (V.A.6) to the geometric classification of Wilder arcs [14] is

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presented. Next (V.B, V.C) two different types of compositions of arcs are considered (each analogous to the composition of tame knots). The algebraic invariants of such composites are shown (V.B.5, V.C.10, V.C.11) to be computable from the algebraic invariants of their factors. In studying such arc composites, we find (V.B.7, V.C.12, V.C.13) that there is very little relation between the algebraic invariants of this paper and the penetration index of Alford and Ball ([1], [2]). Finally in V.D., the algebraic invariants of a number of examples are computed.

B. Preliminaries. Before proceeding to the main body of this paper, a few preliminary definitions are necessary.

DEFINITION. Let k and k' be oriented arcs in S^3 and let p and p' be points of k and k' respectively. k at p is of the same *local type* as k' at p' if there are open neighborhoods U of p and U' of p' and an orientation preserving homeomorphism of U onto U' (where U and U' each inherit their orientation from that of S^3) that maps $(U \cap k, p)$ onto $(U \cap k', p')$.

DEFINITION. Let p be a point of an arc k in S^3 . k is *locally tame* at p if there exists a neighborhood U of p and a homeomorphism h_p of the closure $\bar{U}$ of U into S^3 such that $h_p(\bar{U} \cap k)$ is a polyhedron ([4, p. 146]). If k is locally tame at p , then p is a *tame point* of k . If not, p is a *wild point* of k . A wild point p of k is a *singular point* if there exists a neighborhood of p containing no other wild point of k .

DEFINITION. k at p is of the *trivial local type* if p is a tame point of k . Obviously, there are exactly two trivial local types, corresponding to terminal and interior points of arcs.

In terms of the above definitions the scope of this paper is restricted to the development of algebraic invariants of the local types of oriented arcs at singular or tame interior points.

For purposes of simplification the following two conventions are made. First, only arcs with at most one wild point will be considered. This entails no loss in generality; for if p is a singular point of an arc k , then there is a subarc k' of k of the same local type at p having p as its only wild point. Second, all results are stated for singular points of arcs, although they are obviously also valid for tame points.

1. Ideal topologies. In this and the next two sections Brody's algebraic theory [6] is extended. Throughout these sections R will denote a countable Noetherian unique factorization domain. Since all proofs in this section are of a straightforward nature, they have been omitted.

A. Introduction.

1. DEFINITION. A topology I on R is an *ideal topology* if there exists a collection of ideals $\{I_\alpha\}$ such that the collection $\{a + I_\alpha : a \in R, I_\alpha \in \{I_\alpha\}\}$ forms a basis for the open sets of I .

2. REMARK. Obviously, every ideal neighborhood basis about zero for an ideal topology uniquely determines that topology.

3. REMARK. R together with an ideal topology forms a topological ring.

4. If $\{I_\alpha\}$ is a collection of ideals of R such that the intersection of any two members of this collection contains a third, then $\{I_\alpha\}$ may be considered as a basis of neighborhoods about zero of an ideal topology.

B. *The calculus of ideal topologies.* For notation not defined herein please refer to [6, §1].

Let $\{I_\alpha\}$ be a family of ideal topologies and for each α let $\{I_{\beta\alpha}\}$ be an ideal basis of neighborhoods about zero for I_α .

1. DEFINITION. The *supremum* of the family $\{I_\alpha\}$, written $\bigcap I_\alpha$, is the coarsest ideal topology finer than all I_α , i.e., the supremum with respect to the ordering $\succ$.

It can easily be verified that the collection of all finite intersections of the ideals $\{I_{\beta\alpha}\}$ is an ideal neighborhood basis about zero for $\bigcap I_\alpha$; hence, the notation.

2. DEFINITION. The *product* of the family $\{I_\alpha\}$, written $\prod I_\alpha$, or as $I_1 \cdot I_2 \cdot \dots \cdot I_n$ in case the family is finite, is the ideal topology having as an ideal neighborhood basis about zero the collection of all finite products P_γ of the ideals $\{I_{\beta\alpha}\}$ such that each factor $I_{\beta\alpha}$ of P_γ is from a differently indexed ideal topology.

3. DEFINITION. The *sum* of the family $\{I_\alpha\}$, written $\sum I_\alpha$ or $I_1 + I_2 + \dots + I_n$ in case the family is finite, is the finest ideal topology coarser than all I_α , i.e., the infimum with respect to the ordering $\succ$.

It readily follows that the collection of all ideals of the form $\sum I_\alpha$, $I_\alpha \in \{I_{\beta\alpha}\}_\beta$, summed over all α , is an ideal neighborhood basis about zero for $\sum I_\alpha$; hence, the notation.

C. *Principal ideal topologies.*

1. DEFINITION. A *principal ideal topology* is an ideal topology having a collection of principal ideals as one of its ideal neighborhood bases about zero.

2. NOTATION. Let p be an element of R . Henceforth, p will also denote the principal ideal topology having $\{(p)\}$ as an ideal neighborhood basis about zero, where (p) denotes the ideal generated by p . Hence, $\{(p^n)\}$ is an ideal basis of $p^n = p \cdot p \cdot \dots \cdot p$ (n factors) and $\{(p), (p^2), (p^3), \dots\}$ is an ideal neighborhood basis of $p^\infty = p \cdot p \cdot p \cdot \dots$.

3. DEFINITION. Let I be an ideal topology. An irreducible element p of R is a *prime factor* of I if (p) is open in I . If p is an element of R , then the *exponent of p with respect to I* , written $e(p, I)$, is the largest nonnegative integer n such that (p^n) is open in I . If no such largest nonnegative integer exists, then $e(p, I) = \infty$.

4. Let $\{p_n\}$ be an indexing on the positive integers of all distinct (up to associates) primes of R . Then every principal ideal topology I may be factored uniquely into the form $\prod p_n^{e_n}$ where e_n is a nonnegative integer or ∞ . Moreover, $e_n = e(p_n, I)$.

5. If I and I' are principal ideal topologies such that $I \succ I'$ and if p is an irreducible element of R , then $e(p, I) \geq e(p, I')$.

6. DEFINITION. The *principal sum* of a family of ideal topologies $\{I_\alpha\}$, written $\sum^* I_\alpha$, or $I_1 +^* I_2 +^* I_3 +^* \dots +^* I_n$ in case the family is finite, is the coarsest principal ideal topology finer than $\sum I_\alpha$.

D. *Ideal topology chains.*

1. DEFINITION. An *ideal topology chain* is a monotone decreasing sequence of ideal topologies.

2. DEFINITION. Two ideal topology chains I_n and I'_n are *equivalent* if

(a) for every n there exists an $m(n)$ such that $I_n > I'_{m(n)}$,

(b) for every m there exists an $n(m)$ such that $I'_m > I_{n(m)}$.

3. If I_n and I'_n are two equivalent ideal topology chains, then $\sum^* I_n = \sum^* I'_n$.

4. DEFINITION. Ideal topologies I and I' are *similar* if there exists p and p' in R such that $pI > I'$ and $I < p'I'$.

5. Two principal ideal topologies I and I' are similar if and only if:

(a) I and I' have the same set of prime factors with infinite exponent, and

(b) $e(p, I) = e(p, I')$ for almost all (up to associate) irreducible elements p of R .

6. DEFINITION. An ideal topology chain I_n is *comparable* if there exists a positive integer N such that for every $m, n \geq N$, I_m and I_n are similar. Obviously, if I_n is comparable, then there exists an ideal topology I_∞ such that I_n is similar to I_∞ for almost all n . Any such ideal topology I_∞ is called a *comparing* ideal topology of the chain I_n . It is immediate that any two comparing topologies are similar.

