SUBHARMONIC FUNCTIONS IN THE HALF-PLANE

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1. **Introduction.** Recently, R. P. Boas, Jr. [3] has proved functions of exponential type in $\Re z>0$. But it will also be interesting to study higher classes of functions which are represented as the difference of two subharmonic functions in $\Re z>0$ weakening the condition of exponential type. From this point of view, firstly we will study the behavior of mass distributions, next the representations, and lastly the regularities related to the above functions for $\Re z>0$.

2. **Behavior of mass distributions.**

2.1. GENERALIZATION OF CARLEMAN'S THEOREM. Define a domain

$$D = [\Re z > 0,\ |z| < R] \text{ for } 0 < R < +\infty,$$

and let $u(z)$ be a subharmonic function having a positive harmonic majorant for z in D. Then by the representation theorem of Riesz [10] for $z \in D$,

$$u(z) = -\int g_R(z, \zeta)\, d\mu(e_\zeta)+h(z), \tag{2.1}$$

where $h(z)$ is the least harmonic majorant for $u(z)$ in D and μ is stricken positive mass distribution defined for Borel sets e in D, and

$$g_R(z, \zeta) = \log\left|\frac{z+\bar{\zeta}}{z-\zeta}\right|+\log\left|\frac{R^2-z\bar{\zeta}}{R^2+z\zeta}\right|.$$

By the function $z=R\{(z_1-1)R+((1-z_1)^2R^2+4(1+z_1)^2)^{1/2}\}/2(1+z_1)$, the domain D of a z-plane is conformally mapped onto $|z_1|<1$ of a z_1-plane. If we write

$$h(z(z_1)) = H(z_1),$$

then $H(z_1)$ is harmonic and there exists a positive harmonic majorant for $H(z_1)$ in $|z_1|<1$. Denote this harmonic majorant by $S(z_1)$. Then, if we write

$$T(z_1) = S(z_1)-H(z_1),$$

$T(z_1)$ is a positive harmonic function in $|z_1|<1$. Therefore $H(z_1)$ may be written as the difference of two positive harmonic functions. Therefore, we find

$$\lim_{r\to 1}\int_{-\pi}^{+\pi} |H(re^{i\phi})|\, d\phi = M < +\infty.$$

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Hence the following representation of Nevanlinna [9, p. 187] holds: Except for at most a countable set of θ, $-\pi \leqq \theta < \pi$, the limit $\psi(\theta) = \lim_{r\to 1} \int_0^\theta H(re^{i\phi})\,d\phi$ exists; $\psi(\theta)$ is a function of bounded variation and for $|z_1|<1$,

$$H(z_1) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \Re \frac{e^{i\theta}+z_1}{e^{i\theta}-z_1}\,d\psi(\theta), \tag{2.2}$$

where the integral is the Stieltjes integral. We set

$$H(z_1) = \frac{1}{2\pi}\left(\int_{-\pi}^{\theta_1} + \int_{\theta_1}^{\theta_2} + \int_{\theta}^{+\pi}\right) = J_1 + J_2 + J_3$$

where θ_1 and θ_2 are defined by $\tan(\theta_1/2) = -(2/R)$ and $\tan(\theta_2/2) = (2/R)$ $(-\pi < \theta_1 < \theta_2 < \pi)$ respectively.

Let $-H(\zeta, z_1)$ denote the conjugate harmonic function of the Green function for $|z_1|<1$. Then for $-\pi \leqq \theta < \theta_1$, $\theta_2 < \theta \leqq \pi$ and real t,

$$\frac{\partial H(e^{i\theta}, z_1)}{\partial\theta} = \Re\frac{e^{i\theta}+z_1}{e^{i\theta}-z_1}, \quad \frac{\partial H(e^{i\theta(t)}, z_1(z))}{\partial t} = \frac{\partial H}{\partial\theta}\cdot\frac{d\theta}{dt}$$

and by an elementary calculation

$$\frac{dt}{d\theta} = -[R^4t^2 + (R^2+t^2)^2]/[2R^2(R^2-t^2)],$$

$$\frac{\partial H(e^{i\theta(t)}, z_1(z))}{\partial\theta} = \Re\frac{itR^2z - R^2 + z^2 - t^2 + t^2z^2/R^2}{it(R^2-z^2) - R^2z - t^2z}$$

$$= \frac{r\cos\phi(R^2-r^2)(t^2R^2+(R+t^2/R)^2)}{(t^2+r^2-2tr\sin\phi)(R^4+t^2r^2-2R^2tr\sin\phi)}, \qquad z = re^{i\phi}.$$

Therefore

$$-\frac{\partial H(e^{i\theta(t)}, z_1(z))}{\partial t} = 2\Re\left(\frac{1}{z-it} - \frac{z}{R^2+itz}\right).$$

Thus, if we write

$$\Psi(r) = \int_0^r \frac{dt}{d\theta(t)}\,d\psi(\theta(t)), \tag{2.3}$$

$$J_2(z_1) = \frac{1}{\pi}\int_{-R}^{R} k_1^R(it, z)\,d\Psi(t).$$

And for an arbitrary positive ε,

$$\int_{-R+\varepsilon}^{R-\varepsilon} |d\Psi(r)| \leqq \int_{-R+\varepsilon}^{R-\varepsilon} |dt/d\theta|\,|d\psi(\theta(t))|$$

$$\leqq \max_{|t|<R-\varepsilon} |dt/d\theta| \int_{-\pi}^{+\pi} |d\psi(\theta)| < K(\varepsilon) < +\infty.$$

Therefore, $\Psi(r)$ is a function of bounded variation in the interval $[-R+\varepsilon, R-\varepsilon]$.

Next, we estimate $J_1(z_1)+J_3(z_1)$. For $\theta_1<\theta<\theta_2$ and $-\pi/2<\phi<\pi/2$,

$$\frac{d\theta}{d\phi}=\frac{4R\cos\phi}{R^2+4\sin^2\phi},$$

$$\frac{\partial H(e^{i\theta(\phi)}, z_1(z))}{\partial\theta}=\Re\frac{R^2z+2i\sin\phi(R^2-z^2)}{R(R^2-z^2)+2i\sin\phi zR^2}$$

$$=\frac{r\cos\tau(R^2-r^2)(R^2+4\sin^2\phi)}{(R^2+r^2-2Rr\cos(\phi-\tau))(R^2+r^2+2Rr\cos(\phi+\tau))},\qquad z=re^{i\tau}.$$

Therefore

$$\frac{\partial H(e^{i\theta(\phi)}, z_1(z))}{\partial\phi}=2\Re\left(\frac{z}{Re^{i\phi}-z}+\frac{z}{Re^{-i\phi}+z}\right).$$

Thus, if we define $\Phi(\tau)=\int_0^\tau (d\phi/d\theta(\phi))\,d\psi(\theta(\phi))$, then for an arbitrary positive ε, $\Phi(\tau)$ is a function of bounded variation in the interval $[-\pi/2+\varepsilon, \pi/2-\varepsilon]$. And

$$J_1(z_1)+J_3(z_1)=\frac{1}{\pi}\int_{-\pi/2}^{+\pi/2}K_2^R(Re^{i\phi}, z)\,d\Phi(\phi). \tag{2.4}$$

From (2.1), (2.2), (2.3), and (2.4), we get

Theorem 1. *Define the domain $D=[\Re z>0, |z|<R]$ for $0<R<+\infty$. Let $u(z)$ be a subharmonic function having a positive harmonic majorant for z in D. Then there exist two functions $\Psi(t)$ and $\Phi(\phi)$ which are defined in the intervals $(-R, R)$ and $(-\pi/2, \pi/2)$ respectively and of bounded variation on arbitrary closed intervals included in the above intervals respectively, and a stricken positive mass distribution $\mu(e_\zeta)$ defined for the Borel sets e in D. And for $D\ni z$,*

$$u(z)=\frac{1}{\pi}\int_{-R}^{+R}K_1^R(it, z)\,d\Psi(t)+\frac{1}{\pi}\int_{-\pi/2}^{+\pi/2}K_2^R(Re^{i\phi}, z)\,d\Phi(\phi)$$
$$-\int_{\zeta\in D}g_R(z,\zeta)\,d\mu(e_\zeta)$$

where

$$K_1^R(it, z)=\Re\left(\frac{1}{z-it}-\frac{z}{R^2+itz}\right),$$

$$K_2^R(Re^{i\phi}, z)=\Re\left(\frac{z}{Re^{i\phi}-z}+\frac{z}{Re^{-i\phi}+z}\right)$$

and

$$g_R(z,\zeta)=\log\left|\frac{(z+\bar\zeta)(R^2-z\bar\zeta)}{(z-\zeta)(R^2+z\zeta)}\right|.$$

This theorem is an extension of a theorem of F. and R. Nevanlinna [8].

Let $\mu^*(e_\zeta)$ be a mass distribution which may be written as the difference of two stricken nonnegative mass distributions $\mu_1(e_\zeta)$ and $\mu_2(e_\zeta)$ defined for the Borel sets e in D respectively. Then, from the proof of this theorem, we easily get

Theorem 2. *Let $u(z)$ be a function represented as the difference of two subharmonic* *functions having positive harmonic majorants for z in D. Then there exist two functions* $\Psi(t)$ *and* $\Phi(\phi)$ *as in Theorem* 1 *and for z in D*,

$$u(z) = \frac{1}{\pi}\int_{-R}^{+R} K_1^R(it, z)\, d\Psi(t) + \frac{1}{\pi}\int_{-\pi/2}^{+\pi/2} K_2^R(Re^{i\phi}, z)\, d\Phi(\phi) - \int_{\zeta\in D} g_R(z, \zeta)\, d\mu^*(e_\zeta), \tag{2.6}$$

where K_1^R *and* K_2^R *are defined in Theorem* 1.

