ON A PIECE OF HYPERSURFACE IN A RIEMANNIAN MANIFOLD WITH MEAN CURVATURE BOUNDED AWAY FROM ZERO

BY

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Introduction. In 1965, S. S. Chern proved the following theorem:

THEOREM 0.1. *Let M be a compact piece of an oriented hypersurface of dimension m with smooth boundary ∂M, which is immersed in an euclidean space E of dimension $m+1$. Suppose the mean curvature $H_1 \geqq c > 0$. Let a be a fixed unit vector which makes an angle $\leqq \pi/2$ with the normals of M. Then*

$$mcV_a \leqq L_a,$$

where V_a is the volume of the orthogonal projection of M and L_a that of ∂M in the hyperplane perpendicular to a. If M is defined by the equation

$$z = z(x_1, \ldots, x_m), \qquad x_1^2 + \cdots + x_m^2 \leqq R^2,$$

where $x_1, \ldots, x_m$, z are rectangular coordinates in the space E and the z-axis is chosen in the direction of a, then $cR \leqq 1$ [1, p. 82][2].

This theorem is a generalization of Heinz's theorem for the C^2-surface defined by an equation $z = z(x, y)$ in euclidean three space [2].

The purpose of this paper is to extend this theorem to a hypersurface in a Riemann manifold.

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1. **A certain generalization of Chern's theorem.** We consider a Riemann manifold R^{m+1} $(m+1 \geqq 3)$ of class C^ν $(\nu \geqq 3)$ which admits a one-parameter group G of transformations generated by an infinitesimal transformation:

$$(1.1) \qquad \bar{x}^i = x^i + \xi^i(x)\,\delta\tau$$

(where x^i are local coordinates in R^{m+1} and ξ^i are the components of a contravariant vector ξ. If ξ is a Killing vector, a homothetic Killing vector, a conformal Killing vector, etc. [3, p. 32], then the group G is called isometric, homothetic, conformal, etc.

In R^{m+1}, we consider a domain D. If the domain D is simply covered by the paths of the transformations generated by ξ, and ξ is everywhere of class C^ν and $\neq 0$ in D, then we call D a regular domain with respect to the vector field ξ.

In a regular domain D, we choose a coordinate system such that the paths of the transformations generated by ξ are new x^1-coordinate curves, that is, a coordinate system in which the vector ξ has the components δ_1^i (where δ_j^i denotes the Kronecker delta). Then (1.1) becomes $\bar{x}^i = x^i + \delta_1^i \, \delta\tau$ and the domain D admits the transformation given by

$$(1.1)' \qquad\qquad \bar{x}^i = x^i + \delta_1^i \tau.$$

Now we consider a compact piece M of an oriented hypersurface of dimension m with smooth boundary ∂M, which is immersed in a regular domain D with respect to the vector field ξ. The immersion $x: M \to R^{m+1}$ is locally given by

$$(1.2) \qquad x^i = x^i(u^\alpha), \qquad i = 1, \ldots, m+1, \quad \alpha = 1, \ldots, m.$$

Here and henceforth, Latin indices run from 1 to $m+1$ and Greek indices from 1 to m. Let us consider a hypersurface which is orthogonal to the new x^1-coordinate curves and project $x(M)$ along the x^1-coordinate curve into the orthogonal hypersurface, and let M' be the image of M. If $x'(p)$ is the image of $x(p)$, $p \in M$ under this projection, we have

$$(1.3) \qquad\qquad x'^i = x^i(u^\alpha) - \delta_1^i \tau(u^\alpha),$$

where τ is a function defined on M.

