FUNCTION ALGEBRAS, MEANS, AND FIXED POINTS

BY

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1. Introduction. A semigroup S is said to have the *common fixed point property on compacta* if for each compact Hausdorff space Y , and for each homomorphic representation $\mathcal{S}$ of S as a semigroup of continuous self-maps of Y , there is in Y a common fixed point of $\mathcal{S}$. In [11, Theorem 1], it is shown that S has the common fixed point property on compacta if and only if $m(S)$ has a multiplicative left invariant mean. A natural question is: does this result generalize to the case of a topological semigroup S ?

It is shown in Corollary 1 that if S is a topological semigroup such that $C(S)$ has a multiplicative left invariant mean, then S has the common fixed property on compacta with respect to continuous representations of S . But the proof of the converse encounters difficulties of the kind observed by M. Day in [3], namely that the family of adjoints of left translations on $C(S)$ fails to be a w^* -continuous representation on the desired subsets of $m(S)^*$. To remedy this, we employ E -representations of S , $C(S)$ on compact Hausdorff spaces; an analogue of the slightly continuous representations of S that were introduced in [3].

In the main theorems of this paper, Theorems 1 and 2, it is shown that $C(S)$ has a multiplicative left invariant mean if and only if the pair $S, C(S)$ has the common fixed point property on compacta with respect to E -representations. In these theorems, additional implications are given concerning two other types of representations. In the proofs of Theorems 1 and 2, no use is made of the fact that the algebra $C(S)$ arises from a topology on S , so these results are stated in terms of closed subalgebras of $m(S)$.

Multiplicative left invariant means on $m(S)$ have been studied by the author in [11], and by E. Granirer in [6] and [7]. Subalgebras of $m(S)$ that have multiplicative left invariant means were first considered in [7].

Let Y be a compact convex subset of a locally convex linear topological space, and let S be a semigroup of continuous affine self-maps of Y such that $m(S)$ has a left invariant mean. Day [2, Theorem 1] has shown that then Y contains a common fixed point of S . Let S' be the convex hull (i.e., the set of finite convex combinations) of S . Since S has a common fixed point in Y , then S' must also. In the light of our earlier remarks, it is tempting to conjecture that $m(S')$ has a multiplicative left invariant mean, but this conjecture does not hold. However, a subalgebra of $m(S')$ can be constructed which does have such a multiplicative mean. This algebra

Received by the editors November 25, 1966.

⁽¹⁾ This work was supported by National Science Foundation Grant GP-6327.

could be obtained thus: let y be a specific element of Y , let $\alpha: C(Y) \rightarrow m(S')$ be given by $(\alpha h)s' = h(s'y)$ for $h \in C(Y)$ and $s' \in S'$, then $\alpha(C(Y))$ can be shown to be an algebra with the desired properties. However, a somewhat more general approach than this is employed in §5, where the construction of the algebra is given.

2. Preliminaries and nomenclature. Let S be a semigroup, $m(S)$ the space of all bounded real-valued functions on S , where $m(S)$ has the supremum norm. For $s \in S$, the *left translation* l_s {*right translation* r_s } of $m(S)$ by s is given by $(l_s f)s' = f(ss')$ {($r_s f$) $s' = f(s's)$ }, where $f \in m(S)$ and $s' \in S$. Let X be a subspace of $m(S)$, then X is *left* {*right*} *translation-invariant* if $l_s X \subseteq X$ { $r_s X \subseteq X$ } for all $s \in S$. If X is both left and right translation-invariant, then X is called *translation-invariant*.

Now let X be a left translation-invariant closed subalgebra of $m(S)$ that contains e , the constant 1 function on S . An element $\mu \in X^*$ is a mean on X if $\|\mu\| = 1$ and $\mu(e) = 1$. A mean μ on X is *left invariant* if $\mu(l_s f) = \mu(f)$ for all $f \in X$ and $s \in S$; μ is *multiplicative* if $\mu(f) \cdot \mu(g) = \mu(f \cdot g)$ (the pointwise product) for all $f, g \in X$.