Let $E(\rho, \varepsilon)$ denote a set $\{r \leqq |z| < \rho-\varepsilon,\ \rho+\varepsilon \leqq |z| < R,\ |\arg z| < \pi/2-\varepsilon\ (\varepsilon>0)\}$. By the Carleman method [6] we consider the integral

$$I = (1/2\pi i)\iint_{E(\rho,\varepsilon)\ni\zeta} (z^{-2}+\rho^{-2}) \log\{(z+\bar{\zeta})(R^2-z\bar{\zeta})/(z-\zeta)(R^2+z\zeta)\}\, d\mu^*(e_\zeta)\, dz$$

taken along the contour of the domain $\{|\arg z| < \pi/2,\ r_1 < |z| < \rho\ (r_1 < r)\}$ in the positive sense, starting from the point $z = -i\rho$ with a fixed determination of the logarithm. Then letting $r_1 \to 0$ for fixed r, we get

$$\Re I = -\int_{E(\rho,\varepsilon)\ni\zeta} \left(\Re\frac{1}{\zeta} - \frac{\Re\zeta}{R^2}\right) d\mu^*(e_\zeta) + \frac{1}{\pi\rho}\int_{-\pi/2}^{+\pi/2}\int_{\zeta\in D} g_R(\rho e^{i\phi}, \zeta) \cos\phi\, d\mu^*(e_\zeta)\, d\phi. \tag{2.7}$$

Again, integrating by parts and using the theorem of residues, we get

$$\Re I = \int_{E(\rho,\varepsilon)\ni\zeta;\,|\zeta|<\rho-\varepsilon} (\rho^{-2}-|\zeta|^{-2})\Re\zeta\, d\mu^*(e_\zeta). \tag{2.8}$$

From (2.6) $\int_{|\zeta|<\rho} \Re\zeta |d\mu^*(e_\zeta)| < +\infty$. Therefore from (2.7) and (2.8), we find as $\varepsilon \to 0$ and $r \to 0$

$$\frac{1}{\pi\rho}\int_{-\pi/2}^{+\pi/2}\int_{\zeta\in D} g_R(\rho e^{i\phi}, \zeta)\cos\phi\, d\mu^*(e_\zeta)\, d\phi = \int_{|\zeta|<\rho} (\rho^{-2}-R^{-2})\Re\zeta\, d\mu^*(e_\zeta) + \int_{\rho\leqq|\zeta|<R} (\Re\zeta^{-1} - R^{-2}\Re\zeta)\, d\mu^*(e_\zeta). \tag{2.9}$$

Accordingly, by using the same method as [6, p. 247] or [7] for $h(z)$ we obtain the following equality

$$\begin{aligned}\frac{m(\rho)}{\rho} = & -\int_{|\zeta|<\rho} \rho^{-2}\Re\zeta\, d\mu^*(e_\zeta) - \int_{\rho\leqq|\zeta|<R} \Re\zeta^{-1}\, d\mu^*(e_\zeta) \\ & + \int_{|\zeta|<R} R^{-2}\Re\zeta\, d\mu^*(e_\zeta) + \frac{1}{2\pi}\int_{|t|<\rho} \rho^{-2}\, d\Psi(t) \\ & + \int_{\rho\leqq|t|<R} t^{-2}\, d\Psi(t) - \int_{|t|<R} R^{-2}\, d\Psi(t) + \frac{m^*(R)}{R},\end{aligned} \tag{2.10}$$

where $m(\rho)=(1/\pi)\int_{-\pi/2}^{+\pi/2} u(\rho e^{i\phi})\cos\phi\, d\phi$ and $m^*(R)=(1/\pi)\int_{-\pi/2}^{+\pi/2}\cos\phi\, d\Phi(\phi)$. Thus, using the partial integration for $\int_{\rho\leqq|\zeta|<R}\Re\zeta^{-1}\,d\mu^*(e_\zeta)$ and $\int_{\rho\leqq|t|<R} t^{-2}\,d\Psi(t)$ respectively, we get the following result closely related to Carleman's theorem.

THEOREM 3. *Let $u(z)$ satisfy the hypotheses of Theorem 2. Then for all ρ such that $0<\rho<R$,*

$$\frac{m(\rho)}{\rho} = 2\int_\rho^\rho \frac{A(t)}{t^3}\,dt+\frac{m^*(R)}{R}, \tag{2.11}$$

where $A(t)=-\int_{|\zeta|<t}\Re\zeta\, d\mu^*(e_\zeta)+(1/2\pi)\int_{|\tau|<t} d\Psi(\tau)$, *and* $m(\rho)$ *and* $m^*(R)$ *are defined in* (2.10).

From Theorem 3, we easily obtain

THEOREM 4. *Under the hypotheses of Theorem 2, we can assert the following for all t such that $0<t<R$:*

(I). *If $A(t)\leqq 0$, then $m(t)/t$ is a continuous and nondecreasing function of t.*

(II). *If $A(t)$ is a nonincreasing function of t, then $m(t)/t$ is a convex function of $1/t^2$.*

Proof. Since the equality $m(\rho)/\rho=2\int_\rho^R (A(t)/t^3)\,dt+m^*(R)/R$ for $0<\rho_1<\rho<R$ is obtained from Theorem 3, by using (2.11) we get

$$\frac{m(\rho_1)}{\rho_1} = 2\int_{\rho_1}^{\rho}\frac{A(t)}{t^3}\,dt+\frac{m(\rho)}{\rho}.$$

Therefore (I) holds. Next, from (2.11) we get

$$d\frac{m(\rho)}{\rho}\Big/ d\left(\frac{1}{\rho^2}\right) = A(\rho)$$

for almost every ρ. Hence (II) holds.

By using Theorem 2, we get the following Theorems 5–8 related to the half-plane.

THEOREM 5. *Let $u(z)$ be a function in $\Re z>0$ represented as the difference of two subharmonic functions which have positive harmonic majorants in an arbitrary bounded subdomain in $\Re z>0$. Then there exist a function $\Psi(t)$ of bounded variation defined in the finite imaginary axis and a stricken mass distribution $\mu^*(e_\zeta)$ defined for the Borel sets e in $\Re z>0$ and represented as the difference of two nonnegative mass distributions, and for all ρ and R $(0<\rho<R<+\infty)$,*

$$\frac{m(\rho)}{\rho} = 2\int_\rho^R \frac{A(t)}{t^3}\,dt+\frac{m(R)}{R},$$

where $A(t)=-\int_{|\zeta|<t}\Re\zeta\, d\mu^*(e_\zeta)+(1/2\pi)\int_{|\tau|<t} d\Psi(\tau)$ *and*

$$m(R) = \frac{1}{\pi}\int_{-\pi/2}^{+\pi/2} u(Re^{i\phi})\cos\phi\, d\phi.$$

Proof. Define two domains $D_1\{\Re z>0,\ |z|<R_1\}$ and $D_2\{\Re z>0,\ |z|<R_2\}$ for $0<R_1<R_2<+\infty$. Then applying the proof of Theorem 4 and using the result of the uniqueness of positive mass distribution, there exists the stricken mass distribution $\mu^*(e_\zeta)$ defined in this theorem, and we find for arbitrary positive numbers ρ and R such that $0<\rho<R<R_1$,

$$\frac{m(\rho)}{\rho} = 2\int_\rho^R \frac{A^1(t)}{t^3}\,dt+\frac{m(R)}{R} \tag{2.12}$$

and

$$\frac{m(\rho)}{\rho} = 2\int_\rho^R \frac{A^2(t)}{t^3}\,dt+\frac{m(R)}{R}, \tag{2.13}$$

where

$$A^1(t) = -\int_{|\zeta|<t} \Re\zeta\, d\mu^*(e_\zeta)+\frac{1}{2\pi}\int_{|\tau|<t} d\Psi(\tau)$$

and

$$A^2(t) = -\int_{|\zeta|<t} \Re\zeta\, d\mu^*(e_\zeta)+\frac{1}{2\pi}\int_{|\tau|<t} d\Psi_2(\tau),$$

and $\Psi_1(t)$ and $\Psi_2(t)$ are defined in D_1 and D_2 analogously to $\Psi(t)$ in Theorem 2 respectively. From (2.12) and (2.13), we get

$$\int_\rho^R \frac{A^1(t)-A^2(t)}{t^3}\,dt = 0.$$

Hence $A^1(t)=A^2(t)$ for almost every t $(0<t<R)$. Therefore

$$\int_{|\tau|<t} d\Psi_1(\tau) = \int_{|\tau|<t} d\Psi_2(\tau)$$

for almost every t and $\int_{|\tau|<t} d\Psi_2(\tau)$ is a function of bounded variation of t such that $0\leqq t\leqq R_1$. Thus we complete this proof.

