A DESCRIPTION OF THE TOPOLOGY ON THE DUAL SPACES OF CERTAIN LOCALLY COMPACT GROUPS[1]

BY
LAWRENCE BAGGETT

Introduction. Suppose G is a locally compact group. By the *dual space* $\hat{G}$ of G we mean the set of all equivalence classes of irreducible unitary representations of G equipped with the "hull-kernel" topology. The definition of the hull-kernel topology is a bit unwieldy since it involves the structure of the group C^*-algebra $C^*(G)$. Determining the structure of a C^*-algebra is a very difficult task in itself, and it is hoped that the topology of the dual space of a group can be described in terms of parameters having only to do with the group structure. For example, if G happens to be abelian, then the dual space $\hat{G}$ is the group of all characters on G, and the topology of $\hat{G}$ is given in terms of uniform convergence of the characters on compact subspaces of G. In this paper we prove that the topology on the dual spaces of certain groups can be described in terms of appropriate group-theoretic parameters.

Consider the case when G contains a closed normal subgroup N. Then, if G and N satisfy certain technical conditions, G. W. Mackey has shown, in [13], that the elements of the dual space $\hat{G}$ can be obtained, loosely speaking, as induced representations U^S of G. The situation here is roughly as follows.

Notice first that G acts to the left as a group of continuous transformations on the dual space $\hat{N}$ of N.

If χ is an element of $\hat{N}$, let the *stability subgroup* G_χ for χ be the set of all elements x of G such that $x\chi=\chi$.

Let Y be the set of all pairs (K, S), where K is the stability subgroup G_χ for some element χ of $\hat{N}$, and where S is an irreducible unitary representation of K whose restriction to N is a multiple of χ.

Mackey's result asserts that the inducing map I, which sends an element (K, S) of Y to U^S, is onto the dual space $\hat{G}$ of G. Further, there is a precise equivalence relation $\equiv$ on Y such that $I(K, S)=I(K', S')$ if and only if $(K, S)\equiv(K', S')$. The relation $\equiv$ is, again roughly speaking, that (K, S) and (K', S') are conjugate under some inner automorphism of G.

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Thus, if G and N satisfy some technical assumptions, we may "catalogue" the elements of $\hat{G}$ by means of "cataloguing pairs" (the elements of Y). These cataloguing pairs are "group theoretic" parameters, and the question arises whether we may describe the topology of $\hat{G}$ in terms of this cataloguing.

Now the cataloguing pairs are special kinds of "subgroup-representation pairs." A *subgroup-representation pair* is a pair (K, S), such that K is a closed subgroup of G and S is a unitary representation of K. In [7], J. M. G. Fell has defined a topology on the space of all subgroup-representation pairs. We come now to a conjecture.

CONJECTURE 1. The dual space $\hat{G}$ is homeomorphic to the quotient space obtained from the set of all cataloguing pairs, considered as a subspace of the space of all subgroup-representation pairs, modulo the equivalence relation $\equiv$ mentioned above. This conjecture is false. (See 10.1-D.)

Here is a more sophisticated conjecture.

CONJECTURE 2. Suppose (K, S) is a cataloguing pair for an element W of $\hat{G}$, and suppose B is a subset of $\hat{G}$. Then, W is contained in the closure of B if and only if there exist a subgroup-representation pair (J, T) and a net $[(J^a, T^a)]$ of cataloguing pairs which satisfy:

(i) For each a, the pair (J^a, T^a) catalogues an element of B.

(ii) The net $[(J^a, T^a)]$ converges to (J, T) in the space of all subgroup-representation pairs.

(iii) J is a subgroup of K, and the representation ${}_KU^T$, i.e., the representation of K induced from T, weakly contains S.

We prove the truth of Conjecture 2 in the special case when N is abelian and G is the semidirect product of N with either an abelian group or a compact group. These are our main theorems. (See Theorems 3.3 and 6.2-A.)

Fell, in [6, Theorem 4.3], has shown the truth of Conjecture 2 in the special situation where there exists an element χ' of $\hat{N}$ such that, for each element χ of $\hat{N}$ not equal to χ', the stability subgroup G_χ for χ is N itself. Also, in this connection, Professor Fell has made use of a theorem of J. Glimm, Theorem 2.1 of [9], and has shown the truth of Conjecture 1 in the following special case.

Let M be the subset of $\hat{N}$ on which the function which sends an element χ to its stability subgroup G_χ is continuous. Define $\hat{G}_M$ to be the subset of $\hat{G}$ consisting of those elements W which can be catalogued by elements (K, S) where $S|_N$ is a multiple of some element of M. Then Conjecture 1 is true if we restrict down to the subset $\hat{G}_M$ of $\hat{G}$. This result is stated precisely in 6.4.

In §1 we present the background material and notation which will be used in this paper. Some propositions, which can not be referred to specifically in the literature, are proved here. §§2 through 6 are devoted to the specific development toward and proof of our main theorems. The last sections contain assorted side effects and interesting consequences of the main results. If G is a locally compact group whose dual space is a Hausdorff space, then it has been conjectured that G must be the direct product of an abelian group and a compact group. In §10.3 an example is given which shows that this conjecture is false.

We introduce here the following conventions in notation.

(i) If X is a locally compact topological space, then $L(X)$ denotes the linear space of all continuous complex-valued functions with compact support on X.

(ii) If f is a function on a set X, and if Z is a subset of X, then $f|_Z$ denotes the restriction of the function f to the subset Z.

(iii) Nets and sets will be indicated by square brackets instead of braces, as in the net $[T^\alpha]$ or the set of all $[s_i]$.

(iv) We write e for the identity of all groups. Whenever any confusion could arise, we will be precise.

(v) A topological group G is called *separable* if, as a topological space, G satisfies the second axiom of countability.

(vi) If H is a Hilbert space, we denote the inner product in H by $(\ ,\ )_H$. If E is a normed linear space, let $\|\ \ \|_E$ denote the norm in E.

1. **Preliminaries.**

1.1-A. If G is a locally compact group, then there exists a norm on the linear space $L(G)$ (equipped with the usual convolution and involution) for which the completion of $L(G)$ with respect to this norm is a C^*-algebra called the *group C^*-algebra* and denoted by $C^*(G)$. Further, there is a one-to-one equivalence-preserving correspondence between the set of all unitary representations of G and the set of all nowhere trivial star-representations of $C^*(G)$. There is induced therefore a one-to-one correspondence between the points of the dual space $\hat{G}$ of G, i.e., the set of all equivalence classes of irreducible unitary representations of G, and the points of the dual space $[C^*(G)]^\wedge$ of $C^*(G)$, i.e., the set of all equivalence classes of irreducible star-representations of $C^*(G)$.

Remark. For purposes of simplicity, we often think of the elements of a dual space as being irreducible representations, although this is not logically correct.

In [3], Fell has defined the notion of "weak containment" for star-representations (unitary representations). Then the hull-kernel topology on the dual space can be defined as follows. (Theorem 1.2 of [3].)

Let W be an element of $\hat{G}$ ($[C^*(G)]^\wedge$) and let B be a subset of $\hat{G}$ ($[C^*(G)]^\wedge$). Then W is contained in the closure of B if and only if W is weakly contained in B.

1.1-B. Definition. A locally compact group G is called an *R-group* if the left regular representation R of G weakly contains each irreducible unitary representation of G.

Remark. Abelian and compact groups are R-groups.

1.1-C. Definition. Suppose G is a locally compact group and let N be a closed normal subgroup of G. Denote by K the quotient group G/N. A mapping p of K into G is called a *cross-section of K into G* if:

(i) For each y in K, $p(y)$ belongs to the coset of N which y determines.
(ii) p effects a Borel isomorphism of K and $p(K)$.
(iii) If C is a compact subset of K then $p(C)$ has compact closure in G.

THEOREM. *Let G, K, and N be as in the definition above, and assume that G is separable. Then, a cross-section p of K into G always exists.*

See Proposition 1 of [13].

1.2-A. Suppose G is a locally compact group and suppose K is a closed subgroup of G. If S is a unitary representation of K, we define the *induced representation* U^S (the representation of G induced from S) to be the following unitary representation of G.

The Hilbert space $H(U^S)$ of U^S consists of all functions f on G into $H(S)$ which are locally Bochner measurable (see [2]) and which satisfy the following three conditions.

(i) If x is in G and k is in K, then

$$f(xk) = [\delta_K(k)/\delta_G(k)]^{1/2}S_{(k^{-1})}(f(x)).$$

(δ_K and δ_G refer to the modular functions on K and G respectively.)

(ii) The map $x \to \|f(x)\|^2_{H(S)}$ is locally summable on G.

Before stating condition (iii), we make the following remark. If f and g are locally Bochner measurable functions on G into $H(S)$ which satisfy (i) and (ii) above, then it can be shown that the pair (f, g) defines a regular complex-valued measure $\lambda_{f,g}$ on the locally compact Hausdorff space G/K of left cosets of K. In fact, as a linear functional on $L(G/K)$, $\lambda_{f,g}$ acts as follows. Let h be in $L(G/K)$. Then there exists an element h' in $L(G)$ such that, for all elements x in G,

$$h(xK) = \int_K h'(xk)\, dk.$$

Then $\lambda_{f,g}(h) = \int_G h'(x)(f(x), g(x))_{H(S)}\, dx$.

It turns out that $\lambda_{f,g}(h)$ is independent of the choice of the element h' of $L(G)$. We may now state condition (iii).

(iii) The measure $\lambda_{f,f}$ has finite total mass.

The inner product in $H(U^S)$ is given by:

$$(f, g)_{H(U^S)} = \lambda_{f,g}(G/K).$$

Finally we define the action of U^S on G. If x and y are in G and f is in $H(U^S)$, put

$$[U^S_y(f)](x) = f(y^{-1}x).$$

1.2-B. Now let N be a closed normal subgroup of G, and let χ be an irreducible representation of N whose space $H(\chi)$ is a separable Hilbert space. Assume further that a cross-section p exists from G/N, which we call K, into G. (If G is not separable, assume for convenience that p is actually continuous.)

THEOREM. *The induced representation U^{χ} is equivalent to the representation T defined as follows.*

$H(T)$ is the Hilbert space $L^2(K, H(\chi))$, i.e., the space of all functions h on K into $H(\chi)$ which satisfy:

(i) *For each vector q of $H(\chi)$, the map $k \to (h(k), q)_{H(\chi)}$ is locally measurable on K.*

(ii) *$\int_K \|h(k)\|^2_{H(\chi)}\, dk$ is finite.*

The action of T is given as follows. If k and l are in K, n is in N, and h is in $L^2(K, H(\chi))$, then

$$[T_{p(k)n}(h)](l) = \chi_{[p(l)^{-1}p(k)np(k^{-1}l)]}[h(k^{-1}l)].$$

For a proof, see §1.2 of [1].

1.2-C. Again let G be a locally compact group and let J and K be closed subgroups of G such that J contains K. Then, if S is a unitary representation of K, we may form the representation of J induced from S. In case of confusion, we indicate the group on which an induced representation acts by a subscript to the left. For example, the representation of J induced from S is written ${}_JU^S$.

1.2-D. Let G and K be as above. Assume that T is a representation of G and S is a representation of K. Then $T \otimes U^S$ is equivalent to $U^{[T|_K \otimes S]}$ (see Lemma 4.1 of [1]).

1.2-E. PROPOSITION. *Suppose that G and K are as in the above and suppose that N is a normal subgroup of G which is contained in K. Let π be the natural mapping of G onto G/N, and let S be a representation of K/N. Then*

$$U^{S\cdot\pi} \text{ is equivalent to } [{}_{(G/N)}U^S]\cdot\pi.$$

Again we sketch a proof. We need the following lemmas. The overall hypotheses for the first three lemmas are: N is a closed normal subgroup of the locally compact group G. π denotes the natural mapping of G onto G/N. If X is a locally compact group, then δ_X denotes the modular function on X.

LEMMA 1. *We may fix the left Haar measures on G, N, and G/N such that for each f in $L(G)$,*

$$\int_G f(y)\, dy = \int_{(G/N)} \int_N f(yn)\, dn\, d\pi(y).$$

For the proof, see §33 of [12].

LEMMA 2. *If x is an element of G, then there exists a positive number $d(x)$ such that:*

If f is in $L(N)$, then

$$\int_N f(xnx^{-1})\, dn = d(x) \int_N f(n)\, dn.$$

The proof follows from the uniqueness of left Haar measure, up to a multiplicative constant, on N.

LEMMA 3. *If x is in G, then*

$$\delta_G(x) = d(x)\delta_{(G/N)}(\pi(x)).$$

Proof. Let f be in $L(G)$. Then

$$\delta_G(x)\int_G f(y)\,dy = \int_G f(yx^{-1})\,dy = \int_{(G/N)}\int_N f(ynx^{-1})\,dn\,d\pi(y)$$

$$= \int_{(G/N)}\int_N f(yx^{-1}xnx^{-1})\,dn\,d\pi(y)$$

$$= d(x)\int_{(G/N)}\int_N f(yx^{-1}n)\,dn\,d\pi(y)$$

$$= d(x)\delta_{(G/N)}(\pi(x))\int_{(G/N)}\int_N f(yn)\,dn\,d\pi(y)$$

$$= d(x)\delta_{(G/N)}(\pi(x))\int_G f(y)\,dy.$$

Now the lemma follows from the arbitrariness of f.

LEMMA 4. *Now assume the hypotheses of the proposition. Then if k is in K,*

$$[\delta_K(k)/\delta_G(k)] = [\delta_{(K/N)}(\pi(k))/\delta_{(G/N)}(\pi(k))].$$

This follows from Lemma 3.

To prove the proposition, we need to set up a linear isometry of $H(U^S\cdot\pi)$ and $H([_{(G/N)}U^S]\cdot\pi)$. The latter Hilbert space consists of functions on G/N into $H(S)$, while the former Hilbert space consists of functions on G into $H(S)$. Thus if f is in $H([_{(G/N)}U^S]\cdot\pi)$, define $\theta(f)$ as follows: If x is in G, then $[\theta(f)](x)=f(\pi(x))$.

To show that θ is in fact the intertwining operator we want, we first must show that $\theta(f)$ is in $H(U^{S\cdot\pi})$. Clearly the measurability and summability requirements are satisfied. Lemma 4 is what is needed to show that condition (i) of 1.2-A is satisfied. Notice that the measures $\lambda_{f,f}$ and $\lambda_{[\theta(f),\theta(f)]}$ act on the same measure space G/K, and as linear functionals on $L(G/K)$, $\lambda_{f,f}$ and $\lambda_{[\theta(f),\theta(f)]}$ are identical.

One final comment should be made to verify that θ is onto $H(U^{S\cdot\pi})$. This follows because, if n is an element of N, then $\delta_G(n)=\delta_K(n)=\delta_N(n)$. Hence we may show that θ has an inverse which is defined on all of $H(U^{S\cdot\pi})$.

1.3. We come now to a brief discussion of the cataloguing of elements of $\hat{G}$ by means of subgroup-representation pairs. These results are essentially due to Mackey in [13]. We fix a locally compact group G and a closed normal subgroup N of G.

1.3-A. If x is in G and χ is an irreducible representation of N, define $x\chi$ to be the following representation of N. If n is in N, put $[x\chi]_n=\chi_{[x^{-1}nx]}$.