When Y is a topological space, $C(Y)$ denotes the space of all bounded real-valued continuous functions on Y , where $C(Y)$ has the supremum norm.

Let S be a semigroup, X a subset of $m(S)$, and Y a compact Hausdorff space. Let η be a homomorphism of S onto $\mathcal{S}$, a semigroup (under functional composition) of continuous maps of Y into itself. The set of all $y \in Y$ such that $Ty(C(Y)) \subseteq X$ is denoted by Y' , where the map Ty is given by $(Tyh)s = h((\eta s)y)$, for $h \in C(Y)$, and $s \in S$. The family $\mathcal{S}$ is an *E-representation* of S, X on Y if Y' is nonempty, $\mathcal{S}$ is a *D-representation* if Y' is dense in Y , and $\mathcal{S}$ is an *A-representation* if $Y' = Y$. (The symbol E stands for exists, D for dense, and A for all.) The pair S, X has the *common fixed point property on compacta* with respect to i -representations (where $i = E, D, A$; respectively) if, for each compact Hausdorff space Y , and for each i -representation of S, X on Y , there is in Y a common fixed point of the family $\mathcal{S}$.

3. The main theorems.

THEOREM 1. *Let S be a semigroup, X a translation-invariant closed subalgebra of $m(S)$ that contains the constant functions. Then the following three conditions are equivalent:*

- (1) *X has a multiplicative left invariant mean.*
- (2) *S, X has the common fixed point property on compacta with respect to E -representations.*
- (3) *S, X has the common fixed point property on compacta with respect to D -representations.*

Further, each of the equivalent conditions (1), (2), or (3) implies

- (4) *S, X has the common fixed point property on compacta with respect to A -representations.*

Proof. (1) $\rightarrow$ (2). Let η be a homomorphism of S onto $\mathcal{S}$, an E -representation of S, X on the compact Hausdorff space Y . Then there exists an element $z \in Y$

such that $Tz: C(Y) \rightarrow X$. Let $Tz^*: X^* \rightarrow C(Y)^*$ be the adjoint map of Tz , and let μ be a multiplicative left invariant mean on X . Then since μ is a mean,

$$(Tz^*\mu)1 = \mu(Tz1) = \mu(e) = 1,$$

where 1 designates the constant 1 function on Y . Also, a computation shows that $Tz(h \cdot k) = Tzh \cdot Tzk$ (the pointwise product) for $h, k \in C(Y)$, so $Tz^*\mu$ is a nonzero multiplicative linear functional on $C(Y)$ since μ is a multiplicative mean on X . But Y is a compact Hausdorff space, so by [5, Lemma 25, p. 278], there exists an element $z' \in Y$ such that

$$h(z') = (Tz^*\mu)h = \mu(Tzh)$$

for all $h \in C(Y)$. We will show that z' is the desired common fixed point of $\mathcal{S}$.

For each $s \in S$, define a map $\theta_s: C(Y) \rightarrow C(Y)$ by $(\theta_s h)y = h((\eta s)y)$ for $h \in C(Y)$, $y \in Y$. Then for each $s' \in S$,

$$\begin{aligned} (Tz(\theta_s h))s' &= (\theta_s h)((\eta s')z) = h((\eta s)(\eta s')z) \\ &= h(\eta(ss')z) = (Tzh)(ss') = (l_s(Tzh))s', \end{aligned}$$

where the first and fourth equalities follow from the definition of Tz , and the third follows from the fact that η is a homomorphism. Therefore $Tz(\theta_s h) = l_s(Tzh)$. Thus it follows that for all $h \in C(Y)$ and $s \in S$,

$$\begin{aligned} h((\eta s)z') &= (\theta_s h)z' = \mu(Tz(\theta_s h)) = \mu(l_s(Tzh)) \\ &= \mu(Tzh) = h(z'), \end{aligned}$$

where the fourth equality holds by virtue of the left invariance of μ . But $C(Y)$ distinguishes between the points of Y since Y is a compact Hausdorff space, so $(\eta s)z' = z'$ for all $s \in S$.