If $g(r)$ denotes a real valued function of positive r and $\lim_{r\to+\infty} g(r)=\pm\infty$, and if the derivative of $g(r)$ is evaluated in the interval $[1, +\infty)$ and $g'(r)\neq 0$, by the theorem, we find

THEOREM 6. *If $u(z)$ satisfies the hypotheses of Theorem 5, then we can assert the following*:

(III). *The condition*

$$m(r) \sim rg(r)$$

is equivalent to

$$A(r) \sim -\tfrac{1}{2}r^3g'(r).$$

(IV). *The condition*

$$m(r)/rg(r) \text{ is bounded for } 1<r<+\infty$$

is equivalent to

$$A(r)/r^3g'(r) \text{ is bounded for } 1 < r < +\infty.$$

Proof of (III). The derivative $(d/dR)\int_\rho^R (A(t)/t^3)\,dt = A(R)/R^3$ is evaluated for almost every R such that $\rho < R < +\infty$. Therefore if $\int_\rho^R (A(t)/t^3)\,dt \sim -\frac{1}{2}g(R)$, then $A(R) \sim -\frac{1}{2}R^3 g'(R)$. Thus from (2.12), (III) holds. (IV) is proved analogously.

If we write $A_1(t) = -\int_{|\zeta|<t} \Re\zeta\, d\mu^*(e_\zeta)$, $A_2(t) = (1/\pi)\int_{|\tau|<t} d\Psi(\tau)$ and

$$S(R) = \int_\varepsilon^R \frac{A_2(t)}{t^3} + \frac{m(R)}{R},$$

then by Theorem 5

$$S(\rho) = 2\int_\rho^R \frac{A_1(t)}{t^3}\,dt + S(R),$$

where $R > \rho > \varepsilon > 0$. Therefore we get

THEOREM 7. *If, in Theorem* 5, $u(z)$ *is a function subharmonic in* $\Re z > 0$, *then for* $R > 0$

(V). $S(R)$ *is a continuous and nondecreasing function of* R, *and*

(VI). $S(R)$ *is a convex function of* $1/R^2$.

If we write $T(R) = 2\int_\varepsilon^R (A_1(t)/t^3)\,dt + m(R)/R$, then by Theorem 5,

$$T(P) = \int_\rho^R \frac{A_2(t)}{t^3}\,dt + T(R), \qquad R > \rho > \varepsilon > 0.$$

Hence we get

THEOREM 8. *If, in addition to the hypotheses of Theorem* 5, $u(z)$ *satisfies the Phragmén-Lindelöf boundary condition*; *namely* $\limsup_{z\to i\eta} u(z) \leqq 0$ $(\Re z > 0)$ *for all real finite* η, *then for* $R > 0$,

(VII). $T(R)$ *is a continuous and nondecreasing function of* R, *and*

(VIII). $T(R)$ *is a convex function of* R^{-2}.

Particularly, if $u(z)$ is subharmonic, then $A_1(t) \leqq 0$ and $A_1(t)$ is a nonincreasing function of t. Therefore from (VII), for $R_2 > R_1$,

$$\frac{m(R_1)}{R_1} + 2\int_\varepsilon^{R_1} \frac{A_1(t)}{t^3}\,dt \leqq \frac{m(R_2)}{R_2} + 2\int_\varepsilon^{R_2} \frac{A_1(t)}{t^3}\,dt.$$

Hence we find $m(R_1)/R_1 \leqq m(R_2)/R_2$ which is due to Ahlfors [1] and Heins [5]. Next, $dT(R)/dR^{-2} = (dm(R)/R)/dR^{-2} - A_1(R)$ for almost every R from the definition of $T(R)$. On the other hand, $dT(R)/dR^{-2} = A_2(R)/2$ for almost every R. Hence $2d(m(R)/R)/dR^{-2} = (2A_1(R) + A_2(R))$ for almost every R. Consequently $m(R)/R$ is a convex function of R^{-2}, and the case $A_2(t) \equiv 0$ of this result is due to Tsuji [12].

2.2. HIGHER CLASSES. In this section, we shall improve the conclusion of Theorem 5 and study the behavior of mass distributions of higher classes. For this we start from an application of the representation of (2.6). If $t > |z|$, then Taylor's expansion for $1/(z - it)$ shows that

$$\frac{1}{z-it} = -\frac{1}{it}\sum_{n=0}^{+\infty}\left(\frac{z}{it}\right)^n = -\sum_{n=0}^{+\infty}\frac{r^n}{t^{n+1}}e^{i(n\theta-(n+1)\pi/2)}, \qquad z = re^{i\theta}.$$

Therefore, applying the elementary integration

$$\frac{1}{\pi}\int_{-\pi/2}^{+\pi/2}\Re\frac{1}{re^{i\theta}-it}\sin n(\pi/2-\theta)\,d\theta = \frac{r^n}{2t^{n+1}}, \tag{2.14}$$

and also using the same method as (2.14) we find for $|z|>t$

$$\frac{1}{\pi}\int_{-\pi/2}^{+\pi/2}\Re\frac{1}{re^{i\theta}-it}\sin n(\pi/2-\theta)\,d\theta = \frac{t^{n-1}}{2r^n}. \tag{2.15}$$

Thus from (2.14) and (2.15), we get

$$\begin{aligned}&\frac{1}{\pi}\int_{-\pi/2}^{+\pi/2}\int_{-R}^{+R}\Re\frac{1}{re^{i\theta}-it}\,d\Psi(t)\sin n(\pi/2-\theta)\,d\theta\\ &\qquad = \frac{1}{2}\left(\int_{r\leq|t|<R}\frac{r^n}{t^{n+1}}\,d\Psi(t)+\int_{|t|<r}\frac{t^{n-1}}{r^n}\,d\Psi(t)\right).\end{aligned} \tag{2.16}$$

By the same method as (2.16), we find

$$\frac{1}{\pi}\int_{-\pi/2}^{+\pi/2}\int_{-R}^{+R}\Re\frac{z}{R^2+itz}\,d\Psi(t)\sin n(\pi/2-\theta)\,d\theta = \frac{1}{2}\int_{|t|<R}\frac{t^{n-1}r^n}{R^{2n}}\,d\Psi(t). \tag{2.17}$$

Next, $\Re e^{i\pi(n+1)/2}/-z^n=\sin n(\pi/2-\theta)/r^n$ is harmonic in $\Re z>0$. Therefore we find

$$\begin{aligned}&\int_{-\pi/2}^{+\pi/2}\frac{1}{\pi}\int_{-\pi/2}^{+\pi/2}u(Re^{i\phi})K_2^R(Re^{i\phi},re^{i\theta})\frac{\sin n(\pi/2-\theta)}{r^n}\,d\phi\,d\theta\\ &\qquad = \int_{-\pi/2}^{+\pi/2}u(Re^{i\phi})\frac{1}{\pi}\int_{+\pi/2}^{-\pi/2}\frac{\sin n(\pi/2-\theta)}{r^n}K_2^R(re^{i\theta},Re^{i\phi})\,d\theta\,d\phi\\ &\qquad = \int_{-\pi/2}^{+\pi/2}u(Re^{i\phi})\frac{\sin n(\pi/2-\phi)}{R^n}\,d\phi.\end{aligned} \tag{2.18}$$

Moreover, by using Taylor's expansion for $\log(z+\zeta)$ and $\log(z-\zeta)$, if we write

$$\log\frac{z+\zeta}{z-\zeta} = \sum_{n=0}^{+\infty}A_n,$$

then for $|z|<|\zeta|$,

$$\begin{aligned}A_n &= -\frac{2}{n}\frac{i\Im\zeta^n}{|\zeta|^{2n}}z^n, \qquad n \text{ even};\\ &= \frac{2}{n}\frac{\Re\zeta^n}{|\zeta|^{2n}}z^n, \qquad n \text{ odd}.\end{aligned}$$

Hence

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi/2}^{+\pi/2}\Re A_n\sin n(\pi/2-\theta)\,d\theta &= (-1)^{n/2+1}\frac{r^n}{n}\frac{\Im\zeta^n}{|\zeta|^{2n}}, \qquad n \text{ even};\\ &= (-1)^{(n-1)/2}\frac{r^n}{n}\frac{\Re\zeta^n}{|\zeta|^{2n}}, \qquad n \text{ odd}.\end{aligned} \tag{2.19}$$

And for $|z|>|\zeta|$ we get

$$(2.20)\quad \begin{aligned} \frac{1}{\pi}\int_{-\pi/2}^{+\pi/2} \Re A_n \sin n(\pi/2-\theta)\,d\theta &= (-1)^{n/2+1}\frac{1}{nr^n}\Im\zeta^n, && n \text{ even};\\ &= (-1)^{(n-1)/2}\frac{1}{nr^n}\Re\zeta^n, && n \text{ odd}. \end{aligned}$$

Moreover if we write

$$\log\frac{R^2+z\zeta}{R^2-z\zeta} = \sum_{n=0}^{+\infty} B_n,$$

then we get for $z=re^{i\theta}$,

$$(2.21)\quad \begin{aligned} \frac{1}{\pi}\int_{-\pi/2}^{+\pi/2} \Re B_n \sin n(\pi/2-\theta)\,d\theta &= (-1)^{n/2+1}\frac{r^n}{n}\frac{\Im\zeta^n}{R^{2n}}, && n \text{ even};\\ &= (-1)^{(n-1)/2}\frac{r^n}{n}\frac{\Re\zeta^n}{R^{2n}}, && n \text{ odd}. \end{aligned}$$