It now easily follows that comparability and the comparing ideal topology modulo similarity are invariants of equivalence.

II. Topologies of countably generated modules. In [6] Brody defined the k th divisor topology $\nabla_k(\mathfrak{M})$ and the k th residual topology $\Gamma_k(\mathfrak{M})$ of an arbitrary module $\mathfrak{M}$ for $k \geq 0$. We now investigate these topologies only for countably generated (abbreviated c.g.) modules.

A. *Local topologies and divisor chains.* The author was not able to determine whether or not the residual topologies are principal ideal or even ideal topologies. To circumvent this difficulty the following definition is made.

1. DEFINITION. Let $\mathfrak{M}$ be a c.g. R -module and $\{\mathfrak{M}_\alpha\}$ the collection of all finitely generated (abbreviated f.g.) submodules of $\mathfrak{M}$. The k th local topology $\Lambda_k(\mathfrak{M})$ of $\mathfrak{M}$ is defined as: $\Lambda_k(\mathfrak{M}) = \sum_\alpha^* \nabla_k(\mathfrak{M}/\mathfrak{M}_\alpha)$, i.e., $\Lambda_k(\mathfrak{M})$ is the finest principal ideal topology coarser than $\Gamma_k(\mathfrak{M})$.

The local topologies behave basically like the residual ones.

2. If $\mathfrak{M}$ is a c.g. R -module, then $\Lambda_0(\mathfrak{M}) > \Lambda_1(\mathfrak{M}) > \dots$.

3. If $\mathfrak{M}$ is a homomorphic image of a c.g. R -module $\mathfrak{N}$, then $\Lambda_k(\mathfrak{M}) > \Lambda_k(\mathfrak{N})$, for every $k \geq 0$.

4. DEFINITION. Let $\mathfrak{M}$ be a c.g. R -module and $\mathfrak{M}_n$ an arbitrary monotone increasing sequence of f.g. submodules of $\mathfrak{M}$ whose union is $\mathfrak{M}$. A k th divisor chain of $\mathfrak{M}$ is defined as the descending chain:

$$\nabla_k(\mathfrak{M}/\mathfrak{M}_1) > \nabla_k(\mathfrak{M}/\mathfrak{M}_2) > \nabla_k(\mathfrak{M}/\mathfrak{M}_3) > \dots$$

If Definition I.D. 2 is recalled, it may easily be shown that divisor chains modulo equivalence are invariants of c.g. modules. The divisor chains are related to the local topologies as follows:

5. The principal sum of any k th divisor chain of a c.g. module $\mathfrak{M}$ is the k th local topology of $\mathfrak{M}$, i.e., $\Lambda_k(\mathfrak{M}) = \sum_n^* \nabla_k(\mathfrak{M}/\mathfrak{M}_n)$.

B. Direct sums of modules.

1. If $\mathfrak{M}$ and $\mathfrak{N}$ are f.g. R -modules, then

$$\Delta_k(\mathfrak{M} \oplus \mathfrak{N}) = \sum_{i+j=k}^* \Delta_i(\mathfrak{M}) \Delta_j(\mathfrak{N}), \text{ for every } k \geq 0,$$

where " $\sum^*$ " denotes the smallest principal ideal containing the given collection of ideals.

Proof. Let $k \geq 0$ be fixed and let A and B be relation matrices of $\mathfrak{M}$ and $\mathfrak{N}$ respectively. Let m and n be respectively the number of columns of A and B and let $\{A_i^a\}$ and $\{B_j^b\}$ be the collection of all $(m-i) \times (m-i)$ and all $(n-j) \times (n-j)$ submatrices of A and B , respectively ($i, j \geq 0$).

It may be observed that any nonsingular submatrix of

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \text{ is of the form } \begin{bmatrix} A^* & 0 \\ 0 & B^* \end{bmatrix},$$

where A^* and B^* are nonsingular. Thus, $\Delta_k(\mathfrak{M} \oplus \mathfrak{N})$ is the smallest principal ideal containing all the elements $(\det A_i^a)(\det B_j^b)$ where $i+j=k$, and $i, j \geq 0$. Q.E.D.

2. If $\mathfrak{M}$ is a c.g. R -module and $\{\mathfrak{M}_\alpha\}$ is a collection of f.g. submodules of $\mathfrak{M}$ such that:

(a) $\mathfrak{M}$ is generated by the union of the collection,

(b) The union of any two modules of the collection is contained in a third, then $\{\Delta_k(\mathfrak{M}_\alpha)\}$ is an ideal neighborhood basis about zero for $\nabla_k(\mathfrak{M})$, for every $k \geq 0$.

3. If $\mathfrak{M}$ and $\mathfrak{N}$ are c.g. R -modules, then $\nabla_k(\mathfrak{M} \oplus \mathfrak{N}) = \sum_{i+j=k}^* \nabla_i(\mathfrak{M}) \nabla_j(\mathfrak{N})$, for every $k \geq 0$.

4. If $\mathfrak{M}$ is a c.g. R -module and $\{\mathfrak{M}_\alpha\}$ is a collection of f.g. submodules of $\mathfrak{M}$ satisfying (a) and (b) of II.B.2, then $\Lambda_k(\mathfrak{M}) = \sum_{\alpha}^* \nabla_k(\mathfrak{M}/\mathfrak{M}_\alpha)$, for every $k \geq 0$.

5. If $\mathfrak{M}$ and $\mathfrak{N}$ are c.g. R -modules, then $\Lambda_k(\mathfrak{M} \oplus \mathfrak{N}) = \sum_{i+j=k}^* \Lambda_i(\mathfrak{M}) \Lambda_j(\mathfrak{N})$, for every $k \geq 0$.

Proof. Let $k \geq 0$ be fixed. Let $\{\mathfrak{M}_\alpha\}$ and $\{\mathfrak{N}_\beta\}$ be the collections of all f.g. submodules of $\mathfrak{M}$ and $\mathfrak{N}$ respectively. Then $\Lambda_k(\mathfrak{M} \oplus \mathfrak{N}) = \sum_{\alpha, \beta}^* \nabla_k[(\mathfrak{M} \oplus \mathfrak{N})/(\mathfrak{M}_\alpha \oplus \mathfrak{N}_\beta)]$. Employing II.B.3 and interchanging the order of principal summations, we have the desired result. Q.E.D.

6. If $\mathfrak{M}_n$ is a sequence of f.g. R -modules, then

$$\nabla_k \left(\bigoplus_{n=1}^{\infty} \mathfrak{M}_n \right) = \bigcap_{m=1}^{\infty} \sum_{i_1+i_2+\dots+i_m=k}^* \prod_{n=1}^m \nabla_{i_n}(\mathfrak{M}_n),$$

where $\bigoplus_{n=1}^{\infty} \mathfrak{M}_n$ denotes the weak direct sum.

As corollaries we have:

7. $\nabla_0(\bigoplus_{n=1}^{\infty} \mathfrak{M}_n) = \prod_{n=1}^{\infty} \nabla_0(\mathfrak{M}_n)$ and $\Lambda_0(\bigoplus_{n=1}^{\infty} \mathfrak{M}_n) = \sum_m^* \prod_{n=m+1}^{\infty} \nabla_0(\mathfrak{M}_n)$.

8. If $\mathfrak{M}_n$ is a sequence of f.g. R -modules and p is an irreducible element of R , then p is a prime factor of $\Lambda_0(\bigoplus_{n=1}^{\infty} \mathfrak{M}_n)$ if and only if the ideal generated by p contains $\Delta_0(\mathfrak{M}_n)$ for infinitely many n . Moreover, the exponent of every prime factor of $\Lambda_0(\bigoplus_{n=1}^{\infty} \mathfrak{M}_n)$ is infinite.