(2) $\rightarrow$ (3). Condition (2) is formally stronger than (3).

(3) $\rightarrow$ (1). Let $Z \subseteq X^*$ be the set of all multiplicative means on X , and Q the evaluation map $Q: S \rightarrow Z$ given by $(Qs)f = f(s)$ for $s \in S$, $f \in X$. Let X^* have the w^* -topology, then Z is a norm-bounded (w^* -) closed subset of X^* . Choose Y to be the closure of $Q(S)$ in X^* , then Y is a compact Hausdorff space by [4, Corollary 3, p. 41], and $Q(S) \subseteq Y \subseteq Z \subseteq X^*$ where $Q(S)$ is dense in Y . For each $s \in S$, define a map $L_s: X^* \rightarrow X^*$ by $L_s = l_s^*$. Each L_s is (w^* -) continuous by [4, Theorem 2, p. 18], carries $Q(S)$ into $Q(S)$, and thus carries Y , the closure of $Q(S)$, into Y . Let $\mathcal{S} = \{L_s; s \in S\}$, then $\mathcal{S}$ forms a semigroup, homomorphic to S , of continuous self-maps of Y . We wish to show that $\mathcal{S}$ is a D -representation of S , X on Y .

The space X is a closed subalgebra of $m(S)$ that contains e . By Kakutani's theorem on representations of abstract (M) -spaces (see [10, §24.6, p. 242]), the elements of such a space X evaluated on the space Z yield an isometry of X onto $C(Z)$. By the Tietze extension theorem [5, Theorem 3, p. 15], each function in $C(Y)$ is the restriction to Y of some function in $C(Z)$; hence to each $h \in C(Y)$,

there corresponds an $f_h \in X$ such that $h(\mu) = \mu(f_h)$ for $\mu \in Y$. (It can also be shown that $\|h\| = \|f_h\|$ and consequently that $Y = Z$, but this is not needed for the proof of Theorem 1.) Now suppose that $y \in Q(S)$, so $y = Qs'$ for some $s' \in S$. Then for $h \in C(Y)$ and $s \in S$,

$$\begin{aligned}(Tyh)s &= h(L_sy) = h(l_s^*(Qs')) = (l_s^*(Qs'))f_h \\ &= (Qs')(l_sf_h) = (l_sf_h)s' = f_h(ss') = (r_{s'}f_h)s.\end{aligned}$$

Since X is right translation-invariant, this gives us $Tyh = r_{s'}f_h \in X$; which means that $Ty(C(Y)) \subseteq X$ if $y \in Q(S)$. But $Q(S)$ is dense in Y , thus S is a D -representation of S , X on Y . By (3), there exists $\mu_0 \in Y$ such that $\mu_0 = L_s\mu_0 = l_s^*\mu_0$ for all $s \in S$, hence μ_0 is the required multiplicative left invariant mean on X .

(3) $\rightarrow$ (4). Condition (3) is formally stronger than (4), which proves Theorem 1.

Suppose S is a topological semigroup, that is, S has a Hausdorff topology in which the semigroup product is continuous. It is known that $C(S)$ is a translation-invariant closed subalgebra of $m(S)$ that contains e , so Theorem 1 can be applied to the pair $S, C(S)$. Let Y be a compact Hausdorff space, and η an algebraic homomorphism of S onto $\mathcal{S}$, a semigroup of continuous self-maps of Y . Then $\mathcal{S}$ is a *continuous representation* of S on Y if the map $S \times Y \rightarrow Y$ given by $(\eta s)y$ for $s \in S$ and $y \in Y$, is continuous⁽²⁾. The topological semigroup S has the *common fixed point property on compacta* with respect to continuous representations if for each compact Hausdorff space Y , and each continuous representation of S on Y , there is in Y a common fixed point of $\mathcal{S}$.