Thus G acts to the left as a group of continuous transformations on $\hat{N}$. If χ is in $\hat{N}$, let the *stability subgroup* G_χ for χ be the set of all elements x of G such that $x\chi=\chi$.

DEFINITION. The normal subgroup N is said to be *regularly imbedded* in G if, for each element χ of $\hat{N}$, the natural mapping of $G/(G_\chi)$ onto the orbit (under G) of $\hat{N}$ to which χ belongs is a homeomorphism. (See [10].)

1.3-B. We want now to state two inequivalent sets of hypotheses of the pair (G, N) with which we will deal simultaneously in the following sections.

1. N is abelian and G/N is compact.
2. G is separable, N is of type I, and N is regularly imbedded in G.

When case 2 holds, then, by Theorem 1.1-C, there exists a cross-section p of G/N into G. Henceforth we assume that, if 1 holds and 2 does not hold, there exists a continuous cross-section of G/N into G.

1.3-C. Now assume that (G, N) satisfies 1 or 2 of B above. Let T be an irreducible representation of G. Then there is an orbit θ_T (under G) of $\hat{N}$ which is canonically *associated* with the representation T. Further, if χ is an element of $\hat{N}$, then $T|_N$ weakly contains χ if and only if χ belongs to the closure of the orbit θ_T.

In case hypotheses 2 hold, this is a combination of theorems of Mackey in [13] and Fell in [6]. This procedure can be generalized in a straightforward manner to cover the case when hypotheses 1 hold.

1.3-D. Continue to assume that (G, N) satisfies 1 or 2 of 1.3-B.

PROPOSITION 1. *Let T be an irreducible unitary representation of G and let θ be the orbit of $\hat{N}$ with which T is associated. Suppose χ is an element of θ. Then T is equivalent to a representation of the form U^S, where S is an irreducible unitary representation of the stability subgroup G_χ for χ and $S|_N$ is a multiple of χ.*

See [13].

PROPOSITION 2. *Let T and θ be as in the above proposition. Assume χ and ψ are elements of θ, i.e., $\psi=x\chi$ for some element x of G. Suppose further that T is equivalent to U^S, where S is an irreducible representation of G_χ and $S|_N$ is a multiple of χ, and T is equivalent to $U^{S'}$, where S' is an irreducible representation of G_ψ and $S'|_N$ is a multiple of ψ. Then*:

(i) $G_\psi=xG_\chi x^{-1}$.

(ii) *S' is equivalent to the representation S^x defined by*: *if z is in G_ψ, put*

$$S^x_z = S_{[x^{-1}zx]}.$$

See [13].

PROPOSITION 3. *Let χ be an element of $\hat{N}$ and let S be an irreducible representation of G_χ whose restriction to N is a multiple of χ. Then U^S is irreducible.*

Again, see [13].

DEFINITION. If χ is an element of $\hat{N}$, let G_χ^* denote the set of all irreducible unitary representations S of the stability subgroup G_χ for χ which satisfy $S|_N$ is a multiple of χ.

1.3-E. DEFINITION. An orbit θ is *accommodating* if, for each element χ of θ, there exists a unitary extension of χ to its stability subgroup G_χ, i.e., there exists a unitary representation χ' of G_χ such that $\chi'|_N=\chi$.

PROPOSITION. *Assume (G, N) satisfies one of the sets of hypotheses of* 1.3-B. *Let θ be an accommodating orbit of $\hat{N}$ and let χ be an element of θ. Denote by π the natural mapping of G_χ onto $(G_\chi)/N$. Then the mapping $T \to [\chi' \otimes T\cdot\pi]$ is a one-to-one equivalence-preserving correspondence between $[(G_\chi)/N]^\wedge$ and G_χ^*. (Here, χ' is the unitary extension of χ guaranteed by the definition of "accommodating".)*

See [13] and [15].

1.4. In the last section we have seen that the elements of $\hat{G}$ can be catalogued, under certain technical conditions, by means of pairs (K, S), where K is the stability subgroup G_χ for some element χ of $\hat{N}$, and where S is a particular kind of irreducible unitary representation of K. We present next a discussion of the topology on the space of all such pairs, as given by Fell in [7].

1.4-A. Let G be a locally compact group. By $\mathscr{K}(G)$ we mean the space of all closed subgroups of G equipped with the compact-open topology. (§2 of [7].)

DEFINITION. By a *smooth choice* of left Haar measures on $\mathscr{K}(G)$ we mean an assignment $K \to \lambda_K$ of a particular left Haar measure λ_K to each closed subgroup K of G in such a way that, if f is in $L(G)$, then the mapping $K \to \int_K f|_K \, d\lambda_K$ is continuous on $\mathscr{K}(G)$.

Smooth choices always exist (see the appendix to [9]).

PROPOSITION. *If G is compact, then a smooth choice for $\mathscr{K}(G)$ is the assignment of normalized Haar measure to each subgroup K of G.*

The proof follows from the existence of some smooth choice together with the fact that the function which is identically 1 is in $L(G)$.

1.4-B. Now let Z be the subset of $\mathscr{K}(G)\times G$ consisting of those pairs (K, x) such that x is contained in K. Z is locally compact and Hausdorff, and the linear space $L(Z)$ can be given an algebraic structure (involving convolutions on the subgroups of G) in such a way that it becomes a normed star-algebra. The C^*-completion of this star-algebra is called the *subgroup-C^*-algebra* and is denoted $C_s^*(G)$.

DEFINITION. Let $\mathscr{A}(G)$ denote the set of all pairs (K, T) where K is a closed subgroup of G and T is an irreducible unitary representation of K.

PROPOSITION. *The irreducible star-representations of $C_s^*(G)$ are in a natural one-to-one correspondence with $\mathscr{A}(G)$.*

See §2 of [7].

Thus we may topologize $\mathscr{A}(G)$ by transferring the hull-kernel topology of $[C_s^*(G)]^\wedge$ to $\mathscr{A}(G)$.

1.4-C. In this paragraph we introduce a more useful description of the topology of $\mathscr{A}(G)$.

Let $\mathscr{F}$ be the set of all functions f which satisfy:

(i) The domain of f is a closed subgroup of G.

(ii) f is complex-valued and continuous on its domain.

In §3 of [7], Fell has defined a topology on $\mathscr{F}$.

Definition. The *Fell topology* on $\mathscr{F}$ is that topology, convergence in which is defined as follows. Let $[f^a]$ be a net of elements of $\mathscr{F}$. For each n, let K^a be the domain of f^a. Let f be in $\mathscr{F}$ and denote by K the domain of f. Then the net $[f^a]$ converges to f in the Fell topology if and only if the following two conditions hold.

(i) The net $[K^a]$ converges to K in $\mathscr{K}(G)$.

(ii) For each subnet $[f^{a^b}]$ of the net $[f^a]$, and for each net $[k^b]$ of points of G such that, for each b, k^b is an element of K^{a^b} and such that the net $[k^b]$ converges to an element k of K, the net of complex numbers $[f^{a^b}(k^b)]$ converges to $f(k)$.

Now in §3 of [7], Fell describes the topology on $\mathscr{A}(G)$ in terms of the elements of $\mathscr{F}$ and in terms of the Fell topology on $\mathscr{F}$. Here is one consequence of that description.

Theorem. *Let $[(K^a, S^a)]$ be a net of elements of $\mathscr{A}(G)$ and let (K, S) be an element of $\mathscr{A}(G)$. Assume that, for some function of positive type f associated with S, there exists a net $[f^a]$ of functions which satisfies:*

(i) *Each f^a is a finite sum of functions of positive type associated with S^a.*

(ii) *The net $[f^a]$ converges to f in the Fell topology of $\mathscr{F}$.*

Then the net $[(K^a, S^a)]$ converges to (K, S) in $\mathscr{A}(G)$.

Remark. This theorem is the tool by which we show convergence of a net of subgroup-representation pairs in our main results.

1.4-D. Here are some propositions about the topology of $\mathscr{A}(G)$.

Proposition 1. *Let $[(K^a, S^a)]$ be a net of elements of $\mathscr{A}(G)$ which converges to an element (K, S). Further, suppose that, for each a, the representation $U^{(S^a)}$ is irreducible. Then the net $[U^{(S^a)}]$ converges to each element W of $\hat{G}$ which is weakly contained in U^S.*

The proof of this follows on applying Theorem 4.2 of [7] to each subnet of the net $[(K^a, S^a)]$.

Proposition 2. *Suppose the net $[(K^a, S^a)]$ converges to (K, S), and suppose the net $[(K^a, T^a)]$ converges to (K, T) in $\mathscr{A}(G)$. Further, suppose that, for each a, $S^a \otimes T^a$ is irreducible, and suppose that $S \otimes T$ is irreducible. Then the net $[(K^a, S^a \otimes T^a)]$ converges to $(K, S \otimes T)$ in $\mathscr{A}(G)$.*

The proof follows easily from the description of the topology on $\mathscr{A}(G)$ in terms of the Fell topology on $\mathscr{F}$.

PROPOSITION 3. *Assume that N is a closed normal subgroup of G and denote by π the natural mapping of G onto G/N. Suppose that $[K^a]$ is a net of elements of $\mathscr{K}(G)$ such that, for each a, K^a contains N. Let K be a closed subgroup of G which contains N. For each a, let T^a be an irreducible representation of the quotient group $\pi(K^a)$, and let T be an irreducible representation of $\pi(K)$. Then the net $[(K^a, T^a\cdot\pi)]$ converges to $(K, T\cdot\pi)$ in $\mathscr{A}(G)$ if and only if the net $[(\pi(K^a), T^a)]$ converges to $(\pi(K), T)$ in $\mathscr{A}(G/N)$.*

Again the proof follows immediately from the definitions.

PROPOSITION 4. *Suppose G is a compact group. Let $[K^a]$ be a net of elements of $\mathscr{K}(G)$ which converges to the subgroup K. For each a, let f^a be a continuous complex-valued function on K^a, and assume that the net $[f^a]$ converges in the Fell topology to a function f on K. Then the net of complex numbers $\left[\int_{(K^a)} f^a\right]$ converges to $\int_K f$. (Here all integrals are taken with respect to normalized Haar measure.)*

This follows from Proposition 3.3 of [7] and from 1.4-A above.

1.5-A. PROPOSITION 1. *Let (G, N) satisfy* 1 *of* 1.3-B. *Suppose T is an irreducible representation of G and let θ be the orbit of $\hat{N}$ with which T is associated. Assume further that θ is accommodating. Then T is contained as a direct summand in U^χ for each element χ of θ.*

See Theorem 4.1-B of [1].

PROPOSITION 2. *Let (G, N) satisfy* 2 *of* 1.3-B *and assume that G/N is an R-group. Let T be an irreducible representation of G and let θ be the orbit of $\hat{N}$ with which T is associated. Then T is weakly contained in U^χ for each element χ of θ.*

See Theorem 4.1-A of [1].

1.5-B. THEOREM. *Let (G, N) satisfy* 1 *of* 1.3-B *and denote by K the group G/N. Then*:

(i) *Each of the representations U^χ, for χ in $\hat{N}$, acts in $L^2(K)$.*

(ii) *If f is in $C^*(G)$, then the mapping $\chi \to U_f^\chi$ is continuous on $\hat{N}$ into the C^*-algebra of bounded operators on the Hilbert space $L^2(K)$ with respect to the norm topology of the operators.*

The proof is straightforward.

Theorems on the Topology of $\hat{G}$

2. The semidirect product.

2.1. Let N and K be locally compact groups, and suppose Z is a homomorphism of K into the group of automorphisms of N such that the mapping

$(n, k) \to [Z(k)](n)$ is continuous on $N \times K$ onto N. For simplicity, we write the action of the automorphism $Z(k)$ on an element n of N as $k \cdot n$.

2.1-A. DEFINITION. The *semidirect product* $N * K$ of the groups N and K is to mean the following locally compact group.

The underlying topological space is $N \times K$.

Multiplication in $N * K$ is given as follows:

$$(n, k)(m, l) = ((n+k\cdot m), kl),$$

where we write the operation in N as plus and the operation in K as times.

2.1-B. Let G be the semidirect product $N * K$. Then N can be homeomorphically and isomorphically identified with the closed normal subgroup of G consisting of all pairs (n, e), where n is an element of N. Also, K is homeomorphic and isomorphic to the closed subgroup of G consisting of all pairs $(0, k)$, where k is an element of K. (We write 0 for the additive identity of N, e for the multiplicative identity of K, and $(0, e)$ for the identity of G.)

In certain situations, we will make the above identifications.

Observe that G/N is isomorphic and homeomorphic to K.

The right Haar measure for G can be taken as the product of the right Haar measures for N and K.

2.1-C. We denote the modular functions for N, K, and G by δ_N, δ_K, and δ_G respectively.

To each element k of K there corresponds a positive number $d(k)$, such that, if λ is the right Haar measure for N, and if V is a Borel subset of N, then

$$\lambda(k\cdot V) = d(k^{-1})\lambda(V).$$

The mapping $k \to d(k)$ is a continuous homomorphism into the group of positive real numbers, and we have

$$\delta_G(n, k) = \delta_N(n)\delta_K(k)d(k^{-1}).$$

PROPOSITION. *If N is unimodular and K is compact, then G is unimodular.*

This follows because $d(K)$ is a compact subgroup of the positive reals, hence the set containing only the number one.

2.1-D. PROPOSITION. *Any closed subgroup of G which contains N is of the form $N * J$ where J is a closed subgroup of K. Further, a net $[J^a]$ of closed subgroups of K converges to a subgroup J in $\mathscr{K}(K)$ if and only if the net $[N*(J^a)]$ of subgroups of G converges to the subgroup $N * J$ in $\mathscr{K}(G)$.*

Proof. Let π be the projection of G onto K. Then the proof of the proposition follows from the continuity and openness of π, together with the fact that, for each compact subset Y of K, there exists a compact subset Y' of G such that $\pi(Y') = Y$.

2.2-A. DEFINITION. If n is an element of N, we define the *neutral group* J_n for n as the set of all elements k of K such that $k\cdot n = n$.

PROPOSITION. *The neutral group* $J_{(n+m)}$ *contains the intersection of the neutral groups* J_n *and* J_m.

2.2-B. PROPOSITION. *Let* $[n^a]$ *be a net of elements of* N *which converges to an element* n *of* N. *Suppose that* $J_{(n^a)}$ *is the neutral group for* n^a, *and suppose that the net* $[J_{(n^a)}]$ *converges to a subgroup* J *in* $\mathscr{K}(K)$. *Then* J *is a subgroup of the neutral group* J_n *for* n.

Proof. Let k be an element of J. We will show that $k\cdot n=n$. From the definition of the topology of $\mathscr{K}(K)$, we know that there exists a subnet $[J_{(n^{a^b})}]$ of the net $[J_{(n^a)}]$, and a net $[k^b]$ of elements of K such that:

(i) For each b, k^b is contained in $J_{(n^{a^b})}$.

(ii) The net $[k^b]$ converges to k.

Hence the net $[k^b\cdot(n^{a^b})]$ converges to $k\cdot n$. But by (i), the net $[n^{a^b}]$ converges to $k\cdot n$. Since N is Hausdorff, the net $[n^{a^b}]$ can only converge to the point n. Thus $n=k\cdot n$. Q.E.D.

REMARK. The mapping $n\to J_n$ is not in general continuous, but the proposition above shows that a certain semicontinuity prevails.