Thus, from (2.6), if we write

$$m(r,n) = \frac{1}{\pi}\int_{-\pi/2}^{+\pi/2} u(re^{i\phi})\sin n\left(\frac{\pi}{2}-\theta\right)d\theta,$$

then we get

$$(2.22)\quad \begin{aligned} \frac{m(r,n)}{r^n} &= \frac{1}{2\pi}\left(\int_{|t|<r}\frac{t^{n-1}}{r^{2n}}\,d\Psi(t)+\int_{r\leqq|t|<R}\frac{1}{t^{n+1}}\,d\Psi(t)-\int_{|t|<R}\frac{t^{n-1}}{R^{2n}}\,d\Psi(t)\right)\\ &\quad+\begin{cases} \dfrac{(-1)^{n/2}}{n}\left(\displaystyle\int_{|\zeta|<r}\frac{\Im\zeta^n}{r^{2n}}\,d\mu^*(e_\zeta)+\int_{r\leqq|\zeta|<R}\frac{\Im\zeta^n}{|\zeta|^{2n}}\,d\mu^*(e_\zeta)\right.\\ \qquad\qquad\left.-\displaystyle\int_{|\zeta|<R}\frac{\Im\zeta^n}{R^{2n}}\,d\mu^*(e_\zeta)\right), & n \text{ even};\\ \dfrac{(-1)^{(n+1)/2}}{n}\left(\displaystyle\int_{|\zeta|<r}\frac{\Re\zeta^n}{r^{2n}}\,d\mu^*(e_\zeta)+\int_{r\leqq|\zeta|<R}\frac{\Re\zeta^n}{|\zeta|^{2n}}\,d\mu^*(e_\zeta)\right.\\ \qquad\qquad\left.-\displaystyle\int_{|\zeta|<R}\frac{\Re\zeta^n}{R^{2n}}\,d\mu^*(e_\zeta)\right), & n \text{ odd} \end{cases}\\ &= \frac{m(R,n)}{R^n}. \end{aligned}$$

If we write

$$K(\zeta) = \begin{Bmatrix} (-1)^{n/2}\Im\zeta^n, \ n \text{ even} \\ (-1)^{(n+1)/2}\Re\zeta^n, \ n \text{ odd} \end{Bmatrix} = -|\zeta|^n \sin n(\pi/2-\arg\zeta)$$

and apply integration by parts for

$$\frac{1}{2\pi}\int_{r\leqq|t|<R}\frac{1}{t^{n+1}}\,d\Psi(t) \quad\text{and}\quad \int_{r\leqq|\zeta|<R}|\zeta|^{-2n}K(\zeta)\,d\mu^*(e_\zeta);$$

furthermore, if we set

$$A^*(t, n) = \frac{1}{2\pi}\int_{|\tau|<t} \tau^{n-1}\, d\Psi(\tau)+\frac{1}{n}\int_{|\zeta|<t} K(\zeta)\, d\mu^*(e_\zeta), \tag{2.23}$$

then

$$\frac{m(r, n)}{r^n} = 2n\int_r^R \frac{1}{t^{2n+1}}\, A^*(t)\, dt+\frac{m(R, n)}{R^n}, \tag{2.24}$$

where $A^*(t)=A^*(t, n)$.

Thus we obtain

THEOREM 9. *Under the same hypotheses as in Theorem* 5, *the equality* (2.24) *holds.*

From this Theorem 9 we find directly the following

THEOREM 10. *Suppose that $u(z)$ satisfies the hypotheses of Theorem* 5. *Then we can assert the following for $\sigma=+1$ or -1, $t>0$ and $2n-p>0$:*

(IX). *If $\sigma A^*(t)\leqq 0$, then $\sigma m(t, n)/t^n$ is a continuous and nondecreasing function for t.*

(X). *If $\sigma A^*(t)/t^p$ is a nonincreasing function of t, then $\sigma m(t, n)/t^n$ is a convex function of $t^{-(2n-p)}$.*

(XI). *Let*

$$\int_{r_1}^{r_2} t^{-(2n+1)}A^*(t)\, dt < \varepsilon \tag{2.25}$$

for all $r_2>r_1>N$ if N is selected sufficiently large for an arbitrary positive ε. Then $\lim_{t\to+\infty} (m(t, n)/t^n)=\mu^*$ $(-\infty<\mu^*\leqq+\infty)$ exists, and if $\mu^*<+\infty$, then $\int_1^{+\infty} t^{-(2n+1)}A^*(t)\, dt$ converges.

(XII). *Let $\int_{r_1}^{r_2} t^{-(2n+1)}A^*(t)\, dt$ be bounded above for $1<r_1<r_2<+\infty$ and $\liminf_{r\to+\infty} (m(r, n)/r^n)<+\infty$. Then $m(r, n)/r^n$ and $\int_1^r t^{-(2n+1)}A^*(t)\, dt$ are bounded respectively for $1<r<+\infty$.*

(XIII). *Let $\int_{r_1}^{r_2} t^{-(2n+1)}A^*(t)\, dt<\varepsilon$ for all $0<r_1<r_2<s$ if s is selected sufficiently small for an arbitrary positive ε. Then $\lim_{r\to 0} (m(r, n)/r^n)=\mu_2^*$ $(-\infty\leqq\mu_2^*<+\infty)$ exists, and if $\mu_2^*>-\infty$, then $\int_0^1 t^{-(2n+1)}A^*(t)\, dt$ converges.*

(XIV). *Let $\int_r^1 t^{-(2n+1)}A^*(t)\, dt$ be bounded above for $0<r<1$ and*

$$\limsup_{r\to 0} \frac{m(r, n)}{r^n} > -\infty.$$

Then $\int_r^1 t^{-(2n+1)}A^*(t)\, dt$ and $m(r, n)/r^n$ are bounded respectively for $0<r<1$.

If we write $A_1^*(t)=(1/n)\int_{|\zeta|<t} k(\zeta)\, d\mu^*(e_\zeta)$, $A_2^*(t)=(1/2\pi)\int_{|\tau|<t} \tau^{n-1}\, d\Psi(\tau)$ and $S^*(R)=2n\int_\varepsilon^R (A_2^*(t)/t^{2n+1})\, dt+(m(R, n)/R^n)$ for $R>\varepsilon>0$, then from (2.24), for $R>\rho>\varepsilon$,

$$S^*(\rho) = 2n\int_\rho^R \frac{A_1^*(t)}{t^{2n+1}}\, dt+S^*(R).$$

Hence we get

THEOREM 11. *Let $u(z)$ satisfy the hypotheses of Theorem* 6. *Then we can assert the following*: *For $\sigma=+1$ or -1 and $t>0$,*

(XV). *If $\sigma A_1^*(t)\leqq 0$, then $\sigma S^*(t)$ is a continuous and nondecreasing function for t.*

(XVI). *If $\sigma A_1^*(t)$ is a nonincreasing function of t, then $\sigma S^*(t)$ is a convex function of $1/t^{2n}$.*

If we write $T^*(R)=2n\int_\varepsilon^R (A_1^*(t)/t^{2n+1})\,dt+(m(R,n)/R^n)$ for $0<\varepsilon<R$, then from (2.24), for $R>\rho>\varepsilon$,

$$T^*(\rho) = 2n\int_\rho^R \frac{A_2^*(t)}{t^{2n+1}}\,dt+T^*(R).$$

Hence we get

THEOREM 12. *If, in addition to the hypotheses of Theorem* 8, *n is odd, then we can assert the following*: *For $t>0$,*

(XVII). *$T^*(t)$ is a continuous and nondecreasing function of t, and*

(XVIII). *$T^*(t)$ is a convex function of $1/t^{2n}$.*

Theorems 11 and 12 contain Theorems 7 and 8 respectively.

3. **Representation theorems.** First, we shall state the following fundamental representation theorem.

THEOREM 13. *Let $u(z)$ satisfy the hypotheses of Theorem* 5. *Then for $\varepsilon>0$, $|z|<R<+\infty$ ($\Re z>0$) and $K(\zeta)=K(\zeta,n)$ in Theorem* 9,

$$\begin{aligned} u(z) = {} & \frac{1}{\pi}\int_{\varepsilon<|t|<R}\Re\left\{\left(\frac{z}{it}\right)^n\frac{1}{z-it}-\left(\frac{tz}{iR^2}\right)^{n-1}\frac{z}{R^2+itz}\right\}d\Psi(t)\\ & -\int_{\varepsilon<|\zeta|<R}\Re\left\{\log\frac{(z+\zeta)(R^2-z\zeta)}{(z-\zeta)(R^2+z\zeta)}-\frac{2}{i}\sum_{k=1}^{n-1}\frac{1}{k}\left(\frac{z}{i}\right)^k\right.\\ & \qquad\qquad\left.\times\left(\frac{1}{|\zeta|^{2k}}-\frac{1}{R^{2k}}\right)K(\zeta,k)\right\}d\mu^*(e_\zeta)\\ & +\frac{1}{\pi}\int_{-\pi/2}^{+\pi/2}\Re\left\{\left(\frac{z}{Re^{i\phi}}\right)^{n-1}\frac{z}{Re^{i\phi}-z}+\left(\frac{-z}{Re^{-i\phi}}\right)^{n-1}\frac{z}{Re^{-i\phi}+z}\right\}\\ & \times u(Re^{i\phi})\,d\phi+\Re P(n-1,z,\varepsilon) \end{aligned} \tag{3.1}$$

where $P(n-1,z,\varepsilon)$ denotes

$$\begin{aligned} & \frac{1}{\pi}\int_{|t|<\varepsilon}\left\{\frac{1}{z-it}-\frac{i}{t}\sum_{k=1}^{n-1}\left(\frac{zt}{i}\right)^k\left(\frac{1}{\varepsilon^{2k}}-\frac{1}{R^{2k}}\right)\right\}d\Psi(t)\\ & -\int_{|\zeta|<\varepsilon}\left\{\log\frac{(z+\zeta)(R^2-z\zeta)}{(z-\zeta)(R^2+z\zeta)}+2i\sum_{k=1}^{n-1}\frac{1}{k}\left(\frac{z}{i}\right)^k\left(\frac{1}{\varepsilon^{2k}}-\frac{1}{R^{2k}}\right)K(\zeta,z)\right\}d\mu^*(e_\zeta)\\ & +2i\sum_{k=1}^{n-1}\left(\frac{z}{i\varepsilon}\right)^k m(\varepsilon,k). \end{aligned} \tag{3.2}$$