COROLLARY 1. *Let S be a topological semigroup. If $C(S)$ has a multiplicative left invariant mean, then S has the common fixed point property on compacta with respect to continuous representations.*

Proof. Let $\mathcal{S}$ be a continuous representation of S on a compact Hausdorff space Y . For $y \in Y$, the map $Fy: S \rightarrow Y$ is continuous, where Fy is given by $Fys = (\eta s)y$ for $s \in S$. So for $y \in Y$, $h \in C(Y)$, and $s \in S$,

$$(Tyh)s = h((\eta s)y) = (hFy)s,$$

thus $Tyh \in C(S)$ since hFy is the composition of two continuous functions. Therefore $\mathcal{S}$ is an A -representation of $S, C(S)$ on Y , hence Corollary 1 follows by Theorem 1.

The next corollary generalizes [11, Corollary 1] from the case of finite semigroups to that of compact topological semigroups. An element s_0 of a semigroup S is a *right zero* of S if $Ss_0 = \{s_0\}$.

⁽²⁾ Similarly, $\mathcal{S}$ is a *separately continuous representation* of S on the compact Hausdorff space Y if the map on $S \times Y$ is continuous in each variable. It can be shown that $\mathcal{S}$ is an A -representation of $S, C(S)$ on Y if and only if $\mathcal{S}$ is a separately continuous representation of S on Y .

COROLLARY 2. *Let S be a compact semigroup. Then the following conditions are equivalent:*

- (1) $C(S)$ has a multiplicative left invariant mean.
- (2) S has the common fixed point property on compacta with respect to continuous representations.
- (3) S has a right zero.

Proof. (1) $\rightarrow$ (2). This follows from Corollary 1.

(2) $\rightarrow$ (3). The left multiplications of S by elements $s \in S$ form a continuous representation of S on itself. Since S is compact, then by (2), S has a right zero.

(3) $\rightarrow$ (1). If s_0 is a right zero of S , it follows by a routine computation that Qs_0 is a multiplicative left invariant mean on $C(S)$.

It was shown that condition (4) of Theorem 1 is implied by each of the other three equivalent conditions of the theorem. The converse implication was not shown; I do not know if it holds. However, the implication (3) $\rightarrow$ (1) of Theorem 1 was proven by use of the right translation-invariance of X . If to the hypotheses of the theorem, we add the requirement that X satisfies a certain property that is stronger than right translation-invariance, then conditions (1)–(4) of Theorem 1 can be shown to be equivalent.

Let X be a left invariant closed subspace of $m(S)$. For each $\mu \in X^*$, there is a $\mu_l: X \rightarrow m(S)$ given by $(\mu_l f)s = \mu(l_s f)$ for $f \in X$ and $s \in S$. The space X is *left introverted* if $\mu_l X \subseteq X$ for all $\mu \in X^*$ (see Day [1, §10, p. 540]). Now let X be a closed subalgebra of $m(S)$ which contains e . The algebra X will be called *left M-introverted* if $\mu_l X \subseteq X$ for every multiplicative mean $\mu \in X^*$. It is easily verified that the statements (a) X is left introverted, (b) X is left M -introverted, and (c) X is right translation-invariant, satisfy the relationship (a) $\rightarrow$ (b) $\rightarrow$ (c).

THEOREM 2. *Let S be a semigroup, X a left translation-invariant closed subalgebra of $m(S)$ that contains the constant functions. Let X , in addition, be left M -introverted. Then conditions (1)–(4) of Theorem 1 are equivalent.*

Proof. Since S, X satisfies the hypotheses of Theorem 1, we only need to show that (4) $\rightarrow$ (1). Choose the compact Hausdorff space Y to be the set of all multiplicative means on X , where Y is given the w^* -topology of X^* . Let $\mathcal{S} = \{L_s; s \in S\}$, where $L_s = l_s^*$. By use of [10, §24.6, p. 242] again, it follows that for each $h \in C(Y)$, there exists a unique $f_h \in X$ such that $h(\mu) = \mu(f_h)$ for $\mu \in Y$. Then for $h \in C(Y)$, $\mu \in Y$ and $s \in S$, we have that

$$\begin{aligned} (T\mu h)s &= h(L_s \mu) = h(l_s^* \mu) = (l_s^* \mu)f_h \\ &= \mu(l_s f_h) = (\mu_l f_h)s. \end{aligned}$$

So $T\mu h = \mu_l f_h \in X$, since X is left M -introverted; which means that $Ty(C(Y)) \subseteq X$ for all $y \in Y$. Thus $\mathcal{S}$ is an A -representation of S, X on Y . Since (4) asserts the pair S, X has the common fixed point property on compacta with respect to A -representations, there exists $\mu_0 \in Y$ such that $\mu_0 = l_s^* \mu$ for all $s \in S$.