Let us consider a differential form of degree $m-1$ at a point p of M, defined by

$$(1.4) \quad ((n, (\rho'/\rho)^{m-1}(\delta_1/\rho), \underbrace{dx, \ldots, dx}_{m-1})) \overset{\text{def.}}{=} g^{1/2}(n, (\rho'/\rho)^{m-1}(\delta_1/\rho), dx, \ldots, dx),$$

where the symbol $(\;\;)$ means a determinant of order $m+1$ whose columns are the components of respective vectors or vector-valued differential forms, n is a unit normal vector at a point p of M, dx is a displacement along M, g is the determinant of the metric tensor g_{ij} of R^{m+1}, and the functions ρ and ρ' are defined by

$$\rho(p) = \text{length of the vector } \delta_1 \text{ at } x(p) = (g_{11}(x(p)))^{1/2},$$

$$\rho'(p) = \text{length of the vector } \delta_1 \text{ at } x'(p) = (g_{11}(x'(p)))^{1/2}.$$

Then the exterior differential of the differential form (1.4) divided by $m!$ becomes

$$(1.5) \quad \begin{aligned} &\frac{1}{m!} d((n, (\rho'/\rho)^{m-1}\delta_1/\rho, dx, \ldots, dx)) \\[2mm] &= \frac{1}{m!} \{((Dn, (\rho'/\rho)^{m-1}\delta_1/\rho, dx, \ldots, dx)) + ((n, D(\rho'^{m-1}/\rho_m \, \delta_1), dx, \ldots, dx))\}, \end{aligned}$$

where the symbol D means the covariant differential.

The first term of the right-hand member of the above expression becomes

$$(1.6) \qquad \frac{1}{m!} \left((Dn, (\rho'^{m-1}/\rho^m)\, \delta_1, dx, \ldots, dx) \right) = (-1)^m H_1 (\rho'^{m-1}/\rho^m) n_i \delta_1^i \, dV,$$

where H_1 is the first mean curvature of M and dV is the volume element of M. On the other hand, we easily obtain the following relation:

$$(1.7) \qquad n_i \delta_1^i \, dV = \frac{(-1)^m}{m!} \underset{m}{\big((\delta_1, dx, \ldots, dx) \big)},$$

and from (1.3) we have

$$
\begin{aligned}
\frac{(-1)^m}{m!} \big((\delta_1, dx, \ldots, dx) \big) &= \frac{g^{1/2}}{g'^{1/2}} \frac{(-1)^m}{m!} g'^{1/2} (\delta_1, dx', \ldots, dx') \\
(1.8) \\
&= \frac{g^{1/2}}{g'^{1/2}} n_i' \delta_1^i \, dV_\xi',
\end{aligned}
$$

where n' and dV_ξ' mean the unit normal vector and the volume element at the corresponding point $x'(p)$ of M' respectively and $g' = g(x')$. Consequently from (1.7) and (1.8), we have

$$
\begin{aligned}
\frac{\rho'^{m-1}}{\rho^m} n_i \delta_1^i \, dV &= \frac{g^{1/2}}{(g')^{1/2}} \left(\frac{\rho'}{\rho} \right)^m \frac{\delta_1^i}{\rho'} n_i' \, dV_\xi' \\
(1.9) \\
&= \frac{g^{1/2}}{(g')^{1/2}} \left(\frac{\rho'}{\rho} \right)^m dV_\xi',
\end{aligned}
$$

because δ_1 / ρ' is nothing but the unit normal vector of M'.

Now we suppose that the transformation group G is conformal. Then R^{m+1} admits the infinitesimal transformation given by (1.1) with the additional condition

$$\xi_{i;j} + \xi_{j;i} = 2\phi(x) g_{ij},$$

where the symbol ";" means the covariant derivative. In the new coordinate system of the domain D, the additional condition becomes

$$\frac{\partial g_{ij}}{\partial x^1} = 2\phi(x) g_{ij}.$$

Therefore the metric tensor g_{ij} with respect to the new coordinate system has the form

$$(1.10) \qquad g_{ij} = \exp\left(2 \int \phi(x) \, dx^1 \right) f_{ij}(x^2, \ldots, x^{m+1}).$$

From this, we can easily deduce the following result:

$$(1.11) \qquad \frac{g^{1/2}}{(g')^{1/2}} \left(\frac{\rho'}{\rho} \right)^m = \exp\left(\int_{x'^1}^{x^1} \phi(x) \, dx^1 \right)$$

since $\rho = (g_{11})^{1/2}$ and $\rho' = (g'_{11})^{1/2}$. Therefore from (1.9) and (1.11), (1.6) becomes