Proof. From (2.24) we get

$$\begin{aligned}\frac{2}{i}\sum_{k=1}^{n-1}\left(\frac{z}{i\varepsilon}\right)^k m(\varepsilon, k)\\ &= \frac{4}{i}\sum_{k=1}^{n-1} k\left(\frac{z}{i}\right)^k \int_\varepsilon^R \frac{1}{t^{2k+1}} A^*(t, k)\, dt + \frac{2}{i}\sum_{k=1}^{n-1}\left(\frac{z}{iR}\right)^k m(R, k).\end{aligned} \tag{3.3}$$

Accordingly, by using the equalities

$$\sum_{k=1}^{n-1} \frac{z^k}{(it)^{k+1}} = \frac{(z/it)^n - 1}{z-it} - \frac{1}{it}, \quad \sum_{k=1}^{n-1} \frac{1}{it}\left(\frac{zt}{iR^2}\right)^k = \frac{z((tz/iR^2)^{n-1}-1)}{R^2+itz},$$

$$\begin{aligned}-\frac{2}{i}\sum_{k=1}^{n-1}\left(\frac{z}{iR}\right)^k \sin k(\pi/2-\phi)\\ &= \left\{1-\left(\frac{z}{Re^{i\phi}}\right)^{n-1}\right\}\frac{z}{Re^{i\phi}-z} + \left\{1-\left(\frac{-z}{Re^{-i\phi}}\right)^{n-1}\right\}\frac{z}{Re^{-i\phi}+z},\end{aligned}$$

from (2.22), (2.24), and (3.3), we find (3.1) and (3.2).

THEOREM 14. *Let $u(z)$ satisfy the hypotheses of Theorem 5. Suppose that the conditions* (2.25),

$$\limsup_{R\to+\infty} \frac{1}{R^{n+1}} \int_{-\pi/2}^{+\pi/2} |u(Re^{i\phi})| \cos\phi\, d\phi = 0 \tag{3.4}$$

and

$$\liminf_{R\to+\infty} \frac{1}{R^n} m(R, n) = \mu^* < +\infty \tag{3.5}$$

are satisfied. Then

$$\begin{aligned}u(z) = \lim_{R\to+\infty} \Bigg[\frac{1}{\pi}\int_{\varepsilon<|t|<R} \Re\left\{\left(\frac{z}{it}\right)^n \frac{1}{z-it} - \left(\frac{iz}{iR^2}\right)^{n-1} \frac{z}{R^2+itz}\right\} d\Psi(t)\\ - \int_{\varepsilon<|\zeta|<R} \Re\Bigg\{\log\frac{(z+\zeta)(R^2-z\zeta)}{(z-\zeta)(R^2+z\zeta)} - \frac{2}{i}\sum_{k=1}^{n-1}\frac{1}{k}\left(\frac{z}{i}\right)^k\\ \times\left(\frac{1}{|\zeta|^{2k}} - \frac{1}{R^{2k}}\right) K(\zeta, k)\Bigg\} d\mu^*(e_\zeta)\Bigg]\\ - 2\mu^* \Re \frac{z^n}{i^{n+1}} + \Re Q(n-1, z, \varepsilon)\end{aligned} \tag{3.6}$$

where $K(\zeta, n)$ is defined in Theorem 9 *and*

$$\begin{aligned}Q &= \lim_{R\to+\infty} P(n-1, z, \varepsilon)\\ &= \frac{1}{\pi}\int_{|t|<\varepsilon}\left\{\frac{1}{z-it} - \frac{i}{t}\sum_{k=1}^{n-1}\left(\frac{zt}{i\varepsilon^2}\right)^k\right\} d\Psi(t)\\ &\quad - \int_{|\zeta|<\varepsilon}\left\{\log\frac{z+\zeta}{z-\zeta} + 2i\sum_{k=1}^{n-1}\frac{1}{k}\left(\frac{z}{i\varepsilon^2}\right)^k K(\zeta, k)\right\} d\mu(e_\zeta) + 2i\sum_{k=1}^{n-1}\left(\frac{z}{i\varepsilon}\right)^k m(\varepsilon, k).\end{aligned}$$

Proof. By elementary calculation, we get for $z=re^{i\theta}$, $\zeta=te^{i\phi}$,

$$\Re\left\{\left(\frac{z}{Re^{i\phi}}\right)^n-\left(\frac{-z}{Re^{-i\phi}}\right)^n\right\} = 2\left(\frac{r}{R}\right)^n \sin n\left(\frac{\pi}{2}-\theta\right) \sin n\left(\frac{\pi}{2}-\phi\right)$$

and using the inequality $\sin k(\pi/2-\phi)\leqq k(\pi/2-\phi)\leqq k\tan(\pi/2-\phi)$ $(0<\phi<\pi/2)$, for small $\delta>0$ and large constant M,

$$\begin{aligned}\left|\Re\left\{\frac{z^{n+1}}{(Re^{i\phi})^n(Re^{i\phi}-z)}-\frac{(-z)^{n+1}}{(Re^{-i\phi})^n(Re^{-i\phi}+z)}\right\}\right| \\ = \left|2\sum_{k=n+1}^{+\infty}\left(\frac{r}{R}\right)^k \sin k\left(\frac{\pi}{2}-\theta\right)\sin k\left(\frac{\pi}{2}-\phi\right)\right| \\ \leqq 2\sqrt{2}\pi \sum_{k=n+1}^{+\infty}\left(\frac{r}{R}\right)^k k\frac{\cos\phi}{\sin 2\delta} \leqq \left(\frac{r}{R}\right)^{n+1} M\cos\phi.\end{aligned}$$

Therefore from Theorem 13 and (XI) in Theorem 10, the representation (3.6) holds.

THEOREM 15. *In addition to the hypotheses of Theorem* 5, *let condition* (3.5) *be satisfied. Suppose that*

$$\int_{|\zeta|+\infty} |\zeta|^{-n-2}\Re\zeta|d\mu^*(e_\zeta)| < +\infty \tag{3.7}$$

and that

$$\int_{r_1<|t|<r_2} t^{-n-1}\, d\Psi(t) < \varepsilon, \tag{3.8}$$

$$\int_{r_1<|\zeta|<r_2} |\zeta|^{-2n}K(\zeta, n)\, d\mu^*(e_\zeta) < \varepsilon, \tag{3.9}$$

$$\int_{r_1}^{r_2} \sigma^{n+1}\frac{1}{t^{n+2}}\, d\Psi(\sigma t) < \varepsilon, \qquad \sigma = \pm 1, \tag{3.10}$$

for all $r_2>r_1>N$ if N is selected sufficiently large for an arbitrary positive ε. Finally let

$$\liminf_{R\to+\infty}\frac{1}{R^{n+1}}\int_{-\pi/2}^{+\pi/2}|u(Re^{i\phi})|\cos\phi\, d\phi = 0. \tag{3.11}$$

Then for $z=re^{i\theta}$, $|\theta|<\pi/2$,

$$\begin{aligned}u(z) = \frac{1}{\pi}\int_{\varepsilon<|t|<+\infty}\left(\frac{r}{t}\right)^n \frac{t\sin n(\pi/2-\theta)-r\sin(n-1)(\pi/2-\theta)}{r^2-2tr\sin\theta+t^2}\, d\Psi(t) \\ -\int_{\varepsilon<|\zeta|<+\infty}\left\{\log\left|\frac{z+\bar\zeta}{z-\zeta}\right|+2\sum_{k=1}^{n-1}\frac{K(\zeta,k)}{k|\zeta|^{2k}}r^k\sin k(\pi/2-\theta)\right\}d\mu^*(e_\zeta) \\ +2\mu^* r^n\sin n(\pi/2-\theta)+\Re Q(n-1, z, \varepsilon)\end{aligned} \tag{3.12}$$

where μ^*, $Q(n-1, z, \varepsilon)$ and $K(\zeta, n)$ are defined in Theorem 14.

The case where $n=1$ contains the results by Boas and R. Nevanlinna (see [3]).