4. Examples and remarks. (a) A modified version of Theorem 1 is obtained by dropping the requirement that X is translation-invariant, and replacing it by the hypotheses that X is left translation-invariant, and that the semigroup S contains an identity i . Under these circumstances, it follows that conditions (1)–(4) of Theorem 1 satisfy the relationship:

$$(1) \leftrightarrow (2) \rightarrow (3) \rightarrow (4).$$

The proof of $(1) \rightarrow (2) \rightarrow (3) \rightarrow (4)$ goes through as in Theorem 1; we will indicate the proof of $(2) \rightarrow (1)$. Let Y be the space of multiplicative means on X , where Y has the w^* -topology of X^* , and let $\mathcal{S} = \{l_s^*; s \in S\}$. It can be verified that Y' is nonempty because $Qi \in Y'$, so $\mathcal{S}$ is an E -representation of S , X on Y , hence (1) follows.

(b) An “obvious proof” of the converse to Corollary 1 fails to go through. Let Y be the space of multiplicative means on $C(S)$, where Y is given the w^* -topology of $C(S)^*$, and let $\mathcal{S} = \{l_s^*; s \in S\}$. If the topological semigroup S has the common fixed point property on compacta with respect to continuous representations, and if it can be shown that $\mathcal{S}$ is a continuous representation of S on Y , then it follows that $C(S)$ has a multiplicative left invariant mean. However, $\mathcal{S}$ need not be a continuous representation of S on Y even if $C(S)$ has, in fact, a multiplicative left invariant mean.

For a counterexample (cf. Day [3, Paragraph 1]), let S be the semigroup of nonnegative real numbers under multiplication, where S has the usual topology. Then $Q(0)$ can be verified to be a multiplicative left invariant mean on $C(S)$, so by Corollary 1, S has the common fixed point property on compacta with respect to continuous representations. Let $\mu \in Y$ be a cluster point of Qs as $s \rightarrow \infty$; such a point μ must exist by compactness of Y . Choose $f \in C(S)$ to be $f(s) = \min(s, 1)$ for $s \in S$. It is easily shown that $(l_s^* \mu)f = 1$ if $s \neq 0$, and $(l_s^* \mu)f = 0$ if $s = 0$. Hence if $s \rightarrow 0$ in a deleted neighborhood of 0, then $l_s^* \mu$ does not converge to $l_0^* \mu$, thus $\mathcal{S}$ is not a continuous representation of S on Y . (Additionally, the family $\mathcal{S}$ serves as an example of a D -representation of S , $C(S)$ on Y that is not also an A -representation.)

(c) In [11, Corollary 3], it was shown that if S is a semigroup such that for every $s_1, s_2 \in S$, there exists an $s_3 \in S$ which satisfies $s_1 s_3 = s_2 s_3$, then $m(S)$ has a multiplicative left invariant mean. The converse was obtained [11, Theorems 3 and 4] for the cases where S is Abelian or has left cancellation; this converse was shown to hold for the general case by E. Granirer [6, Theorem 1].

(d) Suppose X_1, X_2 are left translation-invariant closed subalgebras of $m(S)$ that contain the constant functions, where $X_1 \subseteq X_2$. If X_2 has a multiplicative left invariant mean, it can be shown that X_1 does also. This observation, together with (c) above, enables us to construct examples of pairs S, X which have the common fixed point property on compacta with respect to the various representations. For an illustration, let S be the real numbers with the usual metric topology, but with

the semigroup product $s_1 s_2 = \max(s_1, s_2)$. Then by (c), $m(S)$ has a multiplicative left invariant mean, so $C(S)$ does also. By Corollary 1, it follows that the topological semigroup S has the common fixed point property on compacta with respect to continuous representations.