(1.12) $\dfrac{1}{m!} ((Dn, (\rho'^{m-1}/\rho^m)\delta_1, dx, \ldots, dx)) = (-1)^m H_1 \exp \left(\displaystyle\int_{x'^1}^{x^1} \phi \, dx^1 \right) dV'_\xi.$

Put

(1.13) $v_i = \dfrac{\rho'^{m-1}}{\rho^m} \, \delta_1^j g_{ij}.$

Then the second term of the right-hand member of (1.5) may be written as follows:

(1.14) $\dfrac{1}{m!} ((n, D((\rho'^{m-1}/\rho^m)\delta_1), dx, \ldots, dx)) = \dfrac{(-1)^m}{2m} L_v g_{ij} g^{*ij} \, dV,$

where $L_v g_{ij}$ is the Lie derivative of the tensor g_{ij} with respect to v^i, g^{*ij} means $(\partial x^i/\partial u^\alpha)(\partial x^j/\partial u^\beta)g^{\alpha\beta}$, and $g_{\alpha\beta}$ is the metric tensor of M.

Accordingly from (1.12) and (1.14), we can rewrite (1.5) as follows:

(1.15)

$$\dfrac{(-1)^m}{m!} d((n, (\rho'/\rho)^{m-1}\delta_1/\rho, dx, \ldots, dx))$$

$$= \exp \left(\int_{x'^1}^{x^1} \phi \, dx^1 \right) H_1 \, dV'_\xi + \dfrac{1}{2m} L_v g_{ij} g^{*ij} \, dV.$$

Integrating both members of (1.15) over M and applying Stokes' theorem we have

(1.16)

$$\dfrac{(-1)^m}{m!} \int_{\partial M} ((n, (\rho'/\rho)^{m-1}\delta_1/\rho, dx, \ldots, dx))$$

$$= \int_M \exp \left(\int_{x'^1}^{x^1} \phi \, dx^1 \right) H_1 \, dV'_\xi + \dfrac{1}{2m} \int_M L_v g_{ij} g^{*ij} \, dV.$$

On the other hand, from (1.3) and (1.11) we can see that

(1.17)

$$\dfrac{(-1)^m}{m!} ((n, (\rho'/\rho)^{m-1}\delta_1/\rho, dx, \ldots, dx))$$

$$= \dfrac{(-1)^m}{m!} \dfrac{g^{1/2}}{(g')^{1/2}} \left(\dfrac{\rho'}{\rho} \right)^m (g')^{1/2}(n, \delta_1/\rho', dx', \ldots, dx')$$

$$= \dfrac{(-1)^m}{m!} \exp \left(\int_{x'^1}^{x^1} \phi \, dx^1 \right)(g')^{1/2}(n, \delta_1/\rho', dx', \ldots, dx').$$

Now for each point $p \in M$, let us take the small piece $\widetilde{M}_p$ of the hypersurface passing through the corresponding point $x'(p)$ of M', given by the following expression

$$\tilde{x}^i = x^i(u^\alpha) - \delta_1^i \tau_p, \qquad \tau_p = \tau(p) = \text{const},$$

where τ_p means $x^1(p) - x'^1(p)$. Then we have

$$\dfrac{\partial \tilde{x}^i}{\partial u^\alpha} = \dfrac{\partial x^i}{\partial u^\alpha}$$

and we can obtain the following relation:

$$n_i = \frac{|\tilde{g}_{\alpha\beta}|^{1/2}}{|g_{\alpha\beta}|^{1/2}} \frac{g^{1/2}}{(g')^{1/2}} \tilde{n}_i,$$

where $\tilde{g}_{\alpha\beta}$ is the metric tensor of $\tilde{M}_p$ and $\tilde{n}$ is the unit normal vector of $\tilde{M}_p$ at $x'(p)$, $|g_{\alpha\beta}|$ and $|\tilde{g}_{\alpha\beta}|$ mean the determinants of $g_{\alpha\beta}$ and $\tilde{g}_{\alpha\beta}$. Since

$$g_{\alpha\beta} = g_{ij}(x)\frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^j}{\partial u^\beta}, \qquad \tilde{g}_{\alpha\beta} = g_{ij}(x')\frac{\partial \tilde{x}^i}{\partial u^\alpha}\frac{\partial \tilde{x}^j}{\partial u^\beta},$$

making use of (1.10) we have

$$|g_{\alpha\beta}| = \exp\left(2m\int_{x'^1}^{x^1}\phi\,dx^1\right)|\tilde{g}_{\alpha\beta}|.$$