C. *Methods of computation.* We now seek methods for computing divisor and local topologies. Computations of such invariants will essentially be reduced to combinatorial operations on the relation matrices of c.g. modules. We begin by considering Brody's [6] elementary transformations in the countable case.

1. DEFINITION. *The nonsymmetric collection of elementary transformations* is the following:

(0) Permute the rows in any way, or permute the columns in any way.

(I) Adjoin any countable set of rows such that each new row is a finite linear combination of the original rows.

(II) Adjoin a countable set of rows r_n and columns c_n such that the intersection of r_n and c_n is the identity 1 of R , such that all other entries of c_n are zero, and such that there are at most finitely many nonzero entries in r_n and all such nonzero entries lie in the original columns of the matrix.

The proof of the following is essentially that of the Tietze transformation theorem found in [8].

2. THEOREM ([6, p. 145]). *Two c.g. R -modules are isomorphic if and only if each relation matrix of one may be obtained from each relation matrix of the other by a finite sequence of nonsymmetric elementary transformations.*

3. DEFINITION. *The symmetric collection of elementary transformations* is the following:

(0) Permute rows in any way or permute columns in any way.

(I₀) Adjoin any countable set of zero rows.

(II₀) Adjoin a countable set of rows r_n and columns c_n such that the intersection of r_n and c_n is the identity 1 of R and all other entries of r_n and c_n are zero.

(III) Add to each row of a given set of the rows any finite linear combination of the rows in the complementary set.

(III*) Add to each column of a given subset of the columns a (possibly infinite) linear combination of the columns in the complementary set such that each column in the complementary set appears in only finitely many such linear combinations.

The above set of elementary transformations is easily shown to be equivalent to the set of transformations found in II.C.1 [11].

From II.B.2 the computation of a divisor topology of a c.g. module $\mathfrak{M}$ may be reduced to finding the elementary divisors of the terms of a monotone increasing sequence of f.g. submodules $\mathfrak{M}_n$ of $\mathfrak{M}$ whose union is $\mathfrak{M}$. The following indicates how such a sequence may be extracted from a particular relation matrix of $\mathfrak{M}$.

4. Let $\mathfrak{M}$ be a c.g. R -module and $j(1) \leq j(2) \leq \dots$ an arbitrary monotone sequence of positive integers. Then there exists a monotone sequence $i(1) \leq i(2) \leq \dots$ of positive

integers and a relation matrix of $\mathfrak{M}$ such that the first $i(n)$ rows and $j(n)$ columns constitute a relation matrix for a f.g. submodule $\mathfrak{M}_n$ of $\mathfrak{M}$ generated by the elements of $\mathfrak{M}$ corresponding to the first $j(n)$ columns of the relation matrix. The $\mathfrak{M}_n$ form a monotone increasing sequence of f.g. submodules of $\mathfrak{M}$ whose union is $\mathfrak{M}$.

A sufficient condition for a relation matrix to be of the above form is the following.

5. Let $\mathfrak{M}$ be a c.g. R -module, $[a_{ij}]$ a relation matrix, $\mathfrak{F}$ a free R -module whose basis elements correspond to the columns of $[a_{ij}]$, and $\mathfrak{R}$ the submodule of $\mathfrak{F}$ whose generators $r_1, r_2, \dots$ correspond to the rows of $[a_{ij}]$. Hence, $\mathfrak{M} = \mathfrak{F}/\mathfrak{R}$. If $i(1) \leq i(2) \leq \dots$ and $j(1) \leq j(2) \leq \dots$ are monotone increasing sequences of positive integers such that:

(a) $a_{1j} = a_{2j} = \dots = a_{i(n),j} = 0$, for $j > j(n)$,

(b) the row vectors $r_{i(n)+1}, \dots, r_{i(n+1)}$ are linearly independent modulo $\mathfrak{F}_{j(n)}$, where $\mathfrak{F}_{j(n)}$ denotes the submodule of $\mathfrak{F}$ generated by the first $j(n)$ columns of $[a_{ij}]$, then the first $i(n)$ rows and $j(n)$ columns of $[a_{ij}]$ constitute a relation matrix of a f.g. submodule $\mathfrak{M}_m$ of $\mathfrak{M}$ generated by the elements of $\mathfrak{M}$ corresponding to the first $j(n)$ columns of $[a_{ij}]$.

Propositions II.C.4 and 5 are slight generalizations of those found in [6]. Finally, the local topologies are computed by use of the following.

6. If $\mathfrak{M}$ is a c.g. R -module, A one of its relation matrices, and $\mathfrak{M}_m$ the f.g. submodule of $\mathfrak{M}$ generated by the elements of $\mathfrak{M}$ corresponding to the first n columns of A , then a relation matrix of $\mathfrak{M}/\mathfrak{M}_m$ results if the first n columns of A are omitted.

III. $\mathbb{Z}$ -groups modulo local equivalence. In this section all groups are assumed to be countably presentable.

A. $\mathbb{Z}$ -groups and local equivalence.

1. DEFINITION. A $\mathbb{Z}$ -group $[G, \alpha]$ is a group G together with a homomorphism α of G onto the free cyclic group $\mathbb{Z}$.

2. DEFINITION. Let $[G, \alpha]$ and $[H, \beta]$ be $\mathbb{Z}$ -groups. Then a $\mathbb{Z}$ -homomorphism $\phi: [G, \alpha] \rightarrow [H, \beta]$ of $[G, \alpha]$ into $[H, \beta]$ is a homomorphism $\phi: G \rightarrow H$ of G into H such that $\beta\phi = \alpha$.

3. DEFINITION. A normal subgroup N of a group G is *finitely implicated* if there exists a finite collection of elements of G for which N is the smallest normal subgroup of G containing this collection.

4. DEFINITION. Two $\mathbb{Z}$ -groups $[G, \alpha]$ and $[H, \beta]$ are *locally equivalent* if there exists a finite sequence of $\mathbb{Z}$ -groups $[G, \alpha] = [G_0, \alpha_0], [G_1, \alpha_1], [G_2, \alpha_2], \dots, [G_n, \alpha_n] = [H, \beta]$ such that for every $i, 0 < i \leq n$, there exists either a $\mathbb{Z}$ -homomorphism with finitely implicated kernel of $[G_{i-1}, \alpha_{i-1}]$ onto $[G_i, \alpha_i]$ or one from $[G_i, \alpha_i]$ onto $[G_{i-1}, \alpha_{i-1}]$.

It readily follows that:

5. Two $\mathbb{Z}$ -groups $[G, \alpha]$ and $[H, \beta]$ are locally equivalent if and only if there exist a $\mathbb{Z}$ -group $[K, \gamma]$ and $\mathbb{Z}$ -homomorphisms with finitely implicated kernels of $[G, \alpha]$ onto $[K, \gamma]$ and of $[H, \beta]$ onto $[K, \gamma]$.

It is worth noting that finitely generated Z -groups are trivial modulo local equivalence; for:

6. Let $[G, \alpha]$ be a Z -group. If G is finitely generated, then the kernel of α is finitely implicated. Thus, $[G, \alpha]$ is locally equivalent to the trivial Z -group $[Z, \text{identity}]$.

Proof. Let $\{g_1, g_2, \dots, g_n\}$ be a finite set of generators of G . Without loss of generality it may be assumed that $\alpha(g_i) = t$, $i = 1, 2, \dots, n$, where t is the generator of Z . All that need be shown is that the kernel of α is implicated by $\{g_i \cdot g_j^{-1}\}$. Q.E.D.