(e) Let S be a semigroup. An example of an algebra $X \subseteq m(S)$ which has a multiplicative left invariant mean (even if $m(S)$ does not) is given by the trivial case where X consists of all the constant functions on S .

(f) Another example of an algebra $X \subseteq m(S)$, where $m(S)$ does not have a multiplicative left invariant mean but X does, is given by the case where S is the positive integer under addition, and X is the space of convergent functions on S . (For other examples, see Granirer [7].) Define $\mu \in X^*$ by $\mu(f) = \lim_{s \rightarrow \infty} f(s)$ for $f \in X$, then μ is a multiplicative left invariant mean on X . The pair S, X can be shown to satisfy the hypotheses of Theorem 2; in particular, the space X is left M -introverted.

(g) Now let S be the group of integers under addition, and X the space of all $f \in m(S)$ such that $\mu(f)$ exists, where $\mu(f) = \lim_{s \rightarrow +\infty} f(s)$. As in (f), μ is a multiplicative left invariant mean on X . But the pair S, X now satisfies the hypotheses of Theorem 1, but not those of Theorem 2, since X is not left M -introverted. In fact, it can be shown that there exists a multiplicative mean $\mu' \in X^*$ such that $\mu'_i X = m(S)$.

(h) Suppose S is the group of integers under addition, but now let X be the space of all $f \in m(S)$ such that both $\mu(f)$ and $\mu'(f)$ exist, where $\mu(f)$ and $\mu'(f)$ are the limits of $f(s)$ as $s \rightarrow +\infty$ and $s \rightarrow -\infty$, respectively. Then the pair S, X satisfies the hypotheses of Theorem 2, and X has two distinct multiplicative left invariant means, μ and μ' .

(i) Let S be a nontrivial regular Hausdorff space with the property that every continuous real-valued function on S is constant; such a space exists by a result of E. Hewitt [9, Theorem 1, p. 503]. As in Granirer [8, p. 108], define a product on S by the rule $ss' = s$ for all $s, s' \in S$. It is shown in [8] that S is a topological semigroup. By remarks (c) and (e), S provides an example of a topological semigroup for which $C(S)$ has a multiplicative left invariant mean, but $m(S)$ does not. By Corollary 1, S has the common fixed point property on compacta with respect to continuous representations. We can say even more than this, for by the proof of Corollary 1, if $\mathcal{S}$ is a continuous representation of S on the compact Hausdorff space Y , then $\mathcal{S}$ is an A -representation of S on Y . Thus $Tyh \in C(S)$ for $y \in Y$ and $h \in C(Y)$, so Tyh is a constant function. Therefore $h((\eta s)y) = h((\eta s')y)$ for $h \in C(Y)$, $y \in Y$, and $s, s' \in S$. But $C(Y)$ distinguishes between points of Y , so $\eta s = \eta s'$ for all $s, s' \in S$. Hence $\mathcal{S}$ is a singleton set $\mathcal{S} = \{\sigma\}$, where σ is a retraction on Y (recall $\mathcal{S}$ is a semigroup, so σ is an idempotent). Thus σY is the set of all common fixed points of $\mathcal{S}$.

5. Semigroups of finite means. Throughout this section, S designates a semigroup and X denotes a translation-invariant closed subspace of $m(S)$ that contains

the constant functions. It will be shown that a new pair P, W can be constructed from S, X such that W has a multiplicative left invariant mean if and only if X has a left invariant mean. This construction furnishes a source of examples of pairs P, W for which W has a multiplicative left invariant mean, but for which $m(P)$ does not.