Proof. By using Taylor's expansion, we write

$$\nu(z, R) = \int_{\varepsilon<|\zeta|<R} \Re\left\{\log \frac{R^2 - z\bar{\zeta}}{R^2 + z\zeta} + \frac{2}{i} \sum_{k=1}^{n} \frac{1}{k} \left(\frac{z}{i}\right)^k \frac{1}{R^{2k}} K(\zeta, k)\right\} d\mu^*(e_\zeta)$$

$$(3.13) \qquad = -\sum_{k=n+1}^{+\infty} \frac{2}{k} \int_{\varepsilon<|\zeta|<R} \frac{t^k r^k}{R^{2k}} \sin k(\pi/2 - \theta) \sin k(\pi/2 - \phi)\, d\mu^*(e_\zeta),$$

$$\zeta = te^{i\phi}.$$

Then by using (3.7) and the inequality $(1/K) \sin K(\pi/2 - \phi) \leqq \pi/2 - \phi \leqq \tan(\pi/2 - \phi)$, $0 < \phi < \pi/2$, we get

$$(3.14) \qquad \lim_{R\to+\infty} \nu(z, R) = 0.$$

Next, by conditions (3.8) and (3.9), condition (2.25) holds obviously. An application of (XI) in Theorem 10 shows that $\int_\rho^{+\infty} (1/t^{2n+1})A^*(t)\, dt$ converges. Therefore $\int_\rho^{+\infty} (1/t^{2n+1})A_1^*(t)\, dt$ and $\int_\rho^{+\infty} (1/t^{2n+1})A_2^*(t)\, dt$ converge, where $A_1^*(t)$ and $A_2^*(t)$ are defined in Theorem 11. As $\int_\rho^{+\infty} (1/t^{2n+1})A_2^*(t)\, dt$ converges,

$$(3.15) \qquad \left| \int_{|t|<r_1} t^{n-1}(1/r_1^{2n} - 1/r_2^{2n})\, d\Psi(t) + \int_{r_1<|t|<r_2} (1/t^{n+1} - t^{n-1}/r_2^{2n})\, d\Psi(t) \right| < \varepsilon$$

for all $r_2 > r_1 > N$ if N is selected sufficiently large for an arbitrary positive ε. Hence, using (3.8), (3.15) shows

$$(3.16) \qquad \lim_{r_1\to+\infty} \int_{|t|<r_1} t^{n-1} \frac{1}{r_1^{2n}}\, d\Psi(t) = 0.$$

Thus by (3.15) and (3.16), we get the existence of the finite limit, i.e.,

$$(3.17) \qquad \lim_{r\to+\infty} \int_{|t|<r} \frac{1}{t^{n+1}}\, d\Psi(t) \text{ exists} \qquad \text{(finite limit).}$$

Using the same method as the above, we have

$$(3.18) \qquad \lim_{r\to+\infty} \int_{|\zeta|<r} r^{-2n} K(\zeta, n)\, d\mu^*(e_\zeta) = 0.$$

On the other hand, by conditions (3.7), (3.10), and (3.11), we find also for $\sigma = \pm 1$

$$(3.19) \qquad \lim_{r\to+\infty} \int_1^r \sigma^{n+1} \frac{1}{t^{n+2}}\, d\Psi(\sigma t) \text{ exist} \qquad \text{(finite limit).}$$

Now, we find

$$\int_{r_1<|t|<r_2} \Re\left\{\left(\frac{z}{it}\right)^n \frac{1}{z - it}\right\} d\Psi(t)$$

$$= \int_{r_1<|t|<r_2} \Re\left\{\left(\frac{z}{it}\right)^{n+1} \frac{1}{z - it} - \left(\frac{z}{it}\right)^{n+1} \frac{1}{z}\right\} d\Psi(t)$$

$$= \int_{r_1<|t|<r_2} \left\{\left(\frac{r}{t}\right)^{n+1} \frac{t \sin(n+1)(\pi/2 - \theta) - r \sin n(\pi/2 - \theta)}{r^2 - 2tr \sin\theta + t^2} + \frac{r^n}{t^{n+1}} \sin n\left(\frac{\pi}{2} - \theta\right)\right\} d\Psi(t).$$

Therefore, applying the second mean value theorem (with slight modification) for the first integral of the last side, the absolute value of this integral is less than ε for all $r_2 > r_1 > N$ if N is selected sufficiently large for an arbitrary positive ε. Hence by (3.17),

$$\int_{\varepsilon<|t|<+\infty} \Re\left\{\left(\frac{z}{it}\right)^n \frac{1}{z-it}\right\} d\Psi(t) \text{ converges.} \tag{3.20}$$

Next, we find also for $M < R$,

$$\begin{aligned}
&\int_{M<|t|<R} \Re\left\{\left(\frac{tz}{iR^2}\right)^{n-1} \frac{z}{R^2+itz}\right\} d\Psi(t) \\
&= \int_{M<|t|<R} \Re\left\{\left(\frac{tz}{iR^2}\right)^n \frac{z}{R^2+itz} - \left(\frac{tz}{irR^2}\right)^n \frac{1}{it}\right\} d\Psi(t) \\
&= \int_{M<|t|<R} \left\{r\left(\frac{tr}{R^2}\right)^n \frac{R^2 \sin (n+1)(\pi/2-\theta) - tr \sin n(\pi/2-\theta)}{R^4-2R^2tr \sin \theta + t^2r^2}\right. \\
&\qquad\qquad \left. + \frac{t^{n-1}r^n}{R^{2n}} \sin n\left(\frac{\pi}{2}-\theta\right)\right\} d\Psi(t).
\end{aligned}$$

Accordingly, by using the same method as the above, from (3.17) and (3.19) we get

$$\lim_{R\to+\infty} \int_{|t|<R} \Re\left\{\left(\frac{tz}{iR^2}\right)^{n-1} \frac{z}{R^2+itz}\right\} d\Psi(t) = 0. \tag{3.21}$$

Thus applying Theorem 13, (3.12) holds.

THEOREM 16. *Let $u(z)$ satisfy the hypotheses of Theorem* 12. *Suppose that conditions* (3.5), (3.7), (3.9), *and* (3.11) *are satisfied. Then* (3.12) *holds.*

Proof. As $u(z)$ satisfies the PL boundary condition and n is odd, (3.8) is obtained. If we use the same method as in the proof of Theorem 15, we find (3.15). Therefore, we get for $\sigma = \pm 1$,

$$\lim_{r\to+\infty} \int_{\varepsilon}^{r} t^{-n-1}\, d\Psi(\sigma t) \text{ exist} \qquad \text{(finite limit);} \tag{3.22}$$

and (3.22) implies (3.8). Hence, applying Theorem 15, we get (3.12).

The following theorem is easily obtained.

THEOREM 17. *Let $u(z)$ be a function in the half-plane $\Re z > 0$ represented as the difference of two harmonic functions having positive harmonic majorants in an arbitrary bounded subdomain of $\Re z > 0$. Suppose that conditions* (3.5), (3.8), (3.10), *and* (3.11) *are satisfied. Then for $\Psi(t)$ defined in Theorem* 5, $|\theta| < \pi/2$,

$$\begin{aligned}
u(z) = \frac{1}{\pi} \int_{\varepsilon<|t|<+\infty} \left(\frac{r}{t}\right)^n \frac{t \sin n(\pi/2-\theta) - r \sin (n-1)(\pi/2-\theta)}{r^2-2tr \sin \theta + t^2} d\Psi(t) \\
+ 2\mu^* r^n \sin n(\pi/2-\theta) + O(|z|^{n-1}),
\end{aligned}$$

where $\mu^* = \lim_{R\to+\infty} (1/R^n) m(R, n)$ for $m(R, n)$ defined in (2.22).

4. **Regularities.** First we state the following main theorem in this section.

THEOREM 18. *Let $u(z)$ satisfy the hypotheses of Theorem 5. Suppose conditions* (3.5) *and* (3.11) *are satisfied and that for $\sigma = \pm 1$,*

$$\int_{r_1}^{r_2} \sigma^n t^{-n-1}\, d\Psi(\sigma t) < \varepsilon \tag{4.1}$$

for all $r_2 > r_1 > N$ if N is selected sufficiently large for an arbitrary positive ε. Finally let

$$\int_{|\zeta| < +\infty} |\zeta|^{-n-1}\, \Re\zeta |d\mu^*(e_\zeta)| < +\infty. \tag{4.2}$$

Then

$$\lim_{r\to+\infty} \frac{u(re^{i\theta})}{r^n} = 2\mu^* \sin n(\pi/2-\theta), \qquad |\mu^*| < +\infty, \tag{4.3}$$

holds uniformly in any angle $|\theta| < \delta < \pi/2$ except for an open set of r of finite logarithmic length; for $|\theta| < \pi/2$ with the exception at most of a set of outer capacity zero. And

$$\lim_{r\to+\infty} \frac{u(re^{i\theta})}{r^{n+1}} = 0 \tag{4.4}$$

holds uniformly in any angle $|\theta| < \delta < \pi/2$ except for an open set of r of finite linear length.

The conclusion (4.3) in the special case where $n=1$ contains the results by Ahlfors and Heins [2], Boas [3] and Itô [7].

Proof. Now we write

$$\begin{aligned} U(z) &= \int_{\varepsilon < |\zeta| < +\infty} \Re\left\{\log\frac{z+\zeta}{z-\zeta} - \frac{2}{i}\sum_{k=1}^{n-1}\left(\frac{z}{i|\zeta|^2}\right)^k \frac{K(\zeta,k)}{k}\right\} d\mu^*(e_\zeta) \\ &= \int_{\varepsilon<|\zeta|<N} + U_1(z) + U_2(z) + U_3(z) \end{aligned} \tag{4.5}$$

where s denotes a positive integer, $U_1(z)$ denotes $N < |\zeta| < 2^{s-2}$, $U_2(z)$ denotes $|\zeta| > 2^{s+1}$ and $U_3(z)$ denotes $2^{s-2} \leqq |\zeta| \leqq 2^{s+1}$.