2.3. We turn our attention to the special case when G is the semidirect product $N * K$ and N is abelian.

2.3-A. In the presence of the semidirect product structure, the existence of a cross-section p of K, which equals G/N, into G is apparent: $p(k)=(0, k)$.

In this case p is a continuous isomorphism of K into G and p clearly satisfies the conditions in 1.1-C.

If x is an element of G, we wish to write x in its unique canonical form $x=p(k)n$, so that we may apply the formulae in 1.2-B.

Thus, if $x=(n, k)$, then

$$x = (0, k)(k^{-1}\cdot n, e) = p(k)(k^{-1}\cdot n).$$

(Here we have made the identification of N with the subgroup of G consisting of all pairs of the form (m, e) for m in N.)

Now we assume, in addition to the commutativity of N, that G and N satisfy one of the sets of hypotheses in 1.3-B. In our present context, we may state these hypotheses as follows:

1. G/N, which equals K, is compact.
2. G is separable and N is regularly imbedded in G.

2.3-B. If χ is in $\hat{N}$, i.e., χ is an irreducible representation (character) of N, then the representation U^χ acts in $L^2(K)$. (See 1.2-B for what follows.)

Thus, if x, which equals (n, k), is in G, h is in $L^2(K)$, and l is in K, we have:

$$\begin{aligned}[U^\chi_x(h)](l) &= [U^\chi_{[p(k)(k^{-1}\cdot n)]}(h)](l) = \chi_{[p(l)^{-1}p(k)(k^{-1}\cdot n)p(k^{-1}l)]}h(k^{-1}l)\\ &= \chi_{[(0,l^{-1}k)(k^{-1}\cdot n,e)(0,k^{-1}l)]}h(k^{-1}l) = \chi_{(l^{-1}\cdot n,e)}h(k^{-1}l)\\ &= \chi_{(l^{-1}\cdot n)}h(k^{-1}l).\end{aligned}$$

(Here again we have thought of N as being identified with the subgroup $N\times e$ of G, and we have let χ act on elements of N and pairs (m, e) without comment.)

2.4-A. PROPOSITION. *Each element k of K defines an automorphism Z'_k of $\hat{N}$, namely*:

If χ is in $\hat{N}$, define $Z_k(\chi)$ of N as follows:

$$[Z'_k(\chi)]_n = \chi_{(k^{-1}\cdot n)}.$$

2.4-B. Let χ be a character of N. Recall the definition of the stability subgroup G_χ of G for χ: G_χ equals the set of all (n, k) in G such that

$$\chi_{[(n,k)^{-1}(m,e)(n,k)]} = \chi_m$$

for all elements m of N. (See 1.3-A.)

Thus G_χ equals the set of all (n, k) in G such that $\chi_{(k^{-1}\cdot m)}=\chi_m$ for all m in N.

Hence we see that the neutral group J_χ for χ (see 2.2-A) is the intersection of K with the stability subgroup G_χ for χ. Further, $G_\chi=N*(J_\chi)$.

2.4-C. Let χ be in $\hat{N}$.

PROPOSITION. *There exists a unitary representation χ' of the stability subgroup G_χ for χ which extends χ, i.e., $\chi'|_N=\chi$.*

Proof. If (n, k) is in G_χ, define $\chi'_{(n,k)}$ as χ_n.

REMARK. The above proposition asserts that every orbit θ of $\hat{N}$ is accommodating.

2.4-D. As usual, denote by π the natural mapping of G onto K and also denote by π the restriction of that natural mapping to any closed subgroup of G which contains N.

DEFINITION. By a *cataloguing triple* we mean a triple (χ, J, T), where χ is a character of N, J is the neutral group J_χ for χ, and T is an irreducible unitary representation of J.

DEFINITION. If (χ, J, T) is a cataloguing triple, let $W(\chi, J, T)$ be the element of $\hat{G}$ determined by the representation $U^{(\chi'\otimes T\cdot\pi)}$, where χ' is the unitary extension of χ constructed in C above.

REMARK. In order for this definition to be valid, we must be sure that $U^{(\chi'\otimes T\cdot\pi)}$ is irreducible. But this follows from C above and 1.3-E.

PROPOSITION. *The mapping $(\chi, J, T)\to W(\chi, J, T)$ is onto $\hat{G}$.*

Proof. Let W be in $\hat{G}$. Then by Proposition 2 of 1.3-D there exists a character χ of N and an irreducible representation S of the stability subgroup G_χ such that W is determined by the representation U^S, i.e., U^S is contained in the equivalence class W. Then by 1.3-E the representations S which occur as in the preceding sentence can be assumed to be of the form $S=\chi'\otimes T\cdot\pi$, where χ' is the extension of χ constructed in C above and where T is an irreducible representation of G_χ/N, which equals J_χ. But this says that W is equal to $W(\chi, J, T)$. Q.E.D.

2.4-E. Suppose (χ, J, T) and (χ, J, S) are cataloguing triples such that T and S are equivalent. Then $W(\chi, J, T)$ equals $W(\chi, J, S)$. Hence we may think of the elements of $\hat{G}$ as being catalogued by triples (χ, J, T'), where χ is a character of N, J is the neutral group J_χ, and T' is an element of $\hat{J}$. Further, we may think of the triple (χ, J, T') as $(\chi, (J, T'))$, where χ is an element of $\hat{N}$ and where (J, T') is an element of the subgroup-representation space $\mathscr{A}(K)$.

Hence, we have catalogued the members of $\hat{G}$ by elements of the topological space $\hat{N} \times \mathscr{A}(K)$. We want to relate the topology of $\hat{G}$ with the topology of these parameters.

2.5. We are now ready to prove the easier half of our main results.

2.5-A. Let $[\chi^a]$ be the net of characters of N which converges to the character χ in $\hat{N}$. Let the net of pairs $[(J^a, T^a)]$ converge to the pair (J', T) in $\mathscr{A}(K)$, and suppose that, for each a, J^a is the neutral group $J_{(\chi^a)}$ for χ^a. Denote by J the neutral group J_χ for χ. By 2.2-B, J' is contained in J.

2.5-B. We preserve the notation of A above.

Theorem. *The net* $[W(\chi^a, (J^a, T^a))]$ *converges to every element* $W(\chi, (J, S))$ *of* $\hat{G}$ *such that the element* S *of* $\hat{J}$ *is weakly contained in* ${}_J U^T$.

Proof. We let π be the natural mapping of G onto G/N, and we write π for the restriction of this natural mapping to any closed subgroup of G which contains N.

Let $\chi^{a'}$ denote the extension of χ^a to its stability subgroup $G_{(\chi^a)}$ (2.4-C) and let χ' be the extension of χ to its stability subgroup G_χ. Then:

1. The net of pairs $[(G_{(\chi^a)}, \chi^{a'})]$ converges to the pair $((N * J'), \chi'|_{(N*J')})$ in $\mathscr{A}(G)$. This follows immediately from the description of the topology of $\mathscr{A}(K)$ in terms of the Fell topology.

2. The net of pairs $[(G_{(\chi^a)}, (T^a \cdot \pi))]$ converges to the pair $((N * J'), (T \cdot \pi))$ in $\mathscr{A}(G)$ (Proposition 3 of 1.4-D).

3. The net of pairs $[(G_{(\chi^a)}, (\chi^{a'} \otimes (T^a \cdot \pi)))]$ converges to the pair $((N * J'), (\chi'|_{(N*J')} \otimes (T \cdot \pi)))$ in $\mathscr{A}(G)$ (Proposition 2 of 1.4-D and 1 and 2 above).

4. The net $[W(\chi^a, (J^a, T^a))]$, which is determined by the net $[U^{(\chi^{a'} \otimes (T^a \cdot \pi))}]$, converges to every element W of $\hat{G}$ such that W is weakly contained in the representation $U^{[\chi'|_{(N*J')} \otimes (T \cdot \pi)]}$ (Proposition 1 of 1.4-D).

Hence our theorem depends on the analysis of the representation $U^{[\chi'|_{(N*J')} \otimes (T \cdot \pi)]}$. But $U^{[\chi'|_{(N*J')} \otimes (T \cdot \pi)]}$ is equivalent to $U^{[\chi' \otimes [{}_{(G_\chi)} U^T \cdot \pi]]}$, by 1.4-D, which is equivalent to $U^{[\chi' \otimes [[{}_J U^T] \cdot \pi]]}$, by 1.2-E.

Now, if S is an irreducible representation of J which is weakly contained in ${}_J U^T$, then $U^{\chi' \otimes S \cdot \pi}$ is weakly contained in $U^{[\chi' \otimes [[{}_J U^T] \cdot \pi]]}$. (Compare Theorem 1 of [8].) Hence the net $[W(\chi^a, (J^a, T^a))]$ converges to every element $W(\chi, (J, S))$ such that S is weakly contained in ${}_J U^T$. Q.E.D.

Remark. This is half of a theorem which describes the topology of $\hat{G}$ in terms of the topology of the cataloguing triples. There are no conditions on K in this theorem other than our general sets of hypotheses (see 1.3-B). To prove the other half of this theorem, we will need to make some restrictions on K.

3. The special case when K is abelian.

3.1. In this section we prove a theorem which is complementary to Theorem 2.5-B in the special situation where G is separable, N and K are both abelian, and N is regularly imbedded in G. (See Theorem 3.3.) However, we present first some very general constructions which will be needed in later sections as well as throughout §3.

3.1-A. Throughout subsection 3.1 we do not require that G be the semidirect product $N * K$, but we merely require that G and N satisfy one of the two sets of hypotheses in 1.3-B, and we require that G/N, which we shall denote by K, be an R-group (see 1.1-B).

Suppose W is an element of $\hat{G}$ which is contained in the closure of a subset B of $\hat{G}$. Let $[W^a]$ be a net of elements of B which converges to W in $\hat{G}$. Suppose χ is an element of $\hat{N}$ which is contained in the orbit with which W is associated. Then $W|_N$ weakly contains χ by 1.2-C. Hence, by Theorem 3.2 of [7], there exists a subnet $[W^{a^b}]$ of the net $[W^a]$ and, for each b, there exists an element χ^b of $\hat{N}$ such that:

(i) The net $[\chi^b]$ converges to χ.

(ii) For each b, $(W^{a^b})|_N$ weakly contains χ^b.

In view of the fact that each χ^b is contained in the closure of the orbit θ_b with which W^{a^b} is associated, we may actually choose the net $[\chi^b]$ such that:

(iii) For each b, χ^b is contained in the orbit of $\hat{N}$ with which W^{a^b} is associated. Hence,

(iv) For each b, W^{a^b} is contained in $U^{(\chi^b)}$. ((iv) follows from the propositions in 1.5-A.)

3.1-B. We maintain the same notation as developed in A above. Now the net $[G_{(\chi^b)}]$ of stability subgroups for the elements $[\chi^b]$ is a net in the compact Hausdorff space $\mathscr{K}(G)$. Therefore we may choose a subnet $[G_{(\chi^{b^c})}]$ of the net $[G_{(\chi^b)}]$ such that the net $[G_{(\chi^{b^c})}]$ converges to some unique subgroup G' of G.

3.1-C. Proposition. *Assume that G and N satisfy hypotheses* 1 *or* 2 *of* 1.3-B *and assume that G/N is an R-group. Let W be an element of $\hat{G}$ which is contained in the closure of a subset B of $\hat{G}$. Let χ be an element of $\hat{N}$ which is contained in the orbit of $\hat{N}$ with which W is associated. Then there exists a net of pairs $[(W^a, \chi^a)]$ such that:*

(i) *Each element W^a is contained in B, and the net $[W^a]$ converges to W in $\hat{G}$.*

(ii) *For each a, χ^a is an element of $\hat{N}$ which is contained in the orbit of $\hat{N}$ with which W^a is associated, whence, $(W^a)|_N$ weakly contains χ^a.*

(iii) *For each a, W^a is weakly contained in $U^{(\chi^a)}$ and W is weakly contained in U^χ.*

(iv) *The net $[\chi^a]$ converges to χ in $\hat{N}$.*

(v) *The net $[G_{(\chi^a)}]$ of stability subgroups for the elements $[\chi^a]$ converges to a subgroup G' of G.*

REMARK. The subgroup G' mentioned in property (v) is not in general the stability subgroup G_χ. Yet, this is precisely the difficulty we overcome in some special cases in this paper.

The proof of this proposition is contained in A and B above.

3.2. We present two propositions in this subsection, one a very general result which has particular application to the problem to be considered in this section and the other a special proposition relating to the present situation.

3.2-A. Let G be an arbitrary locally compact group, and let N be a closed normal subgroup of G such that G/N, which we call K, is an R-group. Denote by π the natural mapping of G onto K.

PROPOSITION. *Let T be an irreducible representation of G. Then T is weakly contained in $U^{(T|_N)}$.*

Proof. $U^{(T|_N)}$ equals $U^{(T|_N\otimes I)}$, where I is the trivial one-dimensional representation of N. Thus $U^{(T|_N)}$ is equivalent to $T \otimes (U^I)$ by 1.2-D.

But I equals $I'\cdot\pi$, where I' is the trivial one-dimensional representation of the one-element subgroup of K. So, by 1.2-E, U^I is equivalent to $[{}_KU^{I'}]\cdot\pi$. Thus $U^{(T|_N)}$ is equivalent to $T \otimes [[{}_KU^{I'}]\cdot\pi]$. Now since K is an R-group, ${}_KU^{I'}$, which is the regular representation of K, weakly contains the trivial one-dimensional representation I'' on K. Therefore, observing that $I''\cdot\pi$ is the trivial one-dimensional representation I''' of G, we have

$$U^{(T|_N)} \text{ weakly contains } T \otimes I''',$$

and this latter representation is precisely T. This completes the proof.

REMARK. The proof in the separable case follows from Corollary 1, p. 260, of [6].

COROLLARY 1. *Let G be an R-group, N any closed normal subgroup and T any irreducible representation of G. Then T is weakly contained in $U^{(T|_N)}$.*

This follows because G/N is an R-group.

COROLLARY 2. *Let J be an abelian group. If ϕ is a character of J and K' is a closed subgroup of J, then ϕ is weakly contained in $U^{(\phi|_{K'})}$.*

3.2-B. Now assume that G and N satisfy hypotheses 2 of 1.3-B. Further, let G be the semidirect product $N * K$ of two abelian groups. Then G/N, which equals K, is an abelian group, and hence an R-group (1.1-B).

Let T be an irreducible representation of G, θ the orbit of $\hat{N}$ with which T is associated, and χ an element of θ. Write χ' for the unitary representation of the stability subgroup G_χ for χ (2.4-C). Let J_χ denote the neutral group for χ. Then, by 1.3-E, there exists an irreducible representation ϕ of J_χ such that T is equivalent to $U^{(\chi'\otimes\phi')}$, where ϕ' is the irreducible representation of G_χ lifted from ϕ as follows: $\phi'_{(n,j)}=\phi_j$, where (n, j) is in G_χ, whence j is in J_χ.

Of course in this case ϕ is a character of the abelian group J_χ, and the representation $\chi' \otimes \phi'$ acts as follows:

If (n, j) is in G_χ, then $(\chi' \otimes \phi')_{(n,j)} = \chi_n \phi_j$.

3.2-C. We preserve the notation in B above.