Let P be the set of all real-valued functions p on S that satisfy (1) $p(s) \geq 0$ for all $s \in S$, (2) $p(s) > 0$ for at most a finite number of elements of S , and (3) $\sum_{s \in S} p(s) = 1$. Define a product on P by

$$pp'(s) = \sum_{ab=s} p(a)p'(b)$$

for $p, p' \in P$ and $a, b, s \in S$. In other words, P is the semigroup of finite means on S with the convolution operation of $l^1(S)$ as a product. Let $\tau: X \rightarrow m(P)$ be given by

$$(\tau f)p = \sum_{s \in S} p(s)f(s)$$

for $f \in X$ and $p \in P$. Then the closed subalgebra of $m(P)$ that is generated by τX is denoted by W , i.e., W is the smallest closed subalgebra of $m(P)$ that contains τX . Let $I: S \rightarrow P$ be the map where Is is the characteristic function of s .

We list some useful facts about P and W ; see Day [1, §5] for some of these items.

- (1) $I(ss') = (Is)(Is')$ for $s, s' \in S$.
- (2) $p = \sum_{s \in S} p(s)Is$ for $p \in P$.
- (3) $pp' = \sum_{a \in S} \sum_{b \in S} p(a)p'(b)I(ab)$ for $p, p' \in P$.
- (4) $(\tau f)(Is) = f(s)$ for $f \in X$ and $s \in S$.
- (5) $\tau e = e'$, the constant 1 function on P .
- (6) $\|\tau\| = 1$.

LEMMA 1. *The algebra W is translation-invariant.*

Proof. This is shown only for left translation-invariance; the proof is similar for right translation-invariance. For $f \in X$ and $p, p' \in P$, it follows that

$$\begin{aligned} (l_p \tau f)p' &= (\tau f)(pp') = (\tau f)(\sum_{a \in S} \sum_{b \in S} p(a)p'(b)I(ab)) \\ &= \sum_{a \in S} \sum_{b \in S} p(a)p(b)f(ab) = \sum_{a \in S} p(a) \sum_{b \in S} p'(b)((l_a f)b) \\ &= \sum_{a \in S} p(a)((\tau l_a f)p') = (\sum_{a \in S} p(a)\tau l_a f)p'. \end{aligned}$$

Thus we have

$$(7) \quad l_p \tau f = \sum_{a \in S} p(a)\tau l_a f = \tau(\sum_{a \in S} p(a)l_a f).$$

But $l_a f \in X$ since X is left translation-invariant, so $l_p \tau f \in W$ for $f \in X$ and $p \in P$. This means that τX is left translation-invariant. However, l_p is a bounded multiplicative linear operator on $m(P)$, hence the Banach algebra W generated by τX is also left translation-invariant.

Since W is a translation-invariant closed subalgebra of $m(P)$ that contains the constant functions, it is meaningful to speak of a multiplicative left invariant mean on W .

THEOREM 3. *The algebra W has a multiplicative left invariant mean if and only if X has a left invariant mean.*

Proof. From equations (5) and (6), it follows that if μ is a mean on W , then $\tau^* \mu$ is a mean on X . For $f \in X$, $s \in S$, we have

$$l_{Is} \tau f = \tau (\sum_{a \in S} ((Is)a) l_a f) = \tau l_s f$$

by equation (7). Let μ be a left invariant mean on W , then

$$(\tau^* \mu)(l_s f) = \mu(\tau l_s f) = \mu(l_{Is} \tau f) = \mu(\tau f) = (\tau^* \mu)f$$

for all $f \in X$ and all $s \in S$. Hence $\tau^* \mu$ is a left invariant mean on X , which shows the "only if" implication.

For the converse, let λ be a left invariant mean on X . Let $q: P \rightarrow X^*$ be the map given by

$$(qp)f = \sum_{s \in S} p(s)f(s) = (\tau f)p,$$

where $p \in P$ and $f \in X$. Then qP is w^* -dense in the set of means on X since P is the set of finite means on S (see [1, §10, p. 540]), so there exists a net $\{qp_\delta\}$ which is w^* -convergent to λ . Let Y be the set of all multiplicative means on W , and let $Q: P \rightarrow Y$ be the evaluation map $(Qp)h = h(p)$ for $p \in P$ and $h \in W$. By compactness of Y in the w^* -topology of X^* , the net $\{Qp_\delta\}$ has a subnet $\{Qp_\delta\}$ that converges w^* to some $\mu \in Y$. We will show that μ is the desired multiplicative left invariant mean on W .