First we estimate $U_1(z)$ and $U_2(z)$. By using Taylor's expansion, we find for $2^{s-1} < |z| < 2^s$, $z = re^{i\theta}$ and $\zeta = te^{i\phi}$,

$$\begin{aligned} U_1(z) = -2\int_{N<|\zeta|<2^{s-2}} &\left\{\sum_{k=1}^{+\infty} \frac{1}{kr^k} K(\zeta,k)\sin k(\pi/2-\theta)\right. \\ &\left. - \sum_{k=1}^{n-1} \frac{r^k}{kt^{2k}} K(\zeta,k)\sin k(\pi/2-\theta)\right\} d\mu^*(e_\zeta), \end{aligned} \tag{4.6}$$

$$U_2(z) = -2\int_{|\zeta|>2^{s+1}} \sum_{k=n}^{+\infty} \frac{r^k}{kt^{2k}} K(\zeta,k)\sin k(\pi/2-\theta)\, d\mu^*(e_\zeta).$$

Since $(1/k)|K(\zeta, k)| \leqq 2\sqrt{2}t^k/(\sin 2\delta) \sin(\pi/2-\phi)$, $0<\delta<\pi/2$, by the inequalities $\sin k\phi < k\phi < k \tan \phi$, $0<\phi<\pi/2$, there exists a positive constant M such that

$$(4.7)\qquad \begin{aligned} \left|\frac{U_1(z)}{r^n}\right| &< M \int_{N<|\zeta|<2^s-2} \Re\zeta \frac{1}{|\zeta|^{n+1}} |d\mu^*(e_\zeta)|, \\ \left|\frac{U_2(z)}{r^n}\right| &< M \int_{|\zeta|>2^{s+1}} \Re\zeta \frac{1}{|\zeta|^{n+1}} |d\mu^*(e_\zeta)|. \end{aligned}$$

Next we estimate $U_3(z)$. For this we write for a small positive ε,

$$(4.8)\qquad \begin{aligned} &\int_{2^{s-1}}^{2^s} \frac{1}{r^{n+1}} \left\{ \log\left|\frac{z+\zeta}{z-\zeta}\right| + 2\sum_{k=1}^{n-1} \frac{1}{k}\left(\frac{r}{t^2}\right)^k K(\zeta, k) \sin k(\pi/2-\theta)\right\} dr \\ &= \int_{2^{s-1}}^{t-\varepsilon} + \int_{t-\varepsilon}^{t+\varepsilon} + \int_{t+\varepsilon}^{2^s}. \end{aligned}$$

By term-by-term integration of the series we get

$$\begin{aligned} \left|\int_{2^{s-1}}^{t-\varepsilon}\right| &= \left|\int_{2^{s-1}}^{t-\varepsilon} -\frac{2}{r^{n+1}} \sum_{k=n}^{+\infty} \frac{1}{k}\left(\frac{r}{t^2}\right)^k K(\zeta, k) \sin k(\pi/2-\theta)\, dr\right| \\ &\leqq \left[\frac{2\log r}{nt^n} + \frac{2}{t^n}\sum_{k=1}^{+\infty} \frac{1}{(n+k)k}\left(\frac{r}{t}\right)^k\right]_{2^{s-1}}^{t-\varepsilon}. \end{aligned}$$

Since $\sum (1/(n+k)k)$ converges there exists a positive constant M such that

$$(4.9)\qquad \lim_{\varepsilon\to 0}\left|\int_{2^{s-1}}^{t-\varepsilon}\right| < M\frac{1}{t^n}.$$

For $\int_{t+\varepsilon}^{2^s}$ we obtain similarly

$$\begin{aligned} \left|\int_{t+\varepsilon}^{2^s}\right| &= \left|\int_{t+\varepsilon}^{2^s} -\left\{\frac{2}{r^{n+1}}\sum_{k=1}^{+\infty}\frac{1}{kr^k} K(\zeta, k)\sin k(\pi/2-\theta)\right.\right. \\ &\qquad \left.\left. -2\sum_{k=1}^{n-1} \frac{r^{k-n-1}}{kt^{2k}} K(\zeta, k)\sin k(\pi/2-\theta)\right\} dr\right| \\ &\leqq \left[2\sum_{k=1}^{+\infty} \frac{1}{(n+k)k}\frac{t^k}{r^{n+k}} + 2\sum_{k=1}^{n-1}\frac{1}{k(n-k)}\left(\frac{r}{t}\right)^k \frac{1}{r^n}\right]_{t+\varepsilon}^{2^s}. \end{aligned}$$

Accordingly there exists a positive constant M such that

$$(4.10)\qquad \lim_{\varepsilon\to 0}\left|\int_{t+\varepsilon}^{2^s}\right| < M\frac{1}{t^n}.$$

Thus for $|\phi|<(\pi/2)-\delta$, $0<\delta<(\pi/2)$,

$$(4.11)\qquad \int_{2^{s-1}}^{2^s} \frac{1}{r^{n+1}} |U_3(z)|\, dr \leqq \frac{2M}{\sin\delta} \int_{2^{s-2}<|\zeta|<2^{s+1};\,|\phi|<\pi/2-\delta} \frac{1}{t^{n+1}} \Re\zeta |d\mu^*(e_\zeta)|.$$

If $E(s, \varepsilon)$ denotes a set of r in the interval $[2^{s-1}, 2^s]$ such that $|U_3(z)|/r^n > \varepsilon$ for an arbitrary positive ε, and if we set $\bigcup_{s>N} E(s, \varepsilon) = E(N, \varepsilon)$, then

$$\int_{E(N,\varepsilon)} d \log r \leqq \frac{2M}{\varepsilon \sin \delta} \int_{|\zeta| > 2^N;\, |\phi| < \pi/2-\delta} \frac{1}{t^{n+1}} \Re\zeta |d\mu^*(e_\zeta)|. \tag{4.12}$$

If we suitably select a sequence $\{N_n\}$ such that $N_n \to +\infty$ as $n \to +\infty$ for a sequence $\{\varepsilon_n\}$ such that $\varepsilon_n \to 0$ as $n \to +\infty$, the inequality

$$\int_{\bigcup_n E(N_n, \varepsilon_n)} d \log r < +\infty \tag{4.13}$$

follows from (4.2).

Next, for $|\phi| \geqq \pi/2 - \delta$, if $|\theta| < \pi/2 - 2\delta$, $\pi/2 > 2\delta > 0$, we easily see that

$$\lim_{r \to +\infty} \frac{1}{r^n} U_3(z) = 0.$$

Thus $\lim_{r\to+\infty} (1/r^n) U(z) = 0$ with the same exceptional condition as the forepart of (4.3).

If $E_1(s, \varepsilon)$ denotes a set of r such that $U_3(z)/r^{n+1} > \varepsilon$ for an arbitrary positive ε, and $\bigcup_{s>N} E_1(s, \varepsilon) = E_1(N, \varepsilon)$, then

$$\int_{E_1(N,\varepsilon)} dr \leqq \frac{2M_1}{\varepsilon \sin \delta} \int_{|\zeta| > 2^N;\, |\phi| < \pi/2-\delta} \frac{1}{t^{n+1}} \Re\zeta |d\mu^*(e_\zeta)|.$$

Thus, for two sequences $\{N_n\}$ and $\{\varepsilon_n\}$ in (4.13) we get

$$\int_{\bigcup_n E_1(N_n, \varepsilon_n)} dr < +\infty. \tag{4.14}$$

Therefore (4.4) holds for $U(z)$ under the same exceptional condition as that of (4.4).

Next, if we write

$$g(z, \zeta) = \log \left| \frac{z+\bar{\zeta}}{z-\zeta} \right|, \tag{4.15}$$

then $g(z, \zeta) \leqq g(e^{i\theta}, e^{i\phi})$ and for $0 < |\theta| < \delta < \pi/2$,

$$g(z, \zeta) \leqq \frac{8}{\cos^2 \delta} \frac{r \Re\zeta}{t^2} \{g(e^{i\theta}, e^{i\phi}) + 1\} \quad \text{(see [2]).}$$

Hence we have for $2^{s-1} < |z| < 2^s$

$$\begin{aligned} |U_3(z)| \leqq & \int_{2^{s-2} < |\zeta| < 2^{s+1}} \left\{ g(z, \zeta) + 2 \sum_{k=1}^{n-1} \frac{r^n}{kt^n} \right. \\ & \qquad \left. \times |\sin k(\pi/2 - \theta) \sin k(\pi/2 - \phi)| \right\} |d\mu^*(e_\zeta)| \\ < & \frac{2 \cdot 4^n r^n}{\cos^2 \delta} \int_{2^{s-2} < |\zeta| < 2^{s+1}} \frac{1}{t^n} \cos \phi \left\{ g(e^{i\theta}, e^{i\phi}) + \frac{\pi(4^{n-1} - 1)}{12\sqrt{2}} \cot \delta + 1 \right\} |d\mu^*(e_\zeta)|. \end{aligned} \tag{4.16}$$

Thus, if we write $K_1(\delta)=\max(M, 2\cdot 4^n/\cos^2\delta)$ and $K_2=\pi(4^{n-1}-1)/12\sqrt{2}\cot\delta+1$ from (4.7) and (4.16), for $0<|\theta|<\delta<\pi/2$,

$$(4.17)\quad |U(z)| \leqq \int_{\varepsilon<|\zeta|<N} + K_1(\delta)r^n \int_{N<|\zeta|<+\infty} \frac{1}{t^n}\cos\phi\{g(e^{i\theta}, e^{i\phi})+K_2\}|d\mu^*(e_\zeta)|.$$

Hence, by condition (4.2), we find

$$(4.18)\qquad \lim_{r\to+\infty}\frac{U(re^{i\theta})}{r^n} = 0$$

under the same exceptional condition as the last part of (4.3) (see [2], [12]).