Therefore we have the following relation:

$$n_i = \exp\left(\int_{x'^1}^{x^1}\phi\,dx^1\right)\tilde{n}_i.$$

Since

$$g_{ij}(x) = \exp\left(2\int_{x'^1}^{x^1}\phi\,dx^1\right)g_{ij}(x'),$$

it follows that

$$g^{ij}(x) = \exp\left(-2\int_{x'^1}^{x^1}\phi\,dx^1\right)g^{ij}(x').$$

Thus we obtain

$$n^i = \exp\left(-\int_{x'^1}^{x^1}\phi\,dx^1\right)\tilde{n}^i.$$

Substituting the above result in (1.17), we get

$$(1.18) \quad \frac{(-1)^m}{m!}\left((n, (\rho^1/\rho)^{m-1}\delta_1/\rho, dx, \ldots, dx)\right) = \frac{(-1)^m}{m!}(g')^{1/2}(\tilde{n}, \delta_1/\rho', dx', \ldots, dx').$$

Let e_A, $A = 1, \ldots, m-1$, be an oriented basis for the tangent plane of $\partial M'$ at a point $x'(p)$ on the image $\partial M'$ of ∂M, and let n'' be the unit vector of the tangent plane of M' at the same point $x'(p)$ which is orthogonal to e_A, $A = 1, \ldots, m-1$, n' and such that

$$((n'', n', e_1, \ldots, e_{m-1})) > 0.$$

If dx' is a displacement along $\partial M'$, then

$$\frac{(-1)^m}{(m-1)!}\left((n'', n', dx', \ldots, dx')\right)$$

is the volume element of $\partial M'$. The unit normal vector $\tilde{n}$ of $\tilde{M}_p$ at $x'(p)$ can be expressed as follows:

$$\tilde{n} = \alpha n'' + \beta n' + \sum_{A=1}^{m-1}\gamma^A e_A,$$

where α, β, γ^A satisfy

$$\alpha^2 + \beta^2 + g_{ij}(x')e_A^i e_B^j \gamma^A \gamma^B = 1.$$

Therefore we have $|\alpha| \leqq 1$. Since δ_1/ρ' is the unit normal vector n', it follows that

$$(1.19) \quad \frac{(-1)^m}{(m-1)!}((\tilde{n}, \delta_1/\rho', dx', \ldots, dx')) = \frac{(-1)^m}{(m-1)!}\alpha((n'', n', dx', \ldots, dx')).$$

By making use of a notation dL'_ξ for the volume element of $\partial M'$, from (1.18) and (1.19), (1.17) becomes

$$(1.20) \quad \frac{(-1)^m}{m!}((n, (\rho'/\rho)^{m-1}\delta_1/\rho, dx, \ldots, dx)) = \frac{1}{m}\alpha\, dL'_\xi, \qquad |\alpha| \leqq 1.$$

Substituting (1.20) in (1.16), we have

$$\frac{1}{m}\int_{\partial M} \alpha\, dL'_\xi = \int_M \exp\left(\int_{x'^1}^{x^1} \phi\, dx^1\right) H_1\, dV'_\xi + \frac{1}{2m}\int_M L_v g_{ij} g^{*ij}\, dV.$$

Consequently it follows that

$$(1.21) \quad \frac{1}{m}\int_{\partial M} dL'_\xi \geqq \int_M \exp\left(\int_{x'^1}^{x^1} \phi\, dx^1\right) H_1\, dV'_\xi + \frac{1}{2m}\int_M L_v g_{ij} g^{*ij}\, dV.$$