PROPOSITION. *$T|_{(J_\chi)}$ is a multiple of the character ϕ. (This is a special case of Mackey's "restriction theorem." See* [13].)

Proof. We prove the assertion for the equivalent representation $U^{(\chi'\otimes\phi')}$, i.e., we prove that $U^{(\chi'\otimes\phi')}|_{(J_\chi)}$ is a multiple of the character ϕ.

The proof hinges on the following property of the functions f in the space of $U^{(\chi'\otimes\phi')}$.

If (n, k) is in G, (m, j) is in G_χ, and f is in $H(U^{(\chi'\otimes\phi')})$, then using property (i) of 1.2-A and 2.1-C,

$$\begin{aligned} f[(n,k)(m,j)] &= [\delta_{(G_\chi)}(m,j)/\delta_G(m,j)]^{1/2}(\chi' \otimes \phi')_{((m,j)^{-1})}(f(n,k)) \\ &= [\delta_N(m)\delta_{(J_\chi)}(j)d(j^{-1})/\delta_N(m)\delta_K(j)d(j^{-1})]^{1/2}(\chi' \otimes \phi')_{(m,j^{-1})}(f(n,k)) \\ &= \chi_{-m}\phi_{(j^{-1})}f(n,k). \end{aligned}$$

Now recalling how $U^{(\chi'\otimes\phi')}$ acts, we have:

If f is in $H(U^{(\chi'\otimes\phi')})$, (n, k) is in G, and j is in J_χ, then

$$\begin{aligned} [U_j^{(\chi'\otimes\phi')}(f)](n,k) &= f[(0,j^{-1})(n,k)] = f(j^{-1}\cdot n, j^{-1}k) \\ &= f[(0,j^{-1}k)(k^{-1}\cdot n, e)] = \chi_{(-k^{-1}\cdot n)}f(0, kj^{-1}) \\ &= \chi_{(-k^{-1}\cdot n)}f[(0,k)(0,j^{-1})] = \chi_{(-k^{-1}\cdot n)}\phi_j f(0,k) = \phi_j f(n,k). \end{aligned}$$

Hence, $U_j^{(\chi'\otimes\phi')}$ equals ϕ_j times the identity operator, and the proposition is proved.

3.3. Continue to assume that G and N satisfy hypotheses 2 of 1.3-B and that G is the semidirect product of two abelian groups N and K. Recall from 2.4-E that each element W of $\hat{G}$ can be catalogued by a triple $(\chi, (J, T))$, where χ is a character of N, J is the neutral group J_χ for χ, and T is an element of $\hat{J}$. (In the present instance, T is actually a character of J.)

THEOREM. *Let B be a subset of $\hat{G}$ and W an element of $\hat{G}$. Then W is contained in the closure of B if and only if there exist: a cataloguing triple $(\chi, (J, \phi))$ for W, a net $[(\chi^a, (J^a, \phi^a))]$ of cataloguing triples, and an element (K', ψ) of $\mathscr{A}(K)$, such that:*

(i) *The net $[(\chi^a, (J^a, \phi^a))]$ converges to the triple (not necessarily a cataloguing triple) $(\chi, (K', \psi))$ in $N \times \mathscr{A}(K)$.*

(ii) *K' is a subgroup of J.*

(iii) *${}_J U^\psi$ weakly contains ϕ.*

(iv) *For each a, $W(\chi^a, (J^a, \phi^a))$ is in B.*

(See 2.4-D for the meaning of $W(\chi^a, (J^a, \phi^a))$.)

Proof. 3.3-A. If the cataloguing triples $(\chi, (J, \phi))$ and $[(\chi^a, (J^a, \phi^a))]$ and the element (K', ψ) can be found so that properties (i)–(iv) are satisfied, then Theorem 2.5-B guarantees that the net $[W(\chi^a, (J^a, \phi^a))]$ converges to W; hence W is in the closure of B.

3.3-B. Assume W is in the closure of B. Choose an element χ of the orbit of $\hat{N}$ with which W is associated. By 3.1-C, choose a net of pairs $[(W^a, \chi^a)]$ which satisfies the five properties of that proposition.

Since χ is in the orbit of $\hat{N}$ with which W is associated, fix a cataloguing triple $(\chi, (J, \phi))$ for W, and similarly, since for each a, χ^a is in the orbit of $\hat{N}$ with which W^a is associated, fix cataloguing triples $(\chi^a, (J^a, \phi^a))$ for each W^a.

Now, by property (v) of 3.1-C and by 2.1-D, the net $[J^a]$ converges to a subgroup of K which we call K'. Then, by 2.2-B, K' is a subgroup of J and this establishes property (ii) of the theorem.

Define ψ as the character $\phi|_{K'}$ of K'. Then, by Corollary 2 of 3.2-A, (iii) of the theorem is satisfied.

Since by property (iv) of 3.1-C the net $[\chi^a]$ converges to χ, we have left to show only that the net $[(J^a, \phi^a)]$ converges to (K', ψ) in $\mathscr{A}(K)$, because property (iv) of the theorem is implied by 3.1-C.

$W|_{K'}$ is equivalent to $(W|_J)_{K'}$ which is equivalent to a multiple of $\phi|_{K'}$, by 3.2-C. Since $(K')^\wedge$ is Hausdorff, the only element of $(K')^\wedge$ which is weakly contained in $W|_{K'}$ is $\phi|_{K'}$.

Now since the net $[J^a]$ converges to K', Theorem 3.2 of [7] proves the existence of a subnet $[W^{a^b}]$ of the net $[W^a]$ such that there exists, for each b, an element ψ^b in $(J^{a^b})^\wedge$ such that

(i) $(W^{a^b})|_{(J^{a^b})}$ weakly contains ψ^b.

(ii) The net $[(J^{a^b}, \psi^b)]$ converges to $(K', \phi|_{K'})$. But, $(W^{a^b})|_{(J^{a^b})}$ is a multiple of ϕ^{a^b}. Hence ψ^b must equal ϕ^b, and we have that the net $[(J^{a^b}, \phi^{a^b})]$ converges to $(K', \phi|_{K'})$.

This completes the proof of the theorem.

4. A convergence theorem.

4.1. Assume now that G is the semidirect product $N * K$ of an abelian group N and a *compact* group K. Recall that, for each χ in $\hat{N}$, the representation U^χ acts in the Hilbert space $L^2(K)$, and also recall that, under the usual definitions of convolution and involution of functions, $L^2(K)$ is an H^*-algebra. (See [12].)

4.1-A. If χ is a character of N, then $U^\chi|_K$ is the left regular representation of K:

$$[U^\chi_k(f)](l) = f(k^{-1}l).$$

PROPOSITION. *Suppose Z is a subspace of $L^2(K)$ which is stable under some representation U^χ, for χ in $\hat{N}$. Then Z is a left ideal of $L^2(K)$.*

Proof. By the comment just preceding this proposition, Z is stable under the left regular representation of K, hence is stable under left translations and therefore is a left ideal. Q.E.D.

4.1-B. Definition. If f is in $L(G)$ and g is in $L(K)$, define the function $g\cdot f$ on G as follows:

$$g\cdot f(n, k) = \int_K g(p^{-1})f(p\cdot n, pk)\,dp.$$

Now, since f is in $L(G)$, we may take as the support of f a compact set of the form $K\times C$, where C is a compact subset of N which is stable under the action of K. Then $g\cdot f$ also vanishes off the set $K\times C$. Since $g\cdot f$ is continuous in the variable (n, k) and has compact support, $g\cdot f$ is in $L(G)$.

4.1-C. Now let V be a vector in $L^2(K)$ of the form $V=[U_f^\chi(h)]$, where f is in $L(G)$ and h is in $L^2(K)$. Then, if g is in $L(K)$, we have:

$$\begin{aligned}
[U^\chi_{g\cdot f}(h)](l) &= \int_K\int_N g\cdot f(n, k)[U^\chi_{(n,k)}(h)](l)\,dn\,dk \\
&= \int_K\int_N\int_K g(p^{-1})f(p\cdot n, pk)\chi_{(l^{-1}\cdot n)}h(k^{-1}l)\,dp\,dn\,dk \\
&= \int_K\int_K\int_N g(p^{-1})f(n, pk)\chi_{(l^{-1}p^{-1}n)}h(k^{-1}l)\,dn\,dk\,dp \\
&= \int_K\int_N\int_K g(p^{-1})f(n, k)\chi_{(l^{-1}p^{-1}\cdot n)}h(k^{-1}pl)\,dk\,dn\,dp \\
&= \int_K g(p^{-1})[U_f^\chi(h)](pl)\,dp = [g*[U_f^\chi(h)]](l).
\end{aligned}$$

Remark 1. The $\cdot$ operation defined in B above is a special case of the convolution of measures on the group G. (See [11].)

Remark 2. We showed in A above that any subspace of $L^2(K)$ which is stable under some representation U^χ must necessarily be a left ideal. From the above calculation, we see somewhat more clearly how left multiplication is accomplished.

4.1-D. We give here a theorem somewhat out of context, but there seems no better place for it.

Theorem. *Let G be the semidirect product of an abelian group N and a compact group K. Then G is a* CCR-*group.*

Proof. Proposition 1 of 1.5-A assures us that each irreducible representation of G occurs as a direct summand of some representation U^χ, for χ in $\hat{N}$. Hence, it will suffice to prove that, for each f in $C^*(G)$ and each representation U^χ of G, the operator U_f^χ is completely continuous.

If K is a compact group, recall that a *representative function* on K is an element g of $L(K)$ such that the two-sided ideal of $L^2(K)$ generated by g is finite dimensional. Further, recall that the set of representative functions constitutes a dense subspace in $L^2(K)$.

Now if f is in $L(G)$ and g is a representative function on K, then the operator $U^{\chi}_{g\cdot f}$ is completely continuous for all χ; in fact, its range is finite dimensional since it is contained in the right ideal of $L^2(K)$ generated by g.

Define B to be the linear span of the set of all functions on G of the form $g\cdot f$, where f is in $L(G)$ and g is a representative function on K. We claim that B is dense in $L^1(G)$.

Proof of the claim. Let h be in $L^\infty(G)$ and such that $(g\cdot f, h)=0$ for all elements f of $L(G)$ and all representative functions g on K. (Here the $(\ ,\ )$ indicates the duality between $L^1(G)$ and $L^\infty(G)$.) Hence, we have:

$$0 = \int_K \int_N \int_K g(p^{-1})f(p\cdot n, pk)h(n,k)\,dp\,dn\,dk$$

$$= \int_K g(p^{-1})\left[\int_K \int_N f(p\cdot n, pk)h(n,k)\,dn\,dk\right] dp.$$

Since these equalities hold for all representative functions g on K, we conclude that

$$0 = \int_K \int_N f(pn, pk)h(n,k)\,dn\,dk$$

$$= \int_K \int_N f_{(0,p)}(n,k)h(n,k)\,dn\,dk = (f_{(0,p)}, h),$$

for almost all p in K and for all elements f in $L(G)$. But the function $p \to (f_{(0,p)}, h)$ is continuous in the variable p, and hence $0=(f_{(0,p)}, h)$ for all elements f in $L(G)$ and all elements p in K. Thus, for all elements f in $L(G)$, $(f, h)=0$, and thus h is the zero element of $L^\infty(G)$. This shows that B is dense in $L^1(G)$. Therefore, B is dense in $C^*(G)$. Hence, $U^{\chi}_{[C^*(G)]}$ is the same as the set $U^{\chi}_{[B]^-}$, which is contained in the norm closure of the set $[U^{\chi}_{[B]}]^-$, which is contained in the norm closure of the set of all finite dimensional operators on $L^2(K)$, which is precisely the set of completely continuous operators. Q.E.D.

4.2-A. Suppose that W is an element of $\hat{G}$ which is contained in the closure of a subset B of $\hat{G}$. (The element W and the set B can be considered as belonging to the dual space $[C^*(G)]^\wedge$ of $C^*(G)$.) Let χ be an element of $\hat{N}$ which is contained in the orbit with which W is associated. Now Proposition 1 of 1.5-A assures us that there exists a subspace P of $H(U^\chi)$, which equals $L^2(K)$, such that $U^\chi|_P$ is contained in the equivalence class W.

By 4.1-A, P is a left ideal of $L^2(K)$, and therefore can be written as the direct sum of minimal left ideals.

Let h be a nonzero vector of $L^2(K)$ which is contained in some minimal left ideal of P. Let I be the minimal two-sided ideal of $L^2(K)$ to which h belongs.

Define ϕ to be the positive functional on $C^*(G)$ given as follows:

If f is in $C^*(G)$, then $\phi(f)=([U^\chi_f(h)], h)$. ϕ is then a positive functional associated with W.

LEMMA. *There exists a net of pairs* $[(W^a, \phi^a)]$ *such that*:

(i) *For each* a, W^a *is an element of* G *and the net* $[W^a]$ *converges to* W.

(ii) *For each* a, ϕ^a *is a positive functional on* $C^*(G)$ *which is associated with* W^a.

(iii) *The net* $[\phi^a]$ *converges to* ϕ *in the weak-star topology of functionals on* $C^*(G)$, *and, for all* a,

$$\|\phi^a\| \leqq \|\phi\|.$$

The proof follows from [3].

PROPOSITION. *Assume the same hypotheses as above. There exists a net of triples* $[(W^a, \phi^a, \chi^a)]$ *which satisfies*:

(i) *For each* a, W^a *is an element of* B *and the net* $[W^a]$ *converges to* W *in* $\hat{G}$.

(ii) *For each* a, ϕ^a *is a positive functional associated with* W^a, *and the net* $[\phi^a]$ *converges to* ϕ *in the weak-star topology of functionals on* $C^*(G)$. *Also*, $\|\phi^a\| \leqq \|\phi\|$ *for all* a.

(iii) *For each* a, χ^a *is contained in the orbit of* $\hat{N}$ *with which* W^a *is associated, whence* $W^a|_N$ *weakly contains* χ^a.

(iv) *The net* $[\chi^a]$ *converges to* χ *in* $\hat{N}$.

(v) *The net* $[G_{(\chi^a)}]$ *of stability subgroups for the element* $[\chi^a]$ *converges to a subgroup* G' *of* G_χ *in* $\mathscr{K}(G)$.

(vi) *For each* a, *there exists a subspace* P_a *of* $L^2(K)$ *such that* $U^{(\chi^a)}|_{(P_a)}$ *is contained in the equivalence class* W^a.

Proof. Properties (i), (iii), (iv), and (v) are consequences of 3.1-C. Property (ii) follows by combining 3.1-C with the lemma above. Property (vi) follows from property (iii) and Proposition 1 of 1.5-A.

4.2-B. We preserve the above notation. For each a, choose a vector h_a of $L^2(K)$ such that:

(i) h_a is in P_a.

(ii) The positive functional ϕ^a is given by: if f is in $C^*(G)$,

$$\phi^a(f) = ([U_f^{(\chi^a)}(h_a)], h_a).$$

PROPOSITION. *The net* $[\phi^a]$ *of positive functionals guaranteed by Proposition* 4.2-A *above may be chosen in such a way that all the vectors* $[h_a]$ *lie in the minimal two-sided ideal* I *of* $L^2(K)$ *which contains* H.