For $p_\delta \in \{p_\delta\}$ and $f \in X$, we have

$$(\tau^* Qp_\delta)f = (Qp_\delta)(\tau f) = (\tau f)p_\delta = (qp_\delta)f,$$

hence $\tau^* Qp_\delta = qp_\delta$. But $\{Qp_\delta\}$ and $\{qp_\delta\}$ converge w^* to μ and λ respectively, so $\tau^* \mu = \lambda$ by w^* -continuity of τ^* . If $p \in P$ and $f \in X$, we obtain

$$\begin{aligned} \mu(l_p \tau f) &= \mu(\sum_{a \in S} p(a) \tau l_a f) = \sum_{a \in S} p(a) ((\tau^* \mu) l_a f) \\ &= \sum_{a \in S} p(a) \lambda(l_a f) = \sum_{a \in S} p(a) \lambda(f) \\ &= \lambda(f) = (\tau^* \mu)f = \mu(\tau f), \end{aligned}$$

where the first equality follows from equation (7), and the fourth from the left invariance of λ . This means that μ is left invariant when restricted to τX . But μ

and l_p are multiplicative, continuous, and linear; hence μ is left invariant on W , the Banach algebra generated by τX , which proves Theorem 3.

It can also be shown that τ^* is a w^*-w^* homeomorphism of Y onto the set of means of X , but this is not needed for the proof above.

By use of Theorem 3, examples can be obtained of pairs P, W which have a multiplicative left invariant mean, and so by Theorem 1, have the common fixed point property on compacta with respect to E, D , and A -representations. Any topological semigroup S for which $C(S)$ has a left invariant mean will serve admirably as a source of a suitable pair S, X if X is taken to be $C(S)$. Some well-known examples are Abelian topological semigroups, and compact topological groups. Another possibility is the use of appropriate subspaces of $C(S)$; for example, if S is a topological group, then X can be taken to be the space of almost periodic functions on S .

If S is a semigroup such that $m(S)$ has a left invariant mean, then the space $m(P)$ need not have a multiplicative left invariant mean. To illustrate this, let S be the group of integers under addition, and let $X=m(S)$. A computation shows that P is an infinite Abelian cancellation semigroup. But if $m(S')$ has a multiplicative left invariant mean, where S' is a cancellation semigroup, then S' is a singleton set by [11, Theorem 2]; hence $m(P)$ does not have a multiplicative left invariant mean. It follows that W must be a proper subset of $m(P)$, since W does have such a mean by Theorem 3.

BIBLIOGRAPHY

1. M. M. Day, *Amenable semigroups*, Illinois J. Math. **1** (1957), 509–544.
2. ———, *Fixed-point theorems for compact convex sets*, Illinois J. Math. **5** (1961), 585–590.
3. ———, *Correction to my paper "Fixed-point theorems for compact convex sets,"* Illinois J. Math. **8** (1964), 713.
4. ———, *Normed linear spaces*, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer, Berlin, 1962.
5. N. Dunford and J. T. Schwartz, *Linear operators*, Part I, Pure and Appl. Math. No. 7, Interscience, New York, 1958.
6. E. Granirer, *Extremely amenable semigroups*, Math. Scand. **17** (1965), 177–197.
7. ———, *Extremely amenable semigroups. II*, Math. Scand. (to appear).
8. ———, *On the invariant mean on topological semigroups and on topological groups*, Pacific J. Math. **15** (1965), 107–140.
9. E. Hewitt, *On two problems of Urisohn*, Ann. of Math. (2) **47** (1946), 503–509.
10. J. L. Kelley and I. Namioka (et al.), *Linear topological spaces*, Van Nostrand, Princeton, N. J., 1963.
11. T. Mitchell, *Fixed points and multiplicative left invariant means*, Trans. Amer. Math. Soc. **122** (1966), 195–202.

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