From the above inequality, we can obtain the following

THEOREM 1.1 (MAIN THEOREM). *Let M be a compact piece of an oriented hypersurface of dimension m with smooth boundary ∂M, which is immersed in a regular domain of a Riemann manifold of dimension $m+1$ admitting a conformal Killing vector field ξ (i.e., $\xi_{i;j} + \xi_{j;i} = 2\phi g_{ij}$). Suppose the mean curvature $H_1 \geqq c > 0$ ($c=$const). Let us suppose that the $\xi(x)$ makes an angle $\leqq \pi/2$ with the normals of M, and let*

$$(I) \qquad\qquad \int_{x'^1}^{x^1} \phi\, dx^1 \geqq 0 \qquad \textit{everywhere on M,}$$

$$(II) \qquad\qquad \int_M L_v g_{ij} g^{*ij}\, dV \geqq 0.$$

Then

$$mcV'_\xi \leqq L'_\xi,$$

where V'_ξ and L'_ξ are the volumes of the images M' of M and $\partial M'$ of ∂M, by the projection along the paths of the vector field ξ into a hypersurface perpendicular to the paths of ξ.

2. **Some examples of the main theorem.** In this section, we shall give some examples which satisfy the hypotheses (I) and (II) in the main theorem.

The hypothesis (I) means that either ξ is a Killing vector or $\rho \geqq \rho'$ everywhere on M.

As an example in the case satisfying the hypothesis (II), we have the following

THEOREM 2.1. *Let R^{m+1} admit a conformal Killing vector field (i.e., $\xi_{i;j} + \xi_{j;i}$ $= 2\phi_{ij}$) and if the paths of the transformations generated by ξ are geodesics (i.e., the new x^1-coordinate curves are geodesics), and $\phi \geq 0$ everywhere on M, then it follows that*

$$L_v g_{ij} g^{*ij} \geq 0$$

everywhere on M.

Proof. Let us choose a coordinate system such that the family of hypersurfaces perpendicular to the new x^1-coordinate curves are new $x^2 x^3 \cdots x^{m+1}$-coordinate hypersurfaces. Then $g_{1j} = 0$ for $j = 2, \ldots, m+1$.

Since the x^1-coordinate curve is a geodesic, substituting

$$dx^1/ds = 1/(g_{11})^{1/2}, \quad dx^2/ds = \cdots = dx^{m+1}/ds = 0$$

in the differential equation of a geodesic:

$$\frac{d^2 x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0, \qquad i = 1, 2, \ldots, m+1,$$

we have

$$-\frac{1}{2} \frac{1}{g_{11}} \frac{\partial g_{11}}{\partial x^1} + \Gamma^1_{11} = 0, \quad \Gamma^j_{11} = 0, \qquad j = 2, \ldots, m+1.$$

Substituting $\partial g_{ik}/\partial x^1 = 2\phi g_{ik}$ in Γ^j_{11} $(j = 2, \ldots, m+1)$, we obtain

$$\Gamma^j_{11} = \frac{1}{2} g^{jk} \frac{\partial g_{11}}{\partial x^k}, \qquad j = 2, \ldots, m+1.$$

And also from $g^{j1} = 0$, $j = 2, \ldots, m+1$, the conditions $\Gamma^j_{11} = 0$, $j = 2, \ldots, m+1$, become

$$\sum_{k=2}^{m+1} g^{jk} \frac{\partial g_{11}}{\partial x^k} = 0, \qquad j = 2, \ldots, m+1.$$

Since the determinant of g^{jk} $(j, k = 2, \ldots, m+1)$ is not equal to zero, the conditions $\Gamma^j_{11} = 0$ $(j = 2, \ldots, m+1)$ are rewritten as follows:

$$\frac{\partial g_{11}}{\partial x^k} = 0, \qquad k = 2, \ldots, m+1.$$

And the condition

$$-\frac{1}{2} \frac{1}{g_{11}} \frac{\partial g_{11}}{\partial x^1} + \Gamma^1_{11} = 0$$

follows clearly from $g_{1j} = 0$, $j = 2, \ldots, m+1$. Therefore in this case, a necessary and sufficient condition that the x^1-coordinate curve be a geodesic is that $\partial g_{11}/\partial x^k = 0$, $k = 2, \ldots, m+1$.