Proof. Let g be the identity for the minimal two-sided ideal I. Then, if f is in $L(G)$,

$$\begin{aligned}\phi(f) &= ([U_f^\chi(h)], h) = ([U_f^\chi(g * h)], g * h) = (g * [U_f^\chi(g * h)], h)\\ &= ([U_{g\cdot f}^\chi(g * h)], h) = (g * h, [U_{((g\cdot f)^*)}^\chi(h)]) = (h, g * [U_{((g\cdot f)^*)}^\chi(h)])\\ &= (h, [U_{g\cdot((g\cdot f)^*)}^\chi(h)]) = ([U_{((g\cdot((g\cdot f)^*))^*)}^\chi(h)], h) = \phi[((g\cdot((g\cdot f)^*))^*)]\\ &= \lim_a \phi^a[((g\cdot((g\cdot f)^*))^*)] = \lim_a ([U_f^{(\chi^a)}(g * h_a)], g * h_a),\end{aligned}$$

where the last equality follows by reversing some of the above calculations.

Hence we see that the vectors $[g * h_a]$ define positive functionals ϕ'^a on $L(G)$, and, if f is in $L(G)$, we have $\lim_a \phi'^a(f) = \phi(f)$. Also, for each a, and for each element f of $L(G)$,

$$|\phi'^a(f)| \leqq \|g * h_a\|^2 \|f\|_{C^*(G)}.$$

Hence each functional ϕ'^a can be extended to all of $C^*(G)$.

Now for all a,

$$\|\phi'^a\| = \|g * h_a\|^2 \leqq \|h_a\|^2 = \|\phi^a\| \leqq \|\phi\|.$$

This sequence of inequalities tells us, first of all, that the weak-star convergence of the net $[\phi'^a]$ to ϕ on $L(G)$ can be extended to weak-star convergence on all of $C^*(G)$. Secondly, the fact that the set $[\|\phi'^a\|]$ is bounded by $\|\phi\|$ is the remaining condition needed in order to assert that the net $[\phi'^a]$ suffices for the net of positive functionals called for in 4.2-A.

Since the vectors $[g * h_a]$ all lie in I, the proposition is proved.

4.2-C. Lemma. *Let W be an element of $\hat{G}$ which is contained in the closure of a subset B of $\hat{G}$. We assume the results and notation of* A *and* B *above. Then, there exists a subspace Z of $L^2(K)$, for which $U^x|_Z$ is contained in the equivalence class W, such that, for every vector g in Z, there exists a subnet $[P_{a_b}]$ of the net $[P_a]$, and to each b, there corresponds a vector g_b such that:*

(i) *For each b, g_b is contained in P_{a_b}.*

(ii) *The net $[g_b]$ converges to g in $L^2(K)$.*

Proof. Let h be as in Proposition A above, and let the net $[h_a]$ of elements of I be as in B above. Then, for each a,

$$\|h_a\|^2 = \|\phi^a\| \leqq \|\phi\| = \|h\|^2.$$

Thus the set $[h_a]$ of vectors is bounded in norm. Since I is finite dimensional, there exists a subnet $[h_{a_b}]$ of the net $[h_a]$ such that the net $[h_{a_b}]$ converges to a vector v in I.

Define Z to be the closed subspace generated by the vector v under the representation U^x. Let Z' be the dense subspace of Z consisting of vectors of the form $U_f^x(v)$, where f is in $C^*(G)$.

If v', which equals $U_f^x(v)$, is an element of Z', then the net $[U_f^{(x^{a_b})}(h_{a_b})]$ of vectors in $L^2(K)$ satisfies:

(i) For each b, $U^{(x^{a_b})}(h_{a_c})$ is contained in P_{a_b}.

(ii) The net $[U_f^{(x^{a_b})}(h_{a_b})]$ converges to $U_f^x(v)$.

(ii) follows because the net $[h_{a_b}]$ converges to v and by Theorem 1.5-B.

But, Z' is dense in Z, and hence if v'' is an element of Z, then there exists a subnet $[P_{a_{b_c}}]$ of the net $[P_{a_b}]$, and to each c, there corresponds a vector g_c such that:

(i) For each c, g_c is contained in $P_{a_{b_c}}$.

(ii) The net $[g_c]$ converges to v''.

Now the theorem would be complete if we knew that $U^{\chi}|_Z$ were contained in the class W.

We observe that, for any f in $C^*(G)$,

$$([U_f^{\chi}(v)], v) = \lim_b \phi^{a^b}(f)$$
$$= \phi(f).$$

Hence, $U^{\chi}|_P$ and $U^{\chi}|_Z$ are two cyclic representations of $C^*(G)$ whose cyclic vectors define the same positive functional on $C^*(G)$. Therefore $U^{\chi}|_P$ is equivalent to $U^{\chi}|_Z$.

This completes the proof.

4.2-D. THEOREM. *Let W be an element of $\hat{G}$ which is contained in the closure of a subset B of $\hat{G}$. Let χ be an element of the orbit of $\hat{N}$ with which W is associated. Then there exists a subspace Z of $L^2(K)$, for which $U^{\chi}|_Z$ is contained in the equivalence class W, such that, if I is a minimal two-sided ideal of $L^2(K)$, then there exists a net of triples $[(W^a, \chi^a, Z^a)]$ satisfying*:

(i) *Each W^a is an element of B and the net $[W^a]$ converges to W in $\hat{G}$.*

(ii) *For each a, χ^a is a character of N which is contained in the orbit of $\hat{N}$ with which W^a is associated.*

(iii) *For each a, Z^a is a subspace of $L^2(K)$ such that $U^{(\chi^a)}|_{(Z^a)}$ is contained in the equivalence class W^a.*

(iv) *For each element h of $Z \cap I$, there exists a net $[h_a]$ of elements of $L^2(K)$ such that, for each a, h_a is in $Z^a \cap I$, and such that the net $[h_a]$ of functions converges uniformly to h.*

(v) *The net $[\chi^a]$ converges to χ in $\hat{N}$.*

(vi) *The net $[G_{(\chi^a)}]$ of stability subgroups for the elements $[\chi^a]$ of $\hat{N}$ converges to a subgroup G' of G_{χ}.*

Proof. This theorem is mostly a combination of the lemma above and the proposition in 4.2-A. We do need to comment on property (iv) however.

If h belongs to $Z \cap I$, then, by Lemma C above, there exists a net $[h_a]$ such that, for each a, h_a belongs to Z^a and such that $[h_a]$ converges to h in the $L^2(K)$ norm. But, if g is the identity of I, then the net $[g * h_a]$ has the required properties for (iv) of the theorem.

5. **Two final lemmas.**

5.1. Let K be a compact group and J a closed subgroup of K. Let R^K and R^J be the left regular representations of K and J respectively. Suppose M is a finite dimensional left ideal of $L^2(K)$ and denote by T the representation $R^K|_M$.

LEMMA. (i) *The set $M|_J$ of all functions $f|_J$ where f is an element of M, constitutes a left ideal of $L^2(J)$.*

(ii) *If N is a minimal left ideal of $L^2(J)$ contained in $M|_J$, and if S denotes the representation $R^J|_N$, then $T|_J$ contains S as a direct summand.*

Proof. (i) follows because $M|_J$ is a closed subspace of $L^2(J)$ which is closed under left translations by elements of J.

To prove (ii), we define the operator θ on M as follows: If h is in M, then $\theta(h)=h|_J$.

Now observe that, if j and k are in J and h is in M,

$$[\theta[T_j(h)]](k) = [[T_j(h)]|_J](k) = [T_j(h)](k) = h(j^{-1}k) = [R_j^J(\theta(h))](k).$$

Hence θ intertwines $R^J|_{(M|_J)}$ and $T|_{J|_M}$. Also, since M is finite dimensional, θ is continuous.

Write M as $Z \oplus M'$, where Z is the kernel of θ. Then, by Theorem 1.2 of [14], we have:

$$T|_{J|_{M'}} \text{ is equivalent to } R^J|_{(M|_J)},$$

and this latter representation contains S as a direct summand. Q.E.D.

COROLLARY. *If the T in the above lemma is nT', where n is a positive integer, then $T'|_J$ contains S as a direct summand.*

5.2. Assume now that G is the semidirect product of an abelian group N and a compact group K. Suppose χ is an element of $\hat{N}$ and let $N*J$ be the stability subgroup G_χ for χ. Let V be the representation ${}_{(N*J)}U^\chi$ of $N*J$.

Now we know by Theorem 2.2 of [1] that V is equivalent to the representation $\chi' \otimes R'$, where χ' is the unitary extension of χ to $N*J$ (2.4-C), and where R' is the representation of $N*J$ lifted from the left regular representation R of J. By [2] we know that U^V is equivalent to U^χ. We now make this explicit. We examine the representation $U^{(\chi'\otimes R')}$. Recall that $(\chi' \otimes R')_{(n,j)}=\chi_n R_j$.

5.2-A. The space of the representation $U^{(\chi'\otimes R')}$ consists of all locally Bochner measurable functions f on G into $L^2(J)$, which equals $H(\chi' \otimes R')$, which satisfy the following three conditions:

(i) If (n, k) is in G and (m, j) is in $N*J$, then

$$\begin{aligned} f[(n, k)(m, j)] &= (\chi' \otimes R')_{((m,j)^{-1})}[f(n, k)] \\ &= \chi_{-m}R_{(j^{-1})}[f(n, k)]. \end{aligned}$$

(ii) The mapping $(n, k) \to \|f(n, k)\|^2_{L^2(J)}$ must be locally summable on G.

(iii) The measure on $G/(N*J)$, denoted by $\lambda_{f,f}$, must have finite total mass.

5.2-B. We may compute the norm of an element f of $H(U^{(\chi'\otimes R')})$ as follows:

Let h be the function identically one on $G/(N*J)$. Since G/N is compact, $G/(N*J)$ is compact, and hence h is an element of $L(G/(N*J))$. Then the norm squared of an element f of $H(U^{(\chi'\otimes R')})$ is $\lambda_{f,f}(h)$.

Choose an element q of $L(N)$ such that $\int_N q(n)\,dn$ equals one. Define h' as the element of $L(G)$ which sends the pair (n, k) to $q(n)$. Then observe that

$$\begin{aligned}\int_{(N*J)} h'[(n,k)(m,j)]\,d(m,j) &= \int_J\int_N h'((n+k\cdot m), kj)\,dm\,dj \\ &= \int_J\int_N q(n+k\cdot m)\,dm\,dj = \int_J\int_N q(n+m)\,dm\,dj \\ &= \int_J\int_N q(m)\,dm\,dj = 1 = h(\pi(n,k)),\end{aligned}$$

where π denotes the mapping of G onto the space of left cosets of the group $N * J$.

Hence, by 1.2-A, the total mass of $\lambda_{f,f}$ is given by:

$$\begin{aligned}\lambda_{f,f}(h) &= \int_G h'(x)\|f(x)\|^2_{L^2(J)}\,dx = \int_K\int_N h'(n,k)\|f(n,k)\|^2_{L^2(J)}\,dn\,dk \\ &= \int_K\int_N q(n)\,dn\|f(0,k)\|^2_{L^2(J)}\,dk = \int_K\int_J |[f(0,k)](j)|^2\,dj\,dk.\end{aligned}$$

5.2-C. The action of $U^{(\chi'\otimes R')}$ is as follows:

If f is in $H(U^{(\chi'\otimes R')})$, (n, k) and (m, l) are in G,

$$[U^{(\chi'\otimes R')}_{(n,k)}(f)](m,l) = f[(n,k)^{-1}(m,l)].$$

5.3. We now set up the equivalence mentioned above.

5.3-A. Definition. If h is in $L(K)$, define $Q(h)$ to be the following function on G to $L(J)$.

If (n, k) is in G and j is in J,

$$[[Q(h)](n,k)](j) = \chi_{(-k^{-1}\cdot n)}h(kj).$$

5.3-B. One may verify that $Q(h)$ satisfies the three conditions of 5.2-A. Further,

$$\|Q(h)\|^2_{H(U^{\chi'\otimes R'})} = \int_K |h(k)|^2\,d(k).$$

Hence we see that Q is an isometry.

5.3-C. We wish now to show that Q is onto a dense subspace of $H(U^{(\chi'\otimes R')})$.

Since J is compact the representation $(\chi'\otimes R')$ is completely reducible, and in fact $L^2(J)$, which equals $H(\chi'\otimes R')$, is the direct sum $\sum_i (I_i)$, where each subspace I_i is $(\chi'\otimes R')$-stable, and each I_i is a finite dimensional (minimal) two-sided ideal of $L^2(J)$.

Combining this fact with [2], we see that there exists a dense subspace X of $H(U^{(\chi'\otimes R')})$ consisting of functions f which satisfy:

(i) f is continuous from G into $L^2(J)$.

(ii) For each x in G, $f(x)$ is a continuous function on J.

(iii) There exists a finite-dimensional subspace I_f, depending upon f, of $L^2(J)$ such that, for each x, $f(x)$ lies in the subspace I_f.

Now if f is in X, define $P(f)$ to be the following function on K.

If k is in K, then $[P(f)](k) = [f(0,k)](e)$.

Now since I_f is a finite-dimensional topological vector space of continuous functions, evaluation at the identity element of J is continuous on I_f. Hence $P(f)$ is a continuous function on K.

Observe that $Q[P(f)]=f$.

Hence Q maps $L(K)$ onto a dense subspace of $H(U^{(\chi'\otimes R')})$. Since we have seen that Q is an isometry, we may extend Q to all of $L^2(K)$ and conclude that Q maps $L^2(K)$ onto $H(U^{(\chi'\otimes R')})$.

One may now show that Q is the intertwining operator needed to display the equivalence of U^χ and $U^{(\chi'\otimes R')}$.

5.4. Let Z be an irreducible subspace of $L^2(K)$ under the representation U^χ. Choose an irreducible representation T of J so that $U^\chi|_Z$ is equivalent to $U^{(\chi'\otimes T')}$, where T' is the representation of $N * J$ lifted from T. Denote by R^K and R^J the left regular representations of K and J as in 5.1.

5.4-A. Lemma. *Let Y be the set of restrictions to J of the continuous functions in Z. Then $R^J|_Y$ is equivalent to nT, where n is a positive integer.*

Proof. Decompose $L^2(J)$ into a direct sum of its minimal closed two-sided ideals I_i. We may then decompose the space $H(U^{(\chi'\otimes R^{J'})})$ into a direct sum of subspaces M_i, where each M_i consists of the functions on G and in $H(U^{(\chi'\otimes R^{J'})})$ taking values only in the ideal I_i. Let I_0 be the minimal two-sided ideal of $L^2(J)$ associated with the representation T of J. Then the $U^{(\chi'\otimes T')}$-subspace of $H(U^{(\chi'\otimes R^{J'})})$ is exactly M_0.

Now $U^{(\chi'\otimes R^{J'})}|_{(Q(Z))}$ is equivalent to $U^\chi|_Z$, which is equivalent to $U^{(\chi'\otimes T')}$. Hence, by [14], $Q(Z)$ must lie in the subspace M_0. Therefore, if h is a continuous element of Z, then the function $h|_J$, which equals $[Q(h)](0, e)$, must lie in the ideal I_0. Hence Y lies in the ideal I_0 and is clearly a subspace of I_0 which is closed under left translation by elements of J. This completes the proof of the lemma.