B. *The module of a Z -group.* In [6] Brody associated a module with a group as follows.

1. DEFINITION. Let $[G, \alpha]$ be a Z -group and $(x:r)$ one of the countable presentations of G . Then the matrix $[\partial r / \partial x]^{\alpha\phi}$, where $\partial r / \partial x$ denotes the free derivative of r with respect to x [8], may be considered as a relation matrix of a JZ -module, called the *module* of $[G, \alpha]$ and written $\mathfrak{M}[G, \alpha]$, where JZ is the group ring of Z ([8, p. 84]). It is easily verified ([6, p. 145]) that the generalized Tietze transformations of types (0), (I), and (II) of $(x:r)$ induce elementary transformations of types (0), (I), and (II) respectively on the relation matrix $[\partial r / \partial x]^{\alpha\phi}$ of $[G, \alpha]$. Hence, $\mathfrak{M}[G, \alpha]$ depends only on $[G, \alpha]$ and not on the particular presentation of G chosen.

2. DEFINITION. Let $[G, \alpha]$ and $[H, \beta]$ be Z -groups and $\theta: [G, \alpha] \rightarrow [H, \beta]$ a Z -homomorphism. Let $(x; y:r; s)_{\phi_0\phi_1}$ be a countable presentation of θ ([12, p. 412]). Let δx denote the element of $\mathfrak{M}[G, \alpha]$ corresponding to the column $(\partial r / \partial x)^{\alpha\phi_0}$ of the relation matrix $(\partial r / \partial x)^{\alpha\phi_0}$. Similarly, let δx^* denote the element of $\mathfrak{M}[H, \beta]$ corresponding to the column $[\partial(r \cup s) / \partial x]^{\beta\phi_1}$ of $[\partial(r \cup s) / \partial(x \cup y)]^{\beta\phi_1}$. It may easily be shown that a well-defined homomorphism $\theta_*: \mathfrak{M}[G, \alpha] \rightarrow \mathfrak{M}[H, \beta]$ independent of the presentation of θ chosen may be defined by $\theta_*(\delta x) = \delta x^*$. θ_* is called the *homomorphism induced by θ* .

The reader may readily supply the proof for the following.

3. If θ is onto, then so is θ_* . If the kernel of θ is finitely implicated, then the kernel of θ_* is finitely generated.

4. DEFINITION. Two modules $\mathfrak{M}$ and $\mathfrak{N}$ are *locally equivalent* if there exist finitely generated submodules $\mathfrak{M}^*$ and $\mathfrak{N}^*$ of $\mathfrak{M}$ and $\mathfrak{N}$ respectively such that $\mathfrak{M}/\mathfrak{M}^*$ and $\mathfrak{N}/\mathfrak{N}^*$ are isomorphic.

The above is found to be an equivalence relation.

The significance of the above definition can be seen from the following, a consequence of III.B.3.

7. If two Z -groups are locally equivalent, then their corresponding modules are locally equivalent.

C. *Modules modulo local equivalence.*

1. DEFINITION. An elementary transformation of type (IV) adjoins finitely many new rows to the relation matrix, each new row containing only finitely many nonzero entries.

2. Two c.g. modules are locally equivalent if and only if any relation matrix of either module may be obtained from any relation matrix of the other by the application of a finite sequence of elementary transformations of types (0), (I), (II), and (IV).

Proof. A relation matrix of a quotient module $\mathfrak{M}/\mathfrak{M}^*$, where $\mathfrak{M}^*$ is f.g., can be obtained by adjoining finitely many rows to a relation matrix of $\mathfrak{M}$. Q.E.D.

Next we seek invariants of modules modulo local equivalence.

3. If two c.g. modules are locally equivalent, then their k th divisor chains are equivalent, for every $k \geq 0$.

Proof. Let $\mathfrak{M}$ be a c.g. module and $\mathfrak{M}^*$ a f.g. submodule. All that need be demonstrated is that $\mathfrak{M}$ and $\mathfrak{M}/\mathfrak{M}^*$ have equivalent k th divisor chains. We leave this to the reader. Q.E.D.

It now follows from I.D.3 that the local topologies are invariants of modules modulo local equivalence.

IV. Algebraic invariants of local type. Algebraic invariants of the local types of arcs at singular interior points are now constructed. Throughout this section k and k' will denote oriented arcs and p and p' singular interior points of k and k' respectively.

A. Local $\mathbb{Z}$ -groups.

1. There exist arbitrarily small open neighborhoods U of p which satisfy the following conditions with respect to k :

- (a) the boundary ∂U of U is a tame 2-sphere,
- (b) $k \cap \partial U$ consists of finitely many points,
- (c) k pierces [19] ∂U at each point of $k \cap \partial U$,
- (d) the end points of k do not lie in the closure $\bar{U}$ of U .

Such neighborhoods will be termed *acceptable* neighborhoods of p with respect to k .

Proof. From ([14, Theorem 8]) it follows that there exists a locally finite triangulation T of $S^3 - p$ for which $k - p$ is a subcomplex. Let N be an arbitrarily small closed cube with center at p , not containing the end points of k . Then by ([3, p. 56]) there exists for every $\varepsilon > 0$ a homeomorphism f_ε^* of $S^3 - p$ onto itself moving no point by as much as ε which maps the boundary of N onto a polyhedron of $S^3 - p$ under the triangulation T . Extending f_ε^* to a homeomorphism f_ε of S^3 upon itself, it follows that the interior U of $f_\varepsilon(N)$ is an open neighborhood of p which satisfies condition (a). By choosing ε sufficiently small it may be assumed that U also satisfies (d). Finally by applying sufficiently small deformations to $\bar{U}$, U can be made to satisfy (b) and (c). Q.E.D.

2. **DEFINITION.** Let U be an acceptable neighborhood of p with respect to k . Because of (b) and (d) of IV.A.1, $k \cap \bar{U}$ may be considered as a singular relative 1-cycle of $\bar{U}$ modulo $\partial \bar{U}$. By Alexander's theorem ([15, p. 25]) condition (a) implies that $\bar{U}$ is a closed 3-cell. Hence, the linking number of the relative 1-cycle $k \cap \bar{U}$ with any 1-cycle of $U - k$ may be defined [18]. This linking number may then be used to define a homomorphism of the first homology group $H_1(U - k)$ of

$U-k$ onto Z ([21, p. 280]). The composition of this homomorphism with the natural homomorphism of the fundamental group $\Pi(U-k)$ of $U-k$ onto $H_1(U-k)$ will be called the *homomorphism determined by k* .

3. **DEFINITION.** Let U be an acceptable neighborhood of p with respect to k . A *local Z -group* of k at p is the group $\Pi(U-k)$ together with the homomorphism determined by k .

REMARK. IV.A.1 guarantees the existence of at least one local Z -group.

B. *The fundamental invariance theorem.* In this section it will be demonstrated that a local Z -group modulo local equivalence is an invariant of local type.

1. *Let U and V be two acceptable neighborhoods of p with respect to k such that the closure $\bar{V}$ of V is in U . Then $\Pi(U-\bar{V}-k)$ is finitely generated.*

Proof. (By Alexander duality and ([16, p. 48]) $U-\bar{V}-k$ is connected.) $\bar{U}-V$ is obviously a compact 3-manifold. It follows from ([19, Theorem 1]) that $k \cap (\bar{U}-V)$ is a locally tame closed subset of $\bar{U}-V$. Thus, by ([3, p. 154]) $\bar{U}-V$ has a locally finite (hence, finite) triangulation under which $k \cap (\bar{U}-V)$ is a polyhedron. Finally, from ([20, p. 13]) it may be concluded that $\Pi(\bar{U}-V-k)$ and hence $\Pi(U-\bar{V}-k)$ are finitely generated. Q.E.D.