To calculate $L_v g_{ij} g^{*ij}$, we go back to the definition (1.13):

$$v_i = \frac{\rho'^{\,m-1}}{\rho^m}\,\delta_1^j g_{ij},$$

where $\rho = (g_{11}(x))^{1/2}$ and $\rho' = (g_{11}(x'))^{1/2}$. ρ' is constant on the x^1-coordinate curve. Then we have

$$v_{i;j} = \frac{1}{2}\frac{\partial g_{ij}}{\partial x^1}\frac{(g_{11}')^{(m-1)/2}}{(g_{11})^{m/2}} - \frac{m}{2}\frac{(g_{11}')^{(m-1)/2}}{(g_{11})^{(m+2)/2}}\frac{\partial g_{11}}{\partial x^j}\,\delta_1^l g_{il}$$

$$+\frac{m-1}{2}\frac{(g_{11}')^{(m-3)/2}}{(g_{11})^{m/2}}\frac{\partial g_{11}'}{\partial x^j}\,\delta_1^l g_{il},$$

$$L_v g_{ij} = v_{i;j} + v_{j;i}$$

$$= 2\phi g_{ij}\frac{(g_{11}')^{(m-1)/2}}{(g_{11})^{m/2}} - \frac{m}{2}\frac{(g_{11}')^{(m-1)/2}}{(g_{11})^{(m+2)/2}}\left((\partial g_{11}/\partial x^j)g_{1i} + (\partial g_{11}/\partial x^i)g_{1j}\right)$$

$$+\frac{m-1}{2}\frac{(g_{11}')^{(m-3)/2}}{(g_{11})^{m/2}}\left((\partial g_{11}'/\partial x^j)g_{1i} + (\partial g_{11}'/\partial x^i)g_{1j}\right)$$

because $\partial g_{ij}/\partial x^1 = 2\phi g_{ij}$. From the fact that the x^1-coordinate curves are geodesics and $(g_{11}')^{1/2}$ is constant on the x^1-coordinate curve, we have the following relations:

$$\frac{\partial g_{11}}{\partial x^j} = 0, \qquad\qquad j = 2,\ldots,m+1,$$

$$\frac{\partial g_{11}'}{\partial x^1} = 0, \quad \frac{\partial g_{11}'}{\partial x^j} = \frac{\partial g_{11}'}{\partial x'^j} = 0, \quad j = 2,\ldots,m+1,$$

since $x'^j = x^j$, $j = 2,\ldots,m+1$. By making use of the above relations, we get

$$L_v g_{ij} g^{*ij} = 2m\phi\frac{(g_{11}')^{(m-1)/2}}{(g_{11})^{m/2}} - m\frac{(g_{11}')^{(m-1)/2}}{(g_{11})^{(m+2)/2}}\frac{\partial g_{11}}{\partial x^1}g_{11}\frac{\partial x^1}{\partial u^\alpha}\frac{\partial x^1}{\partial u^\beta}g^{\alpha\beta}$$

$$= 2m\phi\frac{(g_{11}')^{(m-1)/2}}{(g_{11})^{m/2}}\left(1 - g_{11}\frac{\partial x^1}{\partial u^\alpha}\frac{\partial x^1}{\partial u^\beta}g^{\alpha\beta}\right),$$

and also we can see that

$$g_{jk}\frac{\delta_1^j}{(g_{11})^{1/2}}\frac{\partial x^k}{\partial u^\alpha} = (g_{11})^{1/2}\frac{\partial x^1}{\partial u^\alpha},$$

since the unit vector $\delta_1^i/(g_{11})^{1/2}$ at a point on M is expressed as follows:

$$\frac{\delta_1^i}{(g_{11})^{1/2}} = \lambda n^i + \mu^\alpha\frac{\partial x^i}{\partial u^\alpha},$$

where

$$\lambda^2 + g_{ij}\frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^j}{\partial u^\beta}\mu^\alpha\mu^\beta = \lambda^2 + g_{\alpha\beta}\mu^\alpha\mu^\beta = 1,$$

we have

$$g_{ij} \frac{\delta_1^i}{(g_{11})^{1/2}} \frac{\partial x^j}{\partial u^\beta} = g_{ij} \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} \mu^\alpha = g_{\alpha\beta}\mu^\alpha.$$