5.4-B. Corollary. *Let X be a minimal left ideal of Z. Then the set X' of restrictions to J of the functions in X is a nonzero left ideal of $L^2(J)$, and, $R^J|_{X'}$ is equivalent to nT, where n is a positive integer.*

6. **The main theorem.**

6.1. Assume that G is the semidirect product $N * K$ of an abelian group N and a compact group K.

Theorem. *Let W be an element of $\hat{G}$ which is contained in the closure of a subset B of $\hat{G}$. Then there exist: an element (K', S') of $\mathscr{A}(K)$, a cataloguing triple $(\chi, (J, T))$ for W and a net of cataloguing triples* $[(\chi^a, (J^a, T^a))]$ (2.4-E), *which satisfy*:

(i) *K' is a subgroup of J.*

(ii) *${}_JU^{S'}$ contains T as a direct summand.*

(iii) *For each a, the triple $(\chi^a, (J^a, T^a))$ catalogues an element W^a of B.*

(iv) *The net $[(\chi^a, (J^a, T^a))]$ converges to $(\chi, (K', S'))$ in the product topological space $\hat{N}\times\mathscr{A}(K)$.*

Proof. Let χ be an element of the orbit of $\hat{N}$ with which W is associated. Choose a subspace Z of $L^2(K)$ as guaranteed by Theorem 4.2-D. Let X be a minimal left ideal of Z and let I be the minimal two-sided ideal of $L^2(K)$ which contains X. Denote as usual by J the neutral group J_χ for χ. For each character χ^a, given by Theorem 4.2-D, let J^a denote the neutral group $J_{(\chi^a)}$ for χ^a. Then by property (vi) of 4.2-D, and by 2.1-D, we know that the net $[J^a]$ converges to a subgroup K' of K, and by 2.2-B, K' is a subgroup of J. This establishes (i) of the theorem.

Denote by R^A the left regular representation of a locally compact group A.

Define M to be the subspace of restrictions to J of the elements of X. For each a, define M^a to be the subspace of restrictions to J^a of the continuous elements of Z^a. Now, by Lemma 5.4, we have:

(i) $R^J|_M$ is equivalent to nT, where T is an irreducible representation of J such that the triple $(\chi, (J, T))$ catalogues W.

(ii) For each a, $R^{(J^a)}|_{(M^a)}$ is equivalent to n^aT^a, where T^a is an irreducible representation of J^a such that the triple $(\chi^a, (J^a, T^a))$ catalogues W^a.

Thus, with these choices of cataloguing pairs, we have established (iii) of the theorem.

Let L be a minimal left ideal of $L^2(K')$ which is contained in $M|_{K'}$, where $M|_{K'}$ is defined in 5.1. Denote by S' the representation $R^{K'}|_L$. Then, by Corollary 5.1, $T|_{K'}$ contains S', and by the Frobenius reciprocity theorem, ${}_JU^{S'}$ contains T. We have thus established (ii) of the theorem.

We have left to prove (iv), and, in view of property (v) of 4.2-D, we need show only that the net $[(J^a, T^a)]$ converges to the pair (K', S') in $\mathscr{A}(K)$. We use 1.4-C.

Choose a nonzero element f in L. Then define the function of positive type ϕ associated with S' as follows:

If k is in K',

$$\phi(k) = \int_{K'} f(k^{-1}t)\overline{f(t)}\, dt.$$

Now $f=h|_{K'}$ for some h in the left ideal X. Now by (iv) of 4.2-D, there exists a net $[h_a]$ of elements of $L^2(K)$ such that:

(i) The net $[h_a]$ converges uniformly on K to h.

(ii) For each a, h_a belongs to $I \cap Z^a$.

For each a, define f^a as $h^a|_{(J^a)}$. Then f^a is in M^a. Let ϕ^a be the following function on J^a.

If j is in J^a,

$$\phi^a(j) = \int_{(J^a)} f^a(j^{-1}t)\overline{f^a(t)}\, dt.$$

Then, for each a, ϕ^a is a function of positive type on J^a associated with the representation n^aT^a, i.e., ϕ^a is a finite sum of functions of positive type associated with T^a. Hence, in order to show the convergence of the net $[(J^a, T^a)]$ to the pair (K', S'), 1.4-C assures us that it is sufficient to show the convergence of the net $[\phi^a]$ to ϕ in the Fell topology of functions on subgroups of K. (See 1.4-C.)

Thus assume, without loss of generality, that the net $[\phi^a]$ has been replaced by a subnet of itself without changing notation. Let $[j^a]$ be a net of elements of K satisfying:

(i) For each a, j^a is contained in J^a.

(ii) The net $[j^a]$ converges to an element k of K'.

We need to show that the net $[\phi^a(j^a)]$ converges to $\phi(k)$. Hence, if $\varepsilon > 0$,

$$|\phi(k)-\phi^a(j^a)| = \left|\int_{K'} f(k^{-1}t)\overline{f(t)}\,dt - \int_{(J^a)} f^a((j^a)^{-1}t)\overline{f^a(t)}\,dt\right|$$

$$= \left|\int_{K'} h(k^{-1}t)\overline{h(t)}\,dt - \int_{(J^a)} h_a((j^a)^{-1}t)\overline{h_a(t)}\,dt\right|$$

$$\leq \left|\int_{K'} h(k^{-1}t)\overline{h(t)}\,dt - \int_{(J^a)} h(k^{-1}t)\overline{h(t)}\,dt\right| \tag{1}$$

$$+\left|\int_{(J^a)} [h(k^{-1}t)-h((j^a)^{-1}t)]\overline{h(t)}\,dt\right| \tag{2}$$

$$+\left|\int_{(J^a)} [h((j^a)^{-1}t)\overline{h(t)}-h_a((j^a)^{-1}t)\overline{h_a(t)}]\,dt\right|. \tag{3}$$

Now (1) is eventually less than $\varepsilon/3$ because of the continuity of H, the definition of a smooth choice, and by Proposition 1.4-A.

(2) is eventually less than $\varepsilon/3$ because of the uniform continuity of h.

(3) is eventually less than $\varepsilon/3$ because of the uniform convergence of the net $[h_a]$ to h.

The theorem is now completely proved.

COROLLARY. *Suppose $[W^a]$ is a net of elements of $\hat{G}$ which converges to an element W. Then there exist: an element (K', S') of $\mathscr{A}(K)$, a cataloguing triple $(\chi, (J, T))$ for W and a net of cataloguing triples $[\chi^b, (J^b, T^b)]$, such that:*

(i) *The net $[W(\chi^b, (J^b, T^b))]$ of elements of $\hat{G}$ is a subnet of the net $[W^a]$.*

(ii) *Properties* (i), (ii), *and* (iv) *of the theorem hold for the pair and triples above.*

6.2. We may now state the main theorem of this paper.

6.2-A. THEOREM. *Suppose G is the semidirect product $N * K$ of an abelian group N and a compact group K. Then the topology of $\hat{G}$ may be described as follows:*

Let B be a subset of $\hat{G}$ and W an element of $\hat{G}$. W is contained in the closure of B if and only if there exist: a cataloguing triple $(\chi, (J, S))$ for W, an element (K', S') of $\mathscr{A}(K)$, and a net $[(\chi^a, (J^a, T^a))]$ of cataloguing triples, such that:

(i) *For each a, the element $W(\chi^a, (J^a, T^a))$ of $\hat{G}$ is an element of B.*

(ii) *The net $[(\chi^a, (J^a, T^a))]$ converges to $(\chi, (K', S'))$ in $\hat{N} \times \mathscr{A}(K)$.*

(iii) *J contains K', and ${}_JU^{S'}$ contains S.*

The proof follows from Theorems 2.5-B and 6.1.

6.2-B. This theorem is stronger than the conjecture in the introduction; added strength is only in the statement about "containment" rather than simply "weak containment." This strength might well be expected in the presence of the compactness hypothesis.

6.3. It might be well to mention the status of the hypotheses N is abelian and K is compact.

6.3-A. The commutativity of N is part of what is needed to prove Theorem 1.5-B. This theorem is needed in §4.2. Without 4.2, the rest of the theorem, i.e., Theorem 6.1, seems stymied.

The fact that N is abelian also crops up in the following way.

Suppose $[\chi^a]$ is a net of characters of N which converges to a character χ. Then we have concluded that if the net of neutral groups $[J_{(\chi^a)}]$ converged to a group K', then K' was contained in the neutral group for χ. This fact depends upon Proposition 2.2-B as applied to the group $\hat{N} * K$. Similar results can be obtained for arbitrary groups N such that $\hat{N}$ is Hausdorff, but this is not a major step forward. (See §§9 and 10.)

6.3-B. It would seem that by far the more important hypothesis is that of the compactness of K. This is essential in theorem 1.5-B, and therefore assists in §4.2. Also, throughout 4.2 the finite dimensionality of the minimal ideals of $L^2(K)$ is crucial.

There are, of course, occasions when compactness and commutativity have been used when, in their absence, something else would have sufficed, but, on the whole, these two hypotheses are extremely strong and cannot be dropped.

6.3-C. One might expect at least that Theorem 6.2-A could be generalized to an arbitrary compact extension of an abelian group. However, difficulties arise. These difficulties seem to hinge on the lack of continuity conditions on the "minimal" functions in $L^2(K)$ under the representations U^χ. In the general compact extension case, we must consider cocycle representations of compact groups, and, despite the fact that the regular a-representation of a compact group is completely reducible, and can be decomposed into irreducible finite-dimensional subspaces, there is no assurance that these finite-dimensional subspaces will consist of continuous functions.

The arbitrary compact extension of an abelian group can be reduced to the semidirect product case in certain instances.

If G is the compact extension of an abelian group N, where N is a vector group, i.e., some Euclidean space, then it has been shown that G is isomorphic and homeomorphic to $N * K$. See [16].

6.4-A. As stated in earlier sections, the difficulty which is overcome in Theorem 3.3 and 6.2-A is the discontinuity of the map $\chi \to G_\chi$, where G_χ denotes the stability subgroup for the element χ. In fact, Fell, in [4], has pointed out that by making use of a theorem of Glimm (Theorem 2.1 of [9]) one can prove a very general theorem which has application to the problem considered here. We state it in B below in the context of our terminology.

First, suppose G and N satisfy the second set of hypotheses of 1.3-B. Then recall that, if W is an element of $\hat{G}$, then there exists a cataloguing subgroup-representation pair (K, S) which satisfies:

(i) K is the stability subgroup G_χ for some element χ of $\hat{N}$.

(ii) S is an element of $\hat{K}$ such that $S|_N$ is equivalent to a multiple of χ.

(iii) The representation U^S is irreducible and is contained in the equivalence class W.

Of course, this is simply a restatement of Proposition 2 of 1.3-D.

6.4-B. THEOREM. *Let G be a separable locally compact group, N a closed normal subgroup of G which is of type* I *and which is regularly imbedded in G. Let M be a subset of $\hat{N}$ which satisfies the following conditions*:

(i) *M is a Borel subset of $\hat{N}$, and the topology of M relativized from the hull-kernel topology of $\hat{N}$ is locally compact and Hausdorff.*

(ii) *The subset M is stable under the action of G, i.e., M is a union of orbits.*

(iii) *The mapping $\chi \to G_\chi$ of M into $K(G)$ is continuous.*

Define $\hat{G}_M$ to be the subset of $\hat{G}$ consisting of those elements W such that the orbit θ_W of $\hat{N}$ associated with W is contained in M.

Then, the topology of $\hat{G}_M$ can be described as follows:

Let W be an element of $\hat{G}_M$ and B a subset of $\hat{G}_M$. Then W is contained in the closure of B if and only if there exists a net $[W^a]$, a net of cataloguing pairs $[(K^a, S^a)]$, and a cataloguing pair (K, S), such that:

(i) *$K = G_\chi$ for some element χ of M, and $S|_N$ is equivalent to a multiple of χ.*

(ii) *Each K^a is the stability subgroup $G_{(\chi^a)}$ for some element χ^a of M, and $S^a|_N$ is equivalent to a multiple of χ^a.*

(iii) *(K, S) is a cataloguing pair for W. For each a, (K^a, S^a) is a cataloguing pair for W^a.*

(iv) *For each a, W^a is an element of B.*

(v) *The net $[(K^a, S^a)]$ converges to (K, S) in $\mathscr{A}(G)$.*

This is the verification of the first conjecture of the introduction.

7. A theorem on the topology of $\mathscr{A}(K)$ for K a compact group.

7.1. We have seen in the last sections that, if G is the semidirect product $N * K$ of an abelian group N and a compact group K, then the topology of $\hat{G}$ can be described in terms of the topological space $\hat{N} \times \mathscr{A}(K)$. In order to show that this reduction is, in some cases, a simplification, we examine the topological space $\mathscr{A}(K)$.

7.1-A. Let K be a compact group and T and S be irreducible representations of K. Denote by dim (T) the dimension of the representation T. Let χ^T be the function on K which sends an element x to the trace of the operator T_x. Note that the function χ^T is constant on equivalence classes of irreducible representations, and therefore we may speak of χ^T, where T is an element of $\hat{K}$. The function χ^T is called the *character* of the element T of $\hat{K}$.

PROPOSITION 1. *The following formulae will be presumed.*

(i) *Let T be an element of $\hat{K}$. Then*

$$\int_K |\chi^T(x)|^2\,dx = 1.$$

(ii) *Let T and S be distinct elements of $\hat{K}$. Then*

$$\int_K \chi^T(x)\overline{\chi^S(x)}\,dx = 0.$$

For the proofs of these formulae, see §§39 and 40 of [12].

If T is an element of $\hat{K}$, let I_T denote the T-subspace of $L^2(K)$ under the left regular representation of K acting in $L^2(K)$. I_T is then a minimal finite dimensional two-sided ideal of $L^2(K)$. Denote by $[I_T]^-$ the ideal of $L^2(K)$ consisting of all functions $\bar{f}$ such that f is in I_T. Then $[I_T]^-$ is a minimal ideal of $L^2(K)$. Further, if g is the identity of I_T under convolution, then $\bar{g}$ is the identity of $[I_T]^-$ under convolution.

PROPOSITION 2. *Let T be an element of $\hat{K}$ and let G be the identity for I_T under convolution. Then,*

(i) *If f is a function of positive type on K associated with T, then f is in $[I_T]^-$.*

(ii) *The function $\chi^T = \bar{g}/[\dim(T)]$, and therefore, χ^T is a finite sum of functions of positive type of K associated with T.*

Again see §§39 and 40 of [12].

7.1-B. Recall that $\mathscr{A}(K)$ is the space of all subgroup-representation pairs (J, T), where J is a closed subgroup of K and T is an element of $\hat{J}$. The topology of $\mathscr{A}(K)$ is that of the dual space $[C_s^*(K)]^\wedge$ of the subgroup C^*-algebra of K (1.4-B).

LEMMA. *Let (J, T) be an element of $\mathscr{A}(K)$ and assume that the net $[(J^a, T^a)]$ of elements of $\mathscr{A}(K)$ converges to (J, T). Then there exists a subnet $[(J^{a^b}, T^{a^b})]$ of the net $[(J^a, T^a)]$ such that the net $[\chi^{(T^{a^b})}]$ of characters converges to χ^T in the Fell topology of functions on subgroups of K. (See §3 of [7].)*

Proof. By Proposition 2 of A above, χ^T is a finite sum of functions of positive type associated with T. Hence, using §3 of [7] finitely many times, we may assume that there exists a subnet $[(J^{a^b}, T^{a^b})]$ of the net $[(J^a, T^a)]$, and, for each b, a function ϕ^b, such that:

(i) For each b, ϕ^b is a finite sum of functions of positive type associated with T^{a^b}. Hence, ϕ^b lies in the ideal $[I_{(T^{a^b})}]^-$.