Since the intersection number of a 1-chain with a 2-chain is merely the number of algebraic penetrations, it follows that:

2. *Let U and V be two acceptable neighborhoods of p with respect to k such that $V \subset U$. Then $v'_* i_* = v$, where $i_*: \Pi(V-k) \rightarrow \Pi(U-k)$ is the homomorphism induced by inclusion and $v: \Pi(V-k) \rightarrow Z$ and $v': \Pi(U-k) \rightarrow Z$ are the homomorphisms determined by k .*

3. **Fundamental invariance theorem.** *If k at p is of the same local type as k' at p' , then each local Z -group of k at p is locally equivalent to each local Z -group of k' at p' .*

The proof is in two parts.

PART ONE. Any two local Z -groups of k at p are locally equivalent.

Proof. Let U and V be acceptable neighborhoods of p with respect to k such that the closure $\bar{V}$ of V is in U . Then the proof of the statement of part one reduces to demonstrating that the local Z -groups corresponding to U and V are locally equivalent.

By IV.A.1 an acceptable neighborhood W of p with respect to k may be selected so that its closure $\bar{W}$ lies in V . Consider the diagram:

$$\begin{array}{ccccc}
 & & \Pi(V-\bar{W}-k) & & \\
 & \swarrow \delta & & \searrow \gamma & \\
 \Pi(U-\bar{W}-k) & & & & \Pi(V-k) \\
 & \searrow \alpha & & \swarrow \beta & \\
 & & \Pi(U-k) & &
 \end{array}$$

where α , β , γ , and δ are the homomorphisms induced by inclusions. By the Van

$\subset \text{Ker } \lambda^*$, there exist homomorphisms $\phi: Z \rightarrow \prod (U-k)/Q^*$ and $\psi: \prod (V-k)/Q \rightarrow \prod (U-k)/Q^*$ such that $\phi\varepsilon = \lambda^*\alpha$ and $\psi\omega = \lambda^*\beta$. By means of ϕ, ψ, χ any homomorphism of the outer system into a group may be factored through $\prod (U-k)/Q^*$. Hence, $\prod (U-k)/Q^*$ is the direct limit of this system. Thus $\text{Ker } \lambda = Q^* = \text{consequence of } [(\alpha \text{ Ker } \varepsilon) \cup (\beta \text{ Ker } \omega)]$.

That the Z -groups corresponding to U and V are locally equivalent can be established as follows. Since $\rho\lambda\alpha = \rho\sigma\varepsilon = \varepsilon = \zeta\alpha$ and $\rho\lambda\beta = \rho\omega = \mu = \zeta\beta$ and $\text{Im}(\alpha) \cup \text{Im}(\beta)$ generates $\prod (U-k)$, $\rho\lambda = \zeta$. From above $\rho\omega = \mu$. Hence, λ and ω are both Z -epimorphisms. By IV.B.1, $\prod (V-\bar{W}-k)$ and $\prod (U-\bar{W}-k)$ are finitely generated. Thus, according to III.A.6, $\text{ker } \nu$ and $\text{ker } \varepsilon$ are finitely implicated. Hence, $\text{Ker } \omega$ (which is the consequence of $\gamma \text{ Ker } \nu$) and $\text{ker } \lambda$ (which is the consequence of $[(\alpha \text{ ker } \varepsilon) \cup (\beta \text{ Ker } \omega)]$) are finitely implicated. The conclusion now follows from III.A.5. This completes the proof of part one.

PART TWO. There is a local Z -group of k at p which is locally equivalent to a local Z -group of k' at p' .

Proof. By hypothesis there exist open neighborhoods U and U' of p and p' respectively and an orientation preserving homeomorphism ϕ of U onto U' which maps $U \cap k$ onto $U' \cap k'$ and p upon p' . By IV.A.1 there exists an acceptable neighborhood V of p with respect to k such that its closure $\bar{V}$ lies in U . Hence, $V' = \phi(V)$ is an open neighborhood of p' which obviously satisfies conditions (b), (c), and (d) of IV.A.1. Since the boundary of V' is locally tame, it follows from [4] that V' satisfies condition (a) of IV.A.1 and hence, V' is an acceptable neighborhood of p' with respect to k' .

Now ϕ induces an isomorphism ϕ_* of $\prod (V-k)$ onto $\prod (V'-k')$. Since ϕ is orientation preserving, it preserves the linking number. Hence, $\nu'\phi_* = \nu$, where ν, ν' are the homomorphisms determined by k and k' , respectively. Thus, ϕ_* is a Z -isomorphism.

C. Further invariants of local type. At this point we would like to employ the results of §III. These results however are only applicable to countably generated Z -groups. Fortunately, every local Z -group is countably generated. This follows readily by applying IV.A.1, IV.B.1, and the Van Kampen theorem [7] and by observing that the direct limit of a countable system of finitely generated groups is countably generated.

1. **DEFINITION.** A *local module* of k at p is the module of a local Z -group of k at p .

From III.B.7 and the fundamental invariance theorem IV.B.3 it follows that a local module modulo local equivalence is an invariant of local type.

2. **DEFINITION.** For every integer $n \geq 0$, an *n th divisor chain* of k at p is an *n th divisor chain* of a local module of k at p .

We may conclude from III.C.3 that each *n th divisor chain modulo equivalence* is an invariant of local type.

Since local topologies are invariants of modules modulo local equivalence, we have yet another invariant of local type, namely:

3. **DEFINITION.** For every integer $n \geq 0$, the n th local topology of k at p , written $\Lambda_n(k, p)$, is defined as the n th local topology of any module of k at p .

V. Applications. It is now possible to consider some of the applications of the algebraic invariants developed in the previous section.

A. Wilder arcs.

1. **DEFINITION.** Let k be an arc and p a point of k . The *penetration index* [1], [2] of k at p , written $P(k, p)$, is the smallest cardinal number n such that there are arbitrarily small 2-spheres enclosing p and containing no more than n points of k .

2. **DEFINITION.** Let p be a singular interior point of an arc k . Then k is *Wilder arc* if $P(k, p) = 2$.

3. *A method of decomposing an arbitrary Wilder arc k with singular interior point p into a sequence K_n of tame knots is now given.*

Since $P(k, p) = 2$ there is a sequence of arbitrarily small 2-spheres S_n , each S_n enclosing p and containing exactly two points a_n and b_n of k such that the oriented subarc from a_n to b_n has the induced orientation of k . By ([1, p. 34]) it may be assumed that each S_n is tame and pierced at a_n and b_n by k . Let U_n denote the complementary domain of S_n containing p . (Hence, by Alexander's theorem ([15, p. 25]) U_n is an open 3-cell.) Thus, the U_n 's form a sequence of acceptable neighborhoods of p with respect to k such that:

- (a) the closure $\bar{U}_n$ of U_n lies in U_{n-1} ,
- (b) $\bigcap U_n = p$,
- (c) k cuts the boundary of U_n in just two points a_n and b_n , and
- (d) by [19] $k \cup S_n$ is locally tame at a_n and b_n .

k is decomposed as follows. $\bar{U}_{n-1} - U_n$ intersects k in two oriented arcs A_n from a_{n-1} to a_n and B_n from b_n to b_{n-1} . On S_n select an arc J_n from a_n to b_n . By (d) the oriented simple closed curve $K_n = A_n \cup J_n \cup B_n \cup J_n^{-1}$ is a tame knot. The sequence of tame knots K_n is the decomposition of k sought.