Therefore it follows that

$$g_{11} \frac{\partial x^1}{\partial u^\alpha} \frac{\partial x^1}{\partial u^\beta} g^{\alpha\beta} = g_{\alpha\gamma}\mu^\gamma g_{\beta\theta}\mu^\theta g^{\alpha\beta} = g_{\gamma\theta}\mu^\gamma\mu^\theta \leqq 1.$$

Consequently if $\phi \geqq 0$ everywhere on M, then it follows that

$$L_v g_{ij} g^{*ij} = 2m\phi \frac{(g_{11}')^{(m-1)/2}}{(g_{11})^{m/2}} \left(1 - g_{11} \frac{\partial x^1}{\partial u^\alpha} \frac{\partial x^1}{\partial u^\beta} g^{\alpha\beta} \right) \geqq 0$$

everywhere on M.

From Theorem 1.1 and Theorem 2.1, we can see

THEOREM 2.2. *Let M be a compact piece of an oriented hypersurface of dimension m with smooth boundary ∂M, which is immersed in a regular domain of a Riemann manifold of dimension $m+1$ admitting a conformal Killing vector field ξ with $\xi_{i;j} + \xi_{j;i} = 2\phi g_{ij}$, $\phi \geqq 0$. Suppose the mean curvature $H_1 \geqq c > 0$ ($c = $const) and suppose that $\xi(x)$ makes an angle $\leqq \pi/2$ with the normal at each point of M. If the paths of the transformations generated by ξ are geodesics, then*

$$mcV_\xi' \leqq L_\xi',$$

where V_ξ' and L_ξ' are the volumes of the image M' of M and the image $\partial M'$ of ∂M, by the projection along the paths into a hypersurface perpendicular to the paths, which never intersects M.

COROLLARY 2.1. *Let M be a compact piece of an oriented hypersurface of dimension m with smooth boundary ∂M, which is immersed in a regular domain of a Riemann manifold of dimension $m+1$ admitting a homothetic Killing vector field $\xi(x)$ with $\xi_{i;j} + \xi_{j;i} = 2c_0 g_{ij}$, $c_0 > 0$, or a Killing vector field. Suppose the mean curvature $H_1 \geqq c > 0$ ($c = $const) and that $\xi(x)$ makes an angle $\leqq \pi/2$ with the normal at each point of M. If the paths of the transformations generated by ξ are geodesics, then*

$$mcV_\xi' \leqq L_\xi',$$

where V_ξ' and L_ξ' are the volumes of the image M' of M and the image $\partial M'$ of ∂M, by the projection along the paths into a hypersurface perpendicular to the paths, which never intersects M.

Especially if our manifold R^{m+1} is an euclidean space E^{m+1} and if ξ is the homothetic Killing vector field on E^{m+1} with components $\xi^i = x^i$, x^i being rectangular coordinates in the space E^{m+1}, then the paths of the transformations generated by

ξ are the lines through O and a hypersurface perpendicular to these paths is a hypersphere of radius, say R_0, around O. The functions ρ and ρ' are given by

$$\rho = \left(\sum_{i=1}^{m+1} (x^i)^2\right)^{1/2} \quad \text{and} \quad \rho' = R_0.$$

Thus, as a special case of Corollary 2.1, we have the following

COROLLARY 2.2. *Let M be a compact piece of an oriented hypersurface of dimension m with smooth boundary ∂M, which is immersed in an euclidean space of dimension $m+1$. Suppose that the position vector $x(u)$ makes an angle $\leq \pi/2$ with the normal of $x(M)$ at $x(u)$ for every $u \in M$ and that the mean curvature H_1 satisfies $H_1 \geq c > 0$ ($c = $ const). Then*

$$mcV'_\xi \leq L'_\xi$$

where V'_ξ and L'_ξ are the volumes of the image M' of M and the image $\partial M'$ of ∂M, by the projection from the origin O into a hypersphere around O not intersecting M.