(ii) The net $[\phi^b]$ converges to χ^T in the Fell topology.

Define θ^b on J^{a^b} as follows: If x is in J^{a^b}, then

$$\theta^b(x) = \int_{(J^{a^b})} \phi^b(t^{-1}xt)\,dt.$$

Now, by §39 of [12], θ^b is central, contained in $[I_{(T^{a^b})}]^-$ and therefore is a multiple $\lambda^b\chi^{(T^{a^b})}$ of the character $\chi^{(T^{a^b})}$ of the element T^{a^b}. Also, note that 1^b is nonnegative for each b. This follows because $\theta^b(e)$ and $\chi^{(T^{a^b})}(e)$ are both nonnegative numbers.

First, let us show that the net $[\theta^b]$ converges to χ^T in the Fell topology.

Thus, assume, without loss of generality, that the net $[\theta^b]$ has been replaced by a subnet of itself without changing notation. Let $[x^b]$ be a net of elements of K such that:

(i) For each b, x^b belongs to J^{a^b}.

(ii) The net $[x^b]$ converges to an element x of J.

We wish to show that the net $[\theta^b(x^b)]$ converges to $\chi^T(x)$.

Define ψ^b as the function on J^{a^b} which sends an element t of J^{a^b} to $\phi^b(t^{-1}x^bt)$.

Now it is easy to see that the net $[\psi^b]$ converges to the function on J which is constant with constant value $\chi^T(x)$. Hence, by Proposition 4 of 1.4-D, we have:

The net $\left[\int_{(J^{a^b})} \psi^b(t)\,dt\right]$, which is the same as the net $\left[\int_{(J^{a^b})} \phi^b(t^{-1}x^bt)\,dt\right]$, which in turn is the same as the net $[\theta^b(x^b)]$, converges to the number $\int_J \chi^T(x)\,dt$, which equals $\chi^T(x)$. This establishes the fact that the net $[\theta^b]$ converges to χ^T in the Fell topology.

Now, the net $[|\theta^b|^2]$ converges to $|\chi^T|^2$ in the Fell topology, and hence, by Proposition 4 of 1.4-D again, the net $\left[\int_{(J^{a^b})} |\theta^b|^2\right]$ converges to $\int_J |\chi^T|^2$. This implies that the net $\left[|\lambda^b|^2 \int_{(J^{a^b})} |\chi^{(T^{a^b})}|^2\right]$, which is the same as the net $[|\lambda^b|^2]$, converges to the number $\int_J |\chi^T|^2$, which equals 1. Therefore, since each λ^b is nonnegative, we may conclude that the net $[\lambda^b]$ converges to 1. Hence, the net $[\chi^{(T^{a^b})}]$ must itself converge to χ^T in the Fell topology. This completes the proof of the lemma.

7.2. THEOREM. *Let K be a compact group. Then the topological space $\mathscr{A}(K)$ is locally compact and Hausdorff.*

Proof. The fact that $\mathscr{A}(K)$ is locally compact follows from the fact that $\mathscr{A}(G)$ is locally compact for an arbitrary locally compact group G. (See 1.4-B and Theorem 2.1 of [3].)

Suppose $\mathscr{A}(K)$ is not Hausdorff. Then let $[(J^a, T^a)]$ be a net of elements of $\mathscr{A}(K)$ which converges to two distinct points (J, T) and (J', S). By Lemma 1.5 of [7], we know that J' equals J.

Let χ and ϕ be the characters of T and S respectively. Then, by passing to a subnet, again without changing notation, we may assume that the net $[\chi^{(T^a)}]$ converges to χ in the Fell topology. Passing once more to a subnet, we may assume that the net $[\chi^{(T^a)}]$ converges to ϕ in the Fell topology.

Now since the Fell topology of functions defined on subgroups is Hausdorff (see §3 of [7]), we conclude that χ and ϕ are the same function. Therefore, T and S have the same character, i.e., T and S are identical. This contradicts the distinctness of the pairs (J, T) and (J, S), and the theorem is proved.

7.3. THEOREM. *Suppose (J, T) is an element of $\mathscr{A}(K)$ which is the limit of two nets $[(J^a, T^a)]$ and $[(J^a, S^a)]$ such that the corresponding elements T^a and S^a in these* *two nets are elements of the dual space* $(J^a)^\wedge$ *of the same subgroup* J^a *of* K. *Then eventually* T^a *must be equivalent to* S^a.

Proof. Assume false. Then, without loss, we may assume that, for each a, T^a is inequivalent to S^a.

Let χ denote the character of T. For each a, denote by χ^a the character of T^a and denote by ϕ^a the character of S^a. Then by passing perhaps twice to subnets, we may assume that:

(i) The net $[\chi^a]$ converges to χ in the Fell topology.

(ii) The net $[\phi^a]$ converges to χ in the Fell topology.

Then, it is easy to see that the net $[\chi^a(\phi^a)^-]$ converges to $\chi\bar{\chi}$ in the Fell topology. Hence, the net $[\int_{(J^a)} \chi^a(\phi^a)^-]$ converges to $\int_J |\chi|^2$. However, for each a, $\int_{(J^a)} \chi^a(\phi^a)^-$ is zero, while $\int_J |\chi|^2$ equals 1. This gives a contradiction and the theorem follows.

REMARK. This theorem shows that, once we know the path of approach to an element (J, T) of $\mathscr{A}(K)$ with respect to the subgroup space $\mathscr{K}(K)$, the path of approach via subgroup-representation pairs is unambiguously determined.

COROLLARY. *Let* K *be a compact abelian group. Suppose* χ *is a character of* K *and that* J *is a closed subgroup of* K. *Then a net* $[(J^a, \phi^a)]$ *of elements of* $\mathscr{A}(K)$ *converges to the element* $(J, \chi|_J)$ *if and only if eventually* ϕ^a *equals* $\chi|_{(J^a)}$.

Proof. This follows from the theorem and Theorem 3.2 of [7].

8. **The quotient space** $Q(G)$.

8.1. Theorem 6.2-A asserts that, if G is the semidirect product $N * K$ of an abelian group N and a compact group K, then the elements of $\hat{G}$ can be catalogued by means of cataloguing triples $(\chi, (J, T))$—elements of the space $\hat{N} \times \mathscr{A}(K)$—and that the cataloguing mapping possesses certain topological properties. Also, by 2.4-E, we know that two cataloguing triples catalogue the same element of $\hat{G}$ if and only if they are related in a certain way. We are led then to study the quotient space mentioned in the Introduction, and in preparation for that, we make the following observations and definitions.

8.1-A. If $(\chi, (J, T))$ is an element of $\hat{N} \times \mathscr{A}(K)$, and if p is an element of K, define T^p to be the element of the dual space $(pJp^{-1})^\wedge$ of the subgroup pJp^{-1} given as follows: Let T' be a representation of J which belongs to the equivalence class T. Define the representation T'^p on pJp^{-1} as the homomorphism which sends an element x of pJp^{-1} to the operator $T'_{(p^{-1}xp)}$. Then define T^p to be the equivalence class of representations to which T'^p belongs.

DEFINITION. If $(\chi, (J, T))$ is an element of $\hat{N} \times \mathscr{A}(K)$ and if p is an element of K, define $p(\chi, (J, T))$ to be the following element of $\hat{N} \times \mathscr{A}(K)$:

$$p(\chi, (J, T)) = (p\chi, ((pJp^{-1}), T^p)).$$

PROPOSITION. *The mapping of* $K \times [\hat{N} \times \mathscr{A}(K)]$ *into* $\hat{N} \times \mathscr{A}(K)$ *which sends the pair* $(p, (\chi, (J, T))$ *to* $p(\chi, (J, T))$ *is continuous, and, for a fixed element* p *of* K, *the mapping* $(\chi, (J, T)) \to p(\chi, (J, T))$ *is a homeomorphism of* $\hat{N} \times \mathscr{A}(K)$ *onto itself.*

This proof is a routine result of the definitions of the topologies in $\mathscr{K}(K)$ and $\mathscr{A}(K)$. It also depends on the continuity of functions of positive type associated with representations.

8.1-B. We define a relation on $\hat{N} \times \mathscr{A}(K)$ as follows:

If x and y are elements of $\hat{N} \times \mathscr{A}(K)$, we say that $x \equiv y$ if and only if $x = py$ for some element p of K.

This relation $\equiv$ is an equivalence relation on $\hat{N} \times \mathscr{A}(K)$. We denote by $R(G)$ the quotient space derived from $\hat{N} \times \mathscr{A}(K)$ by the equivalence relation $\equiv$ defined above, and we let π denote the quotient mapping of $\hat{N} \times \mathscr{A}(K)$ onto $R(G)$.

Let us denote by $Q'(G)$ the subset of $\hat{N} \times \mathscr{A}(K)$ consisting of cataloguing triples. Let $Q''(G)$ be the subset $\pi(Q'(G))$ of $R(G)$, and denote by $Q(G)$ the closure in $R(G)$ of $Q''(G)$.

8.1-C. Proposition 1. *The quotient mapping π is open on $\hat{N} \times \mathscr{A}(K)$.*

Proof. Let U be an open subset of $\hat{N} \times \mathscr{A}(K)$. We must show that $\pi^{-1}(\pi(U))$ is open. But $\pi^{-1}(\pi(U))$ is the set of all elements x of $\hat{N} \times \mathscr{A}(K)$ which are equivalent to elements of U, i.e., $\pi^{-1}(\pi(U))$ equals KU. But KU equals the union over all p in K of pU. By the proposition in A above, pU is open for all p; hence KU is open. Q.E.D.

Proposition 2. *If z is an element of $R(G)$ then $\pi^{-1}(z)$ is compact, hence closed; hence $R(G)$ is a T_1 topological space.*

Proof. $\pi^{-1}(z)$ is the set of all elements x of $\hat{N} \times \mathscr{A}(K)$ which are equivalent to some fixed element z' in $\pi^{-1}(z)$. Hence $\pi^{-1}(z)$ is equal to Kz'. Since K is compact, Kz' is compact by proposition A above. Q.E.D.

8.1-D. Theorem. *$R(G)$ is locally compact and Hausdorff.*

Remark. This theorem is a special case of the more general result: Let K be a compact group of homeomorphisms of a locally compact Hausdorff space X. Define the relation $\equiv$ on X by the following: $x \equiv y$ if and only if there exists an element p of K such that $x = p(y)$. Let R be the quotient space derived from X and $\equiv$. Then R is locally compact and Hausdorff. The proof here goes through in the general case as well.

Proof. By Theorem 7.2, we know that $\hat{N} \times \mathscr{A}(K)$ is locally compact and Hausdorff. We prove the theorem in two steps.

(i) Let z be an element of $R(G)$ and U a neighborhood of z. Then there exists a closed compact neighborhood C of z which is contained in U.

Proof of the first claim. Let z' be an element of $\pi^{-1}(z)$. Choose a compact neighborhood C' of z' in $\hat{N} \times \mathscr{A}(K)$. Define C'' as $\pi^{-1}(\pi(C'))$. Then C'' is the set KC' and hence is compact. Define C to be $\pi(C')$. Then C is compact since C' is compact. C is closed since C'' is closed. C is a neighborhood of z because C' is a neighborhood of z' and π is open. End of proof.

We have established that $R(G)$ is locally compact.

(ii) To prove that $R(G)$ is Hausdorff, let z and y be two distinct points of $R(G)$. Then by Proposition 2 of C above, y is closed, and hence z is contained in the open set $R(G)-y$. Now by (i) above choose a closed compact neighborhood C of z which is contained in $R(G)-y$. Then z is contained in the interior of C, while y is contained in the open set $R(G)-C$. This completes the proof of (ii) and the proof of the theorem.

COROLLARY. *$Q(G)$ is locally compact and Hausdorff, and $Q''(G)$ is Hausdorff.*

8.1-E. PROPOSITION. *If x is an element of $R(G)$ which is contained in the closure of a subset B of $R(G)$, then there exists an element y of $\pi^{-1}(x)$ such that y is contained in the closure of the set $[\pi^{-1}(B)]$.*

Proof. This follows immediately from the openness of the mapping π.

COROLLARY. *If x is in $Q(G)$, then each element y of $\pi^{-1}(x)$ is the limit of a net $[y^a]$ of cataloguing triples.*

8.1-F. If W is an element of $\hat{G}$, choose a cataloguing triple $(\chi, (J, T))$ for W. Define $\theta(W)$ to be the element $\pi(\chi, (J, T))$ of $Q(G)$.

Since any two cataloguing triples for W are equivalent (Proposition 2 of 1.3-D and 2.4-E) we see that θ is a well-defined mapping of $\hat{G}$ into $Q(G)$.

Although θ is not in general continuous, it is one-to-one and maps $\hat{G}$ onto a dense subspace of the locally compact Hausdorff space $Q(G)$. Therefore we have identified $\hat{G}$ with a dense subspace of a locally compact Hausdorff space. This identification is not an imbedding, i.e., the identification map is not a homeomorphism. This process has application to the "regularized dual space" introduced by Fell in [5]. It turns out that $Q(G)$ is not in general the regularized dual space, but the regularized dual space is always the continuous image of $Q(G)$.

8.2. Here is a ramification of Theorem 6.1.

THEOREM. *Let $[(\chi^a, (J^a, T^a))]$ be a net of cataloguing triples which converges to the triple $(\chi, (J', T))$. Suppose further, that the net* $[W(\chi^a, (J^a, T^a))]$ (2.4-D, E), *of elements of $\hat{G}$ converges to an element W. Then there exists a cataloguing triple $(\chi, (J, S))$ for W such that $S|_{J'}$ contains T.*

REMARK. This theorem resembles the corollary in 6.1. That corollary asserts that, if a net $[W^a]$ of elements of $\hat{G}$ converges to an element W, then a net $[(\chi^a, (J^a, T^a))]$ of cataloguing triples can be chosen in accordance with certain conditions. This theorem asserts that, if we begin with a convergent net $[(\chi^a, (J^a, T^a))]$ of cataloguing triples, and if, in addition, the net of elements $[W^a]$ of $\hat{G}$ determined by the triples $[(\chi^a, (J^a, T^a))]$ is also convergent, then the net of triples guaranteed by the corollary in 6.1 may be taken as the net with which we began.

Proof. Write the net $[W(\chi^a, (J^a, T^a))]$ as simply $[W^a]$. Then, by the corollary in 6.1, there exists a subnet $[W^{a^b}]$ of the net $[W^a]$, a cataloguing triple $(\phi, (L, V))$

for W, an element (L', V') of $\mathscr{A}(K)$, and a net $[(\phi^b, (L^b, V^b))]$ of cataloguing triples such that:

(i) L' is a subgroup of L, and the representation ${}_LU^{V'}$ contains V.

(ii) For each b, $(\phi^b, (L^b, V^b))$ is a cataloguing triple for W^{a^b}.

(iii) The net $(\phi^b, (L^b, V^b))$ converges to $(\phi, (L', V'))$ in $\hat{N} \times \mathscr{A}(K)$.