By using Boas' method [3, p. 422] and applying the second mean value theorem (with slight modification), we have

$$(4.19)\qquad \lim_{r\to+\infty}\int_{\varepsilon<|t|<+\infty}\frac{t\sin n(\pi/2-\theta)-r\sin(n-1)(\pi/2-\theta)}{t^n(r^2-2tr\sin\theta+t^2)}\,d\Psi(t) = 0$$

for $|\theta|<\delta<\pi/2$. Thus applying Theorem 15, the proof is complete.

THEOREM 19. *Let $u(z)$ satisfy the hypotheses of Theorem* 5. *Suppose that* (3.5), (3.8), (3.11) *and* (4.2) *are satisfied. Finally let*

$$(4.20)\qquad \limsup_{t\to+\infty}\frac{1}{t^{n+1}}\int_1^t \sigma\, d\Psi(\sigma\tau) < +\infty$$

and

$$(4.21)\qquad \limsup_{t\to+\infty}\frac{1}{t}\int_t^{2t}\frac{\sigma^{n-1}}{\tau^n}\,d\Psi(\sigma\tau) \leqq 0, \qquad \sigma = \pm 1.$$

Then the conclusion of Theorem 18 *holds.*

The special case where $n=1$ in Theorem 19 contains the results by Ahlfors and Heins [2] and Boas [3].

Proof. We easily see (3.17). Hence by using the method of Boas [3, p. 443] and the proof of Theorem 18, the conclusion of this theorem is obtained.

Suppose that $u(z)$ is subharmonic in $\Re z>0$. Then, if we write

$$M(r,\delta) = \sup_{|z|=r,\,|\theta|<\delta} u(z)$$

for $z=re^{i\theta}$ and $0<\delta<\pi/2$, we have

THEOREM 20. *If $u(z)$ is subharmonic in $\Re z>0$ in Theorem* 18 *or* 19, *then* $\lim_{r\to+\infty} M(r,\delta)/r^n$ *exists* (*finite limit*) *and is not negative.*

The special case where $n=1$ includes the results by Heins [4] and Boas [3] (see [3]).

Proof. Set for a finite constant μ,

$$(4.22)\quad \overline{V}(z) = -\int_{\varepsilon<|\zeta|<+\infty}\Re\left\{\log\frac{z+\zeta}{z-\zeta}-\frac{2}{i}\sum_{k=1}^{n-1}\frac{1}{k}\left(\frac{z}{it^2}\right)^k K(\zeta,k)\right\}d\mu(e_\zeta)-2\mu\Re\frac{z^n}{i^{n+1}}.$$

$\bar{V}(z)$ is subharmonic in $\Re z>0$ and identically zero on the imaginary axis. By the inequalities of (4.13) there exists a sequence $\{r_n\}$ of r outside an open set of $\bigcup_n E(N_n, \varepsilon_n)$ such that $\lim_{n\to+\infty} (r_{n+1}/r_n)=1$. And using the maximum principle we find $M_1(r_n)\leqq M_1(r)\leqq M_1(r_{n+1})$ if $r_n\leqq r\leqq r_{n+1}$, where $M_1(r)=\sup_{|z|=r,\Re z>0} \bar{V}(z)$.

On the other hand, if we write

$$\bar{V}_s(z) = \int_{2^{s-2}<|\zeta|<2^{s+1}} 2i \sum_{k=1}^{n-1} \frac{1}{k} \left(\frac{z}{it^2}\right)^k K(\zeta, k)\, d\mu(e_\zeta),$$

and if $2^{s-1}<|z|<2^s$, then for $0<\delta<\pi/2$,

$$|\bar{V}_s(z)| \leqq \frac{4^n(4^{n-1}-1)\Pi}{3\sqrt{2}\sin 2\delta} \int_{2^{s-2}<|\zeta|<2^{s+1}} \frac{r^n}{t^n} \cos\phi\, d\mu(e_\zeta).$$

Thus, by (4.2) $\lim_{r\to+\infty} \bar{V}_s(z)/r^n=0$ for $0<\delta<|\theta|<\pi/2$. Accordingly, as

$$-\int_{2^{s-2}<|\zeta|<2^{s+1}} \log\left|\frac{z+\zeta}{z-\zeta}\right| d\mu(e_\zeta) \leqq 0,$$

by using the estimation of $U(z)$ in Theorem 18 we get

$$\lim_{r\to+\infty} \frac{M_1(r)}{r^n} \text{ exists} \qquad \text{(a nonnegative and finite limit).}$$

Hence by (4.19) the proof is complete.

THEOREM 21. *Let $u(z)$ satisfy the hypotheses of Theorem 5. Suppose that conditions (3.5), (3.11), and (4.2) are satisfied. Finally for $\sigma=\pm 1$, suppose*

(4.23) $$\int_{r_1}^{r_2} \frac{\sigma^n}{t^{n+1}} d\Psi(\sigma t) \text{ are bounded above for } 1 < r_1 < r_2 < +\infty.$$

Then, $u(re^{i\theta})/r^n$ is bounded as r tends to infinity under the same exceptional conditions as those of (4.3), and (4.4) holds with the same exceptional conditions as that of (4.4).

Proof. By conditions (4.2) and (4.23) we find that $\int_{r_1}^{r_2} t^{-2n-1}A^*(t)\,dt$ is bounded above for $1<r_1<r_2<+\infty$. Hence, by applying (XII) in Theorem 10, for $\sigma=\pm 1$,

(4.24) $$\int_{r_1}^{r_2} \sigma^n \frac{1}{t^{n+1}} d\Psi(\sigma t) \text{ are bounded for } 1 < r_1 < r_2 < +\infty$$

from (3.5). Thus we easily see the conclusion of this theorem.

The following theorem is easily obtained.

THEOREM 22. *Suppose that $u(z)$ satisfies the hypotheses of Theorem 5. Let conditions (3.5), (3.11), (4.2), and (4.20) be satisfied, and let*

(4.25) $$\int_{r_1<|t|<r_2} \frac{1}{t^{n+1}} d\Psi(t) \text{ be bounded above for } 1 < r_1 < r_2 < +\infty.$$

Then the conclusion of Theorem 21 holds.

The case where $n=1$ in Theorems 21 and 22 contains results by Boas [3] respectively.

THEOREM 23. *Let* $u(z)$ *satisfy the hypotheses of Theorem* 5. *Suppose that conditions* (3.5), (3.11), *and* (4.1) *are satisfied, and that*

$$\limsup_{r\to+\infty}\int_{|\zeta|<r}\frac{\Re\zeta}{|\zeta|^n}\,|d\mu^*(e_\zeta)| < +\infty \tag{4.26}$$

is satisfied. Then (4.3) *holds with the same exceptional conditions as that of* (4.4).

We may state the following theorems from the proof of Theorem 18.

THEOREM 24. *Let* $u(z)$ *satisfy the hypotheses of Theorem* 5. *Suppose that conditions* (3.5), (3.8), (3.11), (4.20), (4.21), *and* (4.26) *are satisfied. Then the conclusion of Theorem* 23 *holds.*

THEOREM 25. *Let* $u(z)$ *satisfy the hypotheses of Theorem* 5. *Suppose that conditions* (3.5), (3.11), (4.23), *and* (4.26) *are satisfied. Then* $u(z)/r^n$ *is bounded as* r *tends to infinity under the same exceptional conditions as that of* (4.4).

THEOREM 26. *Let* $u(z)$ *satisfy the hypotheses of Theorem* 5. *Suppose that conditions* (3.5), (3.11), (4.20), (4.25), *and* (4.26) *are satisfied. Then the conclusion of Theorem* 25 *holds.*

REFERENCES

1. L. Ahlfors, *On Phragmén-Lindelöf's principle*, Trans. Amer. Math. Soc. **41** (1937), 1–8.

2. L. Ahlfors and M. Heins, *Questions of regularity connected with the Phragmén-Lindelöf principle*, Ann. of Math. (2) **50** (1949), 341–346.

3. R. P. Boas, Jr., *Asymptotic properties of functions of exponential type*, Duke Math. J. **20** (1953), 433–448.

4. M. Heins, *On the Phragmén-Lindelöf principle*, Trans. Amer. Math. Soc. **60** (1946), 238–244.

5. ———, *On some theorems associated with the Phragmén-Lindelöf principle*, Ann. Acad. Fenn. Ser. A I Math.-Phys. (1948), no. 46, 10 pp.

6. J. Itô, *Asymptotic properties of some functions*, Bull. Nagoya Inst. Tech. **7** (1955), 245–252.

7. ———, *Asymptotic properties of subharmonic and analytic functions*, Proc. Amer. Math. Soc. **9** (1958), 763–772.

8. F. Nevanlinna and R. Nevanlinna, *Über die Eigenshaften analytischer Funktionen in der Umgebung einer singulären Stelle oder Linie*, Acta Soc. Sci. Fenn. **50** (1922), no. 5.

9. R. Nevanlinna, *Eindeutige analytische Funktion*, Springer, Berlin, 1936.

10. F. Riesz, *Sur les functions subharmoniques et leur rapport à la théorie du potentiel*, Acta Math. **48** (1926), 329–343; **54** (1930), 321–360.

11. E. C. Titchmarsh, *The theory of functions*, Oxford Univ., 1932.

12. M. Tsuji, *On a non-negative subharmonic function in a half-plane*, Kōdai Math. Sem. Rep. **8** (1956), 134–141.

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