4. *Let k be a Wilder arc and p its singular interior point. Let K_n be a decomposition of k into a sequence of tame knots as accomplished above. Then each local module of k at p is locally equivalent to a direct sum $\bigoplus_{n=1}^{\infty} \mathfrak{M}_n$ of JZ-modules $\mathfrak{M}_n$, where $JZ \oplus \mathfrak{M}_n$ is the Alexander module of K_n .*

Proof. Let U_n be the sequence of open 3-cells constructed in V.A.3. By repeated application of the Van Kampen theorem [7] it may be shown that $\prod (U_0 - k)$ is the infinite amalgamated free product:

$$\prod (U_0 - k) \simeq \prod (U_0 - \bar{U}_1 - k) *_Z \prod (U_1 - \bar{U}_2 - k) *_Z \cdots,$$

where it should be observed that the groups $\prod (U_{n-1} - \bar{U}_n - k)$ are respectively the knot groups of the knots K_n .

Now select a Wirtinger presentation $(a_n, x_n; r_n)$ of $\prod (U_{n-1} - \bar{U}_n - k)$ such that for all n each generator is mapped by the homomorphism determined by k onto

the same generator of Z and such that all a_n are images of the same generator of the amalgamating group Z . Thus, a presentation of $\prod (U_0 - k)$ is:

$$(a_1, x_1, a_2, x_2, \dots : r_1, a_1 a_2^{-1}, r_2, a_2 a_3^{-1}, \dots).$$

Applying generalized Tietze transformations, we have an equivalent presentation:

$$(a, x_1, x_2, \dots : r_1^*, r_2^*, \dots),$$

where r_n^* denotes r_n with a_n replaced by a . The corresponding relation matrix for the module associated with $\prod (U_0 - k)$ is:

$$\begin{matrix} & a & x_1 & x_2 & \cdots \\ \begin{matrix} r_1^* \\ r_2^* \\ \vdots \end{matrix} & \begin{bmatrix} \gamma_1 & M_1 & 0 & \cdots \\ \gamma_2 & 0 & M_2 & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{bmatrix} \end{matrix},$$

where $[\gamma_n M_n]$ is an Alexander matrix of K_n , i.e., a relation matrix of the Alexander module of the knot group $\prod (U_{n-1} - \bar{U}_n - k)$. It follows from the fundamental formula of free calculus ([10, p. 551]) that the sum of the columns of M_n is $-\gamma_n$. Hence performing a matrix transformation of type III, we may replace the first column of the above matrix by a column of zeros. Thus, the module associated with $\prod (U_0 - k)$ is the infinite direct sum $JZ \oplus \mathfrak{M}_1 \oplus \mathfrak{M}_2 \oplus \mathfrak{M}_3 \oplus \cdots$, where $\mathfrak{M}_n$ is the module corresponding to the matrix M_n . Observing that $[0M_n]$ is an Alexander matrix of K_n —hence, that $JZ \oplus \mathfrak{M}_n$ is the Alexander module of K_n , we have the conclusion. Q.E.D.

By II.B.3 and 7,

5. $\prod_{n=1}^{\infty} \nabla_1(K_n) > \prod_{n=2}^{\infty} \nabla_1(K_n) > \prod_{n=3}^{\infty} \nabla_1(K_n) > \cdots$ is a 0th divisor chain of k at p , where $\nabla_1(K_n)$ is the 1st divisor topology of the tame knot K_n . Hence, the 0th divisor chains of k at p are comparable; and thus, a 0th comparing topology of k at p exists. One such comparing topology is $\prod_{n=1}^{\infty} \nabla_1(K_n)$.

Hence, the 0th comparing topology modulo similarity is for Wilder arcs yet another invariant of local type.

V.A.5 together with I.D.5 give the following algebraic analogue to the geometric classification of Wilder arcs of Fox and Harrold [14].

6. Let k and k' be Wilder arcs and σ and σ' comparing topologies of each respectively. If k and k' are of the same global type, then

- (a) σ and σ' have the same set of prime factors with infinite exponent, and
- (b) $e(p, \sigma) = e(p, \sigma')$ for almost all (up to associates) irreducible elements of JZ .

7. An irreducible element of JZ is a prime factor of the 0th local topology of k at p if and only if it divides infinitely many of the Alexander polynomials of the knots K_n . Moreover, each such prime factor has infinite exponent.

B. *Mixed-composites of arcs.* A method of combining arcs is now defined and investigated. Throughout the remainder of this paper every cone in S^3 to be considered will be assumed to be locally tame at all of its points except possibly its vertex.

1. DEFINITION. By the standard *half-cone* $C_k^\#$ will be meant the set of points $\{(x, y, z) : y^2 + z^2 \leq x^2, 0 \leq x \leq 1\}$.

2. Let C be a half-cone. An *axis* of C is an image of $\{(x, 0, 0) : 0 \leq x \leq 1\}$ under a homeomorphism of $C_k^\#$ onto C which maps base upon base and vertex upon vertex.

It is easily verified that the local type of the axis of a cone at the cone's vertex is independent of the homeomorphism chosen.

3. DEFINITION. Let p be a singular interior point of an arc k . A half-cone containing k will be said to *enclose* k at p if the only points of k lying on the cone's boundary are p which is the cone's vertex and the end points of k which lie on the interior of the cone's base. (See k_0 in Figure 2.)

FIGURE 2. A Mixed Composite

4. Let p be a singular interior point of arcs k and k_0 and a singular end point of an arc k_1 . k at p is a *mixed-composite* of k_0 with respect to k_1 at p , written $k_0 \text{ } _p k_1$, if there exists a half-cone C enclosing k at p such that:

(a) there is an orientation preserving homeomorphism f of C onto $C_k^\#$ mapping base onto base and vertex onto vertex such that $f(k)$ at the origin is of the same local type as k_0 at p , and

(b) an axis of C is of the same local type at p as k_1 at p . (See Figure 2.)

REMARK. Let f and g be any two homeomorphisms of C onto $C_k^\#$ mapping base onto base and vertex onto vertex. Since the homeomorphism gf^{-1} of $C_k^\#$ onto itself can be extended in the usual fashion to a homeomorphism of S^3 upon itself, it follows that the above definition is independent of the homeomorphism f selected.

5. Let p be a singular interior point of arcs k and k_0 and a singular end point of k_1 . If k at p is a mixed-composite of k_0 with respect to k_1 at p , then k and k_0 both have the same local Z -groups at p .

V.B.5 is an immediate consequence of:

6. Let C be a half-cone which encloses k at p . Then $\prod (C - k)$ as a Z -group is locally equivalent to each local Z -group of k at p .

Proof. Construct a half-cone C^* containing C such that (a) the only points of C lying on the boundary of C^* are the points of C 's base, all of which lie on the base of C^* , and the vertex of C which coincides with the vertex of C^* , and (b) C is a deformation retract of C^* . Also construct an acceptable neighborhood U of p with respect to k such that the base of C^* and no other points of C^* lie on the boundary of U . This construction is possible since C is locally tame everywhere except possibly its vertex.

It is immediate that $\prod (U - C)$ is trivial, that $\prod [(U - C) \cap ((\text{Int } C^*) - k)]$ is Z (where $\text{Int } C^*$ denotes the interior of C^*), and that $\prod [(\text{Int } C^*) - k]$ and $\prod (C - k)$ are Z -isomorphic. Thus, by the Van Kampen theorem [7], $\prod (U - k)$ is the direct limit of:

$$\begin{array}{ccc} & Z & \\ i \swarrow & & \searrow j \\ 1 & & \prod (C - k) \end{array}$$

where i and j are the homomorphisms induced by inclusions. It now follows that $\prod (C - k)$ and $\prod (U - k)$ are locally equivalent Z -groups. Q.E.D.

The following theorem indicates that there is very little relation between the algebraic invariants of this paper and the penetration index.

7. THEOREM. If k at p is a mixed-composite of k_0 with respect to k_1 at p , then $P(k, p) \geq P(k_0, p)P(k_1, p)$.

C. *Interior composites of arcs.* The following composite of arcs is analogous to the companions of tame knots ([22, p. 192]).

1. DEFINITION. By the *standard cone* $C^\#$ will be meant $\{(x, y, z) : y^2 + z^2 \leq x^2 \leq 1\}$.
2. DEFINITION. Let C be a cone. An *axis* k of C is the image of $\{(x, 0, 0) : |x| \leq 1\}$ under a homeomorphism of $C^\#$ onto C which maps lateral surfaces onto corresponding lateral surfaces.

It is a straightforward exercise to verify that the axis of a cone is uniquely determined up to local type.

3. DEFINITION. Let p be an interior point of an arc k . A point q of k (distinct from p) will be said to lie to the *left* or to the *right* of p accordingly as the oriented subarc from q to p has the same or the opposite orientation of that induced by k . By convention p lies both to the left and to the right of itself. The arc consisting of those points to the left of p and that consisting of those to the right of p will be called respectively the *left half*, written k_L , and the *right half*, written k_R , of k with respect to p .

4. DEFINITION. A cone containing k will be said to *enclose* k at p if the only points of k lying on the cone's boundary are p which coincides with the cone's vertex and the end points of k which lie on the respective bases of C . The half-cones

of C which contain the left and right halves of k are denoted by C_L and C_R , respectively, (see Figure 3).

FIGURE 3. An Interior Composite

5. DEFINITION. Let p be a singular interior point of an arc k such that k pierces a tame disc at p . From this it follows that there exists a tame cone $C = C_L \cup C_R$ enclosing k at p . Remove the right (or left) half of k with respect to p and adjoin at p an axis of C_R (or C_L). The resulting arc k^L (or k^R) is called a *left* (or a *right*) *factor* of k at p (although it is not unique).

6. DEFINITION. Let p be a singular interior point of an arc k . k is *left* (or *right*) *tame* at p if there exists a tame cone C which encloses the left (or right) half of k at p so that the left (or right) half of k is tame in C .

REMARK. Obviously, a left or right tame arc pierces a tame disc.

7. DEFINITION. Let p be a singular interior point of arcs k , k_0 , and k_1 . Then k at p is an *interior composite* of k_0 and k_1 at p , written $k = k_0 \$_p k_1$, if there exists a cone C enclosing k at p such that:

- (a) an axis of C is of the same local type at p as k_0 at p , and
- (b) there exists an orientation preserving homeomorphism f of C onto $C^\#$ mapping lateral surfaces onto corresponding lateral surfaces such that the image of k under this homeomorphism is of the same local type at the origin as k_1 at p . (See Figure 3.)

REMARK. This definition can easily be seen to be independent of the homeomorphism f chosen.

REMARK. The composition “ $\$_p$ ” of two arcs is not a uniquely defined operation, not even modulo local type.

The following easily verifiable “associative law” indicates that it is sensible to speak of multiple interior composites, the definition of which is left to the reader.

8. Let p be a singular interior point of arcs k , k_0 , k_1 , and k_2 . Then $k = (k_0 \$_p k_1) \$_p k_2$ if and only if $k = k_0 \$_p (k_1 \$_p k_2)$.

It is immediate that:

9. Let p be a singular interior point of an arc k such that k pierces a tame disc at p . Then $k = k^R \$_p k^L$ and $k = k^L \$_p k^R$.

10. Let p be a singular interior point of arcs k , k_0 , and k_1 . If $k = k_0 \$_p k_1$, then the free product of any local Z -group of k_0 at p and any local Z -group of k_1 at p is locally equivalent to each local Z -group of k at p .

Proof. Because of V.C.8 and V.C.9, k_1 may be assumed to be either right or left tame. Let us assume that k_1 is left tame at p . (The proof of the other case is essentially the same.)

Let $C = C_R \cup C_L$ be the cone corresponding to considering k at p as an interior composite of k_0 and k_1 at p . Construct a cone $C_R^* \cup C_L^*$ containing $C_R \cup C_L$ such that (a) the only points that the boundary of $C_R \cup C_L$ has in common with the boundary of $C_R^* \cup C_L^*$ are the points of its bases which bases lie on the interior of the corresponding bases of $C_R^* \cup C_L^*$ and its vertex which coincides with the vertex of $C_R^* \cup C_L^*$, and (b) $C_R \cup C_L$ is a deformation retract of $C_R^* \cup C_L^*$. Also construct an acceptable neighborhood U of p with respect to k such that the bases of the cone $C_R^* \cup C_L$ and no other points of $C_R^* \cup C_L$ lie on the boundary of U . Such constructions are possible since $C_R \cup C_L$ is locally tame everywhere except possibly its vertex.

It is immediate that $\prod [(U - C_R - k_L) \cap ((\text{Int } C_R^*) - k_R)]$ is Z (where $\text{Int } C_R^*$ denotes the interior of C_R^*), that $\prod (U - C_R - k_L)$ and $\prod (U - k_0)$ are Z -isomorphic, and that $\prod [(\text{Int } C_R^*) - k_R]$ is Z -isomorphic to $\prod (C_R - k_R)$. Thus, by the Van Kampen theorem [7] $\prod (U - k)$ is the direct limit of the system:

$$\begin{array}{ccc} & Z & \\ i \swarrow & & \searrow j \\ \prod (U - k_0) & & \prod (C_R - k_R) \end{array}$$

where i and j are the homomorphisms induced by inclusions. From this it readily follows that $\prod (U - k)$ is locally equivalent to the free product of $\prod (U - k_0)$ and $\prod (C_R - k_R)$.

It will now be shown that $\prod (C_R - k_R)$ is a Z -group of k_1 at p . Let $f: C_R \cup C_L \rightarrow C^\#$ be a homeomorphism corresponding to considering k at p as a composite of k_0 and k_1 . Since k_1 is left tame, f may be modified so that $f(k_L) = \{(x, 0, 0) : -1 \leq x \leq 0\}$. Let U' be an acceptable neighborhood of the origin with respect to $f(k)$ such that all points of the bases of $C^\#$ and no other points of $C^\#$ lie on the boundary of U' . Since $C_R^\# - f(k_R)$ is a deformation retract of $U' - f(k)$, $\prod (C_R - k_R)$ and $\prod (U' - f(k))$ are Z -isomorphic. And since $f(k)$ at the origin is of the same local type as k_1 at p , $\prod (C_R - k_R)$ is a local Z -group of k_1 at p . Q.E.D.

11. The direct sum of any local module of k_0 at p and any local module of k_1 at p is locally equivalent to each local module of k at p . Hence

$$\Lambda_0(k, p) = \Lambda_0(k_0, p) \cdot \Lambda_0(k_1, p).$$

II.D.5 may be used to compute higher order local topologies of interior composites.

At this point it is appropriate to explore briefly the penetration index of interior composites.

12. THEOREM. Let p be a singular interior point of an arc k . If k pierces a tame disc at p , then $P(k, p) \geq P(k_L, p) + P(k_R, p)$.

13. THEOREM. Let p be a singular interior point of arcs k , k_0 , and k_1 . If $k = k_0 \$_p k_1$, then

$$P(k, p) \geq P(k_{0L}, p)P(k_{1L}, p) + P(k_{0R}, p)P(k_{1R}, p).$$

D. Examples. The following method of describing examples is employed. Divide $C^\#$ into an infinite number of sections by the family of parallel planes $x = \pm 2^{-n}$, $n = 0, 1, 2, \dots$. For each nonnegative integer n , let D_n^+ and D_n^- denote the sections $2^{-n-1} \leq x \leq 2^{-n}$ and $-2^{-n} \leq x \leq -2^{-n-1}$, respectively. For convenience let $D^\pm = D_0^\pm$. For each nonnegative integer n , define orientation preserving homeomorphisms $h_n^+ : D^+ \rightarrow D_n^+$ and $h_n^- : D^- \rightarrow D_n^-$ as

$$h_n^\pm(x, y, z) = (x/2^n, y/2^n, z/2^n).$$

Any arc k to be found in this section will be described as follows. On the larger base of D^+ mark off l_+ (>0) collinear points p_i^+ , $1 \leq i \leq l_+$, and on the smaller base the points $q_i^+ = h_1(p_i^+)$, $1 \leq i \leq l_+$. In D^+ construct l_+ disjoint simple polygonal arcs K_i^+ , $1 \leq i \leq l_+$, with the p_i^+ 's and q_i^+ 's as end points and with no other points in common with the boundary D^+ . (p_i^+ and q_i^+ are not necessarily the end points of K_i^+ .) Let K^+ be the union of the K_i^+ 's. In the same manner in D^- construct l_- (>0) disjoint polygonal arcs K^- with end points p_i^- and q_i^- , $1 \leq i \leq l_-$. An arc k is described by fitting together infinitely many copies of K^+ and K^- as follows:

$$k = \left[\bigcup_{n=1}^{\infty} h_n^+(K^+) \right] \cup p \cup \left[\bigcup_{n=1}^{\infty} h_n^-(K^-) \right],$$

where p denotes the origin $(0, 0, 0)$. (See Figure 4.) In the following all arcs are described simply by giving K^+ in D^+ and K^- in D^- . If such an arc is right or left tame, D^+ or D^- is respectively omitted.

FIGURE 4

Throughout the following examples, the methods of ([13, p. 981]) are used in computing the fundamental group.

EXAMPLE 1. We first consider an arc k_1 at a point p which is of the trivial local type. In this case $[Z, \text{identity}]$ and JZ are respectively a local Z -group and a local module. Thus, $\Lambda_n(k_1, p) = 1$ for every $n \geq 0$.

EXAMPLE 2. Figure 5(a). With II.B.6, II.C.6, and V.A.4 it may be shown that $\Lambda_n(k_2, p) = (1 - t + t^2)^\infty$, for $n \geq 0$. The comparing topology (V.A.5) of k_2 at p is $(1 - t + t^2)^\infty$.

FIGURE 5

EXAMPLE 2'. Let k'_2 be a Wilder arc which may be decomposed (V.A.3) into the sequence of prime tame knots $K_1, K_2, K_3, \dots$. Moreover, assume that the respective Alexander polynomials $p_1, p_2, \dots$ of these knots are relatively prime.

Then $\Lambda_0(k, p) = 1$. But the comparing topology (V.A.5) of k at p is the nontrivial topology $\prod_{n=1}^{\infty} p_n$.

EXAMPLE 3. Figure 5(b). A presentation of $\prod (U - k_3)$ is

$$(a_0, a_1, \dots : s, r_1, r_2, \dots),$$

where $r_n = a_{n+1}a_{n+2}a_{n+3}a_{n+2}^{-1}a_{n+1}^{-1}a_n a_{n+1}a_{n+2}^{-1}a_{n+1}^{-1}a_{n+1}a_{n+2}a_{n+3}a_{n+2}^{-1}$ ($n = 0, 1, 2, \dots$) and s is the relation about the vertex p . (See [13, p. 981].) Omitting s , we obtain a presentation of a locally equivalent group. The corresponding matrix is:

$$\begin{matrix} r_0 \\ r_1 \\ r_2 \\ \dots \end{matrix} \begin{bmatrix} -1+t & 2-2t+t^2 & -1+2t-2t^2 & -t+t^2 & 0 & \dots \\ 0 & -1+t & 2-2t+t^2 & -1+2t-2t^2 & -t+t^2 & \dots \\ 0 & 0 & -1+t & 2-2t+t^2 & -1+2t-2t^2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}.$$

Employing II.C.6, II.A.5, II.C.5, we have $\Lambda_0(k_3, p) = (1-t)^\infty$ and $\Lambda_n(k_3, p) = 1$, for $n \geq 1$. Since by V.A.6 k_3 is not a Wilder arc and since $P(k_3, p) \leq 4$, it follows that $P(k_3, p) = 4$.

EXAMPLE 4. Figure 5(c). Since k_4 is an interior composite of its left and right factors k_3 and k'_3 respectively, it follows from V.C.II that each local module of k_4 at p is locally equivalent to the direct sum of local modules of k_3 and k'_3 at p . Hence, by II.B.5:

$$\Lambda_n(k_4, p) = \sum_{i+j=n}^* \Lambda_i(k_3, p) \Lambda_j(k'_3, p).$$

Since k'_3 at p is the mirror image of k_3 at p , it has the same local topologies as k_3 at p . Thus, $\Lambda_n(k_4, p) = (1-t)^\infty$, for $n = 0$ or 1 , and $\Lambda_n(k_4, p) = 1$, for $n \geq 2$. From V.A.6 and V.C.13, $P(k_4, p) = 6$.

EXAMPLE 5. Figure 5(d). This is an interior composite of k_3 and k_2 at p . Thus, from V.C.II and II.B.5 it follows that $\Lambda_0(k_5, p) = (1-t)^\infty(1-t+t^2)^\infty$ and $\Lambda_n(k_5, p) = (1-t+t^2)^\infty$, for $n \geq 1$. Also from V.A.6 we have the $P(k_5, p) = 4$.

EXAMPLE 6. Figure 5(e). Using the same methods as in Example 3, we have $\Lambda_0(k_6, p) = (1-t)^\infty$ and $e(1-t, \Lambda_1(k_6, p)) \geq 3$. It is reasonable to conjecture that $\Lambda_1(k_6, p) = (1-t)^\infty$ and $\Lambda_n(k_6, p) = 1$, for $n \geq 2$.

EXAMPLE 7. Figure 5(f). k_7 at p has the same invariants as k_3 at p .

EXAMPLE 8. Figure 5(g). In computing the invariants of k_8 at p , matrix transformations must be used to give the relation matrix the form of II.C.5. $\Lambda_0(k_8, p) = (1-t+t^2)^\infty$. It appears that $\Lambda_n(k_8, p) = (1-t+t^2)^\infty$, $n \geq 0$.

EXAMPLE 9. Figure 5(h). $\Lambda_0(k_9, p) = 2^\infty(1-t)^\infty$ and $P(k_9, p) = 4$.

EXAMPLE 10. Figure 5(i). $\Lambda_0(k_{10}, p) = 3^\infty(1-t)^\infty$ and $P(k_{10}, p) = 4$.

EXAMPLE 11. Figure 5(j). $\Lambda_0(k_{11}, p) = (1-t)^\infty(1+t^2)^\infty$ and $P(k_{11}, p) = 4$.

EXAMPLE 12. Figure 5(k). $\Lambda_0(k_{12}, p) = (1-t)^\infty(1+t^2+t^4)^\infty$ and $P(k_{12}, p) = 4$.

EXAMPLE 13. Figure 5(l). $\Lambda_0(k_{13}, p) = (1-t)^\infty(2-t+2t^2)^\infty$ and $P(k_{13}, p) = 4$.

EXAMPLE 14. Figure 5(m). $\Lambda_0(k_{14}, p) = (1-t)^\infty(1+t^4+t^5)^\infty$. The lack of symmetry in $\Lambda_0(k_{14}, p)$ can easily be seen to imply that the arc k_{14} is noninvertible.

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