Now, for each b, we have two cataloguing triples for W^{a^b}; hence for each b there exists an element p^b of K such that $(\chi^b, (J^b, T^b)) = p^b(\phi^b, (L^b, V^b))$.

We also may assume, without loss, that the net $[p^b]$ converges to an element p in the compact group K. Therefore, by 8.1-A, $(\chi, (J', T)) = p(\phi, (L', V'))$. This implies that $p(\phi, (L, V)) = (\chi, (J, V^p))$. This latter, therefore, must be a cataloguing triple for W, which proves the first statement of the theorem.

Also, of course, V^p is contained in ${}_JU^{(V'^p)}$, which is equivalent to ${}_JU^T$.

Now the theorem follows if we define S to be V^p.

COROLLARY. *Suppose* $[W^a]$ *is a net of elements of* $\hat{G}$, *and suppose that a net of cataloguing triples* $[(\chi^a, (J^a, T^a))]$ *can be found such that, for each* a, W^a *is catalogued by the triple* $(\chi^a, (J^a, T^a))$, *and such that the net* $[(\chi^a, (J^a, T^a))]$ *converges to a triple* $(\chi, (J', T))$. *Then the net* $[W^a]$ *is convergent and each element* W, *in the set of limits to the net* $[W^a]$, *is catalogued by a triple of the form* $(\chi, (J, S))$ *where* $S|_{J'}$ *contains* T.

REMARK. This corollary shows us that the description of the topology of $\hat{G}$ in terms of the topology of the cataloguing triples has certain advantages. Namely, we may identify all the limits of a given net of elements of $\hat{G}$ merely by considering one limit of that net.

9. The Hausdorff property of the dual space.

9.1. Of course it is well known that the dual spaces of abelian and compact groups are Hausdorff. It has been a plausible conjecture that every group G whose dual space $\hat{G}$ is a Hausdorff space must in fact be the direct product of an abelian group and a compact group. This is *not* the case. (See Example 10.3.)

In this section we show the equivalence of the Hausdorffness of the dual space and the continuity of the *neutral subgroup mapping*, and using this theorem, we are able to show the validity of the above conjecture under certain circumstances.

9.1-A. LEMMA. *Let* J *be a compact group and* K' *a proper closed subgroup of* J. *Denote by* I *the trivial one-dimensional representation of* K' *and denote by* I' *the one-dimensional representation of* J. *Then the representation* U^I *contains at least two distinct elements of* $\hat{J}$.

Proof. Since K' is proper, U^I is at least two-dimensional. By the Frobenius reciprocity theorem, U^I contains I' exactly once. Therefore U^I must contain some other element of $\hat{J}$. Q.E.D.

9.1-B. We present next the main result of this section. Let G be a semidirect product $N * K$ of an abelian group N and a compact group K. Denote by M the *neutral subgroup mapping*, i.e., the mapping of $\hat{N}$ into $\mathscr{K}(K)$ which sends an element χ of $\hat{N}$ into the neutral group J_χ for χ. (2.2-A)

PROPOSITION. *The dual space $\hat{G}$ is Hausdorff if and only if the mapping M is continuous.*

Proof. Assume first that the mapping M is not continuous. Then there exists a net $[\chi^a]$ of characters of N which converges to a character χ, but for which the net $[J_{(\chi^a)}]$, written simply as $[J^a]$, of neutral subgroups does not converge to the neutral subgroup J_χ, written simply as J. We may, however, assume that the net $[J^a]$ does converge to some subgroup K' in $\mathscr{K}(K)$. By 2.2-B, K' is a subgroup of J, and by assumption, K' is a proper closed subgroup of J.

For each a, let I^a denote the trivial one-dimensional representation of J^a, and denote by I the trivial representation of K'. Then the net of cataloguing triples $[(\chi^a, (J^a, I^a))]$ converges to $(\chi, (K', I))$. Hence, by 6.2-A, the net $[W(\chi^a, (J^a, I^a))]$ of elements of $\hat{G}$ converges to every element $W(\chi, (J, S))$ such that S is contained in the representation ${}_JU^I$. By the lemma in A above, we know that there exist at least two distinct elements S of $\hat{J}$ such that ${}_JU^I$ contains S. Hence the net $[W(\chi^a, (J^a, I^a))]$ converges to at least two distinct points of $\hat{G}$, and thus $\hat{G}$ is not Hausdorff.

Now assume that $\hat{G}$ is not Hausdorff. Then there exists a net $[W^a]$ of elements of $\hat{G}$ which converges to two distinct points W and W' of $\hat{G}$. Let χ be an element of the orbit of $\hat{N}$ with which W is associated. Then, by Corollary 6.1, there exist: a cataloguing triple $(\chi, (J, T))$ for W, an element (K', S') of $\mathscr{A}(K)$, and a net $[(\chi^b, (J^b, T^b))]$ of cataloguing triples, such that:

(i) K' is a subgroup of J and the representation ${}_JU^{S'}$ contains T.

(ii) The net $[W(\chi^b, (J^b, T^b))]$ is a subnet of the net $[W^a]$.

(iii) The net $[(\chi^b, (J^b, T^b))]$ converges to $(\chi, (K', S'))$.

Now, the net $[W(\chi^b, (J^b, T^b))]$ converges to W', and hence, by Theorem 8.2, there exists a cataloguing triple $(\chi, (J, S))$ for W'. Since W and W' are distinct, T and S are inequivalent. Also, T and S are both contained in ${}_JU^{S'}$. Therefore K' must be a proper subgroup of J. Thus, the net $[J^b]$, which is the same as the net $[J_{(\chi^{a^b})}]$, does not converge to the neutral group J_χ. But the net $[\chi^b]$, which is the same as the net $[\chi^{a^b}]$, does converge to χ. Thus, the mapping M is not continuous at the point χ.

Now the proof of the proposition is complete.

9.2-A. DEFINITION. A locally compact abelian group G is called *topologically divisible* if, for each neighborhood W of the identity in G and for each element x of G, $x = y^n$, where y is an element of W and n is an integer.

PROPOSITION 1. *If G is topologically divisible, then G is compactly generated.*

PROPOSITION 2. *Let G be a topologically divisible group. If G contains a nonvoid connected open subset, then G is connected.*

PROPOSITION 3. *A connected abelian Lie group is topologically divisible.*

9.2-B. THEOREM. *Let G be the semidirect product $N * K$, where K is a compact group and where N is an abelian group whose dual group $\hat{N}$ is connected and topologically divisible. Then the dual space $\hat{G}$ is Hausdorff if and only if G equals the direct product $N \times K$ of N and K.*

Proof. Of course the "if" part is obvious. Assume then that $\hat{G}$ is Hausdorff. We will show first that, if χ is an element of $\hat{N}$, then the neutral group J_χ equals K.

Thus, let U be a neighborhood of the element K in $\mathscr{K}(K)$. Now by Proposition 9.1-B, the mapping M which sends each character χ of N to its neutral group J_χ is continuous. Hence choose a neighborhood W of the identity character of N such that for each element ϕ in W, the neutral group J_ϕ lies in U.

Now if χ is an element of $\hat{N}$, then by assumption there exists an element ϕ of W and an integer n such that $\chi = \phi^n$. Hence, by applying Proposition 2.2-A n times, we conclude that J_χ contains J_ϕ.

Finally, observe that, if J is an element of the open set U, and if J' is a subgroup of K which contains J, then J' is also contained in the open set U.

The last two paragraphs now give the fact that, for each χ in $\hat{N}$, the neutral group J_χ is contained in U. Since this is true for arbitrary neighborhoods U of the element K, we conclude that $J_\chi = K$ for all characters χ of N.

Now, let n be an element of N and let k be an element of K. We show that $k \cdot n$ is always equal to n.

For each character χ of N, we have $\chi(k \cdot n) = [(k^{-1} \cdot \chi)](n) = \chi(n)$.

Now since no character separates the points n and $k \cdot n$, it follows that $n = k \cdot n$.

Therefore each automorphism of N defined by an element of K is the identity automorphism. Thus the semidirect product structure is trivial, i.e., G is the direct product $N \times K$.

9.2-C. COROLLARY. *Suppose G is the semidirect product $N * K$, where K is compact and where N is a compactly generated abelian group such that the only compact subgroup of N is the subgroup containing only the identity. Then, the dual space $\hat{G}$ is Hausdorff if and only if G equals $N \times K$.*

Proof. This follows because $N = Z^j \times R^k$, where Z^j denotes the direct product of j copies of the group of additive integers, and R^k denotes the k-dimensional Euclidean space. (See [11], Theorem 24.30.)

10. **Examples.**

10.1. First let N be the additive group of real numbers and let K be the two element group $[e, k]$. Define the automorphism of N corresponding to the element

k of K as follows: If x is in N, then $k \cdot x = -x$. Then we may form the semidirect product $N * K$.

10.1-A. In order to apply the results of the previous chapters, we must determine what are the orbits of $\hat{N}$ and what are the neutral groups for the elements of $\hat{N}$. In this case, $\hat{N}$ is again the group of additive real numbers.

If χ is a nonzero element of $\hat{N}$, then the orbit containing χ is the two element set $[\chi, -\chi]$. The orbit containing the zero character is the set containing only the character 0.

The neutral group J_0 for 0 is of course all of K. The neutral group J_χ for a nonzero element χ of $\hat{N}$ is the trivial group $[e]$.

10.1-B. The cataloguing triples which will occur in this instance are as follows:

(i) $(0, (K, T))$, where T is one of the two elements of $\hat{K}$.

(ii) $(\chi, ([e], I))$, where χ is a nonzero character of N, and I is the one-dimensional trivial representation of the group $[e]$.

10.1-C. The dual space $\hat{G}$ may be described as follows:

Consider the nonnegative real line modified by splitting the point 0 into two copies $0'$ and $0''$. This modified nonnegative real line is in one-to-one correspondence with $\hat{G}$ as follows:

If s is a positive real number, then s corresponds to the element of $\hat{G}$ catalogued by the triple $(s, ([e], I))$. $0'$ and $0''$ correspond to the two elements of $\hat{G}$ catalogued by the triples $(0, (K, S_1))$ and $(0, (K, S_2))$.

A subset B of $\hat{G}$ is closed if and only if B satisfies the following two conditions.

(i) Having identified B as above, we require that the intersection of B with the positive real numbers be closed relative to the positive real numbers.

(ii) If B contains a sequence of positive real numbers which converges, in the usual sense, to 0, then B contains $0'$ and $0''$.

10.1-D. We observe that $\hat{G}$ is not a Hausdorff space. The first conjecture in the introduction asserts that $\hat{G}$ is always homeomorphic to the quotient space $Q''(G)$. (See 8.1-B.) But, by 8.1-D, $Q''(G)$ is Hausdorff, and hence, the above example shows that this conjecture is false.

10.2. Now let N be three-dimensional Euclidean space, and let K be the group of proper rotations of N. Then, since K can be realized as a group of matrices, there is an obvious mapping of K into the group of automorphisms of N. Thus we may form the semidirect product $G = N * K$.

10.2-A. $\hat{N}$ is of course Euclidean three-space. If χ is a character of N, we denote by $\boldsymbol{\chi}$ the vector in three-space determined by χ. The orbit of $\hat{N}$ containing the zero character is the set $[0]$. If χ is a nonzero character, then the orbit of χ is the spherical shell determined by the vector $\boldsymbol{\chi}$.

The neutral group for the zero character is again all of K. If χ is nonzero, then the neutral group J_χ is the subgroup of K consisting of all rotations which leave the vector $\boldsymbol{\chi}$ fixed. Hence, for each nonzero element χ of $\hat{N}$, J_χ is homeomorphic and isomorphic to the circle group C.

10.2-B. The cataloguing triples which occur in this case are as follows:

(i) $(0, (K, S))$, where 0 is the zero character of N and S is an element of $\hat{K}$.

(ii) $(\chi, (J_\chi, \phi))$, where χ is a nonzero character of N, and where ϕ is a character of the abelian group J_χ.

10.2-C. $\hat{G}$ may be described as follows: Let P be the set of positive real numbers and let Z be the set of integers. Then $\hat{G}$ may be identified with the set $[P\times Z] \cup \hat{K}$. In fact, if (s, n) is an element of $P\times Z$, then (s, n) corresponds to the element of $\hat{G}$ catalogued by the triple $(\chi, (J_\chi, \phi))$, where χ lies along the z-axis and has coordinates $(0, 0, s)$, and where ϕ is the character of the group J_χ, i.e., the circle group, given by the integer n. Also, if S is an element of $\hat{K}$, then S corresponds to the element of $\hat{G}$ catalogued by the triple $(0, (K, S))$.

We make the identification of $\hat{G}$ with $[P\times Z] \cup \hat{K}$. Then, a subset B of $\hat{G}$ is closed if and only if B satisfies the following two conditions.

(i) The intersection of B with $P\times Z$ is a closed subset of $P\times Z$.

(ii) If B contains a sequence (s_j, n_j) of elements of $P\times Z$ such that the sequence $[s_j]$ converges, in the usual sense, to 0, and such that the sequence $[n_j]$ is eventually constant with constant value n, then B must contain all points S of $\hat{K}$ such that $S|_C$ contains n. (Here C represents the subgroup of K consisting of all rotations about the z-axis.)

10.3. We come now to the conjecture mentioned in 9.1. The following example shows that this conjecture is false, i.e., there *does* exist a group G whose dual space *is* Hausdorff and such that G is not the direct product $N\times K$ of any abelian group N and any compact group K.

10.3-A. Let N equal $Z\times C$, where Z is the group of integers and C is the multiplicative group of complex numbers of absolute value one. Let K be the two element group $[e, k]$. Define the automorphism of N corresponding to k as follows: If (n, u) is an element of N, then

$$k\cdot(n, u) = (n, (-1^n)u).$$

We may construct therefore the semidirect product $G = N * K$. The elements of G can be thought of as triples (a, n, u), where a is in K, n is in Z, and u is in C.

10.3-B. $\hat{N}$ equals $C\times Z$. It is easy to see that if (v, m) is an element of $\hat{N}$, then the neutral group $J_{(v,m)}$ is all of K if m is even, and the trivial subgroup $[e]$ if m is odd.

We observe then that the mapping $\chi\to J_\chi$ of $\hat{N}$ into $\mathscr{K}(K)$ is continuous. Therefore, by Proposition 9.1-B, $\hat{G}$ is Hausdorff.

10.3-C. Now assume that G is the direct product $N'\times K'$ of an abelian group N' and a compact group K'.

Then, since N' is abelian and is a direct factor, N' must be contained in the center of G. It is easy to see that an element (a, n, u) of G is in the center of G, if and only if n is even.

The mapping of G which sends an element (a, n, u) to n, i.e., the projection of G onto the second coordinate, can be easily seen to be a continuous homomorphism of G onto the group of integers. If θ is this homomorphism, then $\theta(K')$ must be a compact subgroup of Z, i.e., $\theta(K')$ equals [0]. Therefore, if (a, n, u) is an element of K', then n equals 0.

Now every element of G is the product of an element (a, n, u) of N' and an element (b, m, v) of K'. This product is $(ab, (n+m), uv)$, since m and n are both even. Therefore, every element of G has as its second coordinate the sum of two even integers. This is, of course, a contradiction.

Thus G cannot be written as the direct product of an abelian group and a compact group.

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University of Colorado,
Boulder, Colorado