From this result we have (under the same assumptions as Corollary 2.2)

COROLLARY 2.3. *Let $\rho, \theta_1, \ldots, \theta_m$ be polar coordinates in E^{m+1}, i.e.,*

$$x^1 = \rho \sin \theta_m \sin \theta_{m-1} \cdots \sin \theta_2 \sin \theta_1,$$
$$x^2 = \rho \sin \theta_m \sin \theta_{m-1} \cdots \sin \theta_2 \cos \theta_1,$$
$$x^3 = \rho \sin \theta_m \sin \theta_{m-1} \cdots \sin \theta_3 \cos \theta_2,$$
$$\vdots$$
$$x^m = \rho \sin \theta_m \cos \theta_{m-1},$$
$$x^{m+1} = \rho \cos \theta_m.$$

If M is defined by the equation

$$\rho = \rho(\theta_1, \ldots, \theta_m), \quad 0 \leq \theta_m \leq \phi_0 \,(= \text{const}),$$

then

$$cC'_0 \leq 1,$$

where C'_0 is a constant which depends only on R_0 and ϕ_0 (R_0 is the radius of the hypersphere into which M is projected).

Proof. For simplicity, we shall prove in case $m = 2$. Since M is written as follows:

$$x^1 = \rho(\theta_1, \theta_2) \sin \theta_2 \sin \theta_1,$$
$$x^2 = \rho(\theta_1, \theta_2) \sin \theta_2 \cos \theta_1, \quad 0 \leq \theta_2 \leq \phi_0,$$
$$x^3 = \rho(\theta_1, \theta_2) \cos \theta_2,$$

in this case, M' is defined by

$$x^1 = R_0 \sin \theta_2 \sin \theta_1,$$
$$x^2 = R_0 \sin \theta_2 \cos \theta_1, \quad 0 \leq \theta_2 \leq \phi_0,$$
$$x^3 = R_0 \cos \theta_2,$$

and $\partial M'$ is the circle on the sphere S_0 with the center O and radius R_0, given by

$$x^1 = R_0 \sin \phi_0 \sin \theta_1,$$
$$x^2 = R_0 \sin \phi_0 \cos \theta_1,$$
$$x^3 = R_0 \cos \phi_0.$$

Let g^* be the determinant of the metric tensor $g^*_{\alpha\beta}$ on S_0 with respect to the coordinates θ_1, θ_2. Then $g^* = R_0^4 \sin^2 \theta_2$. Therefore the area of M' is given by

$$V'_\xi = \int_0^{2\pi} \int_0^{\phi_0} R_0^2 \sin \theta_2 \, d\theta_2 \, d\theta_1 = 2R_0^2 \pi (1 - \cos \phi_0),$$

and the length of $\partial M'$, which is a circle of radius $R_0 \sin \phi_0$, is given by

$$L'_\xi = 2\pi R_0 \sin \phi_0.$$

From the results above, we have the following relation:

$$2V'_\xi = 2R_0 \frac{1 - \cos \phi_0}{\sin \phi_0} L'_\xi.$$

Putting

$$C'_0 = 2R_0 \frac{1 - \cos \phi_0}{\sin \phi_0},$$

and making use of Corollary 2.2, we obtain

$$cC'_0 L'_\xi \leqq L'_\xi.$$

Thus $cC'_0 \leqq 1$.

Finally we consider a Killing vector field ξ on E^{m+1} which generates translations so that the paths of the transformations generated by ξ are parallel lines in the direction of ξ and a hypersurface perpendicular to these paths is a hyperplane perpendicular to ξ. In this case, Corollary 2.1 is nothing but Theorem 0.1. Thus Chern's theorem is included as a special case in Corollary 2.